

## NEW RESULTS FOR A SYSTEM OF TWO FRACTIONAL DIFFERENTIAL EQUATIONS INVOLVING $n$ CAPUTO DERIVATIVES

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ABSTRACT. This paper studies a coupled system of two differential equations of arbitrary orders using Caputo approach with  $n$  derivatives,  $n \in N^*$ ,  $n \neq 1$ . New existence and uniqueness results are established using Banach contraction principle. Other existence results are obtained using Schaefer and Krasnoselskii fixed point theorems. Some illustrative examples are also presented.

### 1. INTRODUCTION

In recent years, the subject of fractional differential equations has gained a considerable attention and it has emerged as an interesting and popular field of research. For some recent development on this theory, we refer the reader to [1, 2, 3, 4, 5, 6, 9, 19, 20] and references therein. On the other hand, the study of coupled systems involving fractional differential equations is also important as such systems occur in various problems of applied nature, for instance, see [7, 11, 13, 21, 27]. Some recent results on coupled systems of fractional differential equations on a finite interval can be found in [8, 12, 16, 17, 18, 25, 26, 27]. In [8, 10, 22, 23, 24], the existence and uniqueness of solutions were investigated for a coupled system nonlinear fractional differential equations by using Banach and/or Schauder fixed point theorems.

This paper deals with the existence of solutions for the following problem

$$(1.1) \quad \begin{cases} D^{\alpha_0}x(t) = f_1(t, y(t), D^{\alpha_1}y(t), D^{\alpha_2}y(t), \dots, D^{\alpha_{n-1}}y(t)), t \in [0, 1], \\ D^{\beta_0}y(t) = f_2(t, x(t), D^{\beta_1}x(t), D^{\beta_2}x(t), \dots, D^{\beta_{n-1}}x(t)), t \in [0, 1], \\ x(0) = x^*, x'(0) = x''(0) = \dots = x^{(n-2)}(0) = 0, D^p x(1) = \lambda_1 D^p x(\eta), \\ y(0) = y^*, y'(0) = y''(0) = \dots = y^{(n-2)}(0) = 0, D^q y(1) = \lambda_2 D^q y(\xi), \end{cases}$$

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where  $D^{\alpha_i}, D^{\beta_i}, i = 0, 1, 2, \dots, n-1, D^p$  and  $D^q$  denote the Caputo fractional derivatives, with  $n-1 < \alpha_{n-1} < \dots < \alpha_1 < \alpha_0 < n$  and  $n-1 < \beta_{n-1} < \dots < \beta_1 < \beta_0 < n, p < \alpha_0, q < \beta_0, n \in \mathbb{N}^*, n \neq 1, J = [0, 1], \lambda_1, \lambda_2 \neq 0$  are real constants,  $x^*, y^* \in \mathbb{R}, 0 < \eta, \xi < 1$  are real numbers and  $f_1, f_2$  are two functions which will be specified later.

The rest of this paper is organized as follows: in section 2, we present some preliminaries and lemmas. Section 3 is devoted to existence of solution of the system (1.1). To illustrate our main results, in section 4, three examples are treated.

### 2. PRELIMINARIES

Let us begin this section with some basic concepts of fractional calculus that will be used throughout this paper.

**Definition 2.1.** The Riemann-Liouville fractional integral operator of order  $\alpha \geq 0$ , for a continuous function  $f$  on  $[0, \infty[$  is defined as

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(\tau) d\tau, \alpha > 0,$$

$$J^0 f(t) = f(t),$$

where  $\Gamma(\alpha) := \int_0^\infty e^{-u} u^{\alpha-1} du$ .

**Definition 2.2.** The fractional derivative of  $f \in C^n([0, \infty[)$  in the Caputo's sense is defined as

$$D^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t (t - \tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, n - 1 < \alpha, n \in \mathbb{N}^*.$$

For more details, we refer the reader to [18, 20].

Let us now introduce the following Banach space

$$X := \{x : x \in C([0, 1], \mathbb{R}); D^{\alpha_1}x, D^{\alpha_2}x, \dots, D^{\alpha_{n-1}}x \in C([0, 1], \mathbb{R})\},$$

endowed with the norm  $\|x\|_X = \|x\| + \|D^{\alpha_1}x\| + \|D^{\alpha_2}x\| + \dots + \|D^{\alpha_{n-1}}x\|$ ;  $\|x\| = \sup_{t \in J} |x(t)|, \|D^{\alpha_1}x\| = \sup_{t \in J} |D^{\alpha_1}x(t)|, \|D^{\alpha_2}x\| = \sup_{t \in J} |D^{\alpha_2}x(t)|, \dots, \|D^{\alpha_{n-1}}x\| = \sup_{t \in J} |D^{\alpha_{n-1}}x(t)|$ . Similarly, we can define the space

$$Y := \{y : y \in C([0, 1], \mathbb{R}); D^{\beta_1}y, D^{\beta_2}y, \dots, D^{\beta_{n-1}}y \in C([0, 1], \mathbb{R})\},$$

with the norm  $\|y\|_Y = \|y\| + \|D^{\beta_1}y\| + \|D^{\beta_2}y\| + \dots + \|D^{\beta_{n-1}}y\|$ ;  $\|y\| = \sup_{t \in J} |y(t)|, \|D^{\beta_1}y\| = \sup_{t \in J} |D^{\beta_1}y(t)|, \|D^{\beta_2}y\| = \sup_{t \in J} |D^{\beta_2}y(t)|, \dots, \|D^{\beta_{n-1}}y\| = \sup_{t \in J} |D^{\beta_{n-1}}y(t)|$ .

For  $(x, y) \in X \times Y$ , let  $\|(x, y)\|_{X \times Y} = \|x\|_X + \|y\|_Y$ . It is clear that the product space  $(X \times Y, \|(x, y)\|_{X \times Y})$  is a Banach space.

We give the following lemmas [14, 15].

**Lemma 2.1.** *Let  $r, s > 0, f \in L_1([a, b])$ . Then  $I^r I^s f(t) = I^{r+s} f(t), D^s I^s f(t) = f(t), t \in [a, b]$ .*

**Lemma 2.2.** *Let  $s > r > 0, f \in L_1([a, b])$ . Then  $D^r I^s f(t) = I^{s-r} f(t), t \in [a, b]$ .*

Also, we present the following two lemmas [14].

**Lemma 2.3.** *For  $\alpha > 0$ , the general solution of the fractional differential equation  $D^\alpha x(t) = 0$  is given by  $x(t) = c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1} t^{n-1}$ , where  $c_i \in \mathbb{R}, i = 0, 1, 2, \dots, n - 1, n = [\alpha] + 1$ .*

**Lemma 2.4.** *Let  $\alpha > 0$ . Then  $J^\alpha D^\alpha x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1} t^{n-1}$ , for some  $c_i \in \mathbb{R}, i = 0, 1, 2, \dots, n - 1, n = [\alpha] + 1$ .*

The following auxiliary result is crucial to prove our main results.

**Lemma 2.5.** *Let  $g \in C([0, 1])$ . The solution of the equation*

$$(2.1) \quad D^{\alpha_0} x(t) = g(t), t \in J, n - 1 < \alpha_0 < n, n > 0,$$

*subject to the conditions  $x(0) = x^*, x'(0) = x''(0) = \dots = x^{(n-2)}(0) = 0$  and  $D^p x(1) = \lambda_1 D^p x(\eta)$ , is given by*

$$\begin{aligned} x(t) = & \frac{1}{\Gamma(\alpha_0)} \int_0^t (t-s)^{\alpha_0-1} g(s) ds + x^* \\ & - \frac{\Gamma(n-p)t^{n-1}}{(1-\lambda_1 \eta^{n-p-1})\Gamma(n)\Gamma(\alpha_0-p)} \int_0^1 (1-s)^{\alpha_0-p-1} g(s) ds \\ & + \frac{\lambda_1 \Gamma(n-p)t^{n-1}}{(1-\lambda_1 \eta^{n-p-1})\Gamma(n)\Gamma(\alpha_0-p)} \int_0^\eta (\eta-s)^{\alpha_0-p-1} g(s) ds. \end{aligned}$$

*Proof.* We use Lemma 2.3 and Lemma 2.4 to generate the general solution of (2.1). We have

$$(2.2) \quad x(t) = \frac{1}{\Gamma(\alpha_0)} \int_0^t (t-s)^{\alpha_0-1} g(s) ds - c_0 - c_1 t - c_2 t^2 - \dots - c_{n-1} t^{n-1}.$$

By  $x(0) = x^*$ , and  $x'(0) = \dots = x^{(n-2)}(0) = 0$ , we can obtain  $c_0 = -x^*$  and  $c_1 = c_2 = \dots = c_{n-2} = 0$ .

