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NEW INTEGRAL INEQUALITIES OF FENG QI TYPE VIA RIEMANN-LIOUVILLE FRACTIONAL INTEGRATION

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Abstract. In this paper, we use the Riemann-Liouville integral operator to generate recent fractional integral inequalities of Qi type. Other inequalities are also presented.

1. Introduction

In [6], Ngo et al. proved that

$$\int_0^1 f^{\delta+1}(\tau)d\tau \geq \int_0^1 \tau^\delta f(\tau)d\tau \tag{1.1}$$

and

$$\int_0^1 f^{\delta+1}(\tau)d\tau \geq \int_0^1 \tau f^\delta(\tau)d\tau, \tag{1.2}$$

where $\delta > 0$ and f is a positive continuous function on $[0,1]$ satisfying

$$\int_x^1 f(\tau)d\tau \geq \int_x^1 \tau d\tau, x \in [0,1].$$

s Then, in [4], W.J. Liu, G.S. Cheng and C.C. Li have established a more general result:

$$\int_a^b f^{\alpha+\beta}(\tau)d\tau \geq \int_a^b (\tau-a)^\alpha f^\beta(\tau)d\tau, \tag{1.3}$$

where $\alpha > 0, \beta > 0$ and f is a positive continuous function on $[a,b]$ such that

$$\int_x^b f^\gamma(\tau)d\tau \geq \int_x^b (\tau-a)^\gamma d\tau;\ \gamma := min(1,\beta), x \in [a,b].$$

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Recently, in [12], using the time scales theory, L. Yin and F. Qi proved the following result:

Theorem 1.1: *Let $a, b \in T$. If $f \in C_{rd}(T, \mathbb{R})$ is positive and*

$$\int_a^b f(x)\, \Delta x \geq (b-a)^{p-1}, \tag{1.4}$$

then

$$\int_a^b [f(x)]^p\, \Delta x \geq \left[\int_a^b f(x)\, \Delta x\right]^{p-1}, \tag{1.5}$$

where $p > 1$ or $p < 0$.
If the function f satisfies

$$0 < m \leq (f(x))^p \leq M < \infty,\ x \in [a, b], \tag{1.6}$$

the authors proved

$$\left[\int_a^b [f(x)]^p\, \Delta x\right]^{\frac{1}{p}} \leq (b-a)^{-\frac{p+1}{q}} \left(\frac{M}{m}\right)^{\frac{2}{pq}} \left[\int_a^b (f(x))^{\frac{1}{p}}\, \Delta x\right]^p, \tag{1.7}$$

where $p > 1, \frac{1}{p} + \frac{1}{q} = 1$.

On the other hand, in [11], W.T. Sulaiman established the following result:

Theorem 1.2: *Suppose $f \geq 0, g \geq 0$ on $[a, b]$ and g is non-decreasing. If*

$$\int_x^b f(t)\, dt \geq \int_x^b g(t)\, dt, x \in [a, b], \tag{1.8}$$

then the inequality holds

$$\int_a^b [f(x)]^{\gamma-\delta}\, dx \leq \int_a^b [f(x)]^{\gamma} [g(x)]^{-\delta}\, dx,\ \gamma, \delta > 0, \gamma - \delta > 1. \tag{1.9}$$

The following result, as well, was established in [11]:

Theorem 1.3: *Let $f \geq 0, g \geq 0$ on $[a, b]$ such that f is non-decreasing and g is non-increasing or conversely, then we have the following reverse Chebyshev inequality*

$$\int_a^b [f(x)]^{\gamma} [g(x)]^{\delta}\, dx \leq \frac{1}{b-a} \int_a^b [f(x)]^{\gamma}\, dx \int_a^b [g(x)]^{\delta}\, dx. \tag{1.10}$$

Many researchers have given considerable attention to (1.5) (1.9) and (1.10) and

several inequalities related to these functionals have appeared in the literature, to mention a few, see [1,2,4-7] and the references cited therein.
The main purpose of this paper is to derive some new inequalities using the fractional integral theory. Our results have some relationship with [5, 9, 11, 12]. Some interested inequalities of these references can be deduced as some special cases.

2. Preliminaries

In this section, we give some necessary definitions and properties which will be used in this paper. For more details, see [3, 8, 10].

