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Analytic Approximate Solutions to The Coupled Lotka-Volterra Equations with Fractional Derivatives

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Abstract: By introducing the fractional derivative in the sense of Caputo, we apply the Adomian decomposition method for the coupled Lotka-Volterra equations with time-and space-fractional derivative. As a result, numerical solutions are obtained in a form of rapidly convergent series with easily computable components. The behavior of Adomian solutions are shown graphically for some examples.

Keywords: Adomian method; caputo fractional derivative; coupled Lotka-Volterra equations

1 Introduction

Since Adomian firstly proposed the decomposition method [4, 5] at the begin of 1980s, the algorithm has been widely used for obtaining analytic solutions of physically significant equations [6, 15, 25–30]. With this method, we can easily obtain approximate solutions in the form of a rapidly convergent infinite series with each term computed conveniently[1, 2, 8, 13].

As we all know, for the nonlinear equations with derivatives of integer order, many methods are used to derive approximation solutions [3–5, 7, 11, 18, 23, 24]. However, for the fractional differential equations, there are only limited approaches, such as Laplace transform method [20], the Fourier transform method [16], the iteration method [22] and the operational method [19].

In recent years, considerable interest in fractional differential equations has been stimulated due to their numerous applications in the area of physics and engineering [9, 14, 31], like phenomena in electromagnetic theory, acoustics, electrochemistry and material science [10, 12, 20, 22, 31].

We introduce Caputo fractional derivative and apply the ADM to derive numerical solutions of the coupled Lotka-Volterra equations with time and space fractional derivatives:

$$\begin{cases} D_t^{\alpha} u = \left(D_x^{\beta} u\right)^2 + u u_{xx} + b_1 u^2 + a_1 u + c_1 v + h_1 \\ D_t^{\alpha} v = \left(D_x^{\beta} v\right)^2 + v v_{xx} + b_2 v^2 + a_2 v + c_2 u + h_2, \end{cases} \tag{1.1}$$

where $a_1, a_2, b_1, b_2, c_1, c_2, h_1, h_2$ are arbitrary constants such that $b_1 b_2 \neq 0$ and $c_1 c_2 \neq 0$.

When $\alpha = \beta = 1$, the fractional equations reduce to the coupled Lotka–Volterra equations :

$$\begin{cases} u_t = (u u_x)_x + u\,(a_1 + b_1 u) + c_1 v + h_1 \\ v_t = (v v_x)_x + v\,(a_2 + b_2 v) + c_2 u + h_2. \end{cases} \tag{1.2}$$

The paper is organized as follows: In Sec. II, some necessary details on the fractional calculus are provided. In Sec. III, the coupled Lotka-Volterra equations with time-and-space-fractional derivatives are studied with the Adomian method and figures are used to show the efficiency as well as the accuracy of the approximate results are achieved. Finally, conclusions are followed.

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2 Description of Fractional Calculus

There are several mathematical definitions about fractional derivative[20, 22]. Here, we adopt the two usually used definitions: the Caputo and its reverse operator Riemann-Liouville. That is because Caputo fractional derivative allows traditional initial condition assumption and boundary conditions. More details one can consults [20]. In the following, we will give the necessary notation and basic definitions.

Definition 1 *A real valued function $f(x), x > 0$ is said to be in the space $C_\mu, \mu \in R$ if there exists a real number $p > \mu$ such that $f(x) = x^p f_1(x)$ where $f_1(x) \in C([0,\infty))$.*

Definition 2 *A function $f(x), x > 0$ is said to be in the space $C_\mu^n, n \in N$, if $f^{(n)} \in C_\mu$.*

Definition 3 *The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, for a function $f \in C_\mu$, $(\mu \geq -1)$ is defined as*

$$\begin{array}{l} J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt; \quad \alpha > 0, x > 0 \\ J^0 f(x) = f(x). \end{array} \tag{2.1}$$

For the convenience of establishing the results for the fractional Lotka-Volterra equations, we give the properties

$$J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x) \tag{2.2}$$

and

$$J^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} x^{\alpha+\beta}. \tag{2.3}$$

Definition 4 *The fractional derivative of $f \in C_{-1}^n$ in the Caputo's sense is defined as*

$$D^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, & n-1 < \alpha < n, n \in N^*, \\ \frac{d^n}{dt^n} f(t), & \alpha = n. \end{cases} \tag{2.4}$$

According to the Caputo's derivative, we can easily obtain the following expressions:

$$D^\alpha K = 0; \quad K \text{ is a constant}, \tag{2.5}$$

$$D^\alpha t^\beta = \begin{cases} \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}, & \beta > \alpha - 1, \\ 0, & \beta \leq \alpha - 1. \end{cases} \tag{2.6}$$

Details on Caputo's derivative can be found in [20].

Remark 1 *In this paper, we consider equations (1.1) with time-and space-fractional derivative. When $\alpha \in R^+$, we have:*

$$D_t^\alpha u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{\partial^n u(x,\tau)}{\partial \tau^n} d\tau, & n-1 < \alpha < n \\ \frac{\partial^n u(x,t)}{\partial t^n}, & \alpha = n. \end{cases} \tag{2.7}$$

The form of the space fractional derivative is similar to the above.

3 The Analysis of the ADM Method

Consider the coupled equations with time-and space-fractional derivatives Eq.(1.1).

