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A COUPLED SYSTEM OF FRACTIONAL DIFFERENTIAL EQUATIONS INVOLVING TWO FRACTIONAL ORDERS

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Abstract In this paper, we study a coupled system of nonlinear Caputo fractional differential equations involving two fractional orders. We establish sufficient conditions to prove new existence and uniqueness results. Finally, we present two examples illustrating the main results.

Keywords: Caputo derivative, fixed point, differential equation, existence, uniqueness.
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1. INTRODUCTION

In the last few decades, there has been an explosion of research activities on the application of fractional calculus to very diverse scientific fields ranging from the physics of diffusion and advection phenomena, to control systems, finance and economics. For more details, we refer the reader to [5, 6, 8, 13, 14, 15, 17, 21, 22] and the reference therein. Moreover, the study of fractional order differential equations is also important in various problems of applied sciences, and has attracted the attention of many authors. Considerable work has been done in this field of research, for instance, see [1, 2, 3, 4, 7, 9, 10, 11, 12, 16, 18, 19, 20, 23].
In this paper, we discuss the existence and uniqueness of solution for the following fractional coupled system of nonlinear differential equations:

$$\begin{cases} D^{\alpha}x(t)=\sum\limits_{i=1}^{m}f_i\begin{pmatrix} t,x(t),y(t),D^{\alpha-1}x(t),D^{\alpha-2}x(t),...,D^{\alpha-(n-1)}x(t),\\ D^{\beta-1}y(t),D^{\beta-2}y(t),...,D^{\beta-(n-1)}y(t)\end{pmatrix}, \\ \qquad t\in J, \\ D^{\beta}y(t)=\sum\limits_{i=1}^{m}g_i\begin{pmatrix} t,x(t),y(t),D^{\alpha-1}x(t),D^{\alpha-2}x(t),...,D^{\alpha-(n-1)}x(t),\\ D^{\beta-1}y(t),D^{\beta-2}y(t),...,D^{\beta-(n-1)}y(t)\end{pmatrix}, \\ \qquad t\in J, \\ \qquad \alpha,\beta\in\,]n-1,n[\,, \\ D^{\gamma}x(0)+D^{\gamma}x(1)=D^{\rho}y(0)+D^{\rho}y(1)=0;\ \gamma,\ \rho\in\,]0,1[\,, \\ \sum\limits_{i=1}^{n-2}\left|x^{(i)}(0)\right|=\sum\limits_{i=1}^{n-2}\left|y^{(i)}(0)\right|=0, \\ x(1)+D^{\delta}x(1)=y(1)+D^{\sigma}y(1)=0,\ \delta,\sigma\in\,]n-2,n-1[\,,\ n\in\mathbb{N}^*-\{1\}, \end{cases} \tag{1}$$

where $n \in \mathbb{N}^* - \{1\}$. The derivatives D^{α}, D^{β}, $D^{\alpha-k}$, $D^{\beta-k}$, $k = 1, 2, ..., n-1$, are in the sense of Caputo and $J := [0, 1]$. For each $i = 1, ..., m$, $m \in \mathbb{N}^*$ the functions f_i and g_i : $J \times \mathbb{R}^{2n} \to \mathbb{R}$ will be specified later.

2. BASIC DEFINITIONS AND LEMMAS

In this section, we recall some basic definitions and lemmas which are used throughout this paper [14]:

Definition 2.1. *The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, for a continuous function f on $[0, \infty[$ is defined as:*

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds, \ \alpha > 0, \ t \geq 0,$$

$$J^0 f(t) = f(t), \ t \geq 0, \tag{2}$$

where $\Gamma(\alpha) := \int_0^{\infty} e^{-x} x^{\alpha-1} dx$.

Definition 2.2. *The Caputo derivative of order α for a function $x : [0, \infty) \to \mathbb{R}$, which is at least n-times differentiable can be defined as:*

$$D^{\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} x^{(n)}(s)\, ds = J^{n-\alpha} x^{(n)}(t), \ \alpha > 0, \tag{3}$$

for $n - 1 < \alpha < n, n \in \mathbb{N}^*$.