Definition 3: The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, for a continuous function on $[0,\infty[$ is defined as

$$\begin{aligned} &J^{\alpha}f(t) = \tfrac{1}{\Gamma(\alpha)}\int_a^b (t-x)^{\alpha-1}f(x)dx; \quad \alpha > 0, t > 0 \\ &J^0 f(t) = f(t). \end{aligned} \tag{2.1}$$

For the convenience of establishing the results, we give one basic property

$$J^{\alpha}J^{\beta}f(t) = J^{\alpha+\beta}f(t). \tag{2.2}$$

For the expression (2.1), when $f(t) = t^{\beta}$ we get another expression that will be used later:

$$J^{\alpha}t^{\beta} = \tfrac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}t^{\alpha+\beta}. \tag{2.3}$$

3. Main Results

We start with the following lemmas:

Lemma 3.1: *Let f, g be two positive functions on $[0,\infty[$, then for all $\alpha > 0$, we have*

$$J^{\alpha}\left[\frac{(f(t))^p}{(g(t))^{\frac{p}{q}}}\right] \geq \frac{(J^{\alpha}f(t))^p}{(J^{\alpha}g(t))^{\frac{p}{q}}}, \tag{3.1}$$

where $t > 0, p > 1$ and $\frac{1}{p}+\frac{1}{q} = 1$.

Proof:
For tow functions Φ and Ψ, using the fractional Holder inequality, we can write

$$J^{\alpha}\left|\Phi(t)\Psi(t)\right| \leq \left(J^{\alpha}\left|\Phi(t)\right|^p\right)^{\frac{1}{p}}\left(J^{\alpha}\left|\Psi(t)\right|^q\right)^{\frac{1}{q}}, t > 0 \tag{3.2}$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.
Putting $\Phi(t) = \frac{f(t)}{(g(t))^{\frac{1}{q}}}$ and $\Psi(t) = (g(t))^{\frac{1}{q}}$ in (3.2), we obtain

$$J^{\alpha}(f(t)) = J^{\alpha}\left(\frac{f(t)}{(g(t))^{\frac{1}{q}}}(g(t))^{\frac{1}{q}}\right) \leq \left[J^{\alpha}\left(\frac{(f(x))^{p}}{(g(t))^{\frac{p}{q}}}\right)\right]^{\frac{1}{p}}\left[J^{\alpha}(g(t))\right]^{\frac{1}{q}}. \tag{3.3}$$

Lemma 3.1 is thus proved.

Lemma 3.2: *Let $\alpha > 0, p > 1, \frac{1}{p} + \frac{1}{q} = 1$ and let f and g be two positive functions on $[0, \infty[$, such that $J^{\alpha}f^{p}(t) < \infty, J^{\alpha}g^{q}(t) < \infty, t > 0$.*
If

$$0 < m \leq \frac{f(\tau)}{g(\tau)} \leq M < \infty,\ \tau \in [0, t], \tag{3.4}$$

then the inequality

$$\left[J^{\alpha}f(t)\right]^{\frac{1}{p}}\left[J^{\alpha}g(t)\right]^{\frac{1}{q}} \leq \left(\frac{M}{m}\right)^{\frac{1}{pq}} J^{\alpha}\left[(f(t))^{\frac{1}{p}}(g(t))^{\frac{1}{q}}\right] \tag{3.5}$$

holds.
Proof: Since $\frac{f(\tau)}{g(\tau)} \leq M$, $\tau \in [0, t], t > 0$, therefore,

$$\left[g(\tau)\right]^{\frac{1}{q}} \geq M^{-\frac{1}{q}}\left[f(\tau)\right]^{\frac{1}{q}} \tag{3.6}$$

and so,

$$\left[f(\tau)\right]^{\frac{1}{p}}\left[g(\tau)\right]^{\frac{1}{q}} \geq M^{-\frac{1}{q}}\left[f(\tau)\right]^{\frac{1}{q}}\left[f(\tau)\right]^{\frac{1}{p}} = M^{-\frac{1}{q}}f(\tau). \tag{3.7}$$

Hence, we get

$$\frac{1}{\Gamma(\alpha)}\underline{0}\bar{t}\int (t-\tau)^{\alpha-1}\left[f(\tau)\right]^{\frac{1}{p}}\left[g(\tau)\right]^{\frac{1}{q}}d\tau \geq \frac{1}{\Gamma(\alpha)}\underline{0}\bar{t}\int (t-\tau)^{\alpha-1}M^{-\frac{1}{q}}f(\tau)d\tau \tag{3.8}$$

that is,

$$J^{\alpha}\left[(f(t))^{\frac{1}{p}}(g(t))^{\frac{1}{q}}\right] \geq M^{-\frac{1}{q}}J^{\alpha}f(t). \tag{3.9}$$

