NEW APPLICATIONS OF FRACTIONAL CALCULUS ON PROBABILISTIC RANDOM VARIABLES

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ABSTRACT. New results and new applications of fractional calculus for continuous random variables are presented. Some classical integral results are also generalized. On the other hand, some results (corollaries) on the paper [*Fractional integral inequalities for continuous random variables*, Malaya J. Mat. **2**(2014), 172–179] are corrected.

1. INTRODUCTION

The inequality theory plays an important role in differential equations, probability theory and applied sciences. For more details, we refer the reader to [**3, 14, 15, 16, 17**] and the references therein. Moreover, the integral inequalities using fractional integration are also of great importance. For some applications, one can see [**4, 5, 6, 7, 10, 12**].

The idea to develop the present paper is motivated by the following published results: The first one is the paper [**11**], where P. Kumar proposed some results related to random variables with probability density functions (in short: p.d.f.) defined on some finite real lines. The second work is [**8**], where for the first time the author introduced new fractional random variable concepts with some results that generalize some theorems of [**2**]. Other research papers deal with random inequalities motivating this work can be found in [**1, 13**].

The aim of this paper is to present new results and new applications of fractional calculus for continuous random variables. Some classical integral results can be deduced as some special cases. On the other hand, some corollaries on the paper [**8**] are corrected.

2. Preliminaries

Definition 2.1 ([**9**])**.** The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ for a continuous function h on $[a, b]$, is defined as

$$(1) \qquad J^{\alpha}[h(t)] = \frac{1}{\Gamma(\alpha)} \int_a^t (t - \tau)^{\alpha-1} h(\tau) \mathrm{d}\tau; \qquad \alpha > 0, \quad a < t \leq b,$$

$$J^0[h(t)] = h(t),$$

where $\Gamma(\alpha) := \int_0^{\infty} \mathrm{e}^{-u} u^{\alpha-1} \mathrm{d}u$.

For $t = b$, we put

$$J^{\alpha}[h(b)] = \frac{1}{\Gamma(\alpha)} \int_a^b (b - \tau)^{\alpha-1} h(\tau) \mathrm{d}\tau.$$

We give the following property

$$(2) \qquad J^{\alpha} J^{\beta}[h(t)] = J^{\alpha+\beta}[h(t)], \qquad \alpha \geq 0, \quad \beta \geq 0,$$

and

$$(3) \qquad J^{\alpha} J^{\beta}[h(t)] = J^{\beta} J^{\alpha}[h(t)].$$

In the particular case where $h(t) = t$ on $[a, b]$, we have

$$(4) \qquad J^{\alpha}[b] = \frac{(b - a)^{\alpha+1}}{\Gamma(\alpha + 2)} + \frac{a(b - a)^{\alpha}}{\Gamma(\alpha + 1)}$$

and for $h(t) = t^2$, we have

$$(5) \qquad J^{\alpha}[b^2] = \frac{2(b - a)^{\alpha+2}}{\Gamma(\alpha + 3)} + 2a J^{\alpha}[b] - \frac{a^2(b - a)^{\alpha}}{\Gamma(\alpha + 1)}.$$

We recall also the following concepts and definitions [**8**],

Definition 2.2. The fractional expectation, of order $\alpha > 0$, for a random variable X with a positive p.d.f. f defined on $[a, b]$, is defined as

$$(6) \qquad E_{X,\alpha} = \frac{1}{\Gamma(\alpha)} \int_a^b (b - \tau)^{\alpha-1} \tau f(\tau) \mathrm{d}\tau.$$

Definition 2.3. The fractional moment of order (r, α); $r > 0$, $\alpha > 0$, for a random variable X with a positive p.d.f. f defined on $[a, b]$, is defined as

$$(7) \qquad E_{X^r,\alpha} = \frac{1}{\Gamma(\alpha)} \int_a^b (b - \tau)^{\alpha-1} \tau^r f(\tau) \mathrm{d}\tau.$$

For the fractional variance of X, we recall next definition

Definition 2.4. The fractional variance of order $\alpha > 0$ for a random variable X with a p.d.f. $f \colon [a, b] \to \mathbb{R}^+$, is defined as

$$(8) \qquad \sigma_{X,\alpha}^2 = \frac{1}{\Gamma(\alpha)} \int_a^b (b - \tau)^{\alpha-1} (\tau - E(X))^2 f(\tau) \mathrm{d}\tau.$$

Remark 2.5.

