DOI: 10.2478/auom-2013-0040
An. Şt. Univ. Ovidius Constanţa **Vol. 21(3),2013, 33–41**

New Existence and Uniqueness Results for Fractional Differential Equations

Ahmed ANBER, Soumia BELARBI and Zoubir DAHMANI

Abstract

In this paper, we study a class of boundary value problems of nonlinear fractional differential equations with integral boundary conditions. Some new existence and uniqueness results are obtained by using Banach fixed point theorem. Other existence results are also presented by using Krasnoselskii theorem.

1 Introduction

Fractional differential equations have emerged as a new field of applied mathematics by which many physical phenomena can be modeled, (see [2, 6, 9, 10, 11]). This theory has attracted many scientists and mathematicians to work on [3, 4, 13, 14, 16]. The existence and uniqueness problems of fractional nonlinear differential equations are investigated by many authors. For more details, we refer the reader to [5, 7, 15, 17] and the reference therein. Recently, in [1, 5, 19], the authors studied the existence of solutions for some fractional boundary value problems by using Caputo fractional derivative. Motivated by the work of [20], in this paper we are concerned with the following boundary value problem:

$$\begin{gathered} D^{\alpha}x\left(t\right)+f\left(t,x\left(t\right)\right)=\theta;\ \ 0\leq t\leq 1, 1<\alpha\leq 2, \\ x(0)=\int_0^1 g(\tau)x(\tau)d\tau,\ \ x(1)=\theta, \end{gathered} \tag{1}$$

Key Words: Boundary value problem, Caputo derivative, Fixed point
2010 Mathematics Subject Classification: Primary 26A33, 26A42, 34A12.
Received: April, 2012.
Revised: May, 2012.
Accepted: February, 2013.

where D^{α} denotes the fractional derivative of order α in the sense of Caputo, and $f : [0,1] \times E \to E$ is continuous, such that $(E, ||.||)$ is a Banach space and $C([0,1], E)$ is the Banach space of all continuous functions defined on $[0,1] \to E$ endowed with a topology of uniform convergence with the norm denoted by $||.||$.
We prove new existence and uniqueness results for the problem (1) by using the Banach contraction principle. We also establish other existence results for the problem (1) using Krasnoselskii fixed point theorem [12].

2 Preliminaries

In the following, we give the necessary notation and basic definitions and lemmas which will be used in this paper.

Definition 2.1: *A real valued function $f(t), t > 0$ is said to be in the space $C_{\mu}, \mu \in \mathbb{R}$, if there exists a real number $p > \mu$ such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C([0,\infty))$.*

Definition 2.2: *A function $f(t), t > 0$ is said to be in the space $C_{\mu}^n, n \in \mathbb{N}$, if $f^{(n)} \in C_{\mu}$.*

Definition 2.3: *The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, for a function $f \in C_{\mu}, \mu \geq -1$, is defined as*

$$\begin{aligned} &J^{\alpha} f(t) = \tfrac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau; \quad \alpha > 0, t > 0 \\ &J^0 f(t) = f(t). \end{aligned} \tag{2}$$

The fractional derivative of $f \in C_{-1}^n$ in the Caputo's sense is defined as

$$D^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, & n-1 < \alpha < n, n \in \mathbb{N}^*, \\ \frac{d^n}{dt^n} f(t), & \alpha = n. \end{cases} \tag{3}$$

Details on Caputo's derivative can be found in [8, 18].

Lemma 2.4: ([11]) *For $\alpha > 0$, the general solution of the fractional differential equation $D^{\alpha} x = 0$ is given by*

$$x(t) = c_0 + c_1 t + c_2 t^2 + \ldots c_{n-1} t^{n-1}, \tag{4}$$

where $c_i \in \mathbb{R}, i = 0, 1, 2, \ldots n-1, n = [\alpha] + 1$.

Lemma 2.5: ([11]) *Let* $\alpha > 0$, *then*

$$J^{\alpha} D^{\alpha} x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + ... c_{n-1} t^{n-1}, \tag{5}$$

for some $c_i \in \mathbb{R}, i = 0, 1, 2, ...n-1, n = [\alpha] + 1$.

