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RESEARCH ARTICLE

On the Hermite-Hadamard-Fejer-type inequalities for co-ordinated convex functions via fractional integrals

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ABSTRACT

In this paper, using Riemann-Liouville integral operators, we establish new fractional integral inequalities of Hermite-Hadamard-Fejer type for co-ordinated convex functions on a rectangle of $\mathbb{R}^2$. The results presented here would provide extensions of those given in earlier works.

1. Introduction

Let $\Phi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval I of real numbers and $a, b \in I$, with $a < b$. The following double inequality is well known in the literature as the Hermite-Hadamard inequality [13]:

$$\Phi\left(\frac{a+b}{2}\right) \tag{1}$$
$$\leq \frac{1}{b-a}\int_a^b \Phi(x)\,dx$$
$$\leq \frac{\Phi(a)+\Phi(b)}{2}.$$

The most well-known inequalities related to the integral mean of a convex function are the Hermite Hadamard inequalities or its weighted versions, the so-called Hermite-Hadamard-Fejér inequalities (see, [14], [19], [21]). In [11], Fejer gave a weighted generalizatinon of the inequalities (1) as the following:

Theorem 1. *Let $\Phi : [a, b] \to R$ be a convex function. Then the inequality hold:*

$$\Phi\left(\frac{a+b}{2}\right)\int_a^b \Psi(x)\,dx$$
$$\leq \int_a^b \Phi(x)\,\Psi(x)\,dx$$
$$\leq \frac{\Phi(a)+\Phi(b)}{2}\int_a^b \Psi(x)\,dx,$$

where $\Psi : [a, b] \to R$ is nonnegative, integrable and symmetric to $\frac{(a+b)}{2}$.

In the following, we will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper. More details, one can consult [12, 18].

Definition 1. *([4, 12, 18]) Let $\Phi \in L_1([a, b])$. The Riemann-Liouville integrals $J^{\alpha}_{a+}\Phi$ and $J^{\alpha}_{b-}\Phi$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

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$$J_{a+}^{\alpha}\Phi(x)=\frac{1}{\Gamma(\alpha)}\int_{a}^{x}(x-t)^{\alpha-1}\Phi(t)dt,\quad x>a$$

and

$$J_{b-}^{\alpha}\Phi(x)=\frac{1}{\Gamma(\alpha)}\int_{x}^{b}(t-x)^{\alpha-1}\Phi(t)dt,\quad x<b$$

respectively. Here, $\Gamma(\alpha)$ *is the Gamma function and* $J_{a+}^{0}\Phi(x)=J_{b-}^{0}\Phi(x)=\Phi(x)$.

Meanwhile, in [24], Sarikaya et al. gave the following interesting Riemann-Liouville integral inequalities of Hermite-Hadamard-type:

Theorem 2. *Let* $K:[a,b]\to\mathbb{R}$ *be a positive function with* $0\leq a<b$ *and* $K\in L_1([a,b])$. *If* K *is a convex function on* $[a,b]$, *then the following inequalities for fractional integrals hold:*

$$\begin{aligned}&K\left(\frac{a+b}{2}\right)\\&\leq\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}K(b)+J_{b-}^{\alpha}K(a)\right]\\&\leq\frac{K(a)+K(b)}{2}\end{aligned}\tag{2}$$

with $\alpha>0$.

Later, in [14], Iscan presented the following Hermite-Hadamard-Fejer type inequalities for convex functions via Riemann-Liouville fractional integrals:

Theorem 3. *Let* $K:[a,b]\to\mathbb{R}$ *be convex function with* $0\leq a<b$ *and* $K\in(L_1[a,b])$. *If* $L:[a,b]\to\mathbb{R}$ *is nonnegative, integrable and symmetric with respect to* $\frac{a+b}{2}$, *then the following inequalities for fractional integrals hold:*

$$\begin{aligned}&K\left(\frac{a+b}{2}\right)\left[J_{a+}^{\alpha}L(b)+J_{b-}^{\alpha}L(a)\right]\\&\leq\left[J_{a+}^{\alpha}K(b)+J_{b-}^{\alpha}K(a)\right]\\&\leq\frac{K(a)+K(b)}{2}\left[J_{a+}^{\alpha}L(b)+J_{b-}^{\alpha}L(a)\right]\end{aligned}\tag{3}$$

with $\alpha>0$.

Let us now consider a bi-demensional interval which will be used throughout this paper. So, we define $\Delta=:[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$. A mapping $\Phi:\Delta\to\mathbb{R}$ is said to be convex on the co-ordinates Δ if the following inequality:

$$\begin{aligned}&\Phi(tx+(1-t)z,ty+(1-t)r)\\&\leq t\Phi(x,y)+(1-t)\Phi(z,r)\end{aligned}$$

holds, for all $(x,y),(z,r)\in\Delta$ and $t\in[0,1]$.
A function $\Phi:\Delta\to\mathbb{R}$ is said to be convex on the co-ordinates on Δ if the partial mappings $\Phi_y:[a,b]\to\mathbb{R}$, $\Phi_y(u)=\Phi(u,y)$ and $\Phi_x:[c,d]\to\mathbb{R}$, $\Phi_x(v)=\Phi(x,v)$ are convex where defined for all $x\in[a,b]$ and $y\in[c,d]$ (see, [10]).

A formal definition for co-ordinated convex functions may be stated as follows:

Definition 2. *([10]) A function* $\Phi:\Delta\to\mathbb{R}$ *will be called co-ordinated convex on* Δ, *for all* $t,s\in[0,1]$ *and* $(x,y),(u,r)\in\Delta$, *if the following inequality holds:*

$$\begin{aligned}&\Phi(tx+(1-t)y,su+(1-s)r)\\\leq\;&ts\Phi(x,u)+s(1-t)\Phi(y,u)\\&+t(1-s)\Phi(x,r)+(1-t)(1-s)\Phi(y,r).\end{aligned}\tag{4}$$

Clearly, every convex function is a co-ordinated convex. Furthermore, there exists a co-ordinated convex function which is not convex, (see, [10]).

