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Nonexistence of positive solutions to nonlinear nonlocal elliptic systems

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ABSTRACT

In this paper we consider the question of nonexistence of nontrivial solutions for nonlinear elliptic systems involving fractional diffusion operators. Using a weak formulation approach and relying on a suitable choice of test functions, we derive sufficient conditions in terms of space dimension and systems parameters. Also, we present three main results associated to three different classes of systems.

1. Introduction

In this paper we study the question of nonexistence of nontrivial solutions to three different classes of elliptic systems, that are related by the specificity of the fractional powers of the diffusion operators involved in each system.

First, we consider the semi-linear system

$$\begin{cases} (-\Delta)^{\mu/2}u = |v|^q, \\ (-\Delta)^{\nu/2}v = |u|^p, \end{cases} \tag{1}$$

where p, q are positive numbers, $p > 1$, $q > 1$, and $0 < \mu,\ \nu \leqslant 2$. The fractional power of the Laplacian $(-\Delta)^{\alpha/2}$ $(0 < \alpha \leqslant 2)$ stands for diffusion in media with impurities and is defined as $(-\Delta)^{\alpha} v(x) = \mathcal{F}^{-1}(|\xi|^{\alpha}\mathcal{F}(v)(\xi))(x)$, where $\mathcal{F}$ denotes the Fourier transform and $\mathcal{F}^{-1}$ denotes its inverse.

The definition of solutions we adopt for system (1) is:

Definition 1. We say that the pair (u, v) is a weak solution of (1), if

$$(u, v) \in L^p_{loc}(\mathbb{R}^N) \times L^q_{loc}(\mathbb{R}^N),$$

and

$$\int_{\mathbb{R}^N} u(-\Delta)^{\mu/2}\psi\, dx = \int_{\mathbb{R}^N} |v|^q\psi\, dx, \qquad \int_{\mathbb{R}^N} v(-\Delta)^{\nu/2}\psi\, dx = \int_{\mathbb{R}^N} |u|^p\psi\, dx,$$

for any nonnegative test functions $\psi \in C_0^{\infty}(\mathbb{R}^N)$.

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The elliptic system (1) is formally equivalent to the system of integral equations in $\mathbb{R}^N$

$$\begin{cases} u(x) = C_\mu \displaystyle\int_{\mathbb{R}^N} |x-y|^{\mu-N} v(y)^q \, dy, \\ v(x) = C_\nu \displaystyle\int_{\mathbb{R}^N} |x-y|^{\nu-N} u(y)^p \, dy, \end{cases} \tag{2}$$

where C_μ and C_ν are normalizing constants.

In the special case $\mu = \nu = 2$, the problem is called the Lane–Emden system and it is well known that two critical values appear $\frac{N+2}{N-2}$ and $\frac{N}{N-2}$ which is called Serrin's number. In the case of a single equation

$$\Delta u + |u|^p = 0, \quad u \geqslant 0 \text{ in } \mathbb{R}^N \tag{3}$$

it has been proved in [1,6] that the only classical solution of (3) is $u = 0$ when $0 < p < \frac{N+2}{N-2}$. An extensive study was devoted not only to single equation but to the case of system of equations as well. In fact, D.G. De Figueiredo and P.L. Felmer [5], M.A. Souto [12] and J. Serrin and H. Zou [10] showed that, for $0 < p, q \leqslant \frac{N+2}{N-2}$ and $(p,q) \neq (\frac{N+2}{N-2}, \frac{N+2}{N-2})$, the system has no positive classical solution.

In the case $N \geqslant 2$, $0 < \mu = \nu < N$, $\frac{N}{N-\mu} < p, q \leqslant \frac{N+\mu}{N-\mu}$ and $(p,q) \neq (\frac{N+\mu}{N-\mu}, \frac{N+\mu}{N-\mu})$, D. Chen and L. Ma [4] gave a partial generalized result of the work of D.G. De Figueiredo and P.L. Felmer [5] about Liouville type theorem for nonnegative solutions. Their proof uses a new type of moving plane method introduced by Chen, Li and Ou [3].

