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The Variational Iteration Method for Solving the Fractional Foam Drainage Equation

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Abstract: In this paper, by introducing the fractional derivative in the sense of Caputo, we apply the He's Variational Iteration Method (VIM) for the foam drainage equation with time-and space-fractional derivatives. Numerical solutions are obtained for the fractional foam drainage equation to show the nature of solution as the fractional derivative parameters are changed.

Keywords: Caputo fractional derivative; variational iteration method; fractional differential equation; foam drainage equation; Lagrange multiplier

1 Introduction

Variational Iteration method was first proposed by the Chinese mathematician J.H. He [5, 6]. This method has been widely used for solving the analytic solutions of physically significant equations arranging from linear to nonlinear, from ordinary differential to partial differential, form integer to fractional [6, 17, 18]. The idea of VIM is to construct correction functionals using general Lagrange multipliers identified optimally via the variational theory, and the initial approximations can be freely chosen with unknown constants.

As we all know, for the nonlinear equations of integer order, there exist many methods used to derive the explicit solutions [1, 3, 4]. However, for the fractional differential equations, there are only limited approaches, such as Laplace transform method [13], the Fourier transform method [10], the iteration method [14] and the operational method [12, 15]. In recent ten years, the fractional differential equations have been attracted great attention and widely been used in the areas of physics and engineering [19]. Particularly in some interdisciplinary fields, the fractional derivatives are considered to be a very powerful and useful tool [13, 14, 19]. With the help of fractional derivatives, phenomena in electromagnetic , acoustics electrochemistry and material science can be elegantly described [2, 13, 14, 19].
The study of foam drainage equation is very significant for that the equation is a simple model of the flow of liquid through channels (Plateau borders [21]) and nodes (intersection of four channels) between the bubbles, driven by gravity and capillarity [20]. It has been studied by many authors [8, 9, 16]. The study for the foam drainage equation with time and space-fractional derivatives of this form

$$D_t^{\alpha} u = \frac{1}{2} u u_{xx} - 2u^2 D_x^{\beta} u + \left(D_x^{\beta} u\right)^2, \ \ 0 < \alpha, \ \ \beta \leq 1, \ \ x > 0 \tag{1.1}$$

has been investigated by the ADM method in [2]. The fractional derivatives are considered in the Caputo sense. When $\alpha = \beta = 1$, the fractional equation reduces to the foam drainage equation of the form

$$u_t = \frac{1}{2} u u_{xx} - 2u^2 u_x + (u_x)^2 . \tag{1.2}$$

In the present paper, we employ the VIM method to derive numerical solutions of the fractional foam drainage Eq.(1.1); two cases of special interest such as the time-fractional foam drainage equation and the space-fractional foam drainage equation are discussed in details. Further, we give comparative remarks with the results obtained using ADM method (see [2]).

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2 Basic Definitions

There are several mathematical definitions about fractional derivative [13, 14]. In this paper, we adopt the two usually used definitions: the Caputo and its reverse operator Riemann-Liouville. More details one can consults [13].

Definition 1 *A real valued function* $f(x), x > 0$ *is said to be in the space* $C_\mu, \mu \in \mathcal{R}$ *if there exists a real number* $p > \mu$ *such that* $f(x) = x^p f_1(x)$ *where* $f_1(x) \in C([0,\infty))$.

Definition 2 *A function* $f(x), x > 0$ *is said to be in the space* $C_\mu^n, n \in \mathcal{N}$, *if* $f^{(n)} \in C_\mu$.

Definition 3 *The Riemann-Liouville fractional integral operator of order* $\alpha \geq 0$, *for a function* $f \in C_\mu$, $(\mu \geq -1)$ *is defined as*

$$\begin{aligned} &J^\alpha f(x) = \tfrac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt; \quad \alpha > 0, x > 0 \\ &J^0 f(x) = f(x). \end{aligned} \tag{2.1}$$

For the convenience of establishing the results for the Eq.(1.1), we give the following properties:

$$J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x) \tag{2.2}$$

and

$$J^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} x^{\alpha+\beta}. \tag{2.3}$$

The fractional derivative of $f \in C_{-1}^n$ in the Caputo's sense is defined as

$$D^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)d\tau, & n-1 < \alpha < n, n \in \mathcal{N}^*, \\ \frac{d^n}{dt^n} f(t), & \alpha = n. \end{cases} \tag{2.4}$$

According to (2.4), we can obtain:

$$D^\alpha K = 0; \quad K \text{ is a constant} \tag{2.5}$$

and

$$D^\alpha t^\beta = \begin{cases} \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}, & \beta > \alpha - 1, \\ 0, & \beta \leq \alpha - 1. \end{cases} \tag{2.6}$$

In this paper, we consider the equation (1.1). When $\alpha \in \mathcal{R}^+$, we have:

$$D_t^\alpha u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{\partial^n u(x,\tau)}{\partial \tau^n} d\tau, & n-1 < \alpha < n \\ \frac{\partial^n u(x,t)}{\partial t^n}, & \alpha = n. \end{cases} \tag{2.7}$$

The form of the space fractional derivative is similar to the above and we just omit it here.