Take the initial condition as

$$\begin{cases} u(x,0) = f(x) \\ v(x,0) = g(x). \end{cases} \tag{3.1}$$

Applying the operator J^α, the inverse of D^α in (1.1) and using the initial condition (3.1), yields:

$$\begin{cases} u(x,t) = f(x) + J^\alpha (P_1(u(x,t))) + J^\alpha (P_2(u(x,t))) + b_1 J^\alpha (P_3(u(x,t))) \\ +a_1 J^\alpha (u) + c_1 J^\alpha (v) + J^\alpha (h_1), \\ v(x,t) = g(x) + J^\alpha (Q_1(u(x,t))) + J^\alpha (Q_2(u(x,t))) + b_1 J^\alpha (Q_3(u(x,t))) \\ +a_2 J^\alpha (v) + c_2 J^\alpha (u) + J^\alpha (h_2), \end{cases} \tag{3.2}$$

where $P_1(u(x,t)) = \left(D_x^\beta u\right)^2$, $P_2(u(x,t)) = uu_{xx}$, $P_3(u(x,t)) = u^2$, $Q_1(v(x,t)) = \left(D_x^\beta v\right)^2$, $Q_2(v(x,t)) = vv_{xx}$ and $Q_3(v(x,t)) = v^2$.

The solution of Eq.(1.1) is represented as infinite series like

$$\begin{cases} u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \\ v(x,t) = \sum_{n=0}^{\infty} v_n(x,t). \end{cases} \tag{3.3}$$

The nonlinear operators $P_1(u)$, $P_2(u)$, $P_3(u)$, $Q_1(v)$, $Q_2(v)$ and $Q_3(v)$ are decomposed in these forms

$$\begin{aligned} P_1(u) &= \sum_{n=0}^{\infty} A_n, \quad Q_1(v) = \sum_{n=0}^{\infty} D_n, \\ P_2(u) &= \sum_{n=0}^{\infty} B_n, \quad Q_2(v) = \sum_{n=0}^{\infty} E_n, \\ P_3(u) &= \sum_{n=0}^{\infty} C_n, \quad Q_3(v) = \sum_{n=0}^{\infty} F_n, \end{aligned} \tag{3.4}$$

where A_n, B_n, C_n, D_n, E_n and F_n are the so-called Adomian polynomials and have the form

$$\begin{aligned} A_n &= \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[P_1\left(\sum_{k=0}^{\infty}\lambda^k u_k\right)\right]_{\lambda=0} = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[\left(D_x^\beta\left(\sum_{k=0}^{\infty}\lambda^k u_k\right)\right)^2\right]_{\lambda=0}, \\ B_n &= \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[P_2\left(\sum_{k=0}^{\infty}\lambda^k u_k\right)\right]_{\lambda=0} = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[\left(\sum_{k=0}^{\infty}\lambda^k u_k\right)\left(\sum_{k=0}^{\infty}\lambda^k u_{kxx}\right)\right]_{\lambda=0}, \\ C_n &= \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[P_3\left(\sum_{k=0}^{\infty}\lambda^k u_k\right)\right]_{\lambda=0} = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[\left(\sum_{k=0}^{\infty}\lambda^k v_k\right)^2\right]_{\lambda=0}, \\ D_n &= \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[Q_1\left(\sum_{k=0}^{\infty}\lambda^k v_k\right)\right]_{\lambda=0} = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[\left(D_x^\beta\left(\sum_{k=0}^{\infty}\lambda^k v_k\right)\right)^2\right]_{\lambda=0}, \\ E_n &= \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[Q_2\left(\sum_{k=0}^{\infty}\lambda^k v_k\right)\right]_{\lambda=0} = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[\left(\sum_{k=0}^{\infty}\lambda^k v_k\right)\left(\sum_{k=0}^{\infty}\lambda^k v_{kxx}\right)\right]_{\lambda=0}, \\ F_n &= \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[Q_3\left(\sum_{k=0}^{\infty}\lambda^k v_k\right)\right]_{\lambda=0} = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[\left(\sum_{k=0}^{\infty}\lambda^k v_k\right)^2\right]_{\lambda=0}. \end{aligned} \tag{3.5}$$

We give the first four components of these polynomials:

$$\begin{aligned} &A_0 = \left(D_x^\beta u_0\right)^2, A_1 = 2\left(D_x^\beta u_0\right)\left(D_x^\beta u_1\right), A_2 = 2\left(D_x^\beta u_0\right)\left(D_x^\beta u_2\right) + \left(D_x^\beta u_1\right)^2, \\ &A_3 = \left(D_x^\beta u_0\right)\left(D_x^\beta u_3\right) + \left(D_x^\beta u_1\right)\left(D_x^\beta u_2\right). \end{aligned} \tag{3.6}$$

The first four components of B_n are:

$$\begin{aligned} &B_0 = u_0 u_{0xx}, B_1 = u_0 u_{1xx} + u_1 u_{0xx}, B_2 = u_0 u_{2xx} + u_1 u_{1xx} + u_2 u_{0xx}, \\ &B_3 = u_0 u_{3xx} + u_1 u_{2xx} + u_2 u_{1xx} + u_3 u_{0xx}. \end{aligned} \tag{3.7}$$

The first four components of C_n are given by:

$$C_0 = (u_0)^2, C_1 = 2u_0 u_1, C_2 = 2u_0 u_2 + (u_1)^2, C_3 = u_0 u_3 + u_1 u_2. \tag{3.8}$$