Consequently,

$$\left(J^{\alpha}\left[(f(t))^{\frac{1}{p}}(g(t))^{\frac{1}{q}}\right]\right)^{\frac{1}{p}} \geq M^{-\frac{1}{pq}}\left(J^{\alpha}f(t)\right)^{\frac{1}{p}}. \tag{3.10}$$

On the other hand, since $mg(\tau) \leq f(\tau), \tau \in [0, t], t > 0$, then we have

$$\left[f(\tau)\right]^{\frac{1}{p}} \geq m^{\frac{1}{p}}\left[g(\tau)\right]^{\frac{1}{p}} \tag{3.11}$$

and so,

$$[g(\tau)]^{\frac{1}{q}}\,[f(\tau)]^{\frac{1}{p}} \geq m^{\frac{1}{p}}\,[g(\tau)]^{\frac{1}{p}}\,[g(\tau)]^{\frac{1}{q}} = m^{\frac{1}{p}}g(\tau). \tag{3.12}$$

Now, multiplying both sides of (3.12) by $\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}$, $\tau \in (0,t)$, then integrating the resulting inequality with respect to τ over $(0,t)$, we obtain

$$\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\,[f(\tau)]^{\frac{1}{p}}\,[g(\tau)]^{\frac{1}{q}}\,d\tau \geq \frac{m^{\frac{1}{p}}}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\,g(\tau)d\tau. \tag{3.13}$$

Then we have,

$$J^{\alpha}\left[(f(t))^{\frac{1}{p}}\,(g(t))^{\frac{1}{q}}\right] \geq m^{\frac{1}{p}}J^{\alpha}g(t). \tag{3.14}$$

Hence, we can write

$$\left(J^{\alpha}\left[(f(t))^{\frac{1}{p}}\,(g(t))^{\frac{1}{q}}\right]\right)^{\frac{1}{q}} \geq m^{\frac{1}{pq}}\,(J^{\alpha}g(t))^{\frac{1}{q}}. \tag{3.15}$$

Thanks to (3.10) and (3.15), we obtain (3.5).

Remark 3.1: If we take $\alpha = 1$, then Lemma 3.1 becomes Theorem 2.1 in [9] on $[0,t]$.

Lemma 3.3: *Let* $\alpha > 0, f$ *and* g *be two positive functions on* $[0,\infty[$, *such that* $J^{\alpha}f^{p}(t) < \infty, J^{\alpha}g^{q}(t) < \infty; t > 0$.
If

$$0 < m \leq \frac{(f(\tau))^{p}}{(g(\tau))^{q}} \leq M < \infty,\ \tau \in [0,t],$$

then we have

$$[J^{\alpha}f^{p}(t)]^{\frac{1}{p}}\,[J^{\alpha}g^{q}(t)]^{\frac{1}{q}} \leq \left(\frac{M}{m}\right)^{\frac{1}{pq}} J^{\alpha}(f(t)g(t)), \tag{3.16}$$

where $p > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$.
Proof: Replacing $f(\tau)$ and $g(\tau)$ respectively by $(f(\tau))^{p}$ and $(g(\tau))^{q}$, $\tau \in [0,t], t > 0$ in Lemma 3.2, we obtain (3.16).

We give also the following result:

Theorem 3.1: *Let* $\alpha > 0, p > 1$ *and* f *be a positive function on* $[0,\infty[$, *such that for all* $t > 0$,

$$J^{\alpha}(f(t)) \geq \left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right)^{p-1}. \tag{3.17}$$

Then the inequality

$$J^{\alpha}\left(f(t)\right)^{p} \geq \left(J^{\alpha} f(t)\right)^{p-1} \tag{3.18}$$

holds.

Proof: Using Lemma 3.1 and the condition (3.17), we obtain

$$\begin{aligned} J^{\alpha}\left((f(t))^{p}\right) = J^{\alpha}\left(\tfrac{(f(t))^{p}}{1^{p-1}}\right) \geq \tfrac{(J^{\alpha} f(t))^{p}}{(J^{\alpha}1)^{p-1}} = \left(\tfrac{\Gamma(\alpha+1)}{t^{\alpha}}\right)^{p-1}\left(J^{\alpha} f(t)\right)^{p} \\ \geq \left(J^{\alpha} f(t)\right)^{p-1}. \end{aligned} \tag{3.19}$$

This ends the proof of Theorem 3.1.