Lemma 2.1. *For $\alpha > 0$, the general solution of the fractional differential equation $D^{\alpha} x(t) = 0$, is given by*

$$x(t) = c_0 + c_1 t + c_2 t^2 + ... + c_{n-1} t^{n-1}, \tag{4}$$

where $c_i \in \mathbb{R}$, $i = 0, ..., n-1$, $n = [\alpha] + 1$.

Lemma 2.2. *Let $\alpha > 0$. Then*

$$J^{\alpha} D^{\alpha} x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + ... + c_{n-1} t^{n-1}, \tag{5}$$

where $c_i \in \mathbb{R}$, $i = 0, 1, ..., n-1$, $n = [\alpha] + 1$.

Lemma 2.3. *Let $p, q > 0$, $f \in L^1([a, b])$. Then $J^p J^q f(t) = J^{p+q} f(t)$, $D^p J^p f(t) = f(t)$, $t \in [a, b]$*

Lemma 2.4. *Let $q > p > 0$, $f \in L^1([a, b])$. Then $D^p J^q f(t) = J^{q-p} f(t), t \in [a, b]$.*

Lemma 2.5. *Let E be Banach space. Assume that $T : E \to E$ is completely continuous operator. If the set $V = \{x \in E : x = \mu T x, \ 0 < \mu < 1\}$ is bounded, then T has a fixed point in E.*

We prove the following Lemma:

Lemma 2.6. *Suppose that* $(G_i)_{i=1,...,m} \in C([0,1], \mathbb{R})$, *and consider the problem*

$$\begin{cases} D^{\alpha} x(t) = \sum_{i=1}^{m} G_i(t),\ t \in [0,1],\ m \in \mathbb{N}^*, \\ n-1 < \alpha < n,\ n \in \mathbb{N}^* - \{1\}, \end{cases} \tag{6}$$

with the conditions:

$$\begin{aligned} D^{\gamma} x(0) + D^{\gamma} x(1) &= 0,\ \gamma \in]0,1[, \\ \sum_{i=1}^{n-2} \left| x^{(i)}(0) \right| &= 0,\ n \in \mathbb{N}^* - \{1\}, \\ x(1) + D^{\delta} x(1) &= 0,\ \delta \in]n-2, n-1[,\ n \in \mathbb{N}^* - \{1\}. \end{aligned} \tag{6}$$

Then, the integral solution of (6) *is given by:*

$$\begin{aligned} x(t) = &\sum_{i=1}^{m} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} G_i(s)\,ds - \sum_{i=1}^{m} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} G_i(s)\,ds \\ &- \sum_{i=1}^{m} \int_0^1 \frac{(1-s)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)} G_i(s)\,ds \\ &+ \left(\frac{\Gamma(n-\gamma)(\Gamma(n-\delta)+\Gamma(n))}{\Gamma(n)\Gamma(n-\delta)} - \frac{\Gamma(n-\gamma)t^{n-1}}{\Gamma(n)} \right) \sum_{i=1}^{m} \int_0^1 \frac{(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} G_i(s)\,ds. \end{aligned} \tag{8}$$

Proof. We have

$$x(t) = \sum_{i=1}^{m} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} G_i(s)\,ds - c_0 - c_1 t - c_2 t^2 - ... - c_{n-1} t^{n-1}, \tag{9}$$

where $c_i \in \mathbb{R}$, $i = 0, 1, 2, ..., n-1$.

For $k = 1, 2, ..., n-1$, we obtain

$$x^{(k)}(0) = -k! c_k, \tag{10}$$

and thanks to (7), we have

$$c_{n-1} = \frac{\Gamma(n-\gamma)}{\Gamma(n)} \sum_{i=1}^{m} \int_0^1 \frac{(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} G_i(s)\,ds.$$

Using (7) and (10) , we obtain

$$c_1 = c_2 = ... = c_{n-2} = 0.$$

Applying 2.4 and using the last condition of (7) , we get

$$\begin{aligned} c_0 &= \sum_{i=1}^{m}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}G_i(s)\,ds + \sum_{i=1}^{m}\int_0^1 \frac{(1-s)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)}G_i(s)\,ds \\ &\quad -\frac{\Gamma(n-\gamma)(\Gamma(n-\delta)+\Gamma(n))}{\Gamma(n)\Gamma(n-\delta)}\sum_{i=1}^{m}\int_0^1 \frac{(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}G_i(s)\,ds. \end{aligned}$$