($r1*$) If we take $\alpha = 1$ in Definition 2.2, we obtain the classical expectation $E_{X,1} = E(X)$.

($r2*$) If we take $\alpha = 1$ in Definition 2.4, we obtain the classical variance $\sigma^2_{X,1} = \sigma^2(X) := \int_a^b (\tau - E(X))^2 f(\tau) \mathrm{d}\tau$.

3. Main Results

We begin by proving the following generalized property of the p.d.f. of X.

Theorem 3.1. *Let X be a continuous random variable having a p.d.f. $f: [a,b] \to \mathbb{R}^+$. Then we have*

$$(9) \qquad J^{\alpha+1}[f(b)] = \frac{\Gamma(\alpha - n + 1)}{\Gamma(\alpha + 1)} \sum_{i=0}^{n} \left[(-1)^i C_n^i b^{n-i} E_{X^i, \alpha - n + 1} \right],$$

where $\alpha \geq 0$, $n = [\alpha]$.

Proof. For any $\alpha \geq 0$, we can write $\alpha = n + s$; $n = [\alpha]$, $s \in (0,1)$. And then,

$$(10) \quad J^{\alpha+1}[f(b)] = \frac{1}{\Gamma(\alpha + 1)} \int_a^b (b - \tau)^\alpha f(\tau) \mathrm{d}\tau = \frac{1}{\Gamma(\alpha + 1)} \int_a^b (b - \tau)^{n+s} f(\tau) \mathrm{d}\tau.$$

Hence,

$$(11) \qquad J^{\alpha+1}[f(b)] = \frac{1}{\Gamma(\alpha + 1)} \sum_{i=0}^{n} \left[(-1)^i C_n^i b^{n-i} \int_a^b (b - \tau)^s \tau^i f(\tau) \mathrm{d}\tau \right].$$

Thanks to Definition 2.3, we obtain

$$(12) \qquad J^{\alpha+1}[f(b)] = \frac{\Gamma(\alpha - n + 1)}{\Gamma(\alpha + 1)} \sum_{i=0}^{n} \left[(-1)^i C_n^i b^{n-i} E_{X^i, s+1} \right],$$

Theorem 3.1 is thus proved. $\qquad \square$

Remark 3.2.

1. In the above theorem, if we take $\alpha = 0$, we obtain the well known property of a p.d.f. of X, that is $\int_a^b f(u) \mathrm{d}u = 1$.

2. The above theorem implies, in particular, that the property (P3*) in the paper [**8**] is not correct.

3. Thanks to the above theorem, we confirm that of [**8**, Corollary 3.1] is not correct.

In what follows, we will generalise a well know classical variance property. We have

Theorem 3.3. *Let X be a continuous random variable having a p.d.f. $f: [a,b] \to \mathbb{R}^+$. Then, for any $\delta \geq 1$, we have*

$$(13) \quad \sigma^2_{X,\delta} = E_{X^2,\delta} - 2E(X)E_{X,\delta} + E^2(X) \frac{\Gamma(\delta - n)}{\Gamma(\delta)} \sum_{i=0}^{n} \left[(-1)^i C_n^i b^{n-i} E_{X^i, \delta - n} \right]$$

where $n = [\delta - 1]$.

Proof. By Definition 2.4, we can write

$$(14) \qquad \sigma_{X,\delta}^2 = \frac{1}{\Gamma(\delta)} \int_a^b (b-\tau)^{\delta-1} (\tau - E(X))^2 f(\tau) \mathrm{d}\tau.$$

Hence,

$$(15) \qquad \sigma_{X,\delta}^2 = E_{X^2,\delta} - 2E(X)E_{X,\delta} + E^2(X) J^\delta f(b),$$

Using Theorem 3.1 with $\alpha = \delta - 1$, we obtain the desired formula in Theorem 3.3.
$\square$

Remark 3.4. In Theorem 3.3, if we take $\delta = 1$, we obtain the well know property $\sigma^2(X) = E(X^2) - E^2(X)$.