Thanks to Lemma 2.2, we get

$$D^p x(t) = \frac{1}{\Gamma(\alpha_0-p)} \int_0^t (t-s)^{\alpha_0-p-1} g(s) ds - c_{n-1} \frac{\Gamma(n)}{\Gamma(n-p)} t^{n-p-1}.$$

Using the condition  $D^p x(1) = \lambda_1 D^p x(\eta)$ , we get

$$\begin{aligned} c_{n-1} = & \frac{\Gamma(n-p)}{(1-\lambda_1 \eta^{n-p-1})\Gamma(n)\Gamma(\alpha_0-p)} \int_0^1 (1-s)^{\alpha_0-p-1} g(s) ds \\ & - \frac{\lambda_1 \Gamma(n-p)}{(1-\lambda_1 \eta^{n-p-1})\Gamma(n)\Gamma(\alpha_0-p)} \int_0^\eta (\eta-s)^{\alpha_0-p-1} g(s) ds. \end{aligned}$$

Substituting the values of  $c_0, c_1, c_2, \dots, c_{n-2}, c_{n-1}$  in (2.2), we obtain the desired quantity in Lemma 2.5. □

### 3. MAIN RESULTS

In this section, we present the main results of the paper. For the sake of convenience, let us take.

$$\begin{aligned}
 N_0 &:= \frac{1}{\Gamma(\alpha_0+1)} + \frac{\Gamma(n-p)(1+|\lambda_1|\eta^{\alpha_0-p})}{|1-\lambda_1\eta^{n-p-1}|\Gamma(n)\Gamma(\alpha_0-p+1)}, \\
 N_k &:= \frac{1}{\Gamma(\alpha_0-\alpha_k+1)} + \frac{\Gamma(n-p)(1+|\lambda_1|\eta^{\alpha_0-p})}{|1-\lambda_1\eta^{n-p-1}|\Gamma(n-\alpha_k)\Gamma(\alpha_0-p+1)}, k = 1, \dots, n-1, \\
 M_0 &:= \frac{1}{\Gamma(\beta_0+1)} + \frac{\Gamma(n-q)(1+|\lambda_2|\xi^{\beta_0-q})}{|1-\lambda_2\xi^{n-q-1}|\Gamma(n)\Gamma(\beta_0-q+1)}, \\
 M_h &:= \frac{1}{\Gamma(\beta_0-\beta_h+1)} + \frac{\Gamma(n-q)(1+|\lambda_2|\xi^{\beta_0-q})}{|1-\lambda_2\xi^{n-q-1}|\Gamma(n-\beta_h)\Gamma(\beta_0-q+1)}, h = 1, \dots, n-1, \\
 \omega &:= \omega_0 + \omega_1 + \dots + \omega_{n-1}, \\
 \varpi &:= \varpi_0 + \varpi_1 + \dots + \varpi_{n-1}, \\
 \theta &:= \frac{\Gamma(n-p)(1+|\lambda_1|\eta^{\alpha_0-p})}{|1-\lambda_1\eta^{n-p-1}|\Gamma(n)\Gamma(\alpha_0-p+1)} + \sum_{k=1}^{n-1} \frac{\Gamma(n-p)(1+|\lambda_1|\eta^{\alpha_0-p})}{|1-\lambda_1\eta^{n-p-1}|\Gamma(n-\alpha_k)\Gamma(\alpha_0-p+1)}, \\
 \theta' &:= \frac{\Gamma(n-q)(1+|\lambda_2|\xi^{\beta_0-q})}{|1-\lambda_2\xi^{n-q-1}|\Gamma(n)\Gamma(\beta_0-q+1)} + \sum_{h=1}^{n-1} \frac{\Gamma(n-q)(1+|\lambda_2|\xi^{\beta_0-q})}{|1-\lambda_2\xi^{n-q-1}|\Gamma(n-\beta_h)\Gamma(\beta_0-q+1)}.
 \end{aligned}$$

(H1) : We also suppose that the functions  $f_1, f_2 : [0, 1] \times \mathbb{R}^n \rightarrow \mathbb{R}$  are continuous.

(H2) : There exist non negative continuous functions  $a_i, b_i \in C([0, 1]), i = 0, \dots, n-1$ , such that for all  $t \in [0, 1]$  and  $(x_0, x_1, x_2, \dots, x_{n-1}), (y_0, y_1, y_2, \dots, y_{n-1}) \in \mathbb{R}^n$ , we have

$$\begin{aligned}
 &|f_1(t, x_0, x_1, x_2, \dots, x_{n-1}) - f_1(t, y_0, y_1, y_2, \dots, y_{n-1})| \\
 &\leq a_0(t) |x_0 - y_0| + a_1(t) |x_1 - y_1| + a_2(t) |x_2 - y_2| + \dots + a_{n-1}(t) |x_{n-1} - y_{n-1}|,
 \end{aligned}$$

and

$$\begin{aligned}
 &|f_2(t, x_0, x_1, x_2, \dots, x_{n-1}) - f_2(t, y_0, y_1, y_2, \dots, y_{n-1})| \\
 &\leq b_0(t) |x_0 - y_0| + b_1(t) |x_1 - y_1| + b_2(t) |x_2 - y_2| + \dots + b_{n-1}(t) |x_{n-1} - y_{n-1}|,
 \end{aligned}$$

where,

$$\begin{aligned}
 \omega_0 &= \sup_{t \in J} a_0(t), \omega_1 = \sup_{t \in J} a_1(t), \omega_2 = \sup_{t \in J} a_2(t), \dots, \omega_{n-1} = \sup_{t \in J} a_{n-1}(t), \\
 \varpi_0 &= \sup_{t \in J} b_0(t), \varpi_1 = \sup_{t \in J} b_1(t), \varpi_2 = \sup_{t \in J} b_2(t), \dots, \varpi_{n-1} = \sup_{t \in J} b_{n-1}(t).
 \end{aligned}$$

(H3) : There exists non negative functions  $l_1(t)$  and  $l_2(t)$  such that

$$\begin{aligned}
 &|f_1(t, x_0, x_1, x_2, \dots, x_{n-1})| \leq l_1(t), \\
 &|f_2(t, x_0, x_1, x_2, \dots, x_{n-1})| \leq l_2(t)
 \end{aligned}$$

for each  $t \in J$  and all  $x, y \in \mathbb{R}$ , with  $L_1 = \sup_{t \in J} l_1(t), L_2 = \sup_{t \in J} l_2(t)$ .

Our first result is based on Banach contraction principle. It is the following.

**Theorem 3.1.** *Suppose  $\eta^{n-p-1} \neq \frac{1}{\lambda_1}, \xi^{n-q-1} \neq \frac{1}{\lambda_2}$  and assume that the hypothesis (H2) holds. If*

$$(3.1) \quad \left( N_0 + \sum_{k=1}^{n-1} N_k \right) \omega + \left( M_0 + \sum_{h=1}^{n-1} M_h \right) \varpi < 1,$$

then the fractional system (1.1) has a unique solution on  $J$ .

*Proof.* Let us define the operator  $\phi : X \times Y \rightarrow X \times Y$  by

$$\phi(x, y)(t) := (\phi_1(y)(t), \phi_2(x)(t)),$$

where, for each  $t \in [0, 1]$ ,

$$\begin{aligned} \phi_1 y(t) &:= \frac{1}{\Gamma(\alpha_0)} \int_0^t (t-s)^{\alpha_0-1} f_1(s, y(s), D^{\alpha_1}y(s), \dots, D^{\alpha_{n-1}}y(s)) ds + x^* \\ &\quad - \frac{\Gamma(n-p)t^{n-1}}{(1-\lambda_1\eta^{n-p-1})\Gamma(n)\Gamma(\alpha_0-p)} \int_0^1 (1-s)^{\alpha_0-p-1} f_1(s, y(s), D^{\alpha_1}y(s), \dots, D^{\alpha_{n-1}}y(s)) ds \\ &\quad + \frac{\lambda_1\Gamma(n-p)t^{n-1}}{(1-\lambda_1\eta^{n-p-1})\Gamma(n)\Gamma(\alpha_0-p)} \int_0^\eta (\eta-s)^{\alpha_0-p-1} f_1(s, y(s), D^{\alpha_1}y(s), \dots, D^{\alpha_{n-1}}y(s)) ds, \end{aligned}$$

and

$$\begin{aligned} \phi_2 x(t) &:= \frac{1}{\Gamma(\beta_0)} \int_0^t (t-s)^{\beta_0-1} f_2(s, x(s), D^{\beta_1}x(s), \dots, D^{\beta_{n-1}}x(s)) ds + y^* \\ &\quad - \frac{\Gamma(n-q)t^{n-1}}{(1-\lambda_2\eta^{n-q-1})\Gamma(n)\Gamma(\beta_0-q)} \int_0^1 (1-s)^{\beta_0-q-1} f_2(s, x(s), D^{\beta_1}x(s), \dots, D^{\beta_{n-1}}x(s)) ds \\ &\quad + \frac{\lambda_2\Gamma(n-q)t^{n-1}}{(1-\lambda_2\eta^{n-q-1})\Gamma(n)\Gamma(\beta_0-q)} \int_0^\eta (\eta-s)^{\beta_0-q-1} f_2(s, x(s), D^{\beta_1}x(s), \dots, D^{\beta_{n-1}}x(s)) ds. \end{aligned}$$

We shall prove that  $\phi$  is a contraction mapping.