Therefore,

$$\begin{aligned} x(t) &= \sum_{i=1}^{m}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}G_i(s)\,ds - \sum_{i=1}^{m}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}G_i(s)\,ds \\ &\quad - \sum_{i=1}^{m}\int_0^1 \frac{(1-s)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)}G_i(s)\,ds \\ &\quad + \left(\frac{\Gamma(n-\gamma)(\Gamma(n-\delta)+\Gamma(n))}{\Gamma(n)\Gamma(n-\delta)} - \frac{\Gamma(n-\gamma)t^{n-1}}{\Gamma(n)}\right)\sum_{i=1}^{m}\int_0^1 \frac{(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}G_i(s)\,ds. \end{aligned} \tag{11}$$

Lemma 2.6 is thus proved. ∎

To study (1) , we need to introduce the Banach space:

$$S := \left\{(x,y) : x, y \in C([0,1],\mathbb{R}),\ D^{\alpha-k}x,\ D^{\beta-k}y \in C([0,1],\mathbb{R}), k = 1,2,...,n-1\right\}$$

endowed with the norm

$$\|(x,y)\|_S = \max\begin{pmatrix} \|x\|, \left\|D^{\alpha-1}x\right\|, \left\|D^{\alpha-2}x\right\|, ..., \left\|D^{\alpha-(n-1)}x\right\|, \\ \|y\|, \left\|D^{\beta-1}y\right\|, \left\|D^{\beta-2}y\right\|, ..., \left\|D^{\beta-(n-1)}y\right\| \end{pmatrix}, \tag{12}$$

where,

$$\|x\| = \sup_{t\in J}|x(t)|, \|y\| = \sup_{t\in J}|y(t)|, \left\|D^{\alpha-k}x\right\| = \sup_{t\in J}\left|D^{\alpha-k}x(t)\right|,$$
$$\left\|D^{\beta-k}y\right\| = \sup_{t\in J}\left|D^{\beta-k}y(t)\right|, k = 1,2,...,n-1.$$

3. MAIN RESULTS

In this section, we prove some existence and uniqueness results to the nonlinear fractional coupled system (1) .

For the sake of convenience, we impose the following hypotheses:
(H_1) : For each $i = 1, 2, ..., m$, the functions f_i and $g_i : [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}$ are continuous.

(H_2) : There exist nonnegative real numbers η^i_j, υ^i_j, $j = 1, 2, ..., 2n$, $i = 1, ..., m$, such that for all $t \in [0,1]$ and all $(x_1, x_2, ..., x_{2n}), (y_1, y_2, ..., y_{2n}) \in \mathbb{R}^{2n}$, we have

$$|f_i(t, x_1, x_2, ..., x_{2n}) - f_i(t, y_1, y_2, ..., y_{2n})| \leq \sum_{j=1}^{2n} \eta^i_j |x_j - y_j|$$

and

$$|g_i(t, x_1, x_2, ..., x_{2n}) - g_i(t, y_1, y_2, ..., y_{2n})| \leq \sum_{j=1}^{2n} \upsilon^i_j |x_j - y_j|.$$

(H_3) : There exist nonnegative constants $(L_i)_{i=1,...,m}$ and $(K_i)_{i=1,...,m}$, such that: for each $t \in J$ and all $(x_1, x_2, ..., x_{2n}) \in \mathbb{R}^{2n}$,

$$|f_i(t, x_1, x_2, ..., x_{2n})| \leq L_i, \ |g_i(t, x_1, x_2, ..., x_{2n})| \leq K_i, \ i = 1, ..., m.$$