We also need the following auxiliary lemma:

Lemma 2.6: *A solution of the the fractional boundary value problem* (1) *is given by*:

$$\begin{aligned} x(t) = (1-t)\int_0^1 g(\tau)x(\tau)d\tau + \theta\Big(t - \frac{t}{\Gamma(\alpha+1)} + \frac{t^{\alpha}}{\Gamma(\alpha+1)}\Big) \\ + \tfrac{t}{\Gamma(\alpha)} \textstyle\int_0^1 (1-\tau)^{\alpha-1} f(\tau, x(\tau))d\tau - J^{\alpha} f(t, x(t)). \end{aligned} \tag{6}$$

Proof: We have

$$D^{\alpha} x\,(t) = \theta - f\,(t, x\,(t))\,, 0 \le t \le 1. \tag{7}$$

Applying the operator J^{α} for both sides of (7), and using the identity $J^{\alpha} D^{\alpha} x(t) = x(t) + c_0 + c_1 t$, we get

$$x(t) = \frac{\theta t^{\alpha}}{\Gamma(\alpha+1)} - J^{\alpha} f(t, x(t)) - c_0 - c_1 t. \tag{8}$$

In particular, for $t = 0$, we have

$$c_0 = -\textstyle\int_0^1 g(\tau)x(\tau)d\tau, \tag{9}$$

and for $t = 1$, we obtain

$$c_1 = -\theta + \frac{\theta}{\Gamma(\alpha+1)} + \int_0^1 g(\tau)x(\tau)d\tau - \frac{1}{\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1} f(\tau, x(\tau))d\tau. \tag{10}$$

Substituting the values of c_0 and c_1 in (8), we obtain (6). Lemma 2.6 is thus proved.

Now, we define the operator $T : C([0,1], E) \to C([0,1], E)$ as follows:

$$\begin{aligned} T(x) := (1-t)\int_0^1 g(\tau)x(\tau)d\tau + \theta(t + \frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{t}{\Gamma(\alpha+1)}) \\ + \tfrac{t}{\Gamma(\alpha)} \textstyle\int_0^1 (1-\tau)^{\alpha-1} f(\tau, x(\tau))d\tau - \tfrac{1}{\Gamma(\alpha)} \textstyle\int_0^t (1-\tau)^{\alpha-1} f(\tau, x(\tau))d\tau, \end{aligned} \tag{11}$$

where $0 \le t \le 1, 1 < \alpha \le 2$.

3 Main results

We prove the existence and uniqueness of a unique solution for (1), by using the Banach fixed point theorem. The following conditions are essential to prove the result:

(H_1) :

$$||f(t,x)-f(t,y)|| \leq k||x-y||; k>0, x,y \in E, t \in [0,1],$$

(H_2) : Let d and r be two positive real numbers such that $0<d<1$ and

$$M+\frac{2k}{\Gamma(\alpha+1)} \leq d, \theta(1+\frac{2}{\Gamma(\alpha+1)})+\frac{2N}{\Gamma(\alpha+1)} \leq (1-d)r,$$

where $N := sup_{t\in[0,1]}|f(t,0)|$ and $M := sup_{t\in[0,1]}|g(t)|$.

Theorem 3.1: *Suppose that the conditions (H_1) and (H_2) are satisfied. Then the boundary value problem (1) has a unique solution in $C([0,1],E)$.*

Proof: To prove this theorem, we need to prove that the operator T has a fixed point on $B_r := \{x \in E, ||x|| \leq r\}$.
(1∗) Let $x \in B_r$. We have

$$\begin{aligned} ||T(x)|| = ||(1-t)\int_0^1 g(\tau)x(\tau)d\tau + \theta(t+\frac{t^\alpha}{\Gamma(\alpha+1)}-\frac{t}{\Gamma(\alpha+1)}) \\ +\tfrac{t}{\Gamma(\alpha)}\int_0^1(1-\tau)^{\alpha-1}f(\tau,x(\tau))d\tau - \tfrac{1}{\Gamma(\alpha)}\int_0^t(1-\tau)^{\alpha-1}f(\tau,x(\tau))d\tau||. \end{aligned} \tag{12}$$