If we denote  $F(s) = \begin{vmatrix} f_1(s, y(s), D^{\alpha_1}y(s), \dots, D^{\alpha_{n-1}}y(s)) \\ -f_1(s, y_1(s), D^{\alpha_1}y_1(s), \dots, D^{\alpha_{n-1}}y_1(s)) \end{vmatrix}$ , then for each  $t \in J$  and for  $(x, y), (x_1, y_1) \in X \times Y$  we have

$$\begin{aligned} |\phi_1 y(t) - \phi_1 y_1(t)| &\leq \frac{1}{\Gamma(\alpha_0)} \int_0^t (t-s)^{\alpha_0-1} F(s) ds \\ &\quad + \frac{\Gamma(n-p)}{|1-\lambda_1\eta^{n-p-1}|\Gamma(n)\Gamma(\alpha_0-p)} \int_0^1 (1-s)^{\alpha_0-p-1} F(s) ds \\ &\quad + \frac{|\lambda_1|\Gamma(n-p)}{|1-\lambda_1\eta^{n-p-1}|\Gamma(n)\Gamma(\alpha_0-p)} \int_0^\eta (\eta-s)^{\alpha_0-p-1} F(s) ds. \end{aligned}$$

Thus,

$$\begin{aligned} |\phi_1 y(t) - \phi_1 y_1(t)| &\leq \frac{(\omega_0 + \omega_1 + \dots + \omega_{n-1})[\|y - y_1\| + \|D^{\alpha_1} y - D^{\alpha_1} y_1\| + \dots + \|D^{\alpha_{n-1}} y - D^{\alpha_{n-1}} y_1\|]}{\Gamma(\alpha_0 + 1)} \\ &+ \frac{\Gamma(n-p)(\omega_0 + \omega_1 + \dots + \omega_{n-1})[\|y - y_1\| + \|D^{\alpha_1} y - D^{\alpha_1} y_1\| + \dots + \|D^{\alpha_{n-1}} y - D^{\alpha_{n-1}} y_1\|]}{|1 - \lambda_1 \eta^{n-p-1}| \Gamma(n) \Gamma(\alpha_0 - p + 1)} \\ &+ \frac{|\lambda_1| \Gamma(n-p) \eta^{\alpha_0 - p} (\omega_0 + \omega_1 + \dots + \omega_{n-1})[\|y - y_1\| + \|D^{\alpha_1} y - D^{\alpha_1} y_1\| + \dots + \|D^{\alpha_{n-1}} y - D^{\alpha_{n-1}} y_1\|]}{|1 - \lambda_1 \eta^{n-p-1}| \Gamma(n) \Gamma(\alpha_0 - p + 1)}. \end{aligned}$$

Consequently, we have

$$|\phi_1 y(t) - \phi_1 y_1(t)| \leq N_0 \omega (\|y - y_1\| + \|D^{\alpha_1} y - D^{\alpha_1} y_1\| + \dots + \|D^{\alpha_{n-1}} y - D^{\alpha_{n-1}} y_1\|),$$

which implies that

$$(3.2) \quad \begin{aligned} \|\phi_1(y) - \phi_1(y_1)\| &\leq N_0 \omega (\|y - y_1\| + \|D^{\alpha_1} y - D^{\alpha_1} y_1\| \\ &+ \dots + \|D^{\alpha_{n-1}} y - D^{\alpha_{n-1}} y_1\|). \end{aligned}$$

Similarly,

$$(3.3) \quad \begin{aligned} \|\phi_2(x) - \phi_2(x_1)\| &\leq M_0 \varpi (\|x - x_1\| + \|D^{\beta_1} x - D^{\beta_1} x_1\| \\ &+ \dots + \|D^{\beta_{n-1}} x - D^{\beta_{n-1}} x_1\|). \end{aligned}$$

On the other hand, for  $F(s) = \begin{vmatrix} f_1(s, y(s), D^{\alpha_1} y(s), \dots, D^{\alpha_{n-1}} y(s)) \\ -f_1(s, y_1(s), D^{\alpha_1} y_1(s), \dots, D^{\alpha_{n-1}} y_1(s)) \end{vmatrix}$ , for all  $k = 1, \dots, n - 1$  and for each  $t \in [0, 1]$  we have

$$\begin{aligned} |D^{\alpha_k} \phi_1 y(t) - D^{\alpha_k} \phi_1 y_1(t)| &\leq \frac{1}{\Gamma(\alpha_0 - \alpha_k)} \int_0^t (t - s)^{\alpha_0 - \alpha_k - 1} F(s) ds \\ &+ \frac{\Gamma(n-p)t^{n-\alpha_k-1}}{|1 - \lambda_1 \eta^{n-p-1}| \Gamma(n - \alpha_k) \Gamma(\alpha_0 - p)} \int_0^1 (1 - s)^{\alpha_0 - p - 1} F(s) ds \\ &+ \frac{|\lambda_1| \Gamma(n-p)t^{n-\alpha_k-1}}{|1 - \lambda_1 \eta^{n-p-1}| \Gamma(n - \alpha_k) \Gamma(\alpha_0 - p)} \int_0^\eta (\eta - s)^{\alpha_0 - p - 1} F(s) ds. \end{aligned}$$

By (H2), we obtain

$$\begin{aligned} |D^{\alpha_k} \phi_1 y(t) - D^{\alpha_k} \phi_1 y_1(t)| &\leq \frac{(\omega_0 + \omega_1 + \dots + \omega_{n-1})[\|y - y_1\| + \|D^{\alpha_1} y - D^{\alpha_1} y_1\| + \dots + \|D^{\alpha_{n-1}} y - D^{\alpha_{n-1}} y_1\|]}{\Gamma(\alpha_0 - \alpha_k + 1)} \\ &+ \frac{\Gamma(n-p)(\omega_0 + \omega_1 + \dots + \omega_{n-1})[\|y - y_1\| + \|D^{\alpha_1} y - D^{\alpha_1} y_1\| + \dots + \|D^{\alpha_{n-1}} y - D^{\alpha_{n-1}} y_1\|]}{|1 - \lambda_1 \eta^{n-p-1}| \Gamma(n - \alpha_k) \Gamma(\alpha_0 - p + 1)} \\ &+ \frac{|\lambda_1| \Gamma(n-p) \eta^{\alpha_0 - p} (\omega_0 + \omega_1 + \dots + \omega_{n-1})[\|y - y_1\| + \|D^{\alpha_1} y - D^{\alpha_1} y_1\| + \dots + \|D^{\alpha_{n-1}} y - D^{\alpha_{n-1}} y_1\|]}{|1 - \lambda_1 \eta^{n-p-1}| \Gamma(n - \alpha_k) \Gamma(\alpha_0 - p + 1)}. \end{aligned}$$

Hence we have

$$\begin{aligned} |D^{\alpha_k} \phi_1 y(t) - D^{\alpha_k} \phi_1 y_1(t)| &\leq N_k \omega (\|y - y_1\| + \|D^{\alpha_1} y - D^{\alpha_1} y_1\| \\ &+ \dots + \|D^{\alpha_{n-1}} y - D^{\alpha_{n-1}} y_1\|). \end{aligned}$$

Then

$$(3.4) \quad \begin{aligned} \|D^{\alpha_k} \phi_1(y) - D^{\alpha_k} \phi_1(y_1)\| &\leq N_k \omega (\|y - y_1\| + \|D^{\alpha_1} y - D^{\alpha_1} y_1\| \\ &+ \dots + \|D^{\alpha_{n-1}} y - D^{\alpha_{n-1}} y_1\|). \end{aligned}$$

With the same arguments as before, for each  $h = 1, \dots, n - 1$ , we get

$$(3.5) \quad \begin{aligned} \|D^{\beta_h}\phi_2(x) - D^{\beta_h}\phi_2(x_1)\| &\leq M_h\varpi (\|x - x_1\| + \|D^{\beta_1}x - D^{\beta_1}x_1\| \\ &+ \dots + \|D^{\beta_{n-1}}x - D^{\beta_{n-1}}x_1\|). \end{aligned}$$

Thanks to (3.2) and (3.4), we obtain

$$(3.6) \quad \begin{aligned} \|\phi_1(y) - \phi_1(y_1)\|_X &\leq \left(N_0 + \sum_{k=1}^{n-1} N_k\right) \omega (\|y - y_1\| + \|D^{\alpha_1}y - D^{\alpha_1}y_1\| \\ &+ \dots + \|D^{\alpha_{n-1}}y - D^{\alpha_{n-1}}y_1\|). \end{aligned}$$

Using (3.3) and (3.5) we can write

$$(3.7) \quad \begin{aligned} \|\phi_2(x) - \phi_2(x_1)\|_Y &\leq \left(M_0 + \sum_{h=1}^{n-1} M_h\right) \varpi (\|x - x_1\| + \|D^{\beta_1}x - D^{\beta_1}x_1\| \\ &+ \dots + \|D^{\beta_{n-1}}x - D^{\beta_{n-1}}x_1\|). \end{aligned}$$

Combining (3.6) and (3.7), we deduce that

$$\begin{aligned} \|\phi(x, y) - \phi(x_1, y_1)\|_{X \times Y} &\leq \left[\left(N_0 + \sum_{k=1}^{n-1} N_k\right) \omega + \left(M_0 + \sum_{h=1}^{n-1} M_h\right) \varpi\right] \\ &\times \|(x - x_1, y - y_1)\|_{X \times Y}. \end{aligned}$$

Consequently by (3.1) we conclude that  $\phi$  is a contraction mapping. As a consequence of Banach contraction principle, we deduce that  $\phi$  has a unique fixed point which is the solution of (1.1). □

The second main result is given in the following theorem.

**Theorem 3.2.** *Suppose that for all  $\eta^{n-p-1} \neq \frac{1}{\lambda_1}, \xi^{n-q-1} \neq \frac{1}{\lambda_2}$  and assume that the hypotheses (H1) and (H3) are satisfied. Then, the system (1.1) has at least a solution on  $J$ .*

*Proof.* We use Schaefer’s fixed point theorem to prove that  $\phi$  has at least one fixed point on  $X \times Y$ .