For system (1), we give conditions relating the space dimension N with the system parameters μ, ν, p and q for which every weak nonnegative solution is trivial. Inspired by the work of Mitidieri and Pohozaev (see [8,9]), we also present some results using a different approach from those previously adopted.

Second, we consider the following system governed by classical and fractional Laplacian operators

$$\begin{cases} (-\Delta)^{\mu/2} u - \Delta v = |v|^q, \\ (-\Delta)^{\nu/2} v - \Delta u = |u|^p, \end{cases} \tag{4}$$

where the positivity condition of the solutions is maintained as in the previous case. Note that the positivity condition is a must in order to use Ju's inequality [11].

Motivated by [7], a particular attention will be given to this system. We study the question of nonexistence of weak solutions and give conditions for which only trivial solutions exist.

Finally, fractional Laplacian endowed with different indices will be considered for the system

$$\begin{cases} (-\Delta)^{\mu_1/2} |u| + (-\Delta)^{\mu_2/2} |v| = |v|^q, \\ (-\Delta)^{\nu_1/2} |v| + (-\Delta)^{\nu_2/2} |u| = |u|^p, \end{cases} \tag{5}$$

where for $i = 1, 2$, $0 < \mu_i, \nu_i \leqslant 2$ are constants. Note that for this case, the positivity condition on the solutions is omitted by considering the absolute value of u and v.

In this part we use the same strategy as for (4) to determine a bound on N for which only trivial solutions exist. Indeed, we prove that if the space dimension $N < \max\{\gamma, \theta\}$, where $\gamma = \min\{\frac{\nu_2 p}{p-1}, \nu_1 + \frac{\nu_2}{q-1}, (\frac{\mu_1}{q} + \nu_1)\frac{pq}{pq-1}\}$, $\theta = \min\{\frac{\mu_2 q}{q-1}, \mu_1 + \frac{\nu_2}{p-1}, (\frac{\nu_1}{p} + \mu_1)\frac{pq}{pq-1}\}$, then every pair of weak solution (u, v) is trivial.

In our present work, we overcome the difficulties of using fractional powers of the Laplacian by using weak formulation technique.

2. Main results

The first main result for system (1) is

Theorem 2. *Let (u, v) be a weak solution of system (1). If*

$$N < \max\left\{\left(\nu + \frac{\mu}{q}\right)\frac{pq}{pq-1}, \left(\mu + \frac{\nu}{p}\right)\frac{pq}{pq-1}\right\}, \tag{6}$$

then (u, v) is trivial.

Remark 3.

(i) In the case $\mu = \nu$ condition (6) can be rewritten as $p < \frac{\frac{N}{q}+\mu}{N-\mu}$ or $q < \frac{\frac{N}{p}+\mu}{N-\mu}$. If $1 < p, q < \frac{N}{N-\mu}$, then condition (6) is satisfied. Chen and Ma [4] proved a similar result for $\frac{N}{N-\mu} < p, q < \frac{N+\mu}{N-\mu}$. However, their proof does not cover the case $1 < p, q < \frac{N}{N-\mu}$ covered by Theorem 2.

(ii) In the case $\mu = \nu$ and $p = q$, the condition (6) becomes $p < \frac{N}{N-\mu}$.

Our second main result deals with a general class of systems

Theorem 4. *Let* (u,v) *be a weak nonnegative solution of system* (4)*. If*

$$N < \max\{\gamma,\theta\} \tag{7}$$

where

$$\gamma = \min\left\{\frac{2p}{p-1}, \nu + \frac{2}{q-1}, \left(\frac{\mu}{q}+\nu\right)\frac{pq}{pq-1}\right\}$$

and

$$\theta = \min\left\{\frac{2q}{q-1}, \mu + \frac{2}{p-1}, \left(\frac{\nu}{p}+\mu\right)\frac{pq}{pq-1}\right\},$$

then (u,v) *is trivial.*

Remark 5. For system (4), the moving plane method used by Chen, Li and Ou [3] does not apply because the maximum principle cannot be applied to this system.

Finally, the third main result concerning system (5) is given by following theorem.