For several recent results concerning Hermite-Hadamard's inequality for some convex function on the co-ordinates on a rectangle of $\mathbb{R}^2$, we refer the reader to ([1]- [3], [10], [15]- [17], [20], [22], [27]).

In [10], Dragomir established the following inequality of Hermite-Hadamard-type for co-ordinated convex mapping on a rectangle of $\mathbb{R}^2$ similar to (1).

Theorem 4. *Suppose that* $\Phi:\Delta\to\mathbb{R}$ *is co-ordinated convex on* Δ. *Then one has the inequalities:*

$$
\begin{aligned}
&\Phi\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
\leq\ &\frac{1}{2}\left[\frac{1}{b-a}\int_a^b \Phi\left(x,\frac{c+d}{2}\right)dx\right.\\
&\left.+\frac{1}{d-c}\int_c^d \Phi\left(\frac{a+b}{2},y\right)dy\right]\\
\leq\ &\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d \Phi(x,y)\,dydx \qquad (5)\\
\leq\ &\frac{1}{4}\left[\frac{1}{b-a}\int_a^b \Phi(x,c)\,dx+\frac{1}{b-a}\int_a^b \Phi(x,d)\,dx\right.\\
&\left.+\frac{1}{d-c}\int_c^d \Phi(a,y)\,dy+\frac{1}{d-c}\int_c^d \Phi(b,y)\,dy\right]\\
\leq\ &\frac{\Phi(a,c)+\Phi(a,d)+\Phi(b,c)+\Phi(b,d)}{4}.
\end{aligned}
$$

The above inequalities are sharp.

Later, in [27], Sarikaya and Yaldiz proved inequalities of the Hermite-Hadamard type by using the definition of co-ordinated convex functions for L-Lipschitzian mappings.

In [3], a Hermite-Hadamard-Fejer type inequality for co-ordinated convex mappings is established as follows:

Theorem 5. *Let $\Phi:\Delta\to R$ be a co-ordinated convex function. Then the following double inequality hold:*

$$
\begin{aligned}
&\Phi\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
&\leq\frac{\int_a^b\int_c^d \Phi(x,y)\,p(x,y)\,dydx}{\int_a^b\int_c^d p(x,y)\,dydx} \qquad (6)\\
&\leq\frac{\Phi(a,c)+\Phi(a,d)+\Phi(b,c)+\Phi(b,d)}{4},
\end{aligned}
$$

where $p:\Delta\to R$ is positive, integrable and symmetric with respect to $x=\frac{a+b}{2}$ and $y=\frac{c+d}{2}$ on the co-ordinates on Δ. The above inequalities are sharp.

Because of the wide application of Hermite Hadamard type inequalities, Fejer type inequalities and Riemann-Liouville integrals for two-variable functions, many authors extend their studies to Hermite Hadamard type inequalities and Fejer type inequalities involving Riemann-Liouville integrals not limited to integer integrals.

Definition 3. *([12, 18]) Let $\Phi\in L_1(\Delta)$. The Riemann-Liouville integrals $J^{\alpha,\beta}_{a+,c+}$, $J^{\alpha,\beta}_{a+,d-}$, $J^{\alpha,\beta}_{b-,c+}$ and$J^{\alpha,\beta}_{b-,d-}$ of order $\alpha,\beta>0$ with $a,c\geq 0$ are defined by*

$$J^{\alpha,\beta}_{a+,c+}\Phi(x,y)=$$

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x\int_c^y (x-t)^{\alpha-1}(y-s)^{\beta-1}\Phi(t,s)dsdt,$$

$$x>a,y>c$$

$$J^{\alpha,\beta}_{a+,d-}\Phi(x,y)=$$

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x\int_y^d (x-t)^{\alpha-1}(s-y)^{\beta-1}\Phi(t,s)dsdt,$$

$$x>a,y<d$$

$$J^{\alpha,\beta}_{b-,c+}\Phi(x,y)=$$

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_x^b\int_c^y (t-x)^{\alpha-1}(y-s)^{\beta-1}\Phi(t,s)dsdt,$$

$$x<b,y>c$$

and

$$J^{\alpha,\beta}_{b-,d-}\Phi(x,y)=$$

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_x^b\int_y^d (t-x)^{\alpha-1}(s-y)^{\beta-1}\Phi(t,s)dsdt,$$

$$x<b,y<d$$

respectively. Here, Γ is the Gamma function,

$$J^{0,0}_{a+,c+}\Phi(x,y)=J^{0,0}_{a+,d-}\Phi(x,y)$$

$$=J^{0,0}_{b-,c+}\Phi(x,y)=J^{0,0}_{b-,d-}\Phi(x,y)=\Phi(x,y)$$

and

$$J^{1,1}_{a+,c+}\Phi(x,y)=\int_a^x\int_c^y \Phi(t,s)dsdt.$$

Similar to Definition 1 and Definition 3, we introduce the following fractional integrals:

$$
\begin{aligned}
&J^{\alpha}_{a+}\Phi\left(x,\frac{c+d}{2}\right)\\
&=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\Phi\left(t,\frac{c+d}{2}\right)dt,\quad x>a;
\end{aligned}
$$

$$J_{b-}^{\alpha}\Phi\left(x,\frac{c+d}{2}\right)$$
$$=\frac{1}{\Gamma(\alpha)}\int_{x}^{b}(t-x)^{\alpha-1}\Phi\left(t,\frac{c+d}{2}\right)dt, x<b;$$
$$J_{c+}^{\beta}\Phi\left(\frac{a+b}{2},y\right)$$
$$=\frac{1}{\Gamma(\beta)}\int_{c}^{y}(y-s)^{\beta-1}\Phi\left(\frac{a+b}{2},s\right)ds,\quad y>c;$$
$$J_{d-}^{\beta}\Phi\left(\frac{a+b}{2},y\right)$$
$$=\frac{1}{\Gamma(\beta)}\int_{y}^{d}(s-y)^{\beta-1}\Phi\left(\frac{a+b}{2},s\right)ds,\quad y<d.$$

It is remarkable that Sarikaya et al.([26]) and ([28]) gave the following interesting integral inequalities of Hermite-Hadamard-type involving Riemann-Liouville fractional integrals by using convex functions of 2-variables on the co-ordinates.