In what follows, we propose a result for some estimations of the fractional moments. At the same time, we impose this result as a correction of [**8**, Corollary 3.1]. So, we prove next statement

Theorem 3.5. *Let X be a continuous random variable with a p.d.f. $f \colon [a,b] \to \mathbb{R}^+$ and $\alpha \geq 1$; $n = [\alpha - 1]$.*
(i*) If $f \in L_\infty[a,b]$, then
$$(16)$$
$$\left(\frac{\Gamma(\alpha-n)}{\Gamma(\alpha)} \sum_{i=0}^n \left[(-1)^i C_n^i b^{n-i} E_{X^i,\alpha-n} \right] \right)$$

$$\times \left(E_{X^2,\alpha} - 2E(X)E_{X,\alpha} + \frac{\Gamma(\alpha-n)E^2(X)}{\Gamma(\alpha)} \sum_{i=0}^n \left[(-1)^i C_n^i b^{n-i} E_{X^i,\alpha-n} \right] \right)$$

$$\leq (E_{X-E(X),\alpha})^2 + \|f\|_\infty^2 \left[\frac{(b-a)^\alpha}{\Gamma(\alpha+1)} \left(\frac{2(b-a)^{\alpha+2}}{\Gamma(\alpha+3)} + 2aJ^\alpha[b] - \frac{a^2(b-a)^\alpha}{\Gamma(\alpha+1)} \right) \right.$$

$$\left. - \left(\frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{a(b-a)^\alpha}{\Gamma(\alpha+1)} \right)^2 \right].$$

(ii*) We have also

$$(17) \qquad \begin{aligned} &\frac{\Gamma(\alpha-n)}{\Gamma(\alpha)} \sum_{i=0}^n \left[(-1)^i C_n^i b^{n-i} E_{X^i,\alpha-n} \right] \sigma_{X,\alpha}^2 - (E_{X-E(X),\alpha})^2 \\ &\leq \left(\frac{\Gamma(\alpha-n)(b-a)}{\sqrt{2}\Gamma(\alpha)} \sum_{i=0}^n \left[(-1)^i C_n^i b^{n-i} E_{X^i,\alpha-n} \right] \right)^2. \end{aligned}$$

Proof. In [**8**], it was proved that

$$(18) \qquad \begin{aligned} &\frac{1}{\Gamma^2(\alpha)} \int_a^b \int_a^b (b-\tau)^{\alpha-1} (b-\rho)^{\alpha-1} f(\tau) f(\rho) (\tau-\rho)^2 d\tau d\rho \\ &= 2J^\alpha[f(b)] J^\alpha[f(b)(b-E(X))^2] - 2\left(J^\alpha[f(b)(b-E(X))] \right)^2. \end{aligned}$$

On the other hand, we can observe that

$$\frac{1}{\Gamma^2(\alpha)} \int_a^b \int_a^b (b-\tau)^{\alpha-1}(b-\rho)^{\alpha-1}f(\tau)f(\rho)(\tau-\rho)^2 \mathrm{d}\tau\mathrm{d}\rho$$

(19)
$$\leq \|f\|_\infty^2 \frac{1}{\Gamma^2(\alpha)} \int_a^b \int_a^b (b-\tau)^{\alpha-1}(b-\rho)^{\alpha-1}(\tau-\rho)^2 \mathrm{d}\tau\mathrm{d}\rho$$

$$\leq 2\|f\|_\infty^2 \Big[\frac{(b-a)^\alpha}{\Gamma(\alpha+1)} J^\alpha[b^2] - (J^\alpha[b])^2\Big].$$

Therefore, using (4), (5), (18), and (19), we obtain (16).

For the second part of Theorem 3.5, we remark that

(20)
$$\frac{1}{\Gamma^2(\alpha)} \int_a^b \int_a^b (b-\tau)^{\alpha-1}(b-\rho)^{\alpha-1}f(\tau)f(\rho)(\tau-\rho)^2 \mathrm{d}\tau\mathrm{d}\rho$$
$$\leq (b-a)^2(J^\alpha[f(b)])^2.$$

Then, taking into account the expression in Theorem 3.1 and thanks to (18), we get (17). $\qquad\square$

Remark 3.6. Taking $\alpha = 1$ in $(i*)$ of the above theorem, we obtain the first inequality of Theorem 1 in [3], and taking $\alpha = 1$ in $(ii*)$ of the same theorem, we obtain the last part of Theorem 1 in [3].