**Step 1:**  $\phi$  is continuous on  $X \times Y$  : By (H1) we conclude that the operator  $\phi$  is continuous.

**Step 2:** The operator  $\phi$  maps bounded sets into bounded sets in  $X \times Y$  : For  $\sigma > 0$ , we take  $(x, y) \in B_\sigma = \{(x, y) \in X \times Y; \|(x, y)\|_{X \times Y} \leq \sigma\}$ . For each  $t \in J$  and

$\Delta_1 = \frac{\Gamma(n-p)t^{n-1}}{|1-\lambda_1\eta^{n-p-1}|\Gamma(n)\Gamma(\alpha_0-p)}$  we have:

$$\begin{aligned} |\phi_1 y(t)| &\leq \frac{1}{\Gamma(\alpha_0)} \int_0^t (t-s)^{\alpha_0-1} |f_1(s, y(s), D^{\alpha_1}y(s), \dots, D^{\alpha_{n-1}}y(s))| ds + |x^*| \\ &\quad + \Delta_1 \int_0^1 (1-s)^{\alpha_0-p-1} |f_1(s, y(s), D^{\alpha_1}y(s), \dots, D^{\alpha_{n-1}}y(s))| ds \\ &\quad + |\lambda_1| \Delta_1 \int_0^\eta (\eta-s)^{\alpha_0-p-1} |f_1(s, y(s), D^{\alpha_1}y(s), \dots, D^{\alpha_{n-1}}y(s))| ds. \end{aligned}$$

Using (H3), we obtain

$$|\phi_1 y(t)| \leq \sup_{t \in J} l_1(t) \left[ \frac{1}{\Gamma(\alpha_0+1)} + \frac{\Gamma(n-p)(1+|\lambda_1|\eta^{\alpha_0-p})}{|1-\lambda_1\eta^{n-p-1}|\Gamma(n)\Gamma(\alpha_0-p+1)} \right] + |x^*|.$$

Thus,  $|\phi_1 y(t)| \leq N_0 \sup_{t \in J} l_1(t) + |x^*|, t \in J$ , and then

$$(3.8) \quad \|\phi_1(y)\| \leq L_1 N_0 + |x^*|.$$

Similarly, we can write

$$(3.9) \quad \|\phi_2(x)\| \leq L_2 M_0 + |y^*|.$$

On the other hand, for all  $k = 1, 2, \dots, n-1$  and  $\Delta_2 = \frac{\Gamma(n-p)t^{n-\alpha_k-1}}{|1-\lambda_1\eta^{n-p-1}|\Gamma(n-\alpha_k)\Gamma(\alpha_0-p)}$  we have

$$\begin{aligned} |D^{\alpha_k} \phi_1 y(t)| &\leq \frac{1}{\Gamma(\alpha_0-\alpha_k)} \int_0^t (t-s)^{\alpha_0-\alpha_k-1} |f_1(s, y(s), D^{\alpha_1}y(s), \dots, D^{\alpha_{n-1}}y(s))| ds \\ &\quad + \Delta_2 \int_0^1 (1-s)^{\alpha_0-p-1} |f_1(s, y(s), D^{\alpha_1}y(s), \dots, D^{\alpha_{n-1}}y(s))| ds \\ &\quad + |\lambda_1| \Delta_2 \int_0^\eta (\eta-s)^{\alpha_0-p-1} |f_1(s, y(s), D^{\alpha_1}y(s), \dots, D^{\alpha_{n-1}}y(s))| ds. \end{aligned}$$

By (H3), we obtain

$$(3.10) \quad \|D^{\alpha_k} \phi_1(y)\| \leq L_1 \left[ \frac{1}{\Gamma(\alpha_0-\alpha_k+1)} + \frac{\Gamma(n-p)(1+|\lambda_1|\eta^{\alpha_0-p})}{|1-\lambda_1\eta^{n-p-1}|\Gamma(n-\alpha_k)\Gamma(\alpha_0-p+1)} \right],$$

and for all  $h = 1, 2, \dots, n-1$ ,

$$(3.11) \quad \|D^{\beta_h} \phi_2(x)\| \leq L_2 \left[ \frac{1}{\Gamma(\beta_0-\beta_h+1)} + \frac{\Gamma(n-q)(1+|\lambda_2|\xi^{\beta_0-q})}{|1-\lambda_2\xi^{n-q-1}|\Gamma(n-\beta_h)\Gamma(\beta_0-q+1)} \right].$$

Combining (3.8) and (3.10), yields to

$$(3.12) \quad \|\phi_1(y)\|_X \leq L_1 \left( N_0 + \sum_{k=1}^{n-1} N_k \right) + |x^*|.$$



Similarly, it follows from (3.9) and (3.11) that

$$(3.13) \quad \|\phi_2(x)\|_Y \leq L_2 \left( M_0 + \sum_{h=1}^{n-1} M_h \right) + |y^*|.$$

Thanks to (3.12) and (3.13), we have

$$\|\phi(x, y)\|_{X \times Y} \leq L_1 \left( N_0 + \sum_{k=1}^{n-1} N_k \right) + L_2 \left( M_0 + \sum_{h=1}^{n-1} M_h \right) + |x^*| + |y^*|.$$

Consequently  $\|\phi(x, y)\|_{X \times Y} < \infty$ .

**Step 3:** Now we show that  $\phi$  is equi-continuous on  $J$ .

Let us take  $(x, y) \in B_\sigma$ ,  $t_1, t_2 \in J$ , such that  $t_1 < t_2$ . Thanks to (H3), we can write:

$$\begin{aligned} |\phi_1 y(t_2) - \phi_1 y(t_1)| &\leq \frac{\sup_{t \in J} l_1(t)}{\Gamma(\alpha_0)} \int_0^{t_1} [(t_2 - s)^{\alpha_0-1} - (t_1 - s)^{\alpha_0-1}] ds \\ &\quad + \frac{\sup_{t \in J} l_1(t)}{\Gamma(\alpha_0)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha_0-1} ds \\ &\quad + \frac{\sup_{t \in J} l_1(t) \Gamma(n-p) (t_1^{n-1} - t_2^{n-1})}{|1 - \lambda_1 \eta^{n-p-1}| \Gamma(n) \Gamma(\alpha_0-p)} \int_0^1 (1-s)^{\alpha_0-p-1} ds \\ &\quad + \frac{\sup_{t \in J} l_1(t) |\lambda_1| \Gamma(n-p) (t_2^{n-1} - t_1^{n-1})}{|1 - \lambda_1 \eta^{n-p-1}| \Gamma(n) \Gamma(\alpha_0-p)} \int_0^\eta (\eta - s)^{\alpha_0-p-1} ds. \end{aligned}$$

Thus,

$$(3.14) \quad \begin{aligned} |\phi_1 y(t_2) - \phi_1 y(t_1)| &\leq \frac{L_1}{\Gamma(\alpha_0+1)} (t_1^{\alpha_0} - t_2^{\alpha_0}) + \frac{2L_1}{\Gamma(\alpha_0+1)} (t_2 - t_1)^{\alpha_0} \\ &\quad + \frac{L_1 \Gamma(n-p)}{|1 - \lambda_1 \eta^{n-p-1}| \Gamma(n) \Gamma(\alpha_0-p+1)} (t_1^{n-1} - t_2^{n-1}) \\ &\quad + \frac{L_1 |\lambda_1| \Gamma(n-p) \eta^{\alpha_0-p}}{|1 - \lambda_1 \eta^{n-p-1}| \Gamma(n) \Gamma(\alpha_0-p+1)} (t_2^{n-1} - t_1^{n-1}). \end{aligned}$$

Analogously, we can write

$$(3.15) \quad \begin{aligned} |\phi_2 x(t_2) - \phi_2 x(t_1)| &\leq \frac{L_2}{\Gamma(\beta_0+1)} (t_1^{\beta_0} - t_2^{\beta_0}) + \frac{2L_2}{\Gamma(\beta_0+1)} (t_2 - t_1)^{\beta_0} \\ &\quad + \frac{L_2 \Gamma(n-q)}{|1 - \lambda_2 \xi^{n-q-1}| \Gamma(n) \Gamma(\beta_0-q+1)} (t_1^{n-1} - t_2^{n-1}) \\ &\quad + \frac{L_2 |\lambda_2| \Gamma(n-q) \eta^{\beta_0-q}}{|1 - \lambda_2 \xi^{n-q-1}| \Gamma(n) \Gamma(\beta_0-q+1)} (t_2^{n-1} - t_1^{n-1}). \end{aligned}$$