Theorem 6. *Let* (u,v) *be a solution to system* (5)*. If*

$$N < \max\{\gamma,\theta\} \tag{8}$$

where

$$\gamma = \min\left\{\frac{\nu_2 p}{p-1}, \nu_1 + \frac{\mu_2}{q-1}, \left(\frac{\mu_1}{q}+\nu_1\right)\frac{pq}{pq-1}\right\}$$

and

$$\theta = \min\left\{\frac{\mu_2 q}{q-1}, \mu_1 + \frac{\nu_2}{p-1}, \left(\frac{\nu_1}{p}+\mu_1\right)\frac{pq}{pq-1}\right\},$$

then (u,v) *is trivial.*

Note that inequality (6) is an immediate result of Theorem 4 when $\nu_2 = \mu_2 = 2$, $\nu_1 = \nu$, and $\mu_1 = \mu$.

3. Proofs of the theorems

We first recall the following proposition from [11, Proposition 3.3].

Proposition 7. *(See [11].) Suppose that* $\delta \in [0,2]$, $\beta + 1 \geqslant 0$, *and* $\theta \in C_0^\infty(\mathbb{R}^N)$, $\theta \geqslant 0$*. Then, the following point-wise inequality holds*:

$$\theta(x)^{\beta+1}(-\Delta)^{\delta/2}\theta(x) \geqslant \frac{1}{\beta+2}(-\Delta)^{\delta/2}\theta(x)^{\beta+2}.$$

Proof. See Appendix A. □

Note that for a nonnegative function $\psi \in C_0^\infty(\mathbb{R}^N)$, $\delta \in [0,2]$ and $\beta > p'$ (i.e., $(\beta-1)p' - \beta\frac{p'}{p} > 0$), we have the following inequality

$$\int_{\mathbb{R}^N} \psi^{(\beta-1)p'-\beta\frac{p'}{p}}\left|(-\Delta)^{\delta/2}\psi\right|^{p'}dx \leqslant \int_K \psi^{(\beta-1)p'-\beta\frac{p'}{p}}\left|(-\Delta)^{\delta/2}\psi\right|^{p'}dx < \infty,$$

where $K := \operatorname{supp}(\psi)$ stands for support of ψ, and $p + p' = pp'$.

For the proof of our main results, we introduce the "standard cutoff function" ψ_0, that is $\psi_0 \in C_0^\infty(\mathbb{R})$ is a smooth decreasing function such that

$$0 \leqslant \psi_0 \leqslant 1, \ \left|\psi_0'(r)\right| \leqslant C/r, \quad \text{and for any } r > 0,\ \psi_0(r) = \begin{cases} 1 & \text{if } r \leqslant 1, \\ 0 & \text{if } r \geqslant 2. \end{cases}$$

Now we are ready to prove the theorems.

Proof of Theorem 2. By the weak formulation of system (1), we get

$$\int_{\mathbb{R}^N} u(-\Delta)^{\mu/2}\psi^\beta\,dx = \int_{\mathbb{R}^N} |v|^q\psi^\beta\,dx$$

and

$$\int_{\mathbb{R}^N} v(-\Delta)^{\nu/2}\psi^\beta\,dx = \int_{\mathbb{R}^N} |u|^p\psi^\beta\,dx,$$

for any nonnegative test function $\psi^\beta \in C_0^\infty(\mathbb{R}^N)$ and $\beta > \max(p', q')$.

Taking into account Proposition 7, we have

$$(-\Delta)^{\delta/2}\psi^\beta \leqslant \beta\psi^{\beta-1}(-\Delta)^{\delta/2}\psi. \tag{9}$$

Using (9) and the Hölder inequality, we estimate the first integral over K as follows

$$\int_{\mathbb{R}^N} u(-\Delta)^{\mu/2}\psi^\beta\,dx \leqslant \beta\int_K u\psi^{\beta/p}\psi^{\beta-1}\psi^{-\beta/p}(-\Delta)^{\mu/2}\psi\,dx$$
$$\leqslant \beta\left(\int_K |u|^p\psi^\beta\,dx\right)^{1/p}\left(\int_K \psi^{(\beta-1)p'-\beta\frac{p'}{p}}\left|(-\Delta)^{\mu/2}\psi\right|^{p'}dx\right)^{1/p'} < \infty,$$

where $K := \operatorname{supp}(\psi)$ and $p + p' = pp'$.