We shall generalize Theorem 3.5 by proving the following result. We have

Theorem 3.7. *Let X be a continuous random variable having a p.d.f. f: $[a,b] \to \mathbb{R}^+$. Then we have*
(1^*) *For all $\alpha \geq 1$, $\beta \geq 1$*
(21)
$$\Big(\frac{\Gamma(\beta-m)}{\Gamma(\beta)} \sum_{i=0}^m \Big[(-1)^i C_m^i b^{m-i} E_{X^i,\beta-m}\Big]\Big)$$

$$\times \Big(E_{X^2,\alpha} - 2E(X)E_{X,\alpha} + \frac{\Gamma(\alpha-n)E^2(X)}{\Gamma(\alpha)} \sum_{i=0}^n \Big[(-1)^i C_n^i b^{n-i} E_{X^i,\alpha-n}\Big]\Big)$$

$$+ \Big(\frac{\Gamma(\alpha-n)}{\Gamma(\alpha)} \sum_{i=0}^n \Big[(-1)^i C_n^i b^{n-i} E_{X^i,\alpha-n}\Big]\Big)$$

$$\times \Big(E_{X^2,\beta} - 2E(X)E_{X,\beta} + \frac{\Gamma(\beta-m)E^2(X)}{\Gamma(\beta)} \sum_{i=0}^m \Big[(-1)^i C_m^i b^{m-i} E_{X^i,\beta-m}\Big]\Big)$$

$$\leq 2(E_{X-E(X),\alpha})(E_{X-E(X),\beta}) + \|f\|_\infty^2 \Big[\frac{(b-a)^\alpha}{\Gamma(\alpha+1)}\Big(\frac{2(b-a)^{\beta+2}}{\Gamma(\beta+3)} + 2aJ^\beta[b] - \frac{a^2(b-a)^\beta}{\Gamma(\beta+1)}\Big)$$

$$+ \frac{(b-a)^\beta}{\Gamma(\beta+1)}\Big(\frac{2(b-a)^{\alpha+2}}{\Gamma(\alpha+3)} + 2aJ^\alpha[b] - \frac{a^2(b-a)^\alpha}{\Gamma(\alpha+1)}\Big)$$

$$- 2\Big(\frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{a(b-a)^\alpha}{\Gamma(\alpha+1)}\Big)\Big(\frac{(b-a)^{\beta+1}}{\Gamma(\beta+2)} + \frac{a(b-a)^\beta}{\Gamma(\beta+1)}\Big)\Big],$$

where $f \in L_\infty[a,b], m = [\beta-1], n = [\alpha-1]$.

(2*) *For any $\alpha \geq 1, \beta \geq 1$, the inequality*

$$\Big(\frac{\Gamma(\beta - m)}{\Gamma(\beta)} \sum_{i=0}^{m} \Big[(-1)^i C_m^i b^{m-i} E_{X^i, \beta-m}\Big]\Big)$$

$$\times \Big(E_{X^2, \alpha} - 2E(X)E_{X, \alpha} + \frac{\Gamma(\alpha - n)E^2(X)}{\Gamma(\alpha)} \sum_{i=0}^{n} \Big[(-1)^i C_n^i b^{n-i} E_{X^i, \alpha-n}\Big]\Big)$$

(22)
$$+ \Big(\frac{\Gamma(\alpha - n)}{\Gamma(\alpha)} \sum_{i=0}^{n} \Big[(-1)^i C_n^i b^{n-i} E_{X^i, \alpha-n}\Big]\Big)$$

$$\times \Big(E_{X^2, \beta} - 2E(X)E_{X, \beta} + \frac{\Gamma(\beta - m)E^2(X)}{\Gamma(\beta)} \sum_{i=0}^{m} \Big[(-1)^i C_m^i b^{m-i} E_{X^i, \beta-m}\Big]\Big)$$

$$\leq 2(E_{X-E(X), \alpha})(E_{X-E(X), \beta}) + (b-a)^2 \Big(\frac{\Gamma(\beta - m)}{\Gamma(\beta)}$$

$$\times \sum_{i=0}^{m} \Big[(-1)^i C_m^i b^{m-i} E_{X^i, \beta-m}\Big]\Big)\Big(\frac{\Gamma(\alpha - n)}{\Gamma(\alpha)} \sum_{i=0}^{n} \Big[(-1)^i C_n^i b^{n-i} E_{X^i, \alpha-n}\Big]\Big)$$

hold with $m = [\beta - 1]$, $n = [\alpha - 1]$.