Theorem 6. *Let $\Phi:\Delta\to\mathbb{R}$ be co-ordinated convex on Δ, with $0\le a<b$, $0\le c<d$ and $\Phi\in L_1(\Delta)$. Then one has the inequalities:*

$$\Phi\left(\frac{a+b}{2},\frac{c+d}{2}\right) \tag{7}$$
$$\le \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}$$
$$\times\left[J_{a+,c+}^{\alpha,\beta}\Phi(b,d)+J_{a+,d-}^{\alpha,\beta}\Phi(b,c)\right.$$
$$\left.+J_{b-,c+}^{\alpha,\beta}\Phi(a,d)+J_{b-,d-}^{\alpha,\beta}\Phi(a,c)\right]$$
$$\le \frac{\Phi(a,c)+\Phi(a,d)+\Phi(b,c)+\Phi(b,d)}{4}.$$

For some recent results connected with fractional integral inequalities, see ([5]- [9], [23]- [26]).

The main aim of this paper is to establish new results on Hermite-Hadamard-Fejer type inequalities for co-ordinated convex functions on the rectangle Δ introduced in the first section of this paper. We will use the Riemann-Liouville integral operators to prove our main results.

2. Hermite-Hadamard-Fejer type inequalities for fractional integrals

In this section, using Riemann-Liouville fractional integral operators, we establish new results on Hermite-Hadamard-Fejer type inequalities for co-ordinated convex functions. We present evidence by using two different methods. We begin by the following theorem:

Theorem 7. *Let $\Phi:\Delta\to\mathbb{R}$ be a co-ordinated convex function such that $\Phi\in L_1(\Delta)$. If $\Psi:\Delta\to\mathbb{R}$ is nonnegative, integrable and symmetric with respect to $\frac{a+b}{2},\frac{c+d}{2}$ on the co-ordinates, then for any $\alpha,\beta>0$ with $a,c\ge 0$, the following integral inequalities hold*

$$\Phi\left(\frac{a+b}{2},\frac{c+d}{2}\right) \tag{8}$$
$$\times\left[J_{a^+c^+}^{\alpha,\beta}\Psi(b,d)+J_{a^+d^-}^{\alpha,\beta}\Psi(b,c)\right.$$
$$\left.+J_{b^-c^+}^{\alpha,\beta}\Psi(a,d)+J_{b^-d^-}^{\alpha,\beta}\Psi(a,c)\right]$$
$$\le \frac{1}{4}\left[J_{a^+c^+}^{\alpha,\beta}(\Phi\Psi)(b,d)+J_{a^+d^-}^{\alpha,\beta}(\Phi\Psi)(b,c)\right.$$
$$\left.+J_{b^-c^+}^{\alpha,\beta}(\Phi\Psi)(a,d)+J_{b^-d^-}^{\alpha,\beta}(\Phi\Psi)(a,c)\right]$$
$$\le \frac{\Phi(a,c)+\Phi(a,d)+\Phi(b,c)+\Phi(b,d)}{4}$$
$$\times\left[J_{a^+c^+}^{\alpha,\beta}\Psi(b,d)+J_{a^+d^-}^{\alpha,\beta}\Psi(b,c)\right.$$
$$\left.+J_{b^-c^+}^{\alpha,\beta}\Psi(a,d)+J_{b^-d^-}^{\alpha,\beta}\Psi(a,c)\right].$$

Proof. Since Φ is a convex function on Δ, then, for all $(t,s)\in[0,1]\times[0,1]$, we can write:

$$\Phi\left(\frac{a+b}{2},\frac{c+d}{2}\right) \tag{9}$$
$$=\Phi\left(\frac{ta+(1-t)b+(1-t)a+tb}{2},\right.$$
$$\left.\frac{sc+(1-s)d+(1-s)c+sd}{2}\right)$$
$$\le \frac{1}{4}[\Phi(ta+(1-t)b,sc+(1-s)d)$$
$$+\Phi(ta+(1-t)b,(1-s)c+sd)$$
$$+\Phi((1-t)a+tb,sc+(1-s)d)$$
$$+\Phi((1-t)a+tb,(1-s)c+sd)].$$

Multiplying both sides of (9) by $t^{\alpha-1}s^{\beta-1}\Psi((1-t)a+tb,(1-s)c+sd)$, and integrating the resulting inequality with respect to (t,s) on $[0,1]\times[0,1]$, we obtain

$$\Phi\left(\frac{a+b}{2},\frac{c+d}{2}\right)$$

$$\times \int_0^1\int_0^1 t^{\alpha-1}s^{\beta-1}\Psi\left((1-t)a+tb,(1-s)c+sd\right)dsdt$$

$$\leq \frac{1}{4}\int_0^1\int_0^1 t^{\alpha-1}s^{\beta-1}\left[\Phi\left(ta+(1-t)b,sc+(1-s)d\right)\right.$$

$$+\Phi\left(ta+(1-t)b,(1-s)c+sd\right)$$

$$+\Phi\left((1-t)a+tb,sc+(1-s)d\right)$$

$$\left.+\Phi\left((1-t)a+tb,(1-s)c+sd\right)\right]$$

$$\times\Psi\left((1-t)a+tb,(1-s)c+sd\right)dsdt$$

$$= \frac{1}{4}\int_0^1\int_0^1 t^{\alpha-1}s^{\beta-1}\Phi\left(ta+(1-t)b,sc+(1-s)d\right)$$

$$\times\Psi\left((1-t)a+tb,(1-s)c+sd\right)dsdt$$

$$+\int_0^1\int_0^1 t^{\alpha-1}s^{\beta-1}\Phi\left(ta+(1-t)b,(1-s)c+sd\right)$$