Consequently,

$$\begin{aligned} ||T(x)|| \leq M||x|| + \theta\Big(1+\frac{2}{\Gamma(\alpha+1)}\Big) \\ +\tfrac{1}{\Gamma(\alpha)}\int_0^1(1-\tau)^{\alpha-1}||f(\tau,x(\tau))-f(\tau,0)||d\tau + \tfrac{1}{\Gamma(\alpha)}\int_0^1(1-\tau)^{\alpha-1}||f(\tau,0)||d\tau \\ +\tfrac{1}{\Gamma(\alpha)}\int_0^t(1-\tau)^{\alpha-1}||f(\tau,x(\tau))-f(\tau,0)||d\tau \\ +\tfrac{1}{\Gamma(\alpha)}\int_0^t(1-\tau)^{\alpha-1}||f(\tau,0)||d\tau. \end{aligned} \tag{13}$$

Using the condition (H_1), with $y = 0$, we can write:

$$||T(x)|| \leq M||x|| + \theta(1 + \frac{2}{\Gamma(\alpha+1)}) + \frac{2k}{\Gamma(\alpha+1)}||x|| + \frac{2N}{\Gamma(\alpha+1)}. \quad (14)$$

Therefore,

$$||T(x)|| \leq (M + \frac{2k}{\Gamma(\alpha+1)})r + \theta(1 + \frac{2}{\Gamma(\alpha+1)}) + \frac{2N}{\Gamma(\alpha+1)}. \quad (15)$$

Using the two conditions of (H_2), we obtain

$$||T(x)|| \leq dr + (1-d)r, \quad (16)$$

which implies that $TB_r \subset B_r$.
Hence T maps B_r into itself.
(2∗) Now, we shall prove that T is a contraction mapping on B_r. Let $x, y \in B_r$, then we can write

$$\begin{aligned} ||T(x) - T(y)|| &\leq (1-t)\int_0^1 ||g(\tau)(x(\tau) - y(\tau))||d\tau \\ &+ \tfrac{t}{\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1}||(f(\tau, x(\tau)) - f(\tau, y(\tau)))||d\tau \\ &+ \tfrac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1}||(f(\tau, x(\tau)) - f(\tau, y(\tau)))||d\tau. \end{aligned} \quad (17)$$

Using the fact that $0 \leq t \leq 1, |g(\tau)| \leq M, \tau \in [0, t]$, and by (H_1), we get

$$||T(x) - T(y)|| \leq M||(x-y)|| + \frac{2k}{\Gamma(\alpha+1)}||(x-y)||. \quad (18)$$

Now, using the first condition of (H_2), we obtain

$$||T(x) - T(y)|| \leq d||(x-y)||. \quad (19)$$

Hence, the operator T is a contraction. Therefore T has a unique fixed point which is a solution of the problem (1).

The following result is based on Krasnoselskii fixed point theorem [12].
To apply this theorem, we need the following hypotheses:

(H_3) :

$$||f(t, x)|| \leq \nu(t); (t, x) \in [0, 1] \times E, \nu \in L^1([0, 1], \mathbb{R}^+).$$

(H_4) : Let $f:[0,1]\times E\to E$ be a jointly continuous function mapping bounded subsets of $[0,1]\times E$ into relatively compact subsets of E.

Theorem 3.2: *Suppose that the hypotheses* (H_3) *and* (H_4) *are satisfied. If* $M<1$, *then the boundary value problem* (1) *has at least a solution in* $C([0,1],E)$.

Proof: Let us fixe

$$\rho\geq(1-M)^{-1}\Big(\theta(1+\frac{2}{\Gamma(\alpha+1)})+\frac{2||\nu||}{\Gamma(\alpha+1)}\Big), \tag{20}$$

where $||\nu||:=sup_{t\in[0,1]}|\nu(t)|$.