3 Basic Idea of He's Variational Iteration Method

To clarify the basic ideas of VIM, we consider the following differential equation:

$$Lu + Nu = g(x,t), \tag{3.1}$$

where L is a linear operator, N is a nonlinear operator and $g(x,t)$ is a homogeneous term.
According to VIM, we can write down a correction functional as follows:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda \left(Lu(x,\xi) + N\widetilde{u}(x,\xi) - g(x,\xi) \right) d\xi, \tag{3.2}$$

where λ is a general Lagrangian multiplier which can be identified optimally via the variational theory. The subscript n indicates the nth approximation and $\widetilde{u}_n$ is considered as a restricted variation, i.e.$\delta\widetilde{u} = 0$.

It is obvious now that the main steps of the VIM require first the determination of λ that will be identified optimally. Having determined the Lagrangian multiplier, the successive approximations u_n, $n < 0$ of the solution u will be readily obtained upon using any selective function u_0. Consequently, the solution is obtained as:

$$u(x,t) = \lim_{\longrightarrow\infty} u_n(x,t). \tag{3.3}$$

The convergence of the variational iteration method is investigated in [7].

4 Applications of the VIM Method

Consider the following form of the fractional foam drainage equation with time-and space-fractional derivatives:

$$D_t^{\alpha}u - \frac{1}{2}uu_{xx} + 2u^2 D_x^{\beta}u - \left(D_x^{\beta}u\right)^2 = 0;\ \ 0 < \alpha \leq 1,\ \ 0 < \beta \leq 1, \tag{4.1}$$

where the operators D_t^{α} and D_x^{β} stand for the fractional derivative.
Take the initial condition as

$$u(x,0) = f(x). \tag{4.2}$$

The correction functional for Eq.(4.1) can be approximately expressed as follows:

$$\begin{aligned} u_{n+1}(x,t) &= u_n(x,t) + \int_0^t \lambda(\tau)\left[D_{\tau}^{\alpha}\left(u_n(x,\tau)\right) - \tfrac{1}{2}\left(\tfrac{\partial\widetilde{u}_n(x,\tau)}{\partial xx}\right)\left(\widetilde{u}_n(x,\tau)\right)\right. \\ &\left. +2\left(\widetilde{u}_n(x,\tau)\right)^2 D_x^{\beta}\left(\widetilde{u}_n(x,\tau)\right) - \left(D_x^{\beta}\left(\widetilde{u}_n(x,\tau)\right)\right)^2\right]d\tau, \end{aligned} \tag{4.3}$$

where λ is a general Lagrange multiplier, $\widetilde{u}_n(x,\tau)$ is considered as restricted variations and $\delta\widetilde{u}_n$ is considered as a restricted variation. Making the above correction functional stationary and noticing that $\delta\widetilde{u}_n = 0$, we obtain:

$$\begin{aligned} \delta u_{n+1}(x,t) &= u_n(x,t) + \int_0^t \delta\lambda(\tau)\left[D_{\tau}^{\alpha}\left(u_n(x,\tau)\right) - \tfrac{1}{2}\left(\tfrac{\partial\widetilde{u}_n(x,\tau)}{\partial xx}\right)\left(\widetilde{u}_n(x,\tau)\right)\right. \\ &\left. +2\left(\widetilde{u}_n(x,\tau)\right)^2 D_x^{\beta}\left(\widetilde{u}_n(x,\tau)\right) - \left(D_x^{\beta}\left(\widetilde{u}_n(x,\tau)\right)\right)^2\right]d\tau, \end{aligned} \tag{4.4}$$

or

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \int_0^t \delta\lambda(\tau)\left[D_{\tau}^{\alpha}\left(u_n(x,\tau)\right)\right]d\tau \tag{4.5}$$

