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Generalizations of Some Integral Inequalities Using Riemann-Liouville Operator

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Abstract

In this paper, we use the Riemann-Liouville fractional integral operator to generate some new fractional results related to Feng Qi inequality. Our results have some relationships with [W. Liu, Q.A. Ngo and V.N. Huy, Journal of Math. Inequal. Vol. 10, Iss. 2, (2009), 201-212]. Some interested inequalities (Theorems 7,8) of this reference can be deduced as some special cases.

Keywords: *Fractional calculus, Integral inequalities, Integration of non integer order, Qi inequality, Riemann-Liouville integrals.*

1 Introduction

In [11], the following interesting integral inequality is proved: Let $n \in N$ and suppose $f(x)$ has a continuous derivative of the n^{th} order on the interval $[a, b]$ such that $f^{(i)}(a) \geq 0$ for $0 \leq i \leq n-1$. If $f^{(n)}(x) \geq n!$, then

$$\int_a^b [f(\tau)]^{n+2} d\tau \geq \left(\int_a^b f(\tau) d\tau\right)^{n+1}. \tag{1}$$

In [10], T.K. Pogany established the following result:

$$\int_a^b [f(\tau)]^{\beta} d\tau \geq \left(\int_a^b f(\tau) d\tau\right)^{\beta-1}, \tag{2}$$

where $f \in C^1([a,b]), f(a) \geq 0$ and $f'(\tau) > (\beta-2)(\tau-a)^{\beta-3}, \tau \in [a,b]$.
In [6], W.J. Liu, G.S. Cheng and C.C. Li established the following inequality:

$$\int_a^b [f(\tau)]^{\alpha+\beta} d\tau \geq \int_a^b (\tau-a)^\alpha f(\tau)^\beta d\tau, \tag{3}$$

where where f is a positive continuous function on $[a,b]$ satisfying

$$\int_x^b [f(\tau)]^\delta d\tau \geq \int_x^b (\tau-a)^\delta d\tau; min(1,\beta) := \delta, x \in [a,b].$$

They also presented some results of type:

$$\int_a^b [f(\tau)]^{\alpha+\beta} d\tau \geq \int_a^b g(\tau)^\alpha f(\tau)^\beta d\tau, \tag{4}$$

where f and g are positive functions on $[a,b]$.
Many researchers have given considerable attention to (2),(3) and (4) and a number of extensions, generalizations and variants have appeared in the literature, see [1, 2, 3, 4, 6, 8, 9, 12].
The purpose of this paper is to generalize some classical integral inequalities of [7] using the Riemann-Liouville interal operator. For our results, some interested inequalities of [7],(Theorem 7 and Theorem 8), can be deduced as some special cases on $[0,t], t > 0$.

2 Basic Definitions of Fractional Integration

Definition 2.1: A real valued function $f(t), t \geq 0$ is said to be in the space $C_\mu, \mu \in R$ if there exists a real number $p > \mu$ such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C([0,\infty[)$.

Definition 2.2: A function $f(t), t \geq 0$ is said to be in the space $C_\mu^n, \mu \in R$, if $f^{(n)} \in C_\mu$.

Definition 2.3: The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, for a function $f \in C_\mu, (\mu \geq -1)$ is defined as

$$\begin{gathered} J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau; \quad \alpha > 0, t > 0, \\ J^0 f(t) = f(t), \end{gathered} \tag{5}$$

where $\Gamma(\alpha) := \int_0^\infty e^{-u} u^{\alpha-1} du$.

For the convenience of establishing the results, we give the semigroup property:

$$J^\alpha J^\beta f(t) = J^{\alpha+\beta} f(t), \alpha \geq 0, \beta \geq 0, \tag{6}$$

which implies the commutative property

$$J^\alpha J^\beta f(t) = J^\beta J^\alpha f(t). \tag{7}$$

For more details, one can consult [5].

3 Main Results

Theorem 3.1 *Let f, g and h be positive and continuous functions on $[0, \infty[$, such that*

$$\Big(g(\tau) - g(\rho)\Big)\Big(\frac{f(\rho)}{h(\rho)} - \frac{f(\tau)}{h(\tau)}\Big) \geq 0; \tau, \rho \in [0, t], t > 0. \tag{8}$$

Then we have

$$\frac{J^\alpha(f(t))}{J^\alpha(h(t))} \geq \frac{J^\alpha(gf(t))}{J^\alpha(gh(t))}, \tag{9}$$

for any $\alpha > 0, t > 0$.