Similarly, we obtain the estimate for the second integral

$$\int_{\mathbb{R}^N} v(-\Delta)^{\nu/2}\psi^\beta\,dx \leqslant \beta\int_K v\psi^{\beta/q}\psi^{\beta-1}\psi^{-\beta/q}(-\Delta)^{\nu/2}\psi\,dx$$
$$\leqslant \beta\left(\int_K |v|^q\psi^\beta dx\right)^{1/q}\left(\int_K \psi^{(\beta-1)q'-\beta\frac{q'}{q}}\left|(-\Delta)^{\nu/2}\psi\right|^{q'}dx\right)^{1/q'} < \infty, \qquad q + q' = qq'.$$

If we set

$$\mathcal{A}_\beta(r,\delta) := \beta\left(\int_{\mathbb{R}^N} \psi^{(\beta-1)r'-\beta\frac{r'}{r}}\left|(-\Delta)^{\delta/2}\psi\right|^{r'}dx\right)^{1/r'},$$

then we can write

$$\int_{\mathbb{R}^N} |u|^p\psi^\beta\,dx \leqslant \mathcal{A}_\beta(q,\nu)\left(\int_K |v|^q\psi^\beta\,dx\right)^{1/q} \tag{10}$$

and

$$\int_{\mathbb{R}^N} |v|^q\psi^\beta\,dx \leqslant \mathcal{A}_\beta(p,\mu)\left(\int_K |u|^p\psi^\beta\,dx\right)^{1/p}.$$

Therefore,

$$\left(\int_{\mathbb{R}^N} |v|^q\psi^\beta\,dx\right)^{1/q} \leqslant \left(\int_K |u|^p\psi^\beta\,dx\right)^{1/pq}\big(\mathcal{A}_\beta(p,\mu)\big)^{1/q}. \tag{11}$$

Using (10) and (11), we have

$$\int_{\mathbb{R}^N} |u|^p\psi^\beta\,dx \leqslant \left(\int_K |u|^p\psi^\beta\,dx\right)^{1/pq}\big(\mathcal{A}_\beta(q,\nu)\big)\big(\mathcal{A}_\beta(p,\mu)\big)^{1/q},$$

and consequently,

$$\left(\int_{\mathbb{R}^N} |u|^p\psi^\beta\,dx\right)^{1-1/(pq)} \leqslant \big(\mathcal{A}_\beta(q,\nu)\big)\big(\mathcal{A}_\beta(p,\mu)\big)^{1/q}.$$

Similarly, we obtain

$$\left(\int_{\mathbb{R}^N} |v|^q \psi^\beta \, dx\right)^{1-1/(pq)} \leqslant \big(\mathcal{A}_\beta(p,\mu)\big)\big(\mathcal{A}_\beta(q,\nu)\big)^{1/p}.$$

Now, we take $\psi(x) = \psi_0(|y|^2)$, $y = \frac{x}{R}$ and $R > 0$ a real number. Then

$$(-\Delta_x)^{\mu/2}\psi(x) = R^{-\mu}(-\Delta_y)^{\mu/2}\psi_0\big(|y|^2\big)$$

and hence

$$\mathcal{A}_\beta(p,\mu) = \beta\left(\int_K \psi_0^{(\beta-1)p'-\beta\frac{p'}{p}} R^{-\mu p'+N}\big|(-\Delta)^{\mu/2}\psi_0\big|^{p'} dy\right)^{1/p'} \leqslant C R^{-\mu+N/p'}, \tag{12}$$

where

$$C = \beta\left(\int_{[1\leqslant|y|\leqslant 2]} \psi_0^{(\beta-1)p'-\beta\frac{p'}{p}}\big(|y|^2\big)\big|(-\Delta)^{\mu/2}\psi_0\big(|y|^2\big)\big|^{p'} dy\right)^{1/p'}.$$