On the other hand, for all  $k = 1, 2, \dots, n-1$ ,

$$(3.16) \quad \begin{aligned} |D^{\alpha_k} \phi_1 y(t_2) - D^{\alpha_k} \phi_1 y(t_1)| &\leq \frac{L_1}{\Gamma(\alpha_0 - \alpha_k + 1)} (t_1^{\alpha_0 - \alpha_k} - t_2^{\alpha_0 - \alpha_k}) + \frac{2L_1}{\Gamma(\alpha_0 - \alpha_k + 1)} (t_2 - t_1)^{\alpha_0 - \alpha_k} \\ &\quad + \frac{L_1 \Gamma(n-p)}{|1 - \lambda_1 \eta^{n-p-1}| \Gamma(n - \alpha_k) \Gamma(\alpha_0 - p + 1)} (t_1^{n - \alpha_k - 1} - t_2^{n - \alpha_k - 1}) \\ &\quad + \frac{L_1 |\lambda_1| \Gamma(n-p) \eta^{\alpha_0 - p}}{|1 - \lambda_1 \eta^{n-p-1}| \Gamma(n - \alpha_k) \Gamma(\alpha_0 - p + 1)} (t_2^{n - \alpha_k - 1} - t_1^{n - \alpha_k - 1}), \end{aligned}$$

and for all  $h = 1, 2, \dots, n - 1$ ,

$$\begin{aligned}
 |D^{\beta_h} \phi_2 x(t_2) - D^{\beta_h} \phi_2 x(t_1)| &\leq \frac{L_2}{\Gamma(\beta_0 - \beta_h + 1)} \left( t_1^{\beta_0 - \beta_h} - t_2^{\beta_0 - \beta_h} \right) + \frac{2L_2}{\Gamma(\beta_0 - \beta_h + 1)} (t_2 - t_1)^{\beta_0 - \beta_h} \\
 (3.17) \qquad \qquad \qquad &+ \frac{L_2 \Gamma(n - q)}{|1 - \lambda_2 \xi^{n - q - 1}| \Gamma(n - \beta_h) \Gamma(\beta_0 - q + 1)} \left( t_1^{n - \beta_h - 1} - t_2^{n - \beta_h - 1} \right) \\
 &+ \frac{L_2 |\lambda_2| \Gamma(n - q) \eta^{\beta_0 - q}}{|1 - \lambda_2 \xi^{n - q - 1}| \Gamma(n - \beta_h) \Gamma(\beta_0 - q + 1)} \left( t_2^{n - \beta_h - 1} - t_1^{n - \beta_h - 1} \right).
 \end{aligned}$$

By (3.14), (3.15), (3.16) and (3.17), we can state that  $\|\phi(x, y)(t_2) - \phi(x, y)(t_1)\| \rightarrow 0$  as  $t_2 \rightarrow t_1$ . By Arzela-Ascoli theorem, we conclude that  $\phi$  is completely continuous operator.

**Step 4:** Finally, we show that the set  $\Omega$  defined by

$$\Omega = \{(x, y) \in X \times Y, (x, y) = \mu \phi(x, y), 0 < \mu < 1\},$$

is bounded.

Let  $(x, y) \in \Omega$ , then  $(x, y) = \mu \phi(x, y)$ , for some  $0 < \mu < 1$ . Thus, for each  $t \in J$ , we have  $x(t) = \mu \phi_1 y(t)$ ,  $y(t) = \mu \phi_2 x(t)$ . Then for  $\Delta_1 = \frac{\Gamma(n - p)t^{n - 1}}{|1 - \lambda_1 \eta^{n - p - 1}| \Gamma(n) \Gamma(\alpha_0 - p)}$

$$\begin{aligned}
 \frac{1}{\mu} |x(t)| &\leq \frac{1}{\Gamma(\alpha_0)} \int_0^t (t - s)^{\alpha_0 - 1} |f_1(s, y(s), D^{\alpha_1} y(s), \dots, D^{\alpha_{n-1}} y(s))| ds + |x^*| \\
 &+ \Delta_1 \int_0^1 (1 - s)^{\alpha_0 - p - 1} |f_1(s, y(s), D^{\alpha_1} y(s), \dots, D^{\alpha_{n-1}} y(s))| ds \\
 &+ |\lambda_1| \Delta_1 \int_0^\eta (\eta - s)^{\alpha_0 - p - 1} |f_1(s, y(s), D^{\alpha_1} y(s), \dots, D^{\alpha_{n-1}} y(s))| ds.
 \end{aligned}$$

Thanks to (H3), we can write

$$\frac{1}{\mu} |x(t)| \leq \frac{\sup_{t \in J} l_1(t)}{\Gamma(\alpha_0 + 1)} + |x^*| + \frac{\sup_{t \in J} l_1(t) \Gamma(n - p)}{|1 - \lambda_1 \eta^{n - p - 1}| \Gamma(n) \Gamma(\alpha_0 - p + 1)} + \frac{\sup_{t \in J} l_1(t) |\lambda_1| \eta^{\alpha_0 - p} \Gamma(n - p)}{|1 - \lambda_1 \eta^{n - p - 1}| \Gamma(n) \Gamma(\alpha_0 - p + 1)}.$$

Therefore,

$$|x(t)| \leq \mu \sup_{t \in J} l_1(t) \left[ \frac{1}{\Gamma(\alpha_0 + 1)} + \frac{\Gamma(n - p)(1 + |\lambda_1| \eta^{\alpha_0 - p})}{|1 - \lambda_1 \eta^{n - p - 1}| \Gamma(n) \Gamma(\alpha_0 - p + 1)} \right] + |x^*|.$$

Hence,  $|x(t)| \leq \mu L_1 N_0 + |x^*|$ ,  $t \in J$ , which implies that,

$$(3.18) \qquad \qquad \qquad \|x\| \leq \mu L_1 N_0 + |x^*|.$$

Analogously, we have

$$(3.19) \qquad \qquad \qquad \|y\| \leq \mu L_2 M_0 + |y^*|.$$

On the other hand, for all  $k, = 1, 2, \dots, n - 1$ , we have

$$|D^{\alpha_k} x(t)| \leq \mu \sup_{t \in J} l_1(t) \left[ \frac{1}{\Gamma(\alpha_0 - \alpha_k + 1)} + \frac{\Gamma(n - p)(1 + |\lambda_1| \eta^{\alpha_0 - p})}{|1 - \lambda_1 \eta^{n - p - 1}| \Gamma(n - \alpha_k) \Gamma(\alpha_0 - p + 1)} \right], t \in J,$$

Thus,

$$(3.20) \qquad \qquad \qquad \|D^{\alpha_k} x(t)\| \leq \mu L_1 N_k,$$

and for all  $h, = 1, 2, \dots, n - 1,$

$$(3.21) \quad \|D^{\beta_h}y(t)\| \leq \mu L_2 M_h.$$

From (3.18) and (3.20), we get

$$(3.22) \quad \|x\|_X \leq \mu L_1 \left( N_0 + \sum_{k=1}^{n-1} N_k \right) + |x^*|.$$

Analogously, by (3.19) and (3.21), yields

$$(3.23) \quad \|y\|_Y \leq \mu L_2 \left( M_0 + \sum_{h=1}^{n-1} M_h \right) + |y^*|.$$

It follows from (3.22) and (3.23), that

$$\|(x, y)\|_{X \times Y} \leq \mu \left[ L_1 \left( N_0 + \sum_{k=1}^{n-1} N_k \right) + L_2 \left( M_0 + \sum_{h=1}^{n-1} M_h \right) \right] + |x^*| + |y^*|.$$

Hence,  $\|\phi(x, y)\|_{X \times Y} < \infty.$

This shows that  $\Omega$  is bounded.

As consequence of Schaefer’s fixed point theorem, we deduce that  $\phi$  at least a fixed point, which is a solution of the fractional differential system (1.1).  $\square$

Our third result is based on Krasnoselskii theorem [14].

**Theorem 3.3.** *Let  $\eta^{n-p-1} \neq \frac{1}{\lambda_1}, \xi^{n-q-1} \neq \frac{1}{\lambda_2}.$  Suppose that (H1), (H2) and (H3) are satisfied, such that*

$$(3.24) \quad \left( \frac{1}{\Gamma(\alpha_0+1)} + \sum_{k=1}^{n-1} \frac{1}{\Gamma(\alpha_0-\alpha_k+1)} \right) \omega + \left( \frac{1}{\Gamma(\beta_0+1)} + \sum_{h=1}^{n-1} \frac{1}{\Gamma(\beta_0-\beta_h+1)} \right) \varpi < 1.$$

*If there exist  $\delta \in \mathbb{R}$  such that*

$$L_1 \left( N_0 + \sum_{k=1}^{n-1} N_k \right) + L_2 \left( M_0 + \sum_{h=1}^{n-1} M_h \right) + |y^*| + |x^*| \leq \delta,$$

*then the fractional system (1.1) has at least one solution on  $J.$*

*Proof.* We shall prove that  $\phi$  has at least a fixed point on  $X \times Y.$

Suppose that  $L_1(N_0 + \sum_{k=1}^{n-1} N_k) + L_2(M_0 + \sum_{h=1}^{n-1} M_h) + |y^*| + |x^*| \leq \delta$  and let us take

$$\phi(x, y)(t) := T(x, y)(t) + R(x, y)(t) = (T_1y(t), T_2x(t)) + (R_1y(t), R_2x(t)),$$

where

$$T_1y(t) := \frac{1}{\Gamma(\alpha_0)} \int_0^t (t-s)^{\alpha_0-1} f_1(s, y(s), D^{\alpha_1}y(s), \dots, D^{\alpha_{n-1}}y(s)) ds + |x^*|,$$

$$T_2x(t) := \frac{1}{\Gamma(\beta_0)} \int_0^t (t-s)^{\beta_0-1} f_2(s, x(s), D^{\beta_1}x(s), \dots, D^{\beta_{n-1}}x(s)) ds + |y^*|,$$

and

$$\begin{aligned}
 R_1 y(t) &:= \\
 &- \frac{\Gamma(n-p)t^{n-1}}{(1-\lambda_1\eta^{n-p-1})\Gamma(n)\Gamma(\alpha_0-p)} \int_0^1 (1-s)^{\alpha_0-p-1} f_1(s, y(s), D^{\alpha_1}y(s), \dots, D^{\alpha_{n-1}}y(s)) ds \\
 &+ \frac{\lambda_1\Gamma(n-p)t^{n-1}}{(1-\lambda_1\eta^{n-p-1})\Gamma(n)\Gamma(\alpha_0-p)} \int_0^\eta (\eta-s)^{\alpha_0-p-1} f_1(s, y(s), D^{\alpha_1}y(s), \dots, D^{\alpha_{n-1}}y(s)) ds,
 \end{aligned}$$

$$\begin{aligned}
 R_2 x(t) &:= \\
 &- \frac{\Gamma(n-q)t^{n-1}}{(1-\lambda_2\xi^{n-q-1})\Gamma(n)\Gamma(\beta_0-q)} \int_0^1 (1-s)^{\beta_0-q-1} f_2(s, x(s), D^{\beta_1}x(s), \dots, D^{\beta_{n-1}}x(s)) ds \\
 &+ \frac{\lambda_2\Gamma(n-p)t^{n-1}}{(1-\lambda_2\xi^{n-q-1})\Gamma(n)\Gamma(\beta_0-q)} \int_0^1 \int_0^\eta (\eta-s)^{\beta_0-q-1} f_2(s, x(s), D^{\beta_1}x(s), \dots, D^{\beta_{n-1}}x(s)) ds.
 \end{aligned}$$