Remark 3.1: *Applying Theorem* 3.1 *for $\alpha=1$, we obtain Theorem A* of [5].

We further have:

Theorem 3.2: *Let $\alpha>0, p>1$ with $\frac{1}{p}+\frac{1}{q}=1$, such that $t>0, J^{\alpha} f^{p}(t)<\infty, t>$* 0.

If

$$0<m \leq f^{p}(\tau) \leq M<\infty,\ \tau \in[0, t], \tag{3.20}$$

Then

$$\left[J^{\alpha} f^{p}(t)\right]^{\frac{1}{p}} \leq \left(\tfrac{M}{m}\right)^{\frac{2}{pq}}\left(\tfrac{t^{\alpha}}{\Gamma(\alpha+1)}\right)^{-\frac{p+1}{q}}\left(J^{\alpha} f^{\frac{1}{p}}(t)\right)^{p}. \tag{3.21}$$

Proof: Putting $g(\tau)=1$ into lemma 3.3 yields

$$\left[J^{\alpha} f^{p}(t)\right]^{\frac{1}{p}}\left[J^{\alpha}(1^{q})\right]^{\frac{1}{q}} \leq \left(\tfrac{M}{m}\right)^{\frac{1}{pq}} J^{\alpha}(f(t) \times 1) \tag{3.22}$$

which is equivalent to:

$$\left[J^{\alpha} f^{p}(t)\right]^{\frac{1}{p}} \leq \left(\tfrac{M}{m}\right)^{\frac{1}{pq}}\left(\tfrac{t^{\alpha}}{\Gamma(\alpha+1)}\right)^{-\frac{1}{q}} J^{\alpha} f(t). \tag{3.23}$$

Substituting $g(\tau)=1$ into lemma 3.2, we obtain

$$\left[J^{\alpha} f(t)\right]^{\frac{1}{p}}\left[J^{\alpha}(1)\right]^{\frac{1}{q}} \leq \left(\tfrac{M}{m}\right)^{\frac{1}{pq}} J^{\alpha}\left(f^{\frac{1}{p}}(t)\right). \tag{3.24}$$

That is

$$\left[J^{\alpha} f(t)\right]^{\frac{1}{p}} \leq \left(\tfrac{M}{m}\right)^{\frac{1}{p^{2}q}}\left(\tfrac{t^{\alpha}}{\Gamma(\alpha+1)}\right)^{-\frac{1}{q}} J^{\alpha}\left(f^{\frac{1}{p}}(t)\right). \tag{3.25}$$

Hence, we can write

$$J^{\alpha} f(t) \leq \left(\tfrac{M}{m}\right)^{\frac{1}{pq}}\left(\tfrac{t^{\alpha}}{\Gamma(\alpha+1)}\right)^{-\frac{p}{q}}\left[J^{\alpha}\left(f^{\frac{1}{p}}(t)\right)\right]^{p}. \tag{3.26}$$

Combining (3.26) with (3.24), the inequality (3.21) follows.
Remark 3.2: *Applying Theorem* 3.2 *for* $\alpha = 1$, *we obtain the inequality* (1.7) on $[0, t]$.

For the generalization related to [11], we give the following three theorems.

Theorem 3.3: *Let* $f \geq 0, g \geq 0$ *be two functions defined on* $[0, \infty[$ *such that* g *is non-decreasing.*
If

$$J^{\alpha} f(t) \geq J^{\alpha} g(t), t > 0, \tag{3.27}$$

then for all $\gamma, \delta > 0, \alpha > 0, \gamma - \delta \geq 1$, *we have*

$$J^{\alpha} f^{\gamma-\delta}(t) \leq J^{\alpha} f^{\gamma}(t) g^{-\delta}(t). \tag{3.28}$$

Proof:
We use the arithmetic-geometric inequality. For $\gamma > 0, \delta > 0$, we have:

$$\frac{\gamma}{\gamma-\delta} f^{\gamma-\delta}(x) - \frac{\delta}{\gamma-\delta} g^{\gamma-\delta}(x) \leq f^{\gamma}(x) g^{-\delta}(x), x \in [0, t], t > 0. \tag{3.29}$$