The first four components of D_n are:

$$\begin{aligned} &D_0 = \left(D_x^\beta v_0\right)^2, D_1 = 2\left(D_x^\beta v_0\right)\left(D_x^\beta v_1\right), D_2 = 2\left(D_x^\beta v_0\right)\left(D_x^\beta v_2\right) + \left(D_x^\beta v_1\right)^2, \\ &D_3 = \left(D_x^\beta v_0\right)\left(D_x^\beta v_3\right) + \left(D_x^\beta v_1\right)\left(D_x^\beta v_2\right), \end{aligned} \tag{3.9}$$

and those of E_n and F_n are given by:

$$\begin{aligned} &E_0 = v_0 v_{0xx}, E_1 = v_0 v_{1xx} + v_1 v_{0xx}, E_2 = v_0 v_{2xx} + v_1 v_{1xx} + v_2 v_{0xx}, \\ &E_3 = v_0 v_{3xx} + v_1 v_{2xx} + v_2 v_{1xx} + v_3 v_{0xx}, \end{aligned} \tag{3.10}$$

$$F_0 = (v_0)^2, F_1 = 2v_0v_1, F_2 = 2v_2v_0 + (v_1)^2, F_3 = v_0v_3 + v_1v_2. \tag{3.11}$$

Substituting the decomposition series (3.3) and (3.4) into Eq(1.1), yields the following recursive formula:

$$\begin{aligned} &u_0(x,t) = f(x), v_0(x,t) = g(x), \\ &u_{n+1}(x,t) = J^\alpha(A_n) + J^\alpha(B_n) + b_1 J^\alpha(C_n) + a_1 J^\alpha(u_n) + c_1 J^\alpha(v_n) + J^\alpha(h_1), \\ &v_{n+1}(x,t) = J^\alpha(E_n) + J^\alpha(F_n) + b_{\acute{e}} J^\alpha(G_n) + a_2 J^\alpha(v_n) + c_2 J^\alpha(u_n) + J^\alpha(h_2). \end{aligned} \tag{3.12}$$

The Adomian decomposition method converges generally very quickly. Details on its converegence and the convergence speed can be found in [1, 2, 8, 13]. Here according to the above steps, we will derive the numerical solutions for the coupled Eq(1.1) in details.

3.1 Numerical Solutions of Time-Fractional Lotka-Volterra Equations

Consider the following form of the time-fractional equations

$$\begin{cases} D_t^\alpha u = (u_x)^2 + uu_{xx} + b_1u^2 + a_1u + c_1v + h_1 \\ D_t^\alpha v = (v_x)^2 + vv_{xx} + b_2v^2 + a_2v + c_2u + h_2 \end{cases}, 0 < \alpha \leq 1, \tag{3.13}$$

with the initial conditions

$$\begin{cases} u(x,0) = f(x) = \frac{2}{t_0} - 6c_1 + \left(\frac{2}{t_0} - 6c_1 + \frac{2c_1}{b}\right)\cos\left(\sqrt{\frac{b}{2}}x - \beta_0\right), \\ v(x,0) = g(x) = \frac{2}{t_0} - 6c_1 + \frac{4c_1}{b} + \frac{|c_1|}{c_1}\left(\frac{2}{t_0} - 6c_1 + \frac{2c_1}{b}\right)\sin\left(\sqrt{\frac{b}{2}}x - \beta_0\right), \end{cases} \tag{3.14}$$

where $a_1, a_2, b_1, b_2, c_1, c_2, h_1, h_2, \beta_0$ and t_0 are arbitrary constants such that, $b = b_1 = b_2 > 0$, $a_1 = 3c_1$, $a_2 = a_1 - 6c_1$, $c_2 = -c_1$.

The exact solution of the system (3.13) for the special case $\alpha = 1$ is

$$\begin{cases} u(x,t) = \varphi_0(t) \pm \left(\varphi_0(t) + \frac{2c_1}{b}\right)\cos\left(\sqrt{\frac{b}{2}}x \mp |c_1|t - \beta_0\right), \\ v(x,t) = \varphi_0(t) + \frac{4c_1}{b} \pm \frac{|c_1|}{c_1}\left(\varphi_0(t) + \frac{2c_1}{b}\right)\sin\left(\sqrt{\frac{b}{2}}x \mp |c_1|t - \beta_0\right), \end{cases} \tag{3.15}$$

where

$$\varphi_0(t) = \frac{1}{3b}\begin{cases} \frac{2}{t_0 - t} - 6c_1;\ a_1 = 3c_1 \\ |3c_1 - a_1|\tanh\left(\frac{|3c_1 - a_1|}{2}(t_0 - t)\right) - a_1 - 3c_1;\ a_1 \neq 3c_1, \end{cases}$$

$a_1, a_2, b_1, b_2, c_1, c_2, h_1, h_2, \beta_0$ and t_0 are arbitrary constants such that, $b = b_1 = b_2 > 0$, $a_2 = a_1 - 6c_1$, $c_2 = -c_1$, $h_1 = \frac{(2c_1a_1 - 6c_1^2)}{b}$ and $h_2 = h_1 + \frac{4c_1}{b}(3c_1 - a_1)$.