The proof will be given in the following steps:

(1\* :) We shall prove that for any  $(x, y), (x_1, y_1) \in B_\delta$ , then  $T(x, y)(t) + R(x_1, y_1)(t) \in B_\delta$ , such that  $B_\delta = \{(x, y) \in X \times Y; \|(x, y)\|_{X \times Y} \leq \delta\}$ .

For any  $(x, y), (x_1, y_1) \in B_\delta$  and for each  $t \in J$ , we have

$$\begin{aligned}
 |T_1 y(t) + R_1 y_1(t)| &\leq \frac{1}{\Gamma(\alpha_0)} \int_0^t (t-s)^{\alpha_0-1} |f_1(s, y(s), D^{\alpha_1}y(s), \dots, D^{\alpha_{n-1}}y(s))| ds + |x^*| \\
 &+ \frac{\Gamma(n-p)t^{n-1}}{|1-\lambda_1\eta^{n-p-1}|\Gamma(n)\Gamma(\alpha_0-p)} \int_0^1 (1-s)^{\alpha_0-p-1} |f_1(s, y_1(s), D^{\alpha_1}y_1(s), \dots, D^{\alpha_{n-1}}y_1(s))| ds \\
 &+ \frac{\lambda_1\Gamma(n-p)t^{n-1}}{|1-\lambda_1\eta^{n-p-1}|\Gamma(n)\Gamma(\alpha_0-p)} \int_0^\eta (\eta-s)^{\alpha_0-p-1} |f_1(s, y_1(s), D^{\alpha_1}y_1(s), \dots, D^{\alpha_{n-1}}y_1(s))| ds.
 \end{aligned}$$

Using (H3), we obtain

$$|T_1 y(t) + R_1 y_1(t)| \leq \sup_{t \in J} l_1(t) \left[ \frac{1}{\Gamma(\alpha_0+1)} + \frac{\Gamma(n-p)(1+|\lambda_1|\eta^{\alpha_0-p})}{|1-\lambda_1\eta^{n-p-1}|\Gamma(n)\Gamma(\alpha_0-p+1)} \right] + |x^*|.$$

Consequently,  $|T_1 y(t) + R_1 y_1(t)| \leq N_0 \sup_{t \in J} l_1(t) + |x^*|, t \in J$ . Thus,

$$(3.25) \quad \|T_1(y) + R_1(y_1)\| \leq L_1 N_0 + |x^*|.$$

On the other hand, for all  $k = 1, 2, \dots, n-1$ , we have

$$|D^{\alpha_k} T_1 y(t) + D^{\alpha_k} R_1 y_1(t)| \leq \sup_{t \in J} l_1(t) \left[ \frac{1}{\Gamma(\alpha_0-\alpha_k+1)} + \frac{\Gamma(n-p)(1+|\lambda_1|\eta^{\alpha_0-p})}{|1-\lambda_1\eta^{n-p-1}|\Gamma(n-\alpha_k)\Gamma(\alpha_0-p+1)} \right],$$

Hence,

$$(3.26) \quad \|D^{\alpha_k} T_1 y(t) + D^{\alpha_k} R_1 y_1(t)\| \leq L_1 N_k.$$

Combining (3.25) and (3.26), yields

$$(3.27) \quad \|T_1(y) + R_1(y_1)\|_X \leq L_1 \left( N_0 + \sum_{k=1}^{n-1} N_k \right) + |x^*|.$$

Analogously, for all  $h = 1, 2, \dots, n - 1$ , we have

$$(3.28) \quad \|T_2(x) + R_2(x_1)\|_Y \leq L_2 \left( M_0 + \sum_{h=1}^{n-1} M_h \right) + |y^*|.$$

Hence, it follows from (3.27) and (3.28) that

$$\|T(x, y) + R(x_1, y_1)\|_{X \times Y} \leq L_1 \left( N_0 + \sum_{k=1}^{n-1} N_k \right) + L_2 \left( M_0 + \sum_{h=1}^{n-1} M_h \right) + |y^*| + |x^*| \leq \delta.$$

Therefore,  $\|T(x, y) + R(x_1, y_1)\|_{X \times Y} \in B_\delta$ .

(2\* :) We shall prove that  $R$  is continuous and compact. Note that  $R$  is continuous on  $X \times Y$  in view of the continuity of  $f_1$  and  $f_2$  (hypothesis (H1)).

(a\* ) : Now, we prove that  $R$  maps bounded sets into bounded sets of  $X \times Y$ .

For  $(x, y) \in B_\delta$ ,  $\Delta_1 = \frac{\Gamma(n-p)t^{n-1}}{|1-\lambda_1\eta^{n-p-1}|\Gamma(n)\Gamma(\alpha_0-p)}$  and for each  $t \in J$ , we have

$$\begin{aligned} |R_1 y(t)| &\leq \Delta_1 \int_0^1 (1-s)^{\alpha_0-p-1} |f_1(s, y(s), D^{\alpha_1}y(s), \dots, D^{\alpha_{n-1}}y(s))| ds \\ &\quad + |\lambda_1| \Delta_1 \int_0^\eta (\eta-s)^{\alpha_0-p-1} |f_1(s, y(s), D^{\alpha_1}y(s), \dots, D^{\alpha_{n-1}}y(s))| ds. \end{aligned}$$

Thanks to (H3), we obtain

$$|R_1 y(t)| \leq \frac{\sup_{t \in J} l_1(t) \Gamma(n-p) (1+|\lambda_1|\eta^{\alpha_0-p})}{|1-\lambda_1\eta^{n-p-1}|\Gamma(n)\Gamma(\alpha_0-p+1)}, t \in J,$$

Therefore,

$$(3.29) \quad \|R_1(y)\| \leq \frac{L_1 \Gamma(n-p) (1+|\lambda_1|\eta^{\alpha_0-p})}{|1-\lambda_1\eta^{n-p-1}|\Gamma(n)\Gamma(\alpha_0-p+1)}.$$

On the other hand, for all  $k = 1, 2, \dots, n - 1$ , we have

$$(3.30) \quad \|D^{\alpha_k} R_1(y)\| \leq \frac{L_1 \Gamma(n-p) (1+|\lambda_1|\eta^{\alpha_0-p})}{|1-\lambda_1\eta^{n-p-1}|\Gamma(n-\alpha_k)\Gamma(\alpha_0-p+1)}.$$

Using (3.29) and (3.30), we have

$$(3.31) \quad \|R_1(y)\|_X \leq L_1 \left( \frac{\Gamma(n-p) (1+|\lambda_1|\eta^{\alpha_0-p})}{|1-\lambda_1\eta^{n-p-1}|\Gamma(n)\Gamma(\alpha_0-p+1)} + \sum_{k=1}^{n-1} \frac{\Gamma(n-p) (1+|\lambda_1|\eta^{\alpha_0-p})}{|1-\lambda_1\eta^{n-p-1}|\Gamma(n-\alpha_k)\Gamma(\alpha_0-p+1)} \right).$$

Similarly, for all  $h = 1, 2, \dots, n - 1$ ,

$$(3.32) \quad \|R_2(x)\|_Y \leq L_2 \left( \frac{\Gamma(n-q) (1+|\lambda_2|\eta^{\beta_0-q})}{|1-\lambda_2\xi^{n-q-1}|\Gamma(n)\Gamma(\beta_0-q+1)} + \sum_{h=1}^{n-1} \frac{\Gamma(n-q) (1+|\lambda_2|\xi^{\beta_0-q})}{|1-\lambda_2\xi^{n-q-1}|\Gamma(n-\beta_h)\Gamma(\beta_0-q+1)} \right).$$

It follows from (3.31) and (3.32) that  $\|R(x, y)\|_{X \times Y} \leq L_1\theta + L_2\theta' < \infty$ .

(b\* ) : Now, we show that  $R$  is equi-continuous on  $J$ .