We also consider the following quantities:

$$\Sigma_1 = \sum_{i=1}^{m} \left(\eta^i_1 + \eta^i_2 + ... + \eta^i_{2n}\right), \ \Sigma_2 = \sum_{i=1}^{m} \left(\upsilon^i_1 + \upsilon^i_2 + ... + \upsilon^i_{2n}\right),$$

$$A_1 = \frac{\Sigma_1}{\Gamma(\alpha+1)}, \ A_2 = \frac{\Sigma_2}{\Gamma(\beta+1)},$$

$$\begin{aligned} A^k_3 &= \Sigma_1 \left(\frac{1}{\Gamma(k+1)} + \frac{\Gamma(n-\gamma)}{\Gamma(n+k-\alpha)\Gamma(\alpha-\gamma+1)}\right), \ k = 1, 2, ..., n-1, \\ A^k_4 &= \Sigma_2 \left(\frac{1}{\Gamma(k+1)} + \frac{\Gamma(n-\rho)}{\Gamma(n+k-\beta)\Gamma(\beta-\rho+1)}\right), \ k = 1, 2, ..., n-1, \end{aligned}$$

$$\begin{aligned} M_k &= \frac{1}{\Gamma(k+1)} + \frac{\Gamma(n-\gamma)}{\Gamma(n+k-\alpha)\Gamma(\alpha+1-\gamma)}, \ k = 1, 2, ..., n-1, \\ N_k &= \frac{1}{\Gamma(k+1)} + \frac{\Gamma(n-\rho)}{\Gamma(n+k-\beta)\Gamma(\beta+1-\rho)}, \ k = 1, 2, ..., n-1, \end{aligned}$$

$$\Upsilon_1 = \frac{2}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha+1-\delta)} + \frac{1}{\Gamma(\alpha+1-\gamma)} \left| \frac{\Gamma(n-\gamma)(\Gamma(n-\delta)+\Gamma(n))}{\Gamma(n)\Gamma(n-\delta)} - \frac{\Gamma(n-\gamma)}{\Gamma(n)} \right|$$

and

$$\Upsilon_2 = \frac{2}{\Gamma(\beta+1)} + \frac{1}{\Gamma(\beta+1-\sigma)} + \frac{1}{\Gamma(\beta+1-\rho)} \left| \frac{\Gamma(n-\rho)(\Gamma(n-\sigma)+\Gamma(n))}{\Gamma(n)\Gamma(n-\sigma)} - \frac{\Gamma(n-\rho)}{\Gamma(n)} \right|.$$

For the existence and uniqueness of solution for (1), the first main result is the following:

Theorem 3.1. *Assume that* (H_2) *holds. If the inequality*

$$\max\left(A_1, A_2, A_3^1, A_3^2, ..., A_3^{n-1}, A_4^1, A_4^2, ..., A_4^{n-1}\right) < 1 \tag{13}$$

is valid, then system (1) *has a unique solution on* J.

Proof. We define the operator $T : S \to S$ by

$$T(x,y)(t) := (T_1(x,y)(t), T_2(x,y)(t)), \ t \in J,$$

such that,

$$\begin{aligned} T_1(x,y)(t) = &\sum_{i=1}^{m} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \varphi_i(s)\,ds - \sum_{i=1}^{m} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \varphi_i(s)\,ds - \sum_{i=1}^{m} \int_0^1 \frac{(1-s)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)} \varphi_i(s)\,ds \\ &+ \left(\frac{\Gamma(n-\gamma)(\Gamma(n-\delta)+\Gamma(n))}{\Gamma(n)\Gamma(n-\delta)} - \frac{\Gamma(n-\gamma)t^{n-1}}{\Gamma(n)} \right) \sum_{i=1}^{m} \int_0^1 \frac{(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} \varphi_i(s)\,ds \end{aligned} \tag{14}$$

and

$$\begin{aligned} T_2(x,y)(t) = &\sum_{i=1}^{m} \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \psi_i(s)\,ds - \sum_{i=1}^{m} \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \psi_i(s)\,ds - \sum_{i=1}^{m} \int_0^1 \frac{(1-s)^{\beta-\sigma-1}}{\Gamma(\beta-\sigma)} \psi_i(s)\,ds \\ &+ \left(\frac{\Gamma(n-\rho)(\Gamma(n-\sigma)+\Gamma(n))}{\Gamma(n)\Gamma(n-\sigma)} - \frac{\Gamma(n-\rho)t^{n-1}}{\Gamma(n)} \right) \sum_{i=1}^{m} \int_0^1 \frac{(1-s)^{\beta-\rho-1}}{\Gamma(\beta-\rho)} \psi_i(s)\,ds, \end{aligned} \tag{15}$$

where

$$\varphi_i(s) = f_i\left(s, x(s), y(s), D^{\alpha-1}x(s), \ldots, D^{\alpha-(n-1)}x(s), D^{\beta-1}y(s), \ldots, D^{\beta-(n-1)}y(s)\right)$$

and

$$\psi_i(s) = g_i\left(s, x(s), y(s), D^{\alpha-1}x(s), \ldots, D^{\alpha-(n-1)}x(s), D^{\beta-1}y(s), \ldots, D^{\beta-(n-1)}y(s)\right).$$