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \lambda(\tau)\,\delta u_n(x,\tau) + \int_0^t \delta u_n(x,\tau)\,\lambda'(\tau)\,d\tau = 0 \tag{4.6}$$

which produces the stationary conditions:

$$\begin{aligned} \lambda'(\tau) &= 0, (4.7(a)) && (1) \\ 1 + \lambda(\tau)|_{\tau=t} &= 0, (4.7(b)) && (2) \end{aligned}$$

where Eq.(4.7a) is called Lagrange–Euler equation and Eq.(4.7b) natural boundary condition.
The Lagrange multiplier, therefore, can be identified as $\lambda = -1$ and the following variational iteration formula can be obtained:

$$u_{n+1}(x,t) = u_n - \int_0^t \left[D_t^{\alpha}u_n - \tfrac{1}{2}\widetilde{u}_n\widetilde{u}_{nxx} + 2\left(\widetilde{u}_n\right)^2 D_x^{\beta}\widetilde{u}_n - \left(D_x^{\beta}\widetilde{u}_n\right)^2\right]d\tau. \tag{4.8}$$

4.1 Numerical Solutions of Time-Fractional Foam Drainage Equation

Consider the following form of the time-fractional equation

$$D_t^\alpha u - \frac{1}{2} u u_{xx} + 2u^2 u_x - u_x^2 = 0; \;\; 0 < \alpha \le 1, \tag{5.1}$$

with the initial condition

$$u(x,0) = f(x) = -\sqrt{c}\tanh\sqrt{c}(x), \tag{5.2}$$

where c is the velocity of wavefront [15].
The exact solution of (5.1) for the special case $\alpha = \beta = 1$ is

$$u(x,t) = \begin{cases} -\sqrt{c}\tanh(\sqrt{c}(x-ct)); & x \le ct \\ 0; & x > ct. \end{cases} \tag{5.3}$$

In order to obtain numerical solution of the equation (5.1), using the expression (4.8), we get:

$$u_{n+1}(x,t) = u_n - \int_0^t \left[D_\tau^\alpha u_n - \tfrac{1}{2}\tilde{u}_n \tilde{u}_{nxx} + 2\left(\tilde{u}_n\right)^2 \tilde{u}_{nx} - \left(\tilde{u}_{nx}\right)^2 \right] d\tau. \tag{5.4}$$

By the iteration formula (5.4), we can obtain the other components as:

$$\begin{aligned}
&u_0(x,t) = f(x),\\
&u_1(x,t) = u_0 - \int_0^t \left[D_\tau^\alpha u_0 - \tfrac{1}{2}\tilde{u}_0 \tilde{u}_{0xx} + 2\left(\tilde{u}_0\right)^2 \tilde{u}_{0x} - \left(\tilde{u}_{0x}\right)^2 \right] d\tau\\
&= f(x) + t f_1(x), \quad f_1(x) = \tfrac{1}{2} f f_{xx} - 2f^2 f_x + f_x^2,\\
&u_2(x,t) = u_1 - \int_0^t \left[D_\tau^\alpha u_1 - \tfrac{1}{2}\tilde{u}_1 \tilde{u}_{1xx} + 2\left(\tilde{u}_1\right)^2 \tilde{u}_{1x} - \left(\tilde{u}_{1x}\right)^2 \right] d\tau\\
&= f - f_2 t^{2-\alpha} + f_3 t + f_4 \tfrac{t^2}{2} + f_5 \tfrac{t^3}{3} - f_6 \tfrac{t^4}{4};\\
&f_2 = \tfrac{f_1(x)}{(2-\alpha)\Gamma(2-\alpha)}, \qquad f_3 = f_1(x) + \tfrac{1}{2} f_x f_{xx} - 2 f_x f^2 + f_x^2,\\
&f_4 = \tfrac{1}{2}\left(f_x f_{1xx} + f_{xx} f_{1x}\right) - 2\left(f^2 f_{1x} + 2 f f_1 f_x\right) + 2 f_x f_{1x},\\
&f_5 = \tfrac{1}{2} f_{1x} f_{1xx} - 2\left(2 f f_1 f_{1x} + f_x f_1^2\right) + f_{1x}^2, \qquad f_6 = 2 f_1^2 f_{1x}.\\
&u_3(x,t) = u_2 - \int_0^t \left[D_\tau^\alpha u_2 - \tfrac{1}{2}\tilde{u}_2 \tilde{u}_{2xx} + 2\left(\tilde{u}_2\right)^2 \tilde{u}_{2x} - \left(\tilde{u}_{2x}\right)^2 \right] d\tau,
\end{aligned} \tag{5.5}$$

and so on, in the same manner the rest of components of the iteration formula (5.4) can be obtained using the Mathematica Package.