In order to obtain numerical solution of Eq.(3.13), substituting the initial conditions (3.14) and using the Adomian polynomial (3.6)-(3.11) into the expression (3.12), we can compute the results. For simplicity, we only give the first few terms of series:

$$\begin{array}{l} u_0 = f\left(x\right), v_0 = g\left(x\right), \\ u_1 = J^{\alpha}\left(A_0\right) + J^{\alpha}\left(B_0\right) + b_1 J^{\alpha}\left(C_0\right) + a_1 J^{\alpha}\left(u_0\right) + c_1 J^{\alpha}\left(v_0\right) + J^{\alpha}\left(h_1\right) \\ = J^{\alpha}\left(\left(u_{0x}\right)^2\right) + J^{\alpha}\left(u_0 u_{0xx}\right) + b_1 J^{\alpha}\left(\left(u_0\right)^2\right) + a_1 J^{\alpha}\left(u_0\right) + c_1 J^{\alpha}\left(v_0\right) + J^{\alpha}\left(h_1\right) = f_1 \frac{t^{\alpha}}{\Gamma(\alpha+1)}, \\ v_1 = J^{\alpha}\left(D_0\right) + J^{\alpha}\left(E_0\right) + b_2 J^{\alpha}\left(F_0\right) + a_2 J^{\alpha}\left(v_0\right) + c_2 J^{\alpha}\left(u_0\right) + J^{\alpha}\left(h_2\right) \\ = J^{\alpha}\left(\left(v_{0x}\right)^2\right) + J^{\alpha}\left(v_0 v_{0xx}\right) + b_2 J^{\alpha}\left(\left(v_0\right)^2\right) + a_2 J^{\alpha}\left(v_0\right) + c_2 J^{\alpha}\left(u_0\right) + J^{\alpha}\left(h_2\right) = g_1 \frac{t^{\alpha}}{\Gamma(\alpha+1)}, \\ u_2 = J^{\alpha}\left(A_1\right) + J^{\alpha}\left(B_1\right) + b_1 J^{\alpha}\left(C_1\right) + a_1 J^{\alpha}\left(u_1\right) + c_1 J^{\alpha}\left(v_1\right) + J^{\alpha}\left(h_1\right) \\ = J^{\alpha}\left(2u_{0x}u_{1x}\right) + J^{\alpha}\left(u_0 u_{1xx} + u_1 u_{0xx}\right) + b_1 J^{\alpha}\left(2u_0 u_1\right) + a_1 J^{\alpha}\left(u_1\right) + c_1 J^{\alpha}\left(v_1\right) + J^{\alpha}\left(h_1\right) \\ = f_2 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \\ v_2 = J^{\alpha}\left(D_1\right) + J^{\alpha}\left(E_1\right) + b_2 J^{\alpha}\left(F_1\right) + a_2 J^{\alpha}\left(v_1\right) + c_2 J^{\alpha}\left(u_1\right) + J^{\alpha}\left(h_2\right) \\ = J^{\alpha}\left(2v_{0x}v_{1x}\right) + J^{\alpha}\left(v_0 v_{1xx} + v_1 v_{0xx}\right) + b_2 J^{\alpha}\left(2v_0 v_1\right) + a_2 J^{\alpha}\left(v_1\right) + c_2 J^{\alpha}\left(u_1\right) + J^{\alpha}\left(h_2\right) \\ = g_2 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \\ u_3 = J^{\alpha}\left(A_2\right) + J^{\alpha}\left(B_2\right) + J^{\alpha}\left(C_2\right) + a_1 J^{\alpha}\left(u_2\right) + b_1 J^{\alpha}\left(v_2\right) + J^{\alpha}\left(h_1\right) \\ = J^{\alpha}\left(2u_{0x}u_{2x} + \left(u_{1x}\right)^2\right) + J^{\alpha}\left(u_0 u_{2xx} + u_1 u_{1xx} + u_2 u_{0xx}\right) + b_1 J^{\alpha}\left(2u_0 u_2 + \left(u_1\right)^2\right) \\ + a_1 J^{\alpha}\left(u_2\right) + c_1 J^{\alpha}\left(v_2\right) + J^{\alpha}\left(h_1\right) = f_3 \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \\ v_3 = J^{\alpha}\left(D_2\right) + J^{\alpha}\left(E_2\right) + b_2 J^{\alpha}\left(F_2\right) + a_2 J^{\alpha}\left(v_2\right) + c_2 J^{\alpha}\left(u_2\right) + J^{\alpha}\left(h_2\right) \\ = J^{\alpha}\left(2v_{0x}v_{2x} + \left(v_{1x}\right)^2\right) + J^{\alpha}\left(v_0 v_{2xx} + v_1 v_{1xx} + v_2 v_{0xx}\right) + b_2 J^{\alpha}\left(2v_0 v_2 + \left(v_1\right)^2\right) \\ + a_2 J^{\alpha}\left(v_2\right) + c_2 J^{\alpha}\left(u_2\right) + J^{\alpha}\left(h_2\right) = g_3 \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \end{array} \tag{3.16}$$

where

$$\begin{array}{rcl} f_1 & = & f_x^2 + f f_{xx} + b_1 f^2 + a_1 f + c_1 g + h_1, \\ g_1 & = & g_x^2 + g g_{xx} + b_2 g^2 + a_2 g + c_2 f + h_2, \\ f_2 & = & 2f_x f_{1x} + f f_{1xx} + f_1 f_{xx} + 2b_1 f f_1 + a_1 f_1 + c_1 g_1 + h_1, \\ g_2 & = & 2g_x g_{1x} + g g_{1xx} + g_1 g_{xx} + 2b_2 g g_1 + a_2 g_1 + c_2 f_1 + h_2, \\ f_3 & = & 2f_x f_{2x} + f_{1x}^2 + f f_{2xx} + f_1 f_{1xx} + f_2 f_{xx} + b_1\left(2f f_2 + f_1^2\right) + a_1 f_2 + c_1 g_2 + h_1, \\ g_3 & = & 2g_x g_{2x} + g_{1x}^2 + g g_{2xx} + g_1 g_{1xx} + g_2 g_{xx} + b_2\left(2g g_2 + g_1^2\right) + a_2 g_2 + c_2 f_2 + h_2, \end{array} \tag{3.17}$$

and so on in the same manner, the rest of the components of the recursive formulas (3.12) are obtained.