On $B_\rho:=\{x\in E,||x||\leq\rho\}$, we define the operators R and S as

$$\begin{aligned} R(x) &:= (1-t)\int_0^1 g(\tau)x(\tau)d\tau+\theta(t+\frac{t^\alpha}{\Gamma(\alpha+1)}-\frac{t}{\Gamma(\alpha+1)}) \\ S(x) &:= \tfrac{t}{\Gamma(\alpha)}\textstyle\int_0^1(1-\tau)^{\alpha-1}f(\tau,x(\tau))d\tau-J^\alpha f(t,x(t)). \end{aligned} \tag{21}$$

For $x,y\in B_\rho$, we have

$$\begin{aligned} ||R(x)+S(y)|| &\leq ||(1-t)\int_0^1 g(\tau)x(\tau)d\tau+\theta(t+\frac{t^\alpha}{\Gamma(\alpha+1)}-\frac{t}{\Gamma(\alpha+1)})|| \\ &+||\tfrac{t}{\Gamma(\alpha)}\textstyle\int_0^1(1-\tau)^{\alpha-1}f(\tau,y(\tau))d\tau-\tfrac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}f(\tau,y(\tau))d\tau||. \end{aligned} \tag{22}$$

Therefore,

$$\begin{aligned} ||R(x)+S(y)|| &\leq M||x||+\theta(1+\frac{2}{\Gamma(\alpha+1)}) \\ &+||\tfrac{t}{\Gamma(\alpha)}\textstyle\int_0^1(1-\tau)^{\alpha-1}f(\tau,y(\tau))d\tau|| \\ &+||\tfrac{1}{\Gamma(\alpha)}\textstyle\int_0^t(t-\tau)^{\alpha-1}f(\tau,y(\tau))d\tau||. \end{aligned} \tag{23}$$

Using (H_3) and (20), we obtain

$$||R(x)+S(x)||\leq M||x||+\theta(1+\frac{2}{\Gamma(\alpha+1)})+\frac{2||\nu||}{\Gamma(\alpha+1)}\leq M\rho+(1-M)\rho. \tag{24}$$

Hence $R(x) + S(y) \in B_\rho$.
On the other hand, it is easy to see that

$$||R(x) - R(y)|| \le M||x - y||, \tag{25}$$

and since $M < 1$, then R is a contraction mapping.
Moreover, it follows from (H_4) that the operator S is continuous and

$$\begin{array}{c} ||S(x)|| \le \frac{t}{\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} ||f(\tau, x(\tau))|| d\tau \\ + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} ||f(\tau, x(\tau))|| d\tau. \end{array} \tag{26}$$

Since $t \in [0, 1]$, then we can write

$$||S(x)|| \le \frac{2||\nu||}{\Gamma(\alpha+1)}.$$

Hence, S is uniformly bounded on B_ρ.
Let us now take $t_1, t_2 \in [0, 1]$ and $y \in B_\rho$. Then we can write

$$\begin{array}{c} ||Sy(t_1) - Sy(t_2)|| \le ||\frac{t_1 - t_2}{\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} f(\tau, x(\tau)) d\tau|| \\ + || \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - \tau)^{\alpha-1} f(\tau, x(\tau)) d\tau - \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2 - \tau)^{\alpha-1} f(\tau, x(\tau)) d\tau||. \end{array} \tag{27}$$

Thanks to (H_3), we get

$$||Sy(t_1) - Sy(t_2)|| \le \frac{||\nu||}{\Gamma(\alpha+1)} \Big(|t_1 - t_2| + |t_1^\alpha - t_2^\alpha| \Big). \tag{28}$$

The right hand side of (28) is independent of y. Hence S is equicontinuous and as $t_1 \to t_2$, the left hand side of (28) tends to 0; so $S(B_\rho)$ is relatively compact and then by Ascolli-Arzella theorem, the operator S is compact.
Finally, by Krasnoselskii theorem, we conclude that there exists a solution to (1). Theorem 3.2 is thus proved.

References

[1] B. Ahmed, S.K. Ntouyas, A. Alsaedi, *New existence results for nonlinear fractional differential equations with three-point integral boundary conditions.* Advances in Difference Equations, Vol. 2011, (2011), Article ID 107384, 11 pages.