$$\times\Psi\left((1-t)a+tb,(1-s)c+sd\right)dsdt$$

$$+\int_0^1\int_0^1 t^{\alpha-1}s^{\beta-1}\Phi\left((1-t)a+tb,sc+(1-s)d\right)$$

$$\times\Psi\left((1-t)a+tb,(1-s)c+sd\right)dsdt$$

$$+\int_0^1\int_0^1 t^{\alpha-1}s^{\beta-1}\Phi\left((1-t)a+tb,(1-s)c+sd\right)$$

$$\times\Psi\left((1-t)a+tb,(1-s)c+sd\right)dsdt.$$

Setting $x=tb+(1-t)a$, $y=sd+(1-s)c$ and $dx=(b-a)dt$, $dy=(d-c)ds$, we obtain:

$$\frac{1}{(b-a)^{\alpha}(d-c)^{\beta}}\Phi\left(\frac{a+b}{2},\frac{c+d}{2}\right)$$

$$\times\int_a^b\int_c^d (x-a)^{\alpha-1}(y-c)^{\beta-1}\Psi(x,y)\,dydx$$

$$\leq \frac{1}{4(b-a)^{\alpha}(d-c)^{\beta}}$$

$$\times\left\{\int_a^b\int_c^d (x-a)^{\alpha-1}(y-c)^{\beta-1}\right.$$

$$\times\Phi(a+b-x,c+d-y)\Psi(x,y)\,dydx$$

$$+\int_a^b\int_c^d (x-a)^{\alpha-1}(d-y)^{\beta-1}$$

$$\times\Phi(a+b-x,y)\Psi(x,y)\,dydx$$

$$+\int_a^b\int_c^d (b-x)^{\alpha-1}(y-c)^{\beta-1}$$

$$\times\Phi(x,c+d-y)\Psi(x,y)\,dydx$$

$$\left.+\int_a^b\int_c^d (b-x)^{\alpha-1}(d-y)^{\beta-1}\Phi(x,y)\Psi(x,y)\,dydx\right\}$$

$$= \frac{1}{4(b-a)^{\alpha}(d-c)^{\beta}}$$

$$\times\left\{\int_a^b\int_c^d (b-x)^{\alpha-1}(y-c)^{\beta-1}\right.$$

$$\times\Phi(x,c+d-y)\Psi(a+b-x,y)\,dydx$$

$$+\int_a^b\int_c^d (b-x)^{\alpha-1}(d-y)^{\beta-1}$$

$$\times\Phi(x,y)\Psi(a+b-x,y)\,dydx$$

$$+\int_a^b\int_c^d (b-x)^{\alpha-1}(y-c)^{\beta-1}$$

$$\times\Phi(x,c+d-y)\Psi(x,y)\,dydx$$

$$\left.+\int_a^b\int_c^d (b-x)^{\alpha-1}(d-y)^{\beta-1}\Phi(x,y)\Psi(x,y)\,dydx\right\}$$

$$= \frac{1}{4(b-a)^{\alpha}(d-c)^{\beta}} \left\{ \int_a^b \int_c^d (b-x)^{\alpha-1}(y-c)^{\beta-1} \times \Phi(x, c+d-y)\,\Psi(x,y)\,dydx + \int_a^b \int_c^d (b-x)^{\alpha-1}(d-y)^{\beta-1}\Phi(x,y)\,\Psi(x,y)\,dydx + \int_a^b \int_c^d (b-x)^{\alpha-1}(y-c)^{\beta-1} \times \Phi(x, c+d-y)\,\Psi(x,y)\,dydx + \int_a^b \int_c^d (b-x)^{\alpha-1}(d-y)^{\beta-1}\Phi(x,y)\,\Psi(x,y)\,dydx \right\}.$$

Therefore,

$$\begin{aligned} & \frac{\Gamma(\alpha)\Gamma(\beta)}{(b-a)^{\alpha}(d-c)^{\beta}} \Phi\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ & \times \left[J^{\alpha,\beta}_{a^+c^+}\Psi(b,d) + J^{\alpha,\beta}_{a^+d^-}\Psi(b,c) \right. \\ & \left. + J^{\alpha,\beta}_{b^-c^+}\Psi(a,d) + J^{\alpha,\beta}_{b^-d^-}\Psi(a,c) \right] \\ \leq & \frac{\Gamma(\alpha)\Gamma(\beta)}{4(b-a)^{\alpha}(d-c)^{\beta}} \\ & \times \left[J^{\alpha,\beta}_{a^+c^+}(\Phi\Psi)(b,d) + J^{\alpha,\beta}_{a^+d^-}(\Phi\Psi)(b,c) \right. \\ & \left. + J^{\alpha,\beta}_{b^-c^+}(\Phi\Psi)(a,d) + J^{\alpha,\beta}_{b^-d^-}(\Phi\Psi)(a,c) \right]. \end{aligned}$$

The first inequality of (8) is thus proved.
We shall prove the second inequality of (8): Since f is a convex function on Δ, then, for all $(t,s) \in [0,1]\times[0,1]$, it yields

$$\begin{aligned} & \Phi(ta+(1-t)b, sc+(1-s)d) \\ & + \Phi(ta+(1-t)b, (1-s)c+sd) \\ & + \Phi((1-t)a+tb, sc+(1-s)d) \\ & + \Phi((1-t)a+tb, (1-s)c+sd) \qquad (10) \\ \leq & \Phi(a,c)+\Phi(b,c)+\Phi(a,d)+\Phi(b,d). \end{aligned}$$

Then, multiplying both sides of (10) by $t^{\alpha-1}s^{\beta-1}\Psi(tb+(1-t)a, sd+(1-s)c)$ and integrating the resulting inequality with respect to (t,s) over $[0,1]\times[0,1]$, we get

$$\begin{aligned} & \int_0^1\int_0^1 t^{\alpha-1}s^{\beta-1}[\Phi(ta+(1-t)b, sc+(1-s)d) \\ & + \Phi(ta+(1-t)b, (1-s)c+sd) \\ & + \Phi((1-t)a+tb, sc+(1-s)d) \\ & + \Phi((1-t)a+tb, (1-s)c+sd)] \\ & \times \Psi(tb+(1-t)a, sd+(1-s)c)\,dsdt \\ \leq & [\Phi(a,c)+\Phi(b,c)+\Phi(a,d)+\Phi(b,d)] \\ \times & \int_0^1\int_0^1 t^{\alpha-1}s^{\beta-1}\Psi(tb+(1-t)a, sd+(1-s)c)\,dsdt. \end{aligned}$$