Then we have the numerical solution of Eq.(3.13) under the series form

$$\begin{array}{l} u\left(x,t\right) = f\left(x\right) + f_1 \frac{t^{\alpha}}{\Gamma(\alpha+1)} + f_2 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + f_3 \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \ldots \\ v\left(x,t\right) = g\left(x\right) + g_1 \frac{t^{\alpha}}{\Gamma(\alpha+1)} + g_2 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + g_3 \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \ldots \end{array} \tag{3.18}$$

In order to verify the efficiency and accuracy of the proposed ADM method for the time-fractional Lotka-Voltera equations, we draw figures for the numerical solutions (3.18) with $\alpha = 1/2$, as well as the exact solution (3.15) when $\alpha = 1$.

Remark 2 *The accuracy of the numerical solution depend on how many terms we choose. In order to reduce the overall errors, what we need is to add new terms to the decomposition series.*

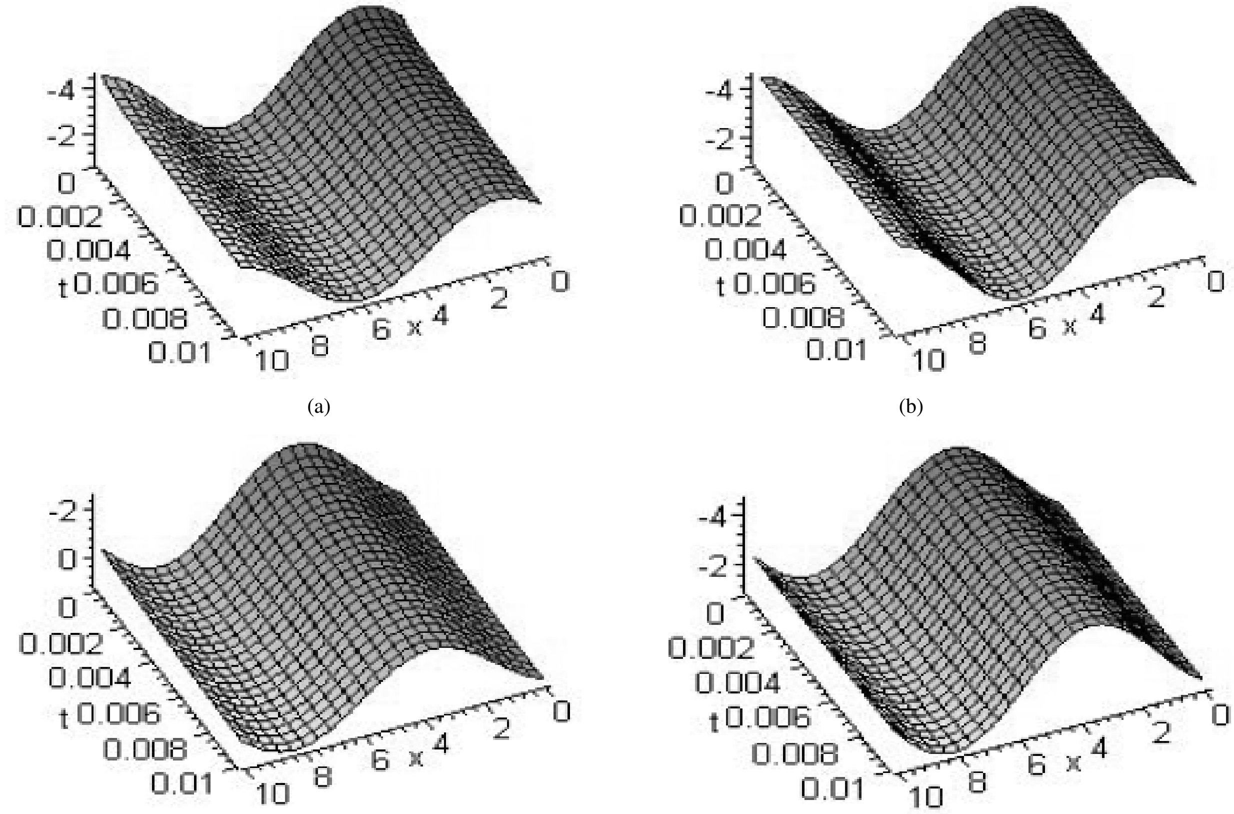
3.2 Numerical Solutions of Space-Fractional Lotka–Volterra Equations

Considering the operator form of the space-fractional equations

$$\left\{ \begin{array}{l} u_t = \left(D_x^{\beta} u\right)^2 + u u_{xx} + b_1 u^2 + a_1 u + c_1 v + h_1 \\ v_t = \left(D_x^{\beta} v\right)^2 + v v_{xx} + b_2 v^2 + a_2 v + c_2 u + h_2, \end{array} \right. \tag{3.19}$$

with the initial conditions

$$\left\{ \begin{array}{l} u\left(x,0\right) = f\left(x\right) = x^2, \\ v\left(x,0\right) = g\left(x\right) = x^3. \end{array} \right. \tag{3.20}$$



(a) (b) (c) (d)

Figure 1: Representing time-fractional solutions of Eq.(3.13). In(a): the numerical solution u(x,t)-(3.18). In(b), the exact solution u(x,t)-(3.15) of Syst.(3.13). In(c): the numerical solution v(x,t)-(3.18). In(d), the exact solution v(x,t)-(3.15) of Eq.(3.13). $a_1 = 3/2; b_1 = 1; c_1 = 1/2; h_1 = 1; a_2 = -3/2; b_2 = 1; c_2 = -1/2; h_2 = 1; \beta_0 = 1; t_0 = 10; \alpha = 1/2;$