[2] K. Balachandran, J.Y. Park, *Nonlocal Cauchy problem for abstract fractional semilinear evolution equations.* Nonlinear Analysis, 71, (2009), 4471-4475.

[3] A. Belarbi, M. Benchohra, A. Ouahab, *Uniqueness results for fractional functional differential equations with infinite delay in Frechet spaces.* Appl. Anal., 85, (2006), 1459-1470.

[4] M. Benchohra, J. Henderson, S.K. Ntouyas, A. Ouahab, *Existence results for fractional order functional differential equations with infinite delay.* J. Math. Anal. Appl., 338(2), (2008), 1340-1350.

[5] M. Benchohra, S. Hamania, S.K. Ntouyas, *Boundary value problems for differential equations with fractional order and nonlocal conditions.* Nonlinear Analysis, 71, (2009), 2391-2396.

[6] Z. Dahmani, M.M. Mesmoudi, R. Bebbouchi, *The foam drainage equation with time and space fractional derivatives solved by the ADM method.* E. J. Qualitative Theory of Diff. Equ., 30, (2008), 1-10.

[7] D. Delbosco, L. Rodino, *Existence and uniqueness for a fractional differential equation.* Journal of Mathematical Analysis and Applications, 204, (1996), 609-625.

[8] R. Gorenflo, F. Mainardi, *Fractional calculus: integral and differential equations of fractional order.* Springer Verlag, Wien, (1997), 223-276.

[9] R. Hilfer, *Applications of fractional calculus in physics.* World Scientific Publishing Co., Inc., River Edge, NJ, (2000).

[10] O.K. Jaradat, A. Al-Omari, S. Momani, *Existence of the mild solution for fractional semilinear initial value problems.* Nonlinear Analysis, 69, (2008), 3153-3159.

[11] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and applications of fractional differential equations.* Elsevier, Amsterdam, (2006).

[12] M.A. Krasnoselskii, *Two remarks on the method of successive approximations.* Uspekhi Matematicheskikh Nauk, 10, (1955), 123-127.

[13] V. Lakshmikantham and A.S. Vatsala, *Theory of fractional differential inequalities and applications.* Commun. Appl. Anal., 11(3-4), (2007), 395-402.

[14] V. Lakshmikantham and A.S. Vatsala, *General uniqueness and monotone iterative technique for fractional differential equations.* Appl. Math. Lett., 21(8), (2008), 828-834.

[15] X. Liu, M. Jia, B. Wu, *Existence and uniqueness of solution for fractional differential equations with integral boundary conditions.* E. J. Qualitative Theory of Diff. Equ., 69, (2009), 1-10.

[16] F. Mainardi, *Fractional calculus: Some basic problems in continuum and statistical mechanics.* Fractals and fractional calculus in continuum mechanics (Udine, 1996), 291-348, CISM Courses and Lectures, 378, Springer, Vienna, (1997).

[17] M.M. Matar, *Boundary Value Problem for Some Fractional Integrodifferential Equations with Nonlocal Conditions.* IJNS, International Journal of Nonlinear Science, 11(1), (2011), 3-9.

[18] I. Podlubny, *Fractional differential equations. An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications.* Mathematics in Science and Engineering, 198. Academic Press, Inc., San Diego, CA, (1999).

[19] Z. Shuqin, *Existence of solution for a boundary value problem of fractional order.* Acta Mathematica Scientia, 26B(2), (2006), 220-228.

[20] P. Zhang, *Existence of positive solutions for nonlocal second-order boundary value problem with variable parameter in Banach spaces.* Fixed Point Theory and Applications, 1(43), (2011), 1-6.

Ahmed ANBER,
Department of Mathematics,
USTO University of Oran,
Oran, Algeria.
Email: ah.anber@gmail.com

Soumia BELARBI,
Faculty of Mathematics,
USTHB of Algiers, Algeria
Email: soumia-math@hotmail.com

Zoubir DAHMANI,
LPAM, Laboratory of Mathematics,
Faculty SEI, UMAB University,
Mostaganem, Algeria.
Email: zzdahmani@yahoo.fr