Multiplying both sides of (3.29) by $\frac{(t-x)^{\alpha-1}}{\Gamma(\alpha)}$, $x \in (0, t)$, yields

$$\frac{(t-x)^{\alpha-1}}{\Gamma(\alpha)} \frac{\gamma}{\gamma-\delta} f^{\gamma-\delta}(x) - \frac{(t-x)^{\alpha-1}}{\Gamma(\alpha)} \frac{\delta}{\gamma-\delta} g^{\gamma-\delta}(x) \leq \frac{(t-x)^{\alpha-1}}{\Gamma(\alpha)} f^{\gamma}(x) g^{-\delta}(x). \tag{3.30}$$

Then integrating (3.30) with respect to x over $(0, t)$, we can write

$$\begin{aligned} \frac{\gamma}{(\gamma-\delta)\Gamma(\alpha)} \left(\int_0^t (t-x)^{\alpha-1} f^{\gamma-\delta}(x)\, dx - \int_0^t (t-x)^{\alpha-1} g^{\gamma-\delta}(x)\, dx \right) \leq \\ \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} f^{\gamma}(x) g^{-\delta}(x)\, dx. \end{aligned} \tag{3.31}$$

Consequently,

$$\frac{\gamma}{(\gamma-\delta)} J^{\alpha} f^{\gamma-\delta}(t) - \frac{\delta}{(\gamma-\delta)} J^{\alpha} g^{\gamma-\delta}(t) \leq J^{\alpha} f^{\gamma}(t) g^{-\delta}(t). \tag{3.32}$$

This implies that,

$$\begin{aligned} \frac{\gamma}{(\gamma-\delta)} J^{\alpha} f^{\gamma-\delta}(t) &\leq J^{\alpha} \left[f^{\gamma}(t) g^{-\delta}(t) \right] + \frac{\delta}{(\gamma-\delta)} J^{\alpha} g^{\gamma-\delta}(t) \\ &\leq J^{\alpha} \left[f^{\gamma}(t) g^{-\delta}(t) \right] + \frac{\delta}{(\gamma-\delta)} J^{\alpha} f^{\gamma-\delta}(t). \end{aligned} \tag{3.33}$$

Thus we get (3.28).

Remark 3.3: *Applying Theorem* 3.3 *for* $\alpha = 1$*, we obtain Theorem* 1.2 on $[0, t]$.

Theorem 3.4: *Let* $\alpha > 0$ *and* f *and* g *be two positive functions on* $[0, \infty[$, *such that* f *is non-decreasing and* g *is non-increasing. Then*

$$J^{\alpha} f^{\gamma}(t)\, g^{\delta}(t) \le \frac{\Gamma(\alpha+1)}{t^{\alpha}} J^{\alpha} f^{\gamma}(t)\, J^{\alpha} g^{\delta}(t) \tag{3.34}$$

for any $t > 0, \gamma > 0, \delta > 0$.
Proof: Let $x, y \in [0, t], t > 0$. For any $\gamma > 0, \delta > 0$, we have

$$\left(f^{\gamma}(x) - f^{\gamma}(y)\right)\left(g^{\delta}(y) - g^{\delta}(x)\right) \ge 0. \tag{3.35}$$

Therefore,

$$f^{\gamma}(x)\, g^{\delta}(y) + f^{\gamma}(y)\, g^{\delta}(x) \ge f^{\gamma}(y)\, g^{\delta}(y) + f^{\gamma}(x)\, g^{\delta}(x). \tag{3.36}$$

And consequently,

$$\begin{aligned} & J^{\alpha} f^{\gamma}(t)\, g^{\delta}(t) + f^{\gamma}(y)\, g^{\delta}(y) \frac{t^{\alpha}}{\alpha\Gamma(\alpha)} \\ & \le g^{\delta}(y)\, J^{\alpha} f^{\gamma}(t) + f^{\gamma}(y)\, J^{\alpha} g^{\delta}(t). \end{aligned} \tag{3.37}$$