In order to estimate the numerical solution of equations (3.19), substituting (3.3), (3.4) and using the initial conditions (3.20) into the expression (3.12), we get the Adomian solution. We give the first few terms of series:

$$
\begin{aligned}
& u_0 = f(x), v_0 = g(x), \\
& u_1 = J(A_0) + J(B_0) + b_1 J(C_0) + c_1 J(v_0) + a_1 J(u_0) + J(h_1) \\
& = J\left(\left(D_x^\beta u_0\right)^2\right) + J(u_0 u_{0xx}) + b_1 J\left((u_0)^2\right) + c_1 J(v_0) + a_1 J(u_0) + J(h_1) \\
& = \left(f_1^2 x^{4-2\beta} + b_1 x^4 + c_1 x^3 + (2 + a_1) x^2 + h_1\right) t, \\
& v_1 = J(D_0) + J(E_0) + b_2 J(F_0) + c_2 J(u_0) + a_2 J(v_0) + J(h_2) \\
& = J\left(\left(D_x^\beta v_0\right)^2\right) + J(v_0 v_{0xx}) + b_2 J\left((v_0)^2\right) + c_2 J(u_0) + a_2 J(v_0) + J(h_2) \\
& = \left(g_1^2 x^{6-2\beta} + b_2 x^6 + 6x^4 + a_2 x^3 + c_2 x^2 + h_2\right) t, \\
& u_2 = J(A_1) + J(B_1) + b_1 J(C_1) + c_1 J(v_1) + a_1 J(u_1) + J(h_1) \\
& = J\left(2\left(D_x^\beta u_0\right)\left(D_x^\beta u_1\right)\right) + J(u_0 u_{1xx} + u_1 u_{0xx}) + b_1 J(2u_0 u_1) + c_1 J(v_1) + a_1 J(u_1) + J(h_1) \\
& = (f_2 x^{8-5\beta} + f_3 x^{8-3\beta} + f_4 x^{7-3\beta} + f_5 x^{6-3\beta} + f_6 x^{4-3\beta} + f_7 x^{6-2\beta} + f_8 x^{4-2\beta} + f_9 x^{2-2\beta} \\
& + f_{10} x^6 + f_{11} x^5 + f_{12} x^4 + f_{13} x^3 + f_{14} x^2 + f_{15} x + f_{16}) \tfrac{t^2}{2}, \\
& v_2 = J(D_1) + J(E_1) + b_2 J(F_1) + c_2 J(u_1) + a_2 J(v_1) + J(h_2) \\
& = J\left(2\left(D_x^\beta v_0\right)\left(D_x^\beta v_1\right)\right) + J(v_0 v_{1xx} + v_1 v_{0xx}) + b_2 J(2v_0 v_1) + c_2 J(u_1) + a_2 J(v_1) + J(h_2). \\
& = (g_2 x^{9-4\beta} + g_3 x^{9-2\beta} + g_4 x^{8-2\beta} + g_5 x^{7-2\beta} + g_6 x^{6-2\beta} + g_7 x^{3-2\beta} + g_8 x^{4-2\beta} + g_9 x^7 \\
& + g_{10} x^9 + g_{11} x^5 + g_{12} x^6 + g_{13} x^4 + g_{14} x^3 + g_{15} x^2 + g_{16} x + g_{17}) \tfrac{t^2}{2}.
\end{aligned} \tag{3.21}
$$

where

$$\begin{aligned}
&f(x)=x^2, f_1=\left(\tfrac{\Gamma(3)}{\Gamma(3-\beta)}\right), f_2=2f_1^2\tfrac{\Gamma(5-2\beta)}{\Gamma(5-3\beta)}, f_3=2b_1f_1^2\tfrac{\Gamma(5)}{\Gamma(5-\beta)}, f_4=2c_1f_1^2\tfrac{\Gamma(4)}{\Gamma(4-\beta)},\\
&f_5=2(2+a_1)f_1^3, f_6=\left(2b_1f_1^2+c_1g_1^2\right), f_7=\left((4-2\beta)(3-2\beta)f_1+a_1f_1^2\right),\\
&f_8=(2(4-2\beta)(3-2\beta)f_1), f_9=\left(2b_1^2+b_2c_1\right), f_{10}=(2b_1c_1),\\
&f_{11}=(12b_1+4b_1+2b_1a_1+6c_1+2b_1), f_{12}=(6c_1+a_2c_1+a_1c_1),\\
&f_{13}=\left(4a_1+24b_1+2b_1h_1+c_1^2+a_1^2+4\right), f_{14}=12c_1, f_{15}=4a_1+c_1h_2+a_1h_1+8,\\
&g(x)=x^3, g_1=\left(\tfrac{\Gamma(4)}{\Gamma(4-\beta)}\right), g_2=2g_1\tfrac{\Gamma(7-2\beta)}{\Gamma(7-3\beta)}, g_3=\left(2b_2g_1\tfrac{\Gamma(6)}{\Gamma(6-\beta)}+2b_2g_1^2\right), g_4=\left(12g_1\tfrac{\Gamma(5)}{\Gamma(5-\beta)}\right),\\
&g_5=\left(2a_2g_1^2+6g_1^2+(6-2\beta)(5-2\beta)g_1^2\right), g_6=\left(2g_2c_2f_1+a_2g_2^2\right),\\
&g_7=c_2f_1^2, g_8=48b_2, g_9=2b_2^2, g_{10}=(108+2c_2b_2), g_{11}=3a_2b_2, g_{12}=(18a_2+c_2b_2),\\
&g_{13}=\left(8c_2+2h_2b_2+c_1c_2+a_2^2\right), g_{14}=(2c_2+2c_2a_1), g_{15}=6h_2, g_{16}=h_1c_2+a_2h_2,
\end{aligned} \tag{3.22}$$