So, we have

$$\left(\int_{\mathbb{R}^N} |u|^p \psi^\beta \, dx\right)^{1-1/(pq)} \leqslant C R^\gamma$$

and

$$\left(\int_{\mathbb{R}^N} |v|^q \psi^\beta \, dx\right)^{1-1/(pq)} \leqslant C R^\theta,$$

where

$$\begin{cases} \gamma = -\nu + \dfrac{N}{q'} - \dfrac{\mu}{q} + \dfrac{N}{p'q}, \\ \theta = -\mu + \dfrac{N}{p'} - \dfrac{\nu}{p} + \dfrac{N}{q'p}. \end{cases}$$

Now, using (6), we can see that if $\gamma < 0$ or $\theta < 0$, then we have

$$\lim_{R\to\infty}\int_{\mathbb{R}^N} |u|^p \psi^\beta \, dx = \int_{\mathbb{R}^N} |u|^p \, dx = 0 \quad \text{or} \quad \lim_{R\to\infty}\int_{\mathbb{R}^N} |v|^q \psi^\beta \, dx = \int_{\mathbb{R}^N} |v|^q \, dx = 0;$$

therefore $(u, v) \equiv (0, 0)$, and this ends the proof. □

Remark 8. In the case of a single equation

$$(-\Delta)^{\mu/2}u = |u|^p, \quad u \geqslant 0 \quad \text{in } \mathbb{R}^N$$

using the same argument as in the proof of Theorem 2, one can verify that if $p < \frac{N}{N-\mu}$, then the solution is trivial.

Proof of Theorem 4. Assume that (u, v) is a weak nontrivial solution to the system (4). As before, we have

$$\int_{\mathbb{R}^N} u(-\Delta)^{\mu/2}\psi^\beta \, dx - \int_{\mathbb{R}^N} v\Delta\psi^\beta \, dx = \int_{\mathbb{R}^N} |v|^q \psi^\beta \, dx$$

and

$$\int_{\mathbb{R}^N} v(-\Delta)^{\nu/2}\psi^\beta \, dx - \int_{\mathbb{R}^N} u\Delta\psi^\beta \, dx = \int_{\mathbb{R}^N} |u|^p \psi^\beta \, dx,$$

where $0 \leqslant \psi^\beta \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ with $\beta > \max(p', q')$.

Using similar arguments as in the proof of Theorem 2, we get

$$\int_{\mathbb{R}^N} |u|^p \psi^\beta \, dx \leqslant \mathcal{A}_\beta(q,\nu)\left(\int_K |v|^q \psi^\beta \, dx\right)^{1/q} + \mathcal{A}_\beta(p,2)\left(\int_K |u|^p \psi^\beta \, dx\right)^{1/p} \tag{13}$$

and

$$\int_{\mathbb{R}^N} |v|^q \psi^\beta\, dx \leqslant \mathcal{A}_\beta(p,\mu)\Big(\int_K |u|^p \psi^\beta\, dx\Big)^{1/p} + \mathcal{A}_\beta(q,2)\Big(\int_K |v|^q \psi^\beta dx\Big)^{1/q}.$$

Setting

$$\mathcal{X} := \Big(\int_{\mathbb{R}^N} |u|^p \psi^\beta\, dx\Big)^{1/p}, \qquad \mathcal{Y} := \Big(\int_{\mathbb{R}^N} |v|^q \psi^\beta\, dx\Big)^{1/q},$$

the last inequalities can be written as

$$\begin{cases} \mathcal{X}^p \leqslant \mathcal{A}_\beta(p,2)\mathcal{X} + \mathcal{A}_\beta(q,\nu)\mathcal{Y}, \\ \mathcal{Y}^q \leqslant \mathcal{A}_\beta(p,\mu)\mathcal{X} + \mathcal{A}_\beta(q,2)\mathcal{Y}. \end{cases}$$

Using Lemma 4 of [7], we obtain

$$\mathcal{X}^{pq} \leqslant C\big\{\big(\mathcal{A}_\beta(p,2)\big)^{\frac{pq}{p-1}} + \big(\mathcal{A}_\beta(q,\nu)\big)^q \big(\mathcal{A}_\beta(q,2)\big)^{\frac{q}{q-1}} + \big(\big(\mathcal{A}_\beta(q,\nu)\big)^q \mathcal{A}_\beta(p,\mu)\big)^{\frac{pq}{pq-1}}\big\}$$

and

$$\mathcal{Y}^{pq} \leqslant C\big\{\big(\mathcal{A}_\beta(q,2)\big)^{\frac{pq}{q-1}} + \big(\mathcal{A}_\beta(p,\mu)\big)^p \big(\mathcal{A}_\beta(p,2)\big)^{\frac{p}{p-1}} + \big(\big(\mathcal{A}_\beta(p,\mu)\big)^p \mathcal{A}_\beta(q,\nu)\big)^{\frac{pq}{pq-1}}\big\}.$$