Let  $t_1, t_2 \in J$ , such that  $t_2 < t_1$  and  $(x, y) \in B_\delta$ . Then by (H3), we obtain:

$$|R_1y(t_1) - R_1y(t_2)| \leq \frac{\sup_{t \in J} l_1(t) \Gamma(n-p) (t_2^{n-1} - t_1^{n-1})}{|1 - \lambda_1 \eta^{n-p-1}| \Gamma(n) \Gamma(\alpha_0 - p)} \int_0^1 (1-s)^{\alpha_0 - p - 1} ds$$

$$+ \frac{\sup_{t \in J} l_1(t) |\lambda_1| \Gamma(n-p) (t_1^{n-1} - t_2^{n-1})}{|1 - \lambda_1 \eta^{n-p-1}| \Gamma(n) \Gamma(\alpha_0 - p)} \int_0^\eta (\eta - s)^{\alpha_0 - p - 1} ds.$$

Thus,

$$(3.33) \quad |R_1y(t_1) - R_1y(t_2)| \leq \frac{L_1 \Gamma(n-p)}{|1 - \lambda_1 \eta^{n-p-1}| \Gamma(n) \Gamma(\alpha_0 - p + 1)} (t_2^{n-1} - t_1^{n-1})$$

$$+ \frac{L_1 |\lambda_1| \Gamma(n-p) \eta^{\alpha_0 - p}}{|1 - \lambda_1 \eta^{n-p-1}| \Gamma(n) \Gamma(\alpha_0 - p + 1)} (t_1^{n-1} - t_2^{n-1}).$$

In the same way, we have

$$(3.34) \quad |R_2x(t_1) - R_2x(t_2)| \leq \frac{L_2 \Gamma(n-q)}{|1 - \lambda_2 \xi^{n-q-1}| \Gamma(n) \Gamma(\beta_0 - q + 1)} (t_2^{n-1} - t_1^{n-1})$$

$$+ \frac{L_2 |\lambda_2| \Gamma(n-q) \xi^{\beta_0 - q}}{|1 - \lambda_2 \xi^{n-q-1}| \Gamma(n) \Gamma(\beta_0 - q + 1)} (t_1^{n-1} - t_2^{n-1}).$$

On the other hand, for all  $k = 1, 2, \dots, n - 1$ ,

$$(3.35) \quad |D^{\alpha_k} R_1y(t_1) - D^{\alpha_k} R_1y(t_2)| \leq \frac{L_1 \Gamma(n-p)}{|1 - \lambda_1 \eta^{n-p-1}| \Gamma(n - \alpha_k) \Gamma(\alpha_0 - p + 1)} (t_2^{n - \alpha_k - 1} - t_1^{n - \alpha_k - 1})$$

$$+ \frac{L_1 |\lambda_1| \Gamma(n-p) \eta^{\alpha_0 - p}}{|1 - \lambda_1 \eta^{n-p-1}| \Gamma(n - \alpha_k) \Gamma(\alpha_0 - p + 1)} (t_1^{n - \alpha_k - 1} - t_2^{n - \alpha_k - 1}),$$

and for all  $h = 1, 2, \dots, n - 1$ ,

$$(3.36) \quad |D^{\beta_h} R_2x(t_1) - D^{\beta_h} R_2x(t_2)| \leq \frac{L_2 \Gamma(n-q)}{|1 - \lambda_2 \xi^{n-q-1}| \Gamma(n - \beta_h) \Gamma(\beta_0 - q + 1)} (t_2^{n - \beta_h - 1} - t_1^{n - \beta_h - 1})$$

$$+ \frac{L_2 |\lambda_2| \Gamma(n-q) \xi^{\beta_0 - q}}{|1 - \lambda_2 \xi^{n-q-1}| \Gamma(n - \beta_h) \Gamma(\beta_0 - q + 1)} (t_1^{n - \beta_h - 1} - t_2^{n - \beta_h - 1}).$$

As  $t_2 \rightarrow t_1$ , the right-hand sides of the inequalities (3.33), (3.34), (3.35) and (3.36) tend to zero. Then, as a consequence of the steps (a\*) and (b\*) and by Arzela-Ascoli theorem, we conclude that  $R$  is completely continuous.

(3\* :) Finally, we prove that  $T$  is a contraction mapping: Let  $(x, y) (x_1, y_1) \in X \times Y$ . Then, for each  $t \in J$  and by (H2), we have

$$(3.37) \quad |T_1y(t) - T_1y_1(t)| \leq \frac{(\omega_0 + \omega_1 + \dots + \omega_{n-1}) [\|y - y_1\| + \|D^{\alpha_1}y - D^{\alpha_1}y_1\| + \dots + \|D^{\alpha_{n-1}}y - D^{\alpha_{n-1}}y_1\|]}{\Gamma(\alpha_0 + 1)}.$$

On the other hand, for all  $k = 1, 2, \dots, n - 1$ , we have

$$(3.38) \quad |D^{\alpha_k} T_1y(t) - D^{\alpha_k} T_1y_1(t)| \leq \frac{(\omega_0 + \omega_1 + \dots + \omega_{n-1}) [\|y - y_1\| + \|D^{\alpha_1}y - D^{\alpha_1}y_1\| + \dots + \|D^{\alpha_{n-1}}y - D^{\alpha_{n-1}}y_1\|]}{\Gamma(\alpha_0 - \alpha_k + 1)}.$$

By (3.37) and (3.38), we obtain

$$(3.39) \quad |T_1y(t) - T_1y_1(t)| \leq \left[ \frac{\omega}{\Gamma(\alpha_0 + 1)} + \omega \sum_{k=1}^{n-1} \frac{1}{\Gamma(\alpha_0 - \alpha_k + 1)} \right]$$

$$\times (\|y - y_1\| + \|D^{\alpha_1}y - D^{\alpha_1}y_1\| + \dots + \|D^{\alpha_{n-1}}y - D^{\alpha_{n-1}}y_1\|).$$

Analogously, for all  $h = 1, 2, \dots, n - 1$ , we can get

$$(3.40) \quad \begin{aligned} |T_2x(t) - T_2x_1(t)| &\leq \left[ \frac{\varpi}{\Gamma(\beta_0+1)} + \varpi \sum_{h=1}^{n-1} \frac{1}{\Gamma(\beta_0-\beta_h+1)} \right] \\ &\times (\|x - x_1\| + \|D^{\beta_1}x - D^{\beta_1}x_1\| + \dots + \|D^{\beta_{n-1}}x - D^{\beta_{n-1}}x_1\|). \end{aligned}$$

Combining (3.39) and (3.40) yields

$$\begin{aligned} \|T(x, y)(t) - T(x_1, y_1)(t)\|_{X \times Y} &\leq (\|(x - x_1, y - y_1)\|_{X \times Y}) \\ &\times \left[ \left( \frac{1}{\Gamma(\alpha_0+1)} + \sum_{k=1}^{n-1} \frac{1}{\Gamma(\alpha_0-\alpha_k+1)} \right) \omega + \left( \frac{1}{\Gamma(\beta_0+1)} + \sum_{h=1}^{n-1} \frac{1}{\Gamma(\beta_0-\beta_h+1)} \right) \varpi \right] \end{aligned}$$

Using the condition (3.24), we deduce that  $T$  is a contraction mapping.

As a consequence of Krasnoselskii’s fixed point theorem we deduce that  $\phi$  has a fixed point which is a solution of the problem (1.1).  $\square$

We give also the following two corollaries.

**Corollary 3.1.** *Assume that  $\eta^{n-p-1} \neq \frac{1}{\lambda_1}, \xi^{n-q-1} \neq \frac{1}{\lambda_2}$  and there exist non negative real numbers  $\theta_i, \Lambda_i, i = 0, 1, \dots, n - 1$  such that for all  $t \in [0, 1]$  and  $(x_0, x_1, \dots, x_{n-1}), (y_0, y_1, \dots, y_{n-1}) \in \mathbb{R}^n$ , we have  $|f_1(t, x_0, x_1, \dots, x_{n-1}) - f_1(t, y_0, y_1, \dots, y_{n-1})| \leq \theta_0|x_0 - y_0| + \dots + \theta_{n-1}|x_{n-1} - y_{n-1}|$  and  $|f_2(t, x_0, x_1, \dots, x_{n-1}) - f_2(t, y_0, y_1, \dots, y_{n-1})| \leq \Lambda_0|x_0 - y_0| + \dots + \Lambda_{n-1}|x_{n-1} - y_{n-1}|$ . If*

$$\left( N_0 + \sum_{k=1}^{n-1} N_k \right) (\theta_0 + \dots + \theta_{n-1}) + \left( M_0 + \sum_{h=1}^{n-1} M_h \right) (\Lambda_0 + \dots + \Lambda_{n-1}) < 1,$$

then the fractional system (1.1) has a unique solution on  $J$ .