Then we obtain a numerical solution of space-fractional equations (3.16) in series form

$$\left\{\begin{aligned}
&u(x,t)=f(x)+\left(f_1^2x^{4-2\beta}+b_1x^4+c_1x^3+(2+a_1)x^2+h_1\right)t+(f_2x^{8-5\beta}\\
&+f_3x^{8-3\beta}+f_4x^{7-3\beta}+f_5x^{6-3\beta}+f_6x^{6-2\beta}+f_7x^{4-2\beta}+f_8x^{2-2\beta}\\
&+f_9x^6+f_{10}x^5+f_{11}x^4+f_{12}x^3+f_{13}x^2+f_{14}x+f_{15})\tfrac{t^2}{2}+...\\
&v(x,t)=g(x)+\left(g_1^2x^{6-2\beta}+b_2x^6+6x^4+a_2x^3+c_2x^2+h_2\right)t+(g_2x^{9-4\beta}\\
&+g_3x^{9-2\beta}+g_4x^{8-2\beta}+g_5x^{7-2\beta}+g_6x^{6-2\beta}+g_7x^{4-2\beta}+g_8x^7+g_9x^9\\
&+g_{10}x^5+g_{11}x^6+g_{12}x^4+g_{13}x^3+g_{14}x^2+g_{15}x+g_{16})\tfrac{t^2}{2}+...
\end{aligned}\right. \tag{3.23}$$

The figures below show the numerical solution (3.23) for the space-fractional Lotka-Voltera Eq.(3.19) with $\alpha=\beta=1/2$ as well as the numerical solution of Eq.(3.19) with $\alpha=\beta=1$.

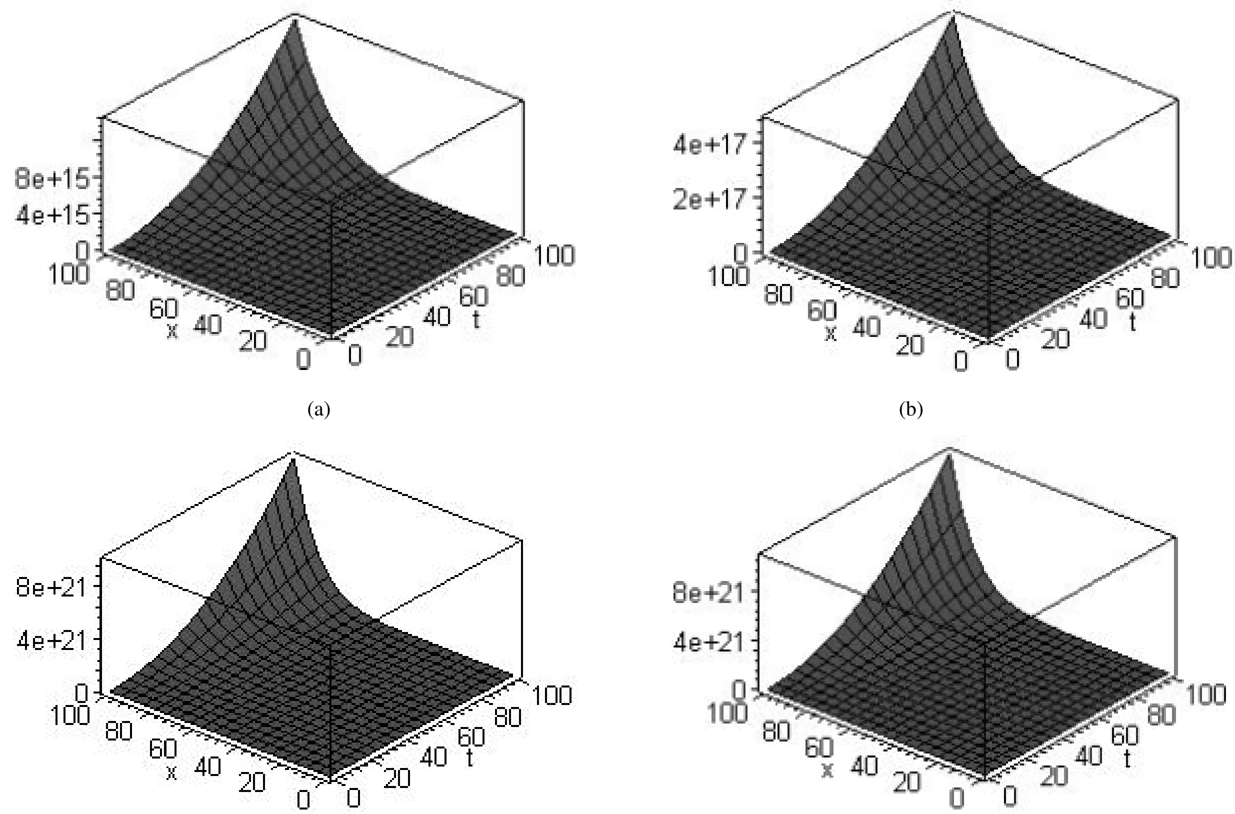


Figure 2: Representing space-fractional solutions of Eq.(3.19). In(a) and (c) solution obtained by the Adomian method of u(x,t) and v(x,t) for β= *1. In(b) and (d), solution obtained by the Adomian method of u(x,t) and v(x,t) for* β= *1/2.*

4 Conclusion

In this paper, the ADM has been successfully applied to derive explicit numerical solutions for the coupled Lotka-Volterra equations with time-and-space fractional derivatives. The above procedure shows that the ADM method is efficient and powerful in solving wide classes of equations in particular evolution fractional order equations.