Using the scaled variable as in the proof of Theorem 2 and the inequality (12), we deduce that

$$\mathcal{X}^{pq} \leqslant C\big(R^{\gamma_1} + R^{\gamma_2} + R^{\gamma_3}\big),$$

where

$$\begin{cases} \gamma_1 = \Big(-2 + \dfrac{N}{p'}\Big)\dfrac{pq}{p-1}, \\ \gamma_2 = \Big(-\nu + \dfrac{N}{q'}\Big)q + \Big(-2 + \dfrac{N}{q'}\Big)\dfrac{q}{q-1}, \\ \gamma_3 = \Big(\Big(-\nu + \dfrac{N}{q'}\Big)q + \Big(-\mu + \dfrac{N}{p'}\Big)\Big)\dfrac{pq}{pq-1}, \end{cases}$$

and

$$\mathcal{Y}^{pq} \leqslant C\big(R^{\theta_1} + R^{\theta_2} + R^{\theta_3}\big),$$

where

$$\begin{cases} \theta_1 = \Big(-2 + \dfrac{N}{q'}\Big)\dfrac{pq}{q-1}, \\ \theta_2 = \Big(-\mu + \dfrac{N}{p'}\Big)p + \Big(-2 + \dfrac{N}{p'}\Big)\dfrac{p}{p-1}, \\ \theta_3 = \Big(\Big(-\mu + \dfrac{N}{p'}\Big)p + \Big(-\nu + \dfrac{N}{q'}\Big)\Big)\dfrac{pq}{pq-1}. \end{cases}$$

Using (7), it is not difficult to verify that, if either $\max(\gamma_1,\gamma_2,\gamma_3) < 0$ or $\max(\theta_1,\theta_2,\theta_3) < 0$, then the proof follows by using the same argument as in the previous problem. □

Proof of Theorem 6. Let (u, v) be a weak solution to the system (5). Following the same method as in the proof of Theorem 4 of system (4) one has

$$\int_{\mathbb{R}^N} |u|^p \psi^\beta\, dx \leqslant \mathcal{A}_\beta(q,\nu_1)\Big(\int_K |v|^q \psi^\beta\, dx\Big)^{1/q} + \mathcal{A}_\beta(p,\nu_2)\Big(\int_K |u|^p \psi^\beta\, dx\Big)^{1/p},$$

and

$$\int_{\mathbb{R}^N} |v|^q \psi^\beta\, dx \leqslant \mathcal{A}_\beta(p,\mu_1)\Big(\int_K |u|^p \psi^\beta\, dx\Big)^{1/p} + \mathcal{A}_\beta(q,\mu_2)\Big(\int_K |v|^q \psi^\beta\, dx\Big)^{1/q}.$$

Similarly, we have

$$\Big(\int_{\mathbb{R}^N} |u|^p \psi^\beta\, dx\Big)^{pq} \leqslant C\big\{\big(\mathcal{A}_\beta(p,\nu_2)\big)^{\frac{pq}{p-1}} + \big(\mathcal{A}_\beta(q,\nu_1)\big)^q \big(\mathcal{A}_\beta(q,\mu_2)\big)^{\frac{q}{q-1}} + \big(\big(\mathcal{A}_\beta(q,\nu_1)\big)^q \mathcal{A}_\beta(p,\mu_1)\big)^{\frac{pq}{pq-1}}\big\},$$

and

$$\left(\int_{\mathbb{R}^N} |v|^q \psi^\beta \, dx\right)^{pq} \leqslant C\left\{\left(\mathcal{A}_\beta(q,\mu_2)\right)^{\frac{pq}{q-1}} + \left(\mathcal{A}_\beta(p,\mu_1)\right)^p \left(\mathcal{A}_\beta(p,\nu_2)\right)^{\frac{p}{p-1}} + \left(\left(\mathcal{A}_\beta(p,\mu_1)\right)^p \mathcal{A}_\beta(q,\nu_1)\right)^{\frac{pq}{pq-1}}\right\}.$$