That is,

$$\begin{aligned} & \frac{1}{4}\left[J^{\alpha,\beta}_{a^+c^+}(\Phi\Psi)(b,d) + J^{\alpha,\beta}_{a^+d^-}(\Phi\Psi)(b,c) \right. \\ & \left. + J^{\alpha,\beta}_{b^-c^+}(\Phi\Psi)(a,d) + J^{\alpha,\beta}_{b^-d^-}(\Phi\Psi)(a,c) \right] \\ \leq & \frac{\Phi(a,c)+\Phi(a,d)+\Phi(b,c)+\Phi(b,d)}{4} \\ & \times \left[J^{\alpha,\beta}_{a^+c^+}\Psi(b,d) + J^{\alpha,\beta}_{a^+d^-}\Psi(b,c) \right. \\ & \left. + J^{\alpha,\beta}_{b^-c^+}\Psi(a,d) + J^{\alpha,\beta}_{b^-d^-}\Psi(a,c) \right]. \end{aligned}$$

The proof of Theorem 7 is thus achieved. □

Remark 1. *In Theorem 7:*
(i) If we take $\alpha=\beta=1$, then the inequality (8) becomes the inequality (6) of Theorem 5.
(ii) If we take $\Psi(x,y)=1$, then (8) becomes (7) of Theorem 6.

We prove also the following result:

Theorem 8. *Let $\Phi : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ be a co-ordinated convex function on Δ, with $a,c \geq 0$, $\alpha,\beta > 0$ and $\Phi \in L_1(\Delta)$. If $\Psi : \Delta \to \mathbb{R}$ is non-negative, integrable and symmetric with respect to $\frac{a+b}{2}$ and $\frac{c+d}{2}$ on the co-ordinates, then we have:*

$$\begin{aligned} & \Phi\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \times \left[J^{\alpha,\beta}_{a+,c+}\Psi(b,d) + J^{\alpha,\beta}_{a+,d-}\Psi(b,c) \right. \\ & \left. + J^{\alpha,\beta}_{b-,c+}\Psi(a,d) + J^{\alpha,\beta}_{b-,d-}\Psi(a,c) \right] \qquad (11) \end{aligned}$$

$$
\begin{aligned}
&\leq J_{a+}^{\alpha}\left[\Phi\left(b,\frac{c+d}{2}\right)J_{c+}^{\beta}\Psi(b,d)\right]\\
&+J_{a+}^{\alpha}\left[\Phi\left(b,\frac{c+d}{2}\right)J_{d-}^{\beta}\Psi(b,c)\right]\\
&+J_{b-}^{\alpha}\left[\Phi\left(a,\frac{c+d}{2}\right)J_{c+}^{\beta}\Psi(a,d)\right]\\
&+J_{b-}^{\alpha}\left[\Phi\left(a,\frac{c+d}{2}\right)J_{d-}^{\beta}\Psi(a,c)\right]\\
&+J_{c+}^{\beta}\left[\Phi\left(\frac{a+b}{2},d\right)J_{a+}^{\alpha}\Psi(b,d)\right]\\
&+J_{c+}^{\beta}\left[\Phi\left(\frac{a+b}{2},d\right)J_{b-}^{\alpha}\Psi(a,d)\right]\\
&+J_{d-}^{\beta}\left[\Phi\left(\frac{a+b}{2},c\right)J_{a+}^{\alpha}\Psi(b,c)\right]\\
&+J_{d-}^{\beta}\left[\Phi\left(\frac{a+b}{2},c\right)J_{b-}^{\alpha}\Psi(a,c)\right]\\
\leq\ & 2\left[J_{a+,c+}^{\alpha,\beta}\left(\Phi\Psi\right)(b,d)+J_{a+,d-}^{\alpha,\beta}\left(\Phi\Psi\right)(b,c)\right.\\
&\left.+J_{b-,c+}^{\alpha,\beta}\left(\Phi\Psi\right)(a,d)+J_{b-,d-}^{\alpha,\beta}\left(\Phi\Psi\right)(a,c)\right]
\end{aligned}
$$

$$
\begin{aligned}
&\leq J_{a+}^{\alpha}\left[\Phi\left(b,c\right)J_{c+}^{\beta}\Psi(b,d)\right]\\
&+J_{a+}^{\alpha}\left[\Phi\left(b,d\right)J_{d-}^{\beta}\Psi(b,c)\right]\\
&+J_{b-}^{\alpha}\left[\Phi\left(a,c\right)J_{c+}^{\beta}\Psi(a,d)\right]\\
&+J_{b-}^{\alpha}\left[\Phi\left(a,d\right)J_{d-}^{\beta}\Psi(a,c)\right]\\
&+J_{c+}^{\beta}\left[\Phi\left(a,d\right)J_{a+}^{\alpha}\Psi(b,d)\right]\\
&+J_{c+}^{\beta}\left[\Phi\left(b,d\right)J_{b-}^{\alpha}\Psi(a,d)\right]\\
&+J_{d-}^{\beta}\left[\Phi\left(a,c\right)J_{a+}^{\alpha}\Psi(b,c)\right]\\
&+J_{d-}^{\beta}\left[\Phi\left(b,c\right)J_{b-}^{\alpha}\Psi(a,c)\right]\\
\leq\ & \frac{\Phi\left(a,c\right)+\Phi\left(a,d\right)+\Phi\left(b,c\right)+\Phi\left(b,d\right)}{4}\\
&\times\left[J_{a+,c+}^{\alpha,\beta}\Psi(b,d)+J_{a+,d-}^{\alpha,\beta}\Psi(b,c)\right.\\
&\left.+J_{b-,c+}^{\alpha,\beta}\Psi(a,d)+J_{b-,d-}^{\alpha,\beta}\Psi(a,c)\right].
\end{aligned}
$$