Proof. Suppose that f, g and h are positive and continuous functions on $[0, \infty[$. Using (8), we can write

$$g(\tau)\frac{f(\rho)}{h(\rho)} + g(\rho)\frac{f(\tau)}{h(\tau)} - g(\rho)\frac{f(\rho)}{h(\rho)} - g(\tau)\frac{f(\tau)}{h(\tau)} \geq 0, \tag{10}$$

for all $\tau, \rho \in [0, t], t > 0$.
That is

$$g(\tau)f(\rho)h(\tau) + g(\rho)f(\tau)h(\rho) - g(\rho)f(\rho)h(\tau) - g(\tau)f(\tau)h(\rho) \geq 0, \tag{11}$$

for all $\tau, \rho \in [0, t], t > 0$.
Now, multiplying both sides of (11) by $\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}$, then integrating the resulting inequality with respect to τ over $(0, t)$, we get

$$f(\rho)J^\alpha gh(t) + g(\rho)h(\rho)J^\alpha f(t) - g(\rho)f(\rho)J^\alpha h(t) - h(\rho)J^\alpha gf(t) \geq 0. \tag{12}$$

Therefore,

$$J^\alpha f(t) J^\alpha gh(t) - J^\alpha h(t) J^\alpha gf(t) \geq 0. \tag{13}$$

Theorem 3.1 is thus proved.
Remark 1: It is clear that on $[0, t]$, Theorem 7 of [7] would follow as a special case of Theorem 3.1 when $\alpha = 1$.

Our second result is the following.

Theorem 3.2 *Let f, g and h be be positive and continuous functions on $[0, \infty[$, such that*

$$\Big(g(\tau) - g(\rho)\Big)\Big(\frac{f(\rho)}{h(\rho)} - \frac{f(\tau)}{h(\tau)}\Big) \geq 0; \tau, \rho \in [0, t], t > 0. \tag{14}$$

Then for all $\alpha > 0, \omega, t > 0$, we have

$$\frac{J^\alpha(f(t))J^\omega(gh(t)) + J^\omega(f(t))J^\alpha(gh(t))}{J^\alpha(h(t))J^\omega(gf(t)) + J^\omega(h(t))J^\alpha(gf(t))} \geq 1. \tag{15}$$

Proof. Suppose that f, g and h are positive and continuous functions on $[0, \infty[$. From (12), we can write

$$\frac{(t-\rho)^{\omega-1}}{\Gamma(\omega)}\Big(f(\rho)J^\alpha gh(t) + g(\rho)h(\rho)J^\alpha f(t) - g(\rho)f(\rho)J^\alpha h(t) - h(\rho)J^\alpha gf(t)\Big) \geq 0. \tag{16}$$

Consequently

$$J^\omega(f(t))J^\alpha(gh(t)) + J^\alpha(f(t))J^\omega(gh(t)) \geq J^\alpha(h(t))J^\omega(gf(t)) + J^\omega(h(t))J^\alpha(gf(t)). \tag{17}$$

Hence, we obtain (15).

Remark 2: (i) Applying Theorem 3.3 for $\alpha = \omega$, we obtain Theorem 3.1.
(ii) Applying Theorem 3.3 for $\alpha = \omega = 1$, we obtain Theorem 7 of [7] on $[0, t]$.

We further have

Theorem 3.3 *Let f and h be two positive continuous functions and $f \leq h$ on $[0, \infty[$. If $\frac{f}{h}$ is decreasing and f is increasing on $[0, \infty[$, then for any $p \geq 1, \alpha > 0, t > 0$, the inequality*

$$\frac{J^\alpha(f(t))}{J^\alpha(h(t))} \geq \frac{J^\alpha(f^p(t))}{J^\alpha(h^p(t))} \tag{18}$$

is valid.