**Corollary 3.2.** *Assume that (H1) holds and  $\eta^{n-p-1} \neq \frac{1}{\lambda_1}, \xi^{n-q-1} \neq \frac{1}{\lambda_2}$ . If there exist  $k_1 > 0$  and  $k_2 > 0$ , such that  $f_1 \leq k_1, f_2 \leq k_2$  on  $J \times \mathbb{R}^n$  then, the coupled system (1.1) has at least a solution on  $J$ .*

#### 4. EXAMPLES

*Example 4.1.* Consider the following fractional differential system, where  $t \in [0, 1]$

$$(4.1) \quad \begin{cases} D^{\frac{7}{2}}x(t) = \frac{|y(t)| + |D^{\frac{9}{4}}y(t)| + |D^{\frac{3}{2}}y(t)| + |D^{\frac{1}{2}}y(t)|}{(t^2 + 32\pi)(e^t + |y(t)| + |D^{\frac{9}{4}}y(t)| + |D^{\frac{3}{2}}y(t)| + |D^{\frac{1}{2}}y(t)|)} + \cosh(2 + t^2), \\ D^{\frac{11}{3}}y(t) = \frac{\sin x(t) + \sin D^{\frac{5}{2}}x(t) + \sin D^{\frac{6}{5}}x(t) + \sin D^{\frac{4}{5}}x(t)}{16(\pi t^2 + 1)} + \arctan(1 + t), \\ x(0) = \sqrt{2}, x'(0) = x''(0) = 0, D^{\frac{1}{2}}x(1) = \frac{4}{5}D^{\frac{1}{2}}x\left(\frac{3}{4}\right), \\ y(0) = \sqrt{3}, y'(0) = y''(0) = 0, D^{\frac{1}{3}}y(1) = \frac{6}{7}D^{\frac{1}{3}}y\left(\frac{2}{5}\right). \end{cases}$$

We have

$$f_1(t, x_1, x_2, x_3, x_4) = \frac{|x_1| + |x_2| + |x_3| + |x_4|}{(t^2 + 32\pi)(e^t + |x_1| + |x_2| + |x_3| + |x_4|)} + \cosh(2 + t^2),$$

$$f_2(t, x_1, x_2, x_3, x_4) = \frac{\sin(x_1) + \sin(x_2) + \sin(x_3) + \sin(x_4)}{16(\pi t^2 + 1)} + \arctan(1 + t)$$

where  $t \in [0, 1], x_1, x_2, x_3, x_4 \in \mathbb{R}$ .

Let  $t \in [0, 1]$  and  $(x_0, x_1, x_2, x_3), (y_0, y_1, y_2, y_3) \in \mathbb{R}^4$ . Then

$$|f_1(t, x_0, x_1, x_2, x_3) - f_1(t, y_0, y_1, y_2, y_3)| \leq \frac{1}{(t^2 + 32\pi)} (|x_0 - y_0| + |x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|),$$

$$|f_2(t, x_0, x_1, x_2, x_3) - f_2(t, y_0, y_1, y_2, y_3)| \leq \frac{1}{16(\pi t^2 + 1)} (|x_0 - y_0| + |x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|).$$

We can take  $a_i(t) = \frac{1}{(t^2 + 32\pi)}, b_i(t) = \frac{1}{16(\pi t^2 + 1)}, i = 0, 1, 2, 3$ . Then  $\omega_i = \sup_{t \in [0,1]} a_i(t) = \frac{1}{32\pi}, \varpi_i = \sup_{t \in [0,1]} b_i(t) = \frac{1}{16}, i = 0, 1, 2, 3$  and  $N_0 = 0.288269, N_1 = 2.203434, N_2 = 1.413119, N_3 = 0.615229, \omega = \frac{1}{8\pi}, M_0 = 0.085713, M_1 = 1.124108, M_2 = 0.322689, M_3 = 0.203869, \varpi = \frac{1}{4}$ . We also have

$$\left(N_0 + \sum_{k=1}^{n-1} N_k\right) \omega + \left(M_0 + \sum_{h=1}^{n-1} M_h\right) \varpi = 0.179938 + 0.434094 = 0.614032 < 1.$$

Hence, by Theorem 3.1, the system (4.1) has a unique solution on  $[0, 1]$ .

*Example 4.2.* The second example is the following system:

$$(4.2) \quad \begin{cases} D^{\frac{10}{3}} x(t) = \frac{\cos y(t) + \sin(D^{\frac{10}{4}} y(t) + D^{\frac{9}{7}} y(t))}{\pi e^{t+15}}, t \in [0, 1], \\ D^{\frac{14}{4}} y(t) = \frac{\sin x(t) + \cos(D^{\frac{5}{2}} x(t) + D^{\frac{4}{3}} x(t))}{t^2 + 20\pi}, t \in [0, 1], \\ x(0) = 3, x'(0) = x''(0) = 0, D^{\frac{4}{5}} x(1) = \frac{3}{4} D^{\frac{4}{5}} x\left(\frac{1}{3}\right), \\ y(0) = \sqrt{5}, y'(0) = y''(0) = 0, D^{\frac{9}{8}} y(1) = \frac{2}{3} D^{\frac{9}{8}} y\left(\frac{2}{5}\right). \end{cases}$$

We have

$$f_1(t, x, y, z) = \frac{\cos x + \sin(y + z)}{\pi e^t + 15}, t \in [0, 1], (x, y, z) \in \mathbb{R}^3,$$

$$f_2(t, x, y, z) = \frac{\sin x + \cos(y + z)}{20\pi + t^2}, t \in [0, 1], (x, y, z) \in \mathbb{R}^3.$$

Let  $x, y, z \in \mathbb{R}$  and  $t \in [0, 1]$ . Then  $|f_1(t, x, y, z)| \leq \frac{2}{\pi e^t + 15}, |f_2(t, x, y, z)| \leq \frac{2}{t^2 + 20\pi}$ . So, we take  $l_1(t) = \frac{2}{\pi e^t + 15}, l_2(t) = \frac{2}{t^2 + 20\pi}$ . Then,  $L_1 = \frac{2}{\pi + 15}, L_2 = \frac{1}{10\pi}$ . By Theorem 3.2, the system (4.2) has at least one solution on  $[0, 1]$ .



*Example 4.3.* Our third example is the following:

$$(4.3) \quad \begin{cases} D^{\frac{17}{5}} x(t) = \frac{e^{-t^2} (|y(t)| + |D^{\frac{15}{6}} y(t)| + |D^{\frac{11}{7}} y(t)| + |D^{\frac{2}{3}} y(t)|)}{(e^{t^2} + 20\pi) (2\pi e^{t^2} + |y(t)| + |D^{\frac{15}{6}} y(t)| + |D^{\frac{11}{7}} y(t)| + |D^{\frac{2}{3}} y(t)|)} + \cos(2 + t^2), \\ D^{\frac{15}{4}} y(t) = \frac{|x(t)| + |D^{\frac{11}{5}} x(t)| + |D^{\frac{4}{3}} x(t)| + |D^{\frac{1}{2}} x(t)|}{(\pi e^t + 18) (e^{-t} + |x(t)| + |D^{\frac{11}{5}} x(t)| + |D^{\frac{4}{3}} x(t)| + |D^{\frac{1}{2}} x(t)|)} + \ln(2 + t^2), \\ x(0) = 2, x'(0) = x''(0) = 0, D^{\frac{5}{4}} x(1) = \frac{5}{6} D^{\frac{5}{4}} x\left(\frac{2}{5}\right), \\ y(0) = 3, y'(0) = y''(0) = 0, D^{\frac{3}{2}} y(1) = \frac{7}{8} D^{\frac{3}{2}} y\left(\frac{3}{5}\right), \end{cases}$$

where  $t \in [0, 1]$ . For this example, for  $t \in [0, 1]$ ,  $x_1, x_2, x_3, x_4 \in \mathbb{R}$  we have

$$f_1(t, x_1, x_2, x_3, x_4) = \frac{e^{-t^2} (|x_1| + |x_2| + |x_3| + |x_4|)}{(e^{t^2} + 20\pi) (2\pi e^{t^2} + |x_1| + |x_2| + |x_3| + |x_4|)} + \cos(2 + t^2),$$

$$f_2(t, x_1, x_2, x_3, x_4) = \frac{|x_1| + |x_2| + |x_3| + |x_4|}{(\pi e^t + 18) (e^{-t} + |x_1| + |x_2| + |x_3| + |x_4|)} + \ln(2 + t^2)$$

Taking  $x_0, x_1, x_2, x_3, y_0, y_1, y_2, y_3 \in \mathbb{R}$ ,  $t \in [0, 1]$ , we have

$$|f_1(t, x_0, x_1, x_2, x_3) - f_1(t, y_0, y_1, y_2, y_3)| \leq \frac{e^{-t^2}}{e^{t^2} + 20\pi} (|x_0 - y_0| + |x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|),$$

$$|f_2(t, x_0, x_1, x_2, x_3) - f_2(t, y_0, y_1, y_2, y_3)| \leq \frac{1}{\pi e^t + 18} (|x_0 - y_0| + |x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|).$$

Hence,  $a_i(t) = \frac{e^{-t^2}}{(e^{t^2} + 20\pi)}$ ,  $b_i(t) = \frac{1}{(\pi e^t + 18)}$ ,  $i = 0, \dots, 3$ .

Then,  $\omega_i = \sup_{t \in [0,1]} a_i(t) = \frac{1}{1+20\pi}$ ,  $\varpi_i = \sup_{t \in [0,1]} b_i(t) = \frac{1}{\pi+18}$ ,  $i = 0, \dots, 3$  and for  $k, h \in \{= 1, 2, 3\}$ , we have

$$\frac{1}{\Gamma(\alpha_0+1)} + \sum_{k=1}^{n-1} \frac{1}{\Gamma(\alpha_0-\alpha_k+1)} = 1.951303,$$

$$\frac{1}{\Gamma(\beta_0+1)} + \sum_{h=1}^{n-1} \frac{1}{\Gamma(\beta_0-\beta_h+1)} = 1.236325.$$

We also have

$$\left( \frac{1}{\Gamma(\alpha_0+1)} + \sum_{k=1}^{n-1} \frac{1}{\Gamma(\alpha_0-\alpha_k+1)} \right) \omega + \left( \frac{1}{\Gamma(\beta_0+1)} + \sum_{h=1}^{n-1} \frac{1}{\Gamma(\beta_0-\beta_h+1)} \right) \varpi = 0.3562179 < 1.$$

By Theorem 3.3, we can state that the problem (4.3) has at least one solution on  $[0, 1]$ .

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