Multiplying both sides of (3.37) by $\frac{(t-y)^{\alpha-1}}{\Gamma(\alpha)}$, $y \in (0, t)$, we get

$$\begin{aligned} & \frac{(t-y)^{\alpha-1}}{\Gamma(\alpha)} J^{\alpha} f^{\gamma}(t)\, g^{\delta}(t) + \frac{(t-y)^{\alpha-1}}{\Gamma(\alpha)} f^{\gamma}(y)\, g^{\delta}(y) \frac{t^{\alpha}}{\alpha\Gamma(\alpha)} \\ & \le \frac{(t-y)^{\alpha-1}}{\Gamma(\alpha)} g^{\delta}(y)\, J^{\alpha} f^{\gamma}(t) + \frac{(t-y)^{\alpha-1}}{\Gamma(\alpha)} f^{\gamma}(y)\, J^{\alpha} g^{\delta}(t). \end{aligned} \tag{3.38}$$

Then integrating (3.38) with respect to y over $(0, t)$, we obtain

$$\begin{aligned} & J^{\alpha} f^{\gamma}(t)\, g^{\delta}(t) \frac{1}{\Gamma(\alpha)} \int_0^t (t-y)^{\alpha-1} dy + \frac{t^{\alpha}}{\alpha\Gamma(\alpha)\Gamma(\alpha)} \int_0^t (t-y)^{\alpha-1} f^{\gamma}(y)\, g^{\delta}(y)\, dy \\ & \le \frac{J^{\alpha} f^{\gamma}(t)}{\Gamma(\alpha)} \int_0^t (t-y)^{\alpha-1} g^{\delta}(y)\, dy + \frac{J^{\alpha} g^{\delta}(t)}{\Gamma(\alpha)} \int_0^t (t-y)^{\alpha-1} f^{\gamma}(y)\, dy. \end{aligned} \tag{3.39}$$

Consequently,

$$\frac{t^{\alpha}}{\alpha\Gamma(\alpha)} J^{\alpha} f^{\gamma}(t)\, g^{\delta}(t) + \frac{t^{\alpha}}{\alpha\Gamma(\alpha)} J^{\alpha} f^{\gamma}(t)\, g^{\delta}(t) \le J^{\alpha} f^{\gamma}(t)\, J^{\alpha} g^{\delta}(t) + J^{\alpha} g^{\delta}(t)\, J^{\alpha} f^{\gamma}(t). \tag{3.40}$$

Then we can write

$$J^{\alpha} f^{\gamma}(t)\, g^{\delta}(t) \le \frac{\alpha\Gamma(\alpha)}{t^{\alpha}} J^{\alpha} f^{\gamma}(t)\, J^{\alpha} g^{\delta}(t). \tag{3.41}$$

Theorem 3.4 is thus proved.
Remark 3.4: It is clear that on $[0, t]$, Theorem 1.3 would follow as a special case

of Theorem 3.4, when $\alpha = 1$.

Theorem 3.5: *Let f and g be two positive functions on $[0, \infty[$, such that f is non-decreasing and g is non-increasing. Then for any $\alpha > 0, \beta > 0, t > 0$, we have*

$$\begin{aligned} &\frac{t^\beta}{\Gamma(\beta+1)} J^\alpha \left(f^\gamma(t)\, g^\delta(t)\right) + \frac{t^\alpha}{\Gamma(\alpha+1)} J^\beta \left(f^\gamma(t)\, g^\delta(t)\right) \\ &\leq \left(J^\alpha f^\gamma(t)\right)\left(J^\beta g^\delta(t)\right) + \left(J^\alpha g^\delta(t)\right)\left(J^\beta f^\gamma(t)\right). \end{aligned} \tag{3.42}$$

Proof:

Using (3.37), we obtain

$$\begin{aligned} &J^\alpha f^\gamma(t)\, g^\delta(t) \frac{1}{\Gamma(\beta)} \int_0^t (t-y)^{\beta-1}\, dy + \frac{t^\alpha}{\alpha\Gamma(\alpha)\Gamma(\beta)} \int_0^t (t-y)^{\beta-1} f^\gamma(y)\, g^\delta(y)\, dy \\ &\leq \frac{J^\alpha f^\gamma(t)}{\Gamma(\beta)} \int_0^t (t-y)^{\beta-1} g^\delta(y)\, dy + \frac{J^\alpha g^\delta(t)}{\Gamma(\beta)} \int_0^t (t-y)^{\beta-1} f^\gamma(y)\, dy \end{aligned} \tag{3.43}$$

which implies (3.44).
Theorem 3.5 is thus proved.
Remark 3.5: *Applying Theorem* 3.5 *for $\alpha = \beta$ we obtain Theorem* 3.4.

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