Also, using the arguments of the previous theorem, we get

$$\left(\int_{\mathbb{R}^N} |u|^p \psi^\beta \, dx\right)^{pq} \leqslant C\left(R^{\gamma_1'} + R^{\gamma_2'} + R^{\gamma_3'}\right),$$

where

$$\begin{cases} \gamma_1' = \left(-\nu_2 + \dfrac{N}{p'}\right)\dfrac{pq}{p-1}, \\ \gamma_2' = \left(-\nu_1 + \dfrac{N}{q'}\right)q + \left(-\mu_2 + \dfrac{N}{q'}\right)\dfrac{q}{q-1}, \\ \gamma_3' = \left(\left(-\nu_1 + \dfrac{N}{q'}\right)q + \left(-\mu_1 + \dfrac{N}{p'}\right)\right)\dfrac{pq}{pq-1}, \end{cases}$$

and

$$\left(\int_{\mathbb{R}^N} |v|^q \psi^\beta \, dx\right)^{pq} \leqslant C\left(R^{\theta_1'} + R^{\theta_2'} + R^{\theta_3'}\right),$$

where

$$\begin{cases} \theta_1' = \left(-\mu_2 + \dfrac{N}{q'}\right)\dfrac{pq}{q-1}, \\ \theta_2' = \left(-\mu_1 + \dfrac{N}{p'}\right)p + \left(-\nu_2 + \dfrac{N}{p'}\right)\dfrac{p}{p-1}, \\ \theta_3' = \left(\left(-\mu_1 + \dfrac{N}{p'}\right)p + \left(-\nu_1 + \dfrac{N}{q'}\right)\right)\dfrac{pq}{pq-1}. \end{cases}$$

Taking either $\max(\gamma_1', \gamma_2', \gamma_3') < 0$ or $\max(\theta_1', \theta_2', \theta_3') < 0$, and using the same arguments as in the previous proofs one can show that $u = v = 0$. □

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Appendix A

Proof of Proposition 7. The proof of this point-wise estimate given in N. Ju [11, Proposition 3.3] for $N = 2$, makes use of the Riesz potential representation of the operator $(-\Delta)^{\delta/2}$ is motivated by the proof of Proposition 3.2 of A. Cordoba and D. Cordoba [2].

We will reproduce Ju's Proof in dimension N just for the convenience of the reader. When $\delta = 0$ or $\delta = 2$, the result is obvious. Now we consider the case $\delta \in (0, 2)$. Then by Proposition 3.3 [11],

$$(-\Delta)^{\delta/2}\theta(x) = C_\delta P.V. \int \frac{\theta(x) - \theta(y)}{|x-y|^{N+\delta}} \, dy.$$

Therefore, for $\theta \geqslant 0$,

$$\left(\theta(x)\right)^{\beta+1}(-\Delta)^{\delta/2}\theta(x) = C_\delta P.V. \int \frac{(\theta(x))^{\beta+2} - (\theta(x))^{\beta+1}\theta(y)}{|x-y|^{N+\delta}} \, dy.$$

By Young's inequality, if $\beta + 1 > 0$, then

$$\left(\theta(x)\right)^{\beta+1}\theta(y) \leqslant \frac{\beta+1}{\beta+2}\left(\theta(x)\right)^{\beta+2} + \frac{1}{\beta+2}\left(\theta(y)\right)^{\beta+2}.$$

Thus

$$\left(\theta(x)\right)^{\beta+1}(-\Delta)^{\delta/2}\theta(x) \geqslant C_\delta \frac{1}{\beta+2} P.V. \int \frac{(\theta(x))^{\beta+2} - (\theta(y))^{\beta+2}}{|x-y|^{N+\delta}} \, dy = \frac{1}{\beta+2}(-\Delta)^{\delta/2}\left(\theta(x)\right)^{\beta+2}.$$

The case $\beta = -1$ is still valid from the above proof, without using Young's inequality. □

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