Proof. Since $\Phi:\Delta\to\mathbb{R}$ is convex on the co-ordinates, it follows that the mapping $F_x:[c,d]\to\mathbb{R}$, $F_x(y)=\Phi(x,y)$, is convex on $[c,d]$ and the mapping $G_x:[c,d]\to\mathbb{R}$, $G_x(y)=\Psi(x,y)$ is nonnegative, integrable and symmetric with respect to $\frac{c+d}{2}$, for all $x\in[a,b]$. Then, thanks to the inequalities (3), we can write

$$
\begin{aligned}
&F_x\left(\frac{c+d}{2}\right)\left[J_{c+}^{\beta}G_x(d)+J_{d-}^{\beta}G_x(c)\right]\\
\leq\ & J_{c+}^{\beta}\left(F_xG_x\right)(d)+J_{d-}^{\beta}\left(F_xG_x\right)(c)\\
\leq\ & \frac{F_x\left(c\right)+F_x\left(d\right)}{2}\left[J_{c+}^{\beta}G_x(d)+J_{d-}^{\beta}G_x(c)\right],\\
&x\in[a,b].
\end{aligned}
$$

That is,

$$
\begin{aligned}
&\Phi\left(x,\frac{c+d}{2}\right)\frac{1}{\Gamma(\beta)}\left[\int_c^d(d-y)^{\beta-1}\Psi\left(x,y\right)dy\right.\\
&\left.+\int_c^d(y-c)^{\beta-1}\Psi\left(x,y\right)dy\right]\\
\leq\ & \frac{1}{\Gamma(\beta)}\left[\int_c^d(d-y)^{\beta-1}\Psi\left(x,y\right)\Phi\left(x,y\right)dy\right.\\
&\left.+\int_c^d(y-c)^{\beta-1}\Psi\left(x,y\right)\Phi\left(x,y\right)dy\right]\\
\leq\ & \frac{\Phi\left(x,c\right)+\Phi\left(x,d\right)}{2}\frac{1}{\Gamma(\beta)}\left[\int_c^d(d-y)^{\beta-1}\Psi\left(x,y\right)dy\right.\\
&\left.+\int_c^d(y-c)^{\beta-1}\Psi\left(x,y\right)dy\right],
\end{aligned}
\tag{12}
$$

for all $x\in[a,b]$.

Multiplying both sides of (12) by $\frac{(b-x)^{\alpha-1}}{\Gamma(\alpha)}$ and $\frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)}$, and integrating with respect to x over $[a,b]$, respectively, we have

$$
\begin{aligned}
&\frac{1}{\Gamma\left(\alpha\right)\Gamma(\beta)}\int_a^b\int_c^d(b-x)^{\alpha-1}(d-y)^{\beta-1}\\
&\times\Phi\left(x,\frac{c+d}{2}\right)\Psi\left(x,y\right)dydx\\
&+\frac{1}{\Gamma\left(\alpha\right)\Gamma(\beta)}\int_a^b\int_c^d(b-x)^{\alpha-1}(y-c)^{\beta-1}\\
&\times\Phi\left(x,\frac{c+d}{2}\right)\Psi\left(x,y\right)dydx
\end{aligned}
\tag{13}
$$

$$\leq \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\left[\int_a^b\int_c^d (b-x)^{\alpha-1}(d-y)^{\beta-1} \times\Phi(x,y)\,\Psi(x,y)\,dydx + \int_a^b\int_c^d (b-x)^{\alpha-1}(y-c)^{\beta-1}\Phi(x,y)\,\Psi(x,y)\,dydx\right] \tag{14}$$

$$\leq \frac{1}{2\Gamma(\alpha)\Gamma(\beta)}\left[\int_a^b\int_c^d (b-x)^{\alpha-1}(d-y)^{\beta-1} \times\Phi(x,c)\,\Psi(x,y)\,dydx + \int_a^b\int_c^d (b-x)^{\alpha-1}(y-c)^{\beta-1}\Phi(x,c)\,\Psi(x,y)\,dydx + \int_a^b\int_c^d (b-x)^{\alpha-1}(d-y)^{\beta-1}\Phi(x,d)\,\Psi(x,y)\,dydx + \int_a^b\int_c^d (b-x)^{\alpha-1}(y-c)^{\beta-1}\Phi(x,d)\,\Psi(x,y)\,dydx\right].$$

and

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d (x-a)^{\alpha-1}(d-y)^{\beta-1} \times\Phi\left(x,\frac{c+d}{2}\right)\Psi(x,y)\,dydx \tag{15}$$

$$+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d (x-a)^{\alpha-1}(y-c)^{\beta-1} \times\Phi\left(x,\frac{c+d}{2}\right)\Psi(x,y)\,dydx$$

$$\leq \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\times\left[\int_a^b\int_c^d (x-a)^{\alpha-1}(d-y)^{\beta-1} \times\Phi(x,y)\,\Psi(x,y)\,dydx + \int_a^b\int_c^d (x-a)^{\alpha-1}(y-c)^{\beta-1} \times\Phi(x,y)\,\Psi(x,y)\,dydx\right]$$

$$\leq \frac{1}{2\Gamma(\alpha)\Gamma(\beta)}\left[\int_a^b\int_c^d (x-a)^{\alpha-1}(d-y)^{\beta-1} \times\Phi(x,c)\,\Psi(x,y)\,dydx + \int_a^b\int_c^d (x-a)^{\alpha-1}(y-c)^{\beta-1}\Phi(x,c)\,\Psi(x,y)\,dydx + \int_a^b\int_c^d (x-a)^{\alpha-1}(d-y)^{\beta-1}\Phi(x,d)\,\Psi(x,y)\,dydx + \int_a^b\int_c^d (x-a)^{\alpha-1}(y-c)^{\beta-1}\Phi(x,d)\,\Psi(x,y)\,dydx\right].$$