Proof. Taking $g := f^{p-1}$, then by Theorem 3.1, we get

$$\frac{J^\alpha(f(t))}{J^\alpha(h(t))} \geq \frac{J^\alpha(ff^{p-1}(t))}{J^\alpha(hf^{p-1}(t))}. \tag{19}$$

Now, since $f \leq h$ on $[0, \infty[$, then

$$\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}hf^{p-1}(\tau) \leq \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}h^p(\tau), \tau \in [0, t], t > 0. \tag{20}$$

Integrating both sides of (20) with respect to τ over $(0,t)$, yields

$$J^{\alpha}(hf^{p-1}(t)) \leq J^{\alpha}(h^{p}(t)). \tag{21}$$

Consequently

$$\frac{J^{\alpha}(ff^{p-1}(t))}{J^{\alpha}(hf^{p-1}(t))} \geq \frac{J^{\alpha}(f^{p}(t))}{J^{\alpha}(h^{p}(t))}. \tag{22}$$

Using (19) and (22), we obtain (18).

Remark 3: Applying Theorem 3.5 for $\alpha = 1$, we obtain Theorem 8 of [7] on $[0,t]$.

Another generalization is the following

Theorem 3.4 *Let f and h be two positive continuous functions and $f \leq h$ on $[0,\infty[$. If $\frac{f}{h}$ is decreasing and f is increasing on $[0,\infty[$, then for any $p \geq 1, \alpha > 0, \omega > 0, t > 0$, we have*

$$\frac{J^{\alpha}(f(t))J^{\omega}(h^{p}(t)) + J^{\omega}(f(t))J^{\alpha}(h^{p}(t))}{J^{\alpha}(h(t))J^{\omega}(f^{p}(t)) + J^{\omega}(h(t))J^{\alpha}(f^{p}(t))} \geq 1. \tag{23}$$

Proof. Taking $g := f^{p-1}$, then by Theorem 3.3, yields

$$\frac{J^{\alpha}(f(t))J^{\omega}(hf^{p-1}(t)) + J^{\omega}(f(t))J^{\alpha}(hf^{p-1}(t))}{J^{\alpha}(h(t))J^{\omega}(f^{p}(t)) + J^{\omega}(h(t))J^{\alpha}(f^{p}(t))} \geq 1. \tag{24}$$

Using the fact that $f \leq h$ on $[0,\infty[$, we can write

$$\frac{(t-\rho)^{\omega-1}}{\Gamma(\omega)}hf^{p-1}(\rho) \leq \frac{(t-\rho)^{\omega-1}}{\Gamma(\omega)}h^{p}(\rho), \rho \in [0,t], t > 0. \tag{25}$$

Integrating both sides of (25) with respect to ρ over $(0,t)$, we obtain

$$J^{\omega}(hf^{p-1}(t)) \leq J^{\omega}(h^{p}(t)). \tag{26}$$

Using (21) and (26), we can write

$$J^{\alpha}f(t)J^{\omega}(hf^{p-1}(t)) + J^{\omega}f(t)J^{\alpha}(hf^{p-1}(t)) \leq J^{\alpha}f(t)J^{\omega}(h^{p}(t)) + J^{\omega}f(t)J^{\alpha}(h^{p}(t)). \tag{27}$$

Now, using (24) and (27), we deduce (23).
Remark 4: (i) Applying Theorem 3.7, for $\alpha = \omega$, we obtain Theorem 3.5.
(ii) Applying Theorem 3.7 for $\alpha = \omega = 1$, we obtain Theorem 8 of [7] on $[0,t]$.

4 Open Problems

In this paper, we have investigated some inequalities of Qi type for fractional integral based on [7]. We will continue exploring other inequalities of this type. At the end, we pose the following problems:
Open Problem 1. Under what conditions does the inequality

$$\frac{J^\alpha(f^{\delta+\beta}(t))}{J^\alpha(f^{\delta+\gamma}(t))} \geq \frac{J^\alpha(t^\mu f^\beta(t))}{J^\alpha(t^\mu f^\gamma(t))} \tag{28}$$

hold for $\alpha, \beta, \gamma, \delta, \mu$?
Open Problem 2. Under what conditions, the inequality

$$J^\alpha(f^{\delta+\beta}(t)) \geq \Big(J^\alpha(t^\delta f^\beta(t))\Big)^\gamma \tag{29}$$

hold for $\alpha, \beta, \gamma, \delta$?
Open Problem 3. Under what conditions does the inequality

$$\frac{J^\alpha(f^{\delta+\beta}(t))}{J^\alpha(f^{\delta+\gamma}(t))} \geq \frac{\Big(J^\alpha(t^\delta f^\beta(t))\Big)^r}{\Big(J^\alpha(t^\delta f^\gamma(t))\Big)^s} \tag{30}$$

hold for $\alpha, \beta, \gamma, \delta, r, s$?

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