Proof. We have

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^b \int_0^b (b-\tau)^{\alpha-1}(b-\rho)^{\beta-1} f(\tau)f(\rho)(\tau-\rho)^2 d\tau d\rho$$

(23)
$$= J^\alpha[f(b)]J^\beta[f(b)(b-E(X))^2] + J^\beta[f(b)]J^\alpha[f(b)(b-E(X))^2]$$

$$- 2J^\alpha[f(b)(b-E(X))]J^\beta[f(b)(b-E(X))].$$

We also have

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^b \int_0^b (b-\tau)^{\alpha-1}(b-\rho)^{\beta-1} f(\tau)f(\rho)(\tau-\rho)^2 d\tau d\rho$$

(24)
$$\leq ||f||_\infty^2 \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^b \int_a^b (b-\tau)^{\alpha-1}(b-\rho)^{\beta-1}(\tau-\rho)^2 d\tau d\rho$$

$$\leq ||f||_\infty^2 \Big[\frac{(b-a)^\alpha}{\Gamma(\alpha+1)} J^\beta[b^2] + \frac{(b-a)^\beta}{\Gamma(\beta+1)} J^\alpha[b^2] - 2J^\alpha[b]J^\beta[b]\Big].$$

Thanks to (23) and (24), we obtain (21).

To obtain (22), we remark that

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^b \int_0^b (b-\tau)^{\alpha-1}(b-\rho)^{\beta-1} f(\tau)f(\rho)(\tau-\rho)^2 d\tau d\rho$$

(25)
$$\leq (b-a)^2 J^\alpha[f(b)]J^\beta[f(b)].$$

Then, thanks to Theorem 3.1 and using (23), we end the proof of this theorem. □

We also prove the following estimation for the fractional variance. This result implies, in particular, that [**8**, Corollary 3.2] is not correct.

Theorem 3.8. *Let f be the p.d.f. of X on $[a,b]$. Then for any $\alpha \geq 1$, we have*

(26)
$$\left(\frac{\Gamma(\alpha-n)}{\Gamma(\alpha)} \sum_{i=0}^{n} \left[(-1)^i C_n^i b^{n-i} E_{X^i,\alpha-n}\right]\right)$$
$$\times \left(E_{X^2,\alpha} - 2E(X)E_{X,\alpha} + \frac{\Gamma(\alpha-n)E^2(X)}{\Gamma(\alpha)} \sum_{i=0}^{n} \left[(-1)^i C_n^i b^{n-i} E_{X^i,\alpha-n}\right]\right)$$
$$\leq (E_{X-E(X),\alpha})^2 + \frac{(b-a)^2}{4}\left(\frac{\Gamma(\alpha-n)}{\Gamma(\alpha)} \sum_{i=0}^{n} \left[(-1)^i C_n^i b^{n-i} E_{X^i,\alpha-n}\right]\right)^2,$$

where $n = [\alpha-1]$.

Proof. Using a fractional Gruss result [**6**] yields the following inequality

(27)
$$J^\alpha[p(b)]J^\alpha[pg^2(b)] - (J^\alpha[pg(b)])^2 \leq \frac{1}{4}\left(J^\alpha[p(b)]\right)^2(M-m)^2.$$

The particular case $p(t) = f(t)$, $g(t) = t - E(X)$, $t \in [a,b]$, $M = b - E(X)$, $m = a - E(X)$ allows us to obtain

(28)
$$J^\alpha[f(b)]\sigma_{X,\alpha}^2 - (E_{X-E(X),\alpha})^2 \leq \frac{1}{4}(J^\alpha[f(b)])^2(b-a)^2.$$

Finally, by Theorem 3.1, we obtain (26). □

Remark 3.9. Taking $\alpha = 1$, we obtain [**3**, Theorem 2].

At the end of this section, we present the reader another estimation for the fractional variance in which we use two parameters $\alpha \geq 1$ and $\beta \geq 1$.

Theorem 3.10. *Let f be the p.d.f. of the random variable X on $[a,b]$. Then for all $\alpha \geq 1$, $\beta \geq 1$, we have*