For the mappings $F_y:[a,b]\to\mathbb{R}$, $F_y(x)=\Phi(x,y)$ and $G_y:[a,b]\to\mathbb{R}$, $G_y(x)=\Psi(x,y)$, we use the same arguments as before. So, we can state that

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d (b-x)^{\alpha-1}(d-y)^{\beta-1} \times\Phi\left(\frac{a+b}{2},y\right)\Psi(x,y)\,dydx \tag{16}$$

$$+ \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d (x-a)^{\alpha-1}(d-y)^{\beta-1} \times\Phi\left(\frac{a+b}{2},y\right)\Psi(x,y)\,dydx$$

$$\leq \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\left[\int_a^b\int_c^d (b-x)^{\alpha-1}(d-y)^{\beta-1} \times\Phi(x,y)\,\Psi(x,y)\,dydx + \int_a^b\int_c^d (x-a)^{\alpha-1}(d-y)^{\beta-1}\Phi(x,y)\,\Psi(x,y)\,dydx\right]$$

$$\leq \frac{1}{2\Gamma(\alpha)\Gamma(\beta)}\left[\int_a^b\int_c^d (b-x)^{\alpha-1}(d-y)^{\beta-1} \times\Phi(a,y)\,\Psi(x,y)\,dydx + \int_a^b\int_c^d (x-a)^{\alpha-1}(d-y)^{\beta-1}\Phi(a,y)\,\Psi(x,y)\,dydx + \int_a^b\int_c^d (b-x)^{\alpha-1}(d-y)^{\beta-1}\Phi(b,y)\,\Psi(x,y)\,dydx + \int_a^b\int_c^d (x-a)^{\alpha-1}(d-y)^{\beta-1}\Phi(b,y)\,\Psi(x,y)\,dydx\right]$$

and

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d (b-x)^{\alpha-1}(y-c)^{\beta-1} \times\Phi\left(\frac{a+b}{2},y\right)\Psi(x,y)\,dydx \tag{17}$$

$$+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d (x-a)^{\alpha-1}(y-c)^{\beta-1} \times\Phi\left(\frac{a+b}{2},y\right)\Psi(x,y)\,dydx$$

$$\leq \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\left[\int_a^b\int_c^d (b-x)^{\alpha-1}(y-c)^{\beta-1} \times\Phi(x,y)\,\Psi(x,y)\,dydx + \int_a^b\int_c^d (x-a)^{\alpha-1}(y-c)^{\beta-1}\Phi(x,y)\,\Psi(x,y)\,dydx\right]$$

$$\leq \frac{1}{2\Gamma(\alpha)\Gamma(\beta)}\left[\int_a^b\int_c^d (b-x)^{\alpha-1}(y-c)^{\beta-1} \times\Phi(a,y)\,\Psi(x,y)\,dydx + \int_a^b\int_c^d (x-a)^{\alpha-1}(y-c)^{\beta-1}\Phi(a,y)\,\Psi(x,y)\,dydx + \int_a^b\int_c^d (b-x)^{\alpha-1}(y-c)^{\beta-1}\Phi(b,y)\,\Psi(x,y)\,dydx + \int_a^b\int_c^d (x-a)^{\alpha-1}(y-c)^{\beta-1} \times\Phi(b,y)\,\Psi(x,y)\,dydx\right].$$

Adding the inequalities (13)-(17), we can write

$$\begin{aligned} & J^{\alpha}_{a+}\left[\Phi\left(b,\frac{c+d}{2}\right)J^{\beta}_{c+}\Psi(b,d)\right] \\ & +J^{\alpha}_{a+}\left[\Phi\left(b,\frac{c+d}{2}\right)J^{\beta}_{d-}\Psi(b,c)\right] \\ & +J^{\alpha}_{b-}\left[\Phi\left(a,\frac{c+d}{2}\right)J^{\beta}_{c+}\Psi(a,d)\right] \\ & +J^{\alpha}_{b-}\left[\Phi\left(a,\frac{c+d}{2}\right)J^{\beta}_{d-}\Psi(a,c)\right] \\ & +J^{\beta}_{c+}\left[\Phi\left(\frac{a+b}{2},d\right)J^{\alpha}_{a+}\Psi(b,d)\right] \\ & +J^{\beta}_{c+}\left[\Phi\left(\frac{a+b}{2},d\right)J^{\alpha}_{b-}\Psi(a,d)\right] \\ & +J^{\beta}_{d-}\left[\Phi\left(\frac{a+b}{2},c\right)J^{\alpha}_{a+}\Psi(b,c)\right] \\ & +J^{\beta}_{d-}\left[\Phi\left(\frac{a+b}{2},c\right)J^{\alpha}_{b-}\Psi(a,c)\right] \\ \leq\ & 2\left[J^{\alpha,\beta}_{a+,c+}(\Phi\Psi)(b,d)+J^{\alpha,\beta}_{a+,d-}(\Phi\Psi)(b,c)\right. \\ & \left.+J^{\alpha,\beta}_{b-,c+}(\Phi\Psi)(a,d)+J^{\alpha,\beta}_{b-,d-}(\Phi\Psi)(a,c)\right] \\ \leq\ & J^{\alpha}_{a+}\left[\Phi(b,c)\,J^{\beta}_{c+}\Psi(b,d)\right] \\ & +J^{\alpha}_{a+}\left[\Phi(b,d)\,J^{\beta}_{d-}\Psi(b,c)\right] \\ & +J^{\alpha}_{b-}\left[\Phi(a,c)\,J^{\beta}_{c+}\Psi(a,d)\right] \\ & +J^{\alpha}_{b-}\left[\Phi(a,d)\,J^{\beta}_{d-}\Psi(a,c)\right] \\ & +J^{\beta}_{c+}\left[\Phi(a,d)\,J^{\alpha}_{a+}\Psi(b,d)\right] \\ & +J^{\beta}_{c+}\left[\Phi(b,d)\,J^{\alpha}_{b-}\Psi(a,d)\right] \\ & +J^{\beta}_{d-}\left[\Phi(a,c)\,J^{\alpha}_{a+}\Psi(b,c)\right] \\ & +J^{\beta}_{d-}\left[\Phi(b,c)\,J^{\alpha}_{b-}\Psi(a,c)\right]. \end{aligned}$$

These give the second and the third inequalities in (11).