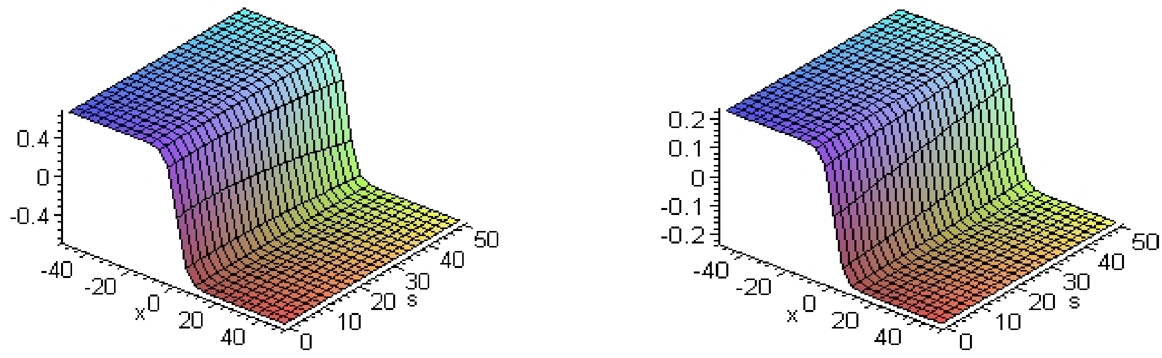
In order to check the efficiency of the VIM method for the equation (5.1), we draw figures for the numerical solution with $\alpha = 0.5$ as well as the exact solution (5.3). Figure (a) stands for the numerical solution $u_3(t,x)$. Figure (b) shows the exact solution (5.3) when $\alpha = \beta = 1$. From these figures, we can appreciate how closely are the two solutions.

Remark 1 *The Eq.*(1) *has been solved by ADM method in [2]. It should be remarked that the graph drawn here (in the case of time-fractional derivative) using the VIM method is agreement with the graph drawn using the ADM method.*

4.2 Numerical Solutions of Space-Fractional Foam Drainage Equation

Considering the operator form of the space-fractional equation

$$u_t - \frac{1}{2} u u_{xx} + 2u^2 D_x^\beta u - (D_x^\beta u)^2 = 0; \;\; 0 < \beta \le 1. \tag{5.6}$$



(a) (b)

Figure 1: Representing time fractional solutions of Eq.(5.1). In (a); solution obtained by the VIM method for $\alpha = \frac{1}{2}$. In (b); the exact solution (5.3); $c = 0.02$.

Assuming the condition as

$$u(x,0) = f(x) = x^2. \tag{5.7}$$

Initial condition has been taken as the above polynomial to avoid heavy calculation of fractional differentiation.
In order to estimate the numerical solution of equation (5.6), using the expression (4.8), we get:

$$u_{n+1}(x,t) = u_n - \int\limits_0^t \left[u_{n\tau} - \tfrac{1}{2}\tilde{u}_n \tilde{u}_{nxx} + 2\left(\tilde{u}_n\right)^2 D_x^\beta \tilde{u}_n - \left(D_x^\beta \tilde{u}_n\right)^2 \right] d\tau. \tag{5.8}$$

By the iteration formula (5.8), we can obtain the other components as:

$$\begin{aligned}
&u_0(x,t) = f(x),\\
&u_1(x,t) = u_0 - \int\limits_0^t \left[u_{0\tau} - \tfrac{1}{2}\tilde{u}_0 \tilde{u}_{0xx} + 2\left(\tilde{u}_0\right)^2 D_x^\beta \tilde{u}_0 - \left(D_x^\beta \tilde{u}_0\right)^2 \right] d\tau\\
&= x^2 + \left[x^2 - 4f_1 x^{6-\beta} + 4f_1^2 x^{4-2\beta}\right] t,\\
&u_2(x,t) = u_1 - \int\limits_0^t \left[u_{1\tau} - \tfrac{1}{2}\tilde{u}_1 \tilde{u}_{1xx} + 2\left(\tilde{u}_1\right)^2 D_x^\beta \tilde{u}_1 - \left(D_x^\beta \tilde{u}_1\right)^2 \right] d\tau\\
&= x^2 + \left[x^2 - 4f_1 x^{6-\beta} + 4f_1^2 x^{4-2\beta}\right] t + [4x^2 - 4f_1\left((6-\beta)(5-\beta)-2\right)x^{6-\beta}\\
&+4f_1^2\left((4-2\beta)(3-2\beta)+2\right)x^{4-2\beta} - 2x^4\left(2f_1x^{2-\beta} + 4f_1f_2x^{6-2\beta} + 4f_1^2x^{4-3\beta}\right)\\
&-8f_1x^{2-\beta}\left(x^4 - 4f_1x^{8-\beta} + 4f_1^2x^{6-2\beta}\right) + 4f_1x^{2-\beta}\left(2f_1x^{2-\beta} + 4f_1f_2x^{6-2\beta} + 4f_1^2f_3x^{4-3\beta}\right)]\tfrac{t^2}{4} \qquad (5.9)\\
&+[(x^2 - 4f_1x^{6-\beta} + 4f_1^2x^{4-2\beta})(2 - 4f_1(6-\beta)(5-\beta)x^{4-\beta} + 4f_1^2(4-2\beta)(3-2\beta)x^{2-2\beta})\\
&-8f_1x^{2-\beta}\left(x^2 - 4f_1x^{6-\beta} + 4f_1^2x^{4-2\beta}\right)^2 - 8\left(x^4 - 4f_1x^{8-\beta} + 4f_1^2x^{6-2\beta}\right)\\
&\left(2f_1x^{2-\beta} + 4f_1f_2x^{6-2\beta} + 4f_1^2x^{4-3\beta}\right) + 2\left(2f_1x^{2-\beta} + 4f_1f_2x^{6-2\beta} + 4f_1^2f_3x^{4-3\beta}\right)^2]\tfrac{t^3}{6}\\
&-2\left(x^2 - 4f_1x^{6-\beta} + 4f_1^2x^{4-2\beta}\right)^2\left(2f_1x^{2-\beta} + 4f_1f_2x^{6-2\beta} + 4f_1^2x^{4-3\beta}\right)\tfrac{t^4}{4}.\\
&u_3(x,t) = u_2 - \int\limits_0^t \left[u_{2\tau} - \tfrac{1}{2}\tilde{u}_2 \tilde{u}_{2xx} + 2\left(\tilde{u}_2\right)^2 D_x^\beta \tilde{u}_2 - \left(D_x^\beta \tilde{u}_2\right)^2 \right] d\tau,
\end{aligned}$$

and so on, in the same manner the rest of components of the iteration formula (5.8) can be obtained using the Mathematica Package.

Note that:

$$f_1 = \frac{1}{\Gamma(3-\beta)}, f_2 = \frac{\Gamma(7-\beta)}{\Gamma(7-2\beta)}, f_3 = \frac{\Gamma(5-2\beta)}{\Gamma(5-3\beta)}. \tag{5.10}$$

Figures (c,d) show respectively the numerical solution $u_3(t,x)$ for the space-fractional Eq. (5.6) for $\beta = 1/2$, respectively $\beta = 1$.

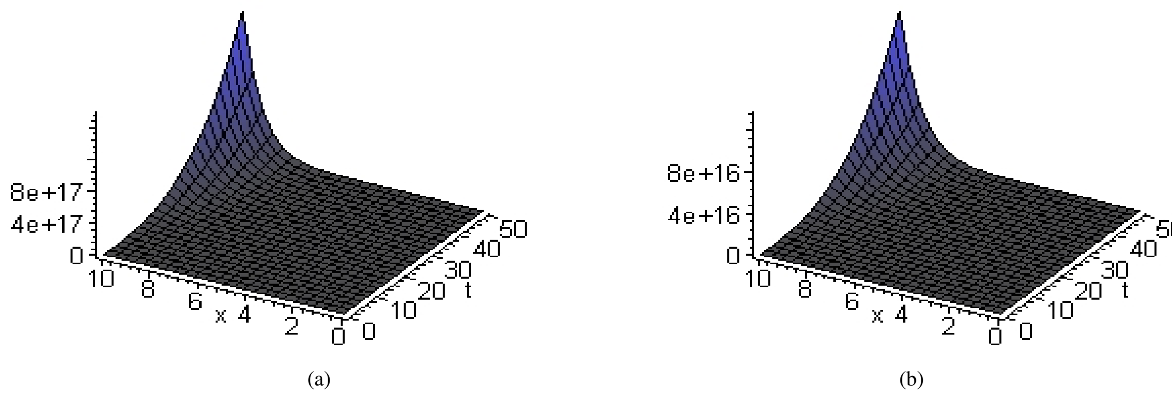


Figure 2: Representing space-fractional solutions of Eq.(5.6). In (c); solution obtained by the VIM method for $\beta = \frac{1}{2}$. In (d); solution obtained by the VIM method for $\beta = 1$.

Remark 2 : *It should be remarked that the graph drawn here (in the case of space fractional derivative) using the VIM method is in excellent agreement with those drawn in [2] using the ADM method.*

5 Conclusion

In this paper, the VIM has been successfully applied to derive explicit numerical solutions for the time-and space-fractional foam drainage equation. The main merits of the VIM are:

1. VIM can overcome difficulties arising in calculation of Adomian polynomials.

2. No linearization is needed; the method is very promising of finding wide application in nonlinear fractional evolution equations.

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