(29)
$$\left(\frac{\Gamma(\alpha-n)}{\Gamma(\alpha)} \sum_{i=0}^{n} \left[(-1)^i C_n^i b^{n-i} E_{X^i,\alpha-n}\right]\right)$$
$$\times \left(E_{X^2,\beta} - 2E(X)E_{X,\beta} + \frac{\Gamma(\beta-m)E^2(X)}{\Gamma(\beta)} \sum_{i=0}^{m} \left[(-1)^i C_m^i b^{m-i} E_{X^i,\beta-m}\right]\right)$$
$$+ \left(\frac{\Gamma(\beta-m)}{\Gamma(\beta)} \sum_{i=0}^{m} \left[(-1)^i C_m^i b^{m-i} E_{X^i,\beta-m}\right]\right)$$
$$\times \left(E_{X^2,\alpha} - 2E(X)E_{X,\alpha} + \frac{\Gamma(\alpha-n)E^2(X)}{\Gamma(\alpha)} \sum_{i=0}^{n} \left[(-1)^i C_n^i b^{n-i} E_{X^i,\alpha-n}\right]\right)$$
$$+ 2(a-E(X))(b-E(X))\left(\frac{\Gamma(\alpha-n)}{\Gamma(\alpha)} \sum_{i=0}^{n} \left[(-1)^i C_n^i b^{n-i} E_{X^i,\alpha-n}\right]\right)$$
$$\times \left(\frac{\Gamma(\beta-m)}{\Gamma(\beta)} \sum_{i=0}^{m} \left[(-1)^i C_m^i b^{m-i} E_{X^i,\beta-m}\right]\right)$$
$$\leq (a+b-2E(X))\left[\left(\frac{\Gamma(\alpha-n)}{\Gamma(\alpha)} \sum_{i=0}^{n} \left[(-1)^i C_n^i b^{n-i} E_{X^i,\alpha-n}\right]\right)E_{X-E(X),\beta}\right.$$
$$\left. + \left(\frac{\Gamma(\beta-m)}{\Gamma(\beta)} \sum_{i=0}^{m} \left[(-1)^i C_m^i b^{m-i} E_{X^i,\beta-m}\right]\right)E_{X-E(X),\alpha}\right],$$

where $m = [\beta-1]$, $n = [\alpha-1]$.

Proof. We use [**6**, Theorem 3.4] and Theorem 3.1.   □

4. Applications

In this section, we present some fractional applications for the uniform random variable X whose $p.d.f.$ is defined for any $x \in [a, b]$ by $f(x) = (b - a)^{-1}$.

a. Fractional Expectation of Order α

We have

$$(30) \qquad E_{X,\alpha} = (b-a)^{-1}\Big[\frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{a(b-a)^{\alpha}}{\Gamma(\alpha+1)}\Big], \qquad \alpha \geq 1,$$

Remark that if we take $\alpha = 1$ in the above formula, then we obtain the well known expectation of X

$$E_{X,1} = \frac{b+a}{2} = E(X).$$

b. Fractional Moment of Orders $(2, \alpha)$

We have

(31)
$$E_{X^2,\alpha} = \frac{2(b-a)^{\alpha+1}}{\Gamma(\alpha+3)} + 2a\Big(\frac{(b-a)^{\alpha}}{\Gamma(\alpha+2)} + \frac{a(b-a)^{\alpha-1}}{\Gamma(\alpha+1)}\Big) - \frac{a^2(b-a)^{\alpha-1}}{\Gamma(\alpha+1)}, \qquad \alpha \geq 1.$$

Taking $\alpha = 1$ in the above formula, we obtain the classical moment of order 2

$$(32) \qquad E_{X^2,1} = \frac{a^2 + b^2 + ab}{3} = E(X^2).$$

c. Fractional Variance of Order α

In this case, the quantity $J^\alpha f(b)$ of Theorem 3.1 is given by

$$(33) \qquad J^\alpha f(b) = \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha+1)}, \alpha \geq 1.$$

Then, thanks to (15) in the proof of Theorem 3.3, we get
(34)
$$\sigma^2_{X,\alpha} = \frac{2(b-a)^{\alpha+1}}{\Gamma(\alpha+3)} + 2a\Big(\frac{(b-a)^{\alpha}}{\Gamma(\alpha+2)} + \frac{a(b-a)^{\alpha-1}}{\Gamma(\alpha+1)}\Big) - \frac{a^2(b-a)^{\alpha-1}}{\Gamma(\alpha+1)}, \qquad \alpha \geq 1.$$

Taking $\alpha = 1$, we obtain $\sigma^2_{X,1} = \sigma^2(X)$, which corresponds to the classical variance of X.

d. Fractional Moment of Orders (r, α)

In the particular case where the p.d.f. of the uniform random X is defined on some positive real interval of type $[0, b]$, the fractional moment of X is given by

$$(35) \qquad E_{X^r,\alpha} = \frac{\Gamma(r+1)}{\Gamma(\alpha+r+1)}b^{r+\alpha-1}.$$

Note that if $\alpha = 1$, we obtain the classical moment of order r for the uniform distribution of X,

$$E_{X^r,1} = \frac{\Gamma(r+1)}{\Gamma(r+2)}b^r = E(X^r).$$

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