Now, by using the first inequality in (3), it yields that

$$\begin{aligned} & \Phi\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\ \times\ & \left[\int_a^b\int_c^d (b-x)^{\alpha-1}(d-y)^{\beta-1}\Psi(x,y)\,dydx\right. \\ +\ & \left.\int_a^b\int_c^d (x-a)^{\alpha-1}(d-y)^{\beta-1}\Psi(x,y)\,dydx\right] \\ \leq\ & \int_a^b\int_c^d (b-x)^{\alpha-1}(d-y)^{\beta-1} \\ & \times\Phi\left(x,\frac{c+d}{2}\right)\Psi(x,y)\,dydx \\ +\ & \int_a^b\int_c^d (x-a)^{\alpha-1}(d-y)^{\beta-1}\Phi\left(x,\frac{c+d}{2}\right)\Psi(x,y)\,dydx \end{aligned}$$

and

$$\begin{aligned} & \Phi\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\ \times\ & \left[\int_a^b\int_c^d (b-x)^{\alpha-1}(d-y)^{\beta-1}\Psi(x,y)\,dydx\right. \\ +\ & \left.\int_a^b\int_c^d (b-x)^{\alpha-1}(y-c)^{\beta-1}\Psi(x,y)\,dydx\right] \\ \leq\ & \int_a^b\int_c^d (b-x)^{\alpha-1}(d-y)^{\beta-1} \\ & \times\Phi\left(\frac{a+b}{2},y\right)\Psi(x,y)\,dydx \\ +\ & \int_a^b\int_c^d (b-x)^{\alpha-1}(y-c)^{\beta-1} \\ & \times\Phi\left(\frac{a+b}{2},y\right)\Psi(x,y)\,dydx. \end{aligned}$$

By addition, and using the fact that Ψ is symmetric, we get

$$
\begin{aligned}
&\Phi\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
\times\ &\left[J^{\alpha,\beta}_{a+,c+}\Psi(b,d)+J^{\alpha,\beta}_{a+,d-}\Psi(b,c)\right.\\
&\left.+J^{\alpha,\beta}_{b-,c+}\Psi(a,d)+J^{\alpha,\beta}_{b-,d-}\Psi(a,c)\right]\\
\leq\ &J^{\alpha}_{a+}\left[\Phi\left(b,\frac{c+d}{2}\right)J^{\beta}_{c+}\Psi(b,d)\right]\\
&+J^{\alpha}_{a+}\left[\Phi\left(b,\frac{c+d}{2}\right)J^{\beta}_{d-}\Psi(b,c)\right]\\
&+J^{\alpha}_{b-}\left[\Phi\left(a,\frac{c+d}{2}\right)J^{\beta}_{c+}\Psi(a,d)\right]\\
&+J^{\alpha}_{b-}\left[\Phi\left(a,\frac{c+d}{2}\right)J^{\beta}_{d-}\Psi(a,c)\right]\\
&+J^{\beta}_{c+}\left[\Phi\left(\frac{a+b}{2},d\right)J^{\alpha}_{a+}\Psi(b,d)\right]\\
&+J^{\beta}_{c+}\left[\Phi\left(\frac{a+b}{2},d\right)J^{\alpha}_{b-}\Psi(a,d)\right]\\
&+J^{\beta}_{d-}\left[\Phi\left(\frac{a+b}{2},c\right)J^{\alpha}_{a+}\Psi(b,c)\right]\\
&+J^{\beta}_{d-}\left[\Phi\left(\frac{a+b}{2},c\right)J^{\alpha}_{b-}\Psi(a,c)\right]
\end{aligned}
$$

which gives the first inequality in (11).

Finally, by using the second inequality in (3), we can state that:

$$
\begin{aligned}
&\frac{\alpha}{2(b-a)^{\alpha}}\left[\int_a^b(b-x)^{\alpha-1}\Phi(x,c)\,dx\right.\\
&\left.+\int_a^b(x-a)^{\alpha-1}\Phi(x,c)\,dx\right]\\
\leq\ &\frac{\Phi(a,c)+\Phi(b,c)}{2},\\
&\frac{\alpha}{2(b-a)^{\alpha}}\left[\int_a^b(b-x)^{\alpha-1}\Phi(x,d)\,dx\right.\\
&\left.+\int_a^b(x-a)^{\alpha-1}\Phi(x,d)\,dx\right]\\
\leq\ &\frac{\Phi(a,d)+\Phi(b,d)}{2},\\
&\frac{\beta}{2(d-c)^{\beta}}\left[\int_c^d(d-y)^{\beta-1}\Phi(a,y)\,dy\right.\\
&\left.+\int_c^d(y-c)^{\beta-1}\Phi(a,y)\,dy\right]\\
\leq\ &\frac{\Phi(a,c)+\Phi(a,d)}{2},\\
&\frac{\beta}{2(d-c)^{\beta}}\left[\int_c^d(d-y)^{\beta-1}\Phi(b,y)\,dy\right.\\
&\left.+\int_c^d(y-c)^{\beta-1}\Phi(b,y)\,dy\right]\\
\leq\ &\frac{\Phi(b,c)+\Phi(b,d)}{2}.
\end{aligned}
$$

By addition, we get the last inequality in (11). □

Remark 2. *In Theorem 8, if we take $\alpha=\beta=1$, then the inequalities (11) become (5).*

3. Conclusion

In this paper, we established the Hermite-Hadamard-Fejer type inequalities for co-ordinated mappings related results to present new type of inequalities involving Riemann-Liouville integral operator. The results presented in this paper would provide generalizations of those given in earlier works. The findings of this study have several significant implications for future applications.

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