References

[1] K. Abbaoui and Y. Cherruault. Convergence of Adomian's method applied to differnetial equations. *Comput. Math. Appl.,* 28(5):(1994),103-109.

[2] K. Abbaoui and Y. Cherruault. New ideas for proving convergence of decomposition method. *Comput. Math. Appl.,* 29:(1995),103-108.

[3] M. J. Ablowitz and P.A. Clarkson. Solitons, nonlinear evolution equations and inverse scatting. *Cambridge University Press, New York* (1991).

[4] G. Adomian. Nonlinear stochastic Systems theory and applications to Physics. *Kluwer Academic Publishers, Dordrecht,*(1989).

[5] G. Adomian. Solving frontier problems of Physics: The decomposition method. *Kluwer Academic Publishers, Boston*,(1994).

[6] E. BABOBLIAN, A. DAVARI, Numerical implementation of Adomian decomposition method for linear Volterra integral equation of the second kind, *Appl. Math. Comput.,* 165 :(2005), 223-227.

[7] Y. Chen and Y. Z. Yan. Weierstrass semi rational expansion method and new doubly periodic solutions of the generalized Hirota-Satsuma coupled KdV system. *Appl. Math. Comput.,* 177:(2006),85-91.

[8] Y. Cherruault.Convergence of Adomian's method.*Kybernetes,* 18:(1989),31-38.

[9] V. Daftardar-Gejji, H. Jafari. Adomian decomposition: a tool for solving a system of fractional differential equations. *J. Math. Anal. Appl.,* 301:(2005), 508-518.

[10] Z. Dahmani, M. M. Mesmoudi and R. Bebbouchi. The foam-drainage equation with time and space fractional derivative solved by the ADM method. *E. J. Qualitative Theory of Diff. Equ.,* 30:(2008),1-10.

[11] E. Fan. Extended tanh function method and its applications to nonlinear equations. *Phys. Lett. A.,*277:(2000), 212-218.

[12] J. H. He. Some applications of nonlinear fractional differential equations and their approximations. *Bull. Sci. Techno.,* 15(2):(1999),86-90.

[13] N. Himoun, K. Abbaoui and Y. Cherruault. New results of convergence of Adomian's method. *Kybernetes,* 28(4-5):(1999),423-429.

[14] H. Jafari and V. Daftardar-Gejji. Solving a system of nonlinear fractional differential equations using Adomian decomposition. *J. Comput. Appl. Math.,* 196:(2006),644-651.

[15] D. Kaya and S. M. El Sayad. A numerical implementation of the decomplosition method for the Lienard equation. *Appl. Math. Comput.,* 171:(2005),1095-1103.

[16] S. Kemple and H. Beyer. Global and causal solutions of fractional differential equations in : Transform Method and Special Functions. *Varna96, Proceding of the 2nd International Workshop (SCTP), Singapore,*(1997).

[17] B. A. Kupershmidt. Mathematics of dispersive water waves. *Communications in Mathematical Physics,* 99:(1995),51-73.

[18] Y. Luchko and R. Gorenflo. The initial value problem for some fractional differential equations with the Caputo derivaitve. *Preprint Series A 08-98, Fachbereich Mathematik und Informatik, Freie Universitat Berlin,*(1998).

[19] Y. Luchko and H. M. Srivastava. The exact solutions of certain differential equations of fractional order by using operational calculus. *Comput. Math. Appl.,* 29:(1995),73-85 .

[20] I. Podlubny. Fractional Differential Equations. *Academic Press, San Diego.,*(1999).

[21] M. Rafei and H. Daniali. Application of the variational iteration method to the Whitham–Broer–Kaup equations. *Appl. Math. Comput.,* (2006).

[22] G. Samko, A. A. Kilbas and O. I. Marichev. Fractional integral and derivative: Theory and Applications. *Gordon and Breach, Yverdon*,(1993).

[23] S. Momani and Z. Odibat. Numerical approach to differential equations of fractional order. *J. Comput. Appl. Math.,* 207:(2007),96-110.

[24] Q. Wang. Numerical solutions for fractional KdV-Burgers equation by Adomian decomposition method. *Appl. Math. Comput.,* 182:(2006),1049-1055.

[25] A. M. Wazwaz. A new approch to the nonlinear advection problem. *Appl. Math. Comput.,* 72:(1995),175-181.

[26] A. M. Wazwaz. A reliable modification of Adomian decomposition method. *Appl. Math. Comput.,* 102:(1999),77-86.

[27] A. M. Wazwaz. A new algorithm for calculating Adomian polynomials for nonlinear operators. *Appl. Math. Comput.,* 111:(2000), 53-69.

[28] A. M. Wazwaz. A computational approach to solitons of the Kadomtsev-Petviashvili equation. *Appl. Math. Comput.,* 123:(2001),205-217.

[29] A. M. Wazwaz. Partial differential equations: Methods and Applications. *Balkema Publishers. The Netherlands,*(2002).

[30] A. M. Wazwaz, A. Georguis. An analytic study of Fisher's equations by using Adomian decomposition method. *Appl. Math. Comput.,* 154:(2004),609-620.

[31] B. J. West, M. Bologna and P. Grigolini. Physics of Fractal Operators. *Springer, New York*(2003).