By lemma 2.4, we obtain

$$\begin{aligned} D^{\alpha-k}T_1(x,y)(t) = &\sum_{i=1}^{m} \int_0^t \frac{(t-s)^{k-1}}{\Gamma(k)} \varphi_i(s)\,ds \\ &- \frac{\Gamma(n-\gamma)t^{n+k-\alpha-1}}{\Gamma(n+k-\alpha)} \sum_{i=1}^{m} \int_0^1 \frac{(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} \varphi_i(s)\,ds \end{aligned} \tag{16}$$

and

$$D^{\beta-k}T_2(x,y)(t) = \sum_{i=1}^{m}\int_0^t \frac{(t-s)^{k-1}}{\Gamma(k)}\Psi_i(s)\,ds$$
$$-\frac{\Gamma(n-\rho)t^{n+k-\beta-1}}{\Gamma(n+k-\beta)}\sum_{i=1}^{m}\int_0^1 \frac{(1-s)^{\beta-\rho-1}}{\Gamma(\beta-\rho)}\Psi_i(s)\,ds, \tag{17}$$

where $k = 1, 2, ..., n-1$.

We shall prove that T is contractive:
Let $(x_1, y_1), (x_2, y_2) \in S$. Then, for each $t \in J$, we have:

$$|T_1(x_1,y_1)(t) - T_1(x_2,y_2)(t)| \leq$$

$$\frac{t^{\alpha}}{\Gamma(\alpha+1)}$$

$$\times \sup_{s\in J}\sum_{i=1}^{m}\left|\begin{array}{c} f_i\left(\begin{array}{c} s, x_1(s), y_1(s), D^{\alpha-1}x_1(s), D^{\alpha-2}x_1(s), ..., D^{\alpha-(n-1)}x_1(s), \\ D^{\beta-1}y_1(s), D^{\beta-2}y_1(s), ..., D^{\beta-(n-1)}y_1(s) \end{array}\right) \\ -f_i\left(\begin{array}{c} s, x_2(s), y_2(s), D^{\alpha-1}x_2(s), D^{\alpha-2}x_2(s), ..., D^{\alpha-(n-1)}x_2(s), \\ D^{\beta-1}y_2(s), D^{\beta-2}y_2(s), ..., D^{\beta-(n-1)}y_2(s) \end{array}\right) \end{array}\right|.$$

By (H_2), it follows that

$$\|T_1(x_1,y_1) - T_1(x_2,y_2)\| \leq$$

$$\frac{1}{\Gamma(\alpha+1)}\sum_{i=1}^{m}\left(\eta_1^i + \eta_2^i + ... + \eta_{2n}^i\right)$$

$$\times \max\left(\begin{array}{c} \|x_1 - x_2\|, \left\|D^{\alpha-1}(x_1-x_2)\right\|, \left\|D^{\alpha-2}(x_1-x_2)\right\|, ..., \left\|D^{\alpha-(n-1)}(x_1-x_2)\right\|, \\ \|y_1 - y_2\|, \left\|D^{\beta-1}(y_1-y_2)\right\|, \left\|D^{\beta-2}(y_1-y_2)\right\|, ..., \left\|D^{\beta-(n-1)}(y_1-y_2)\right\| \end{array}\right).$$

Hence,

$$\|T_1(x_1,y_1) - T_1(x_2,y_2)\| \leq A_1\|(x_1-x_2, y_1-y_2)\|_S. \tag{18}$$

With the same arguments as before, we can show that

$$\|T_2(x_1,y_1) - T_2(x_2,y_2)\| \leq A_2\|(x_1-x_2, y_1-y_2)\|_S. \tag{19}$$

On the other hand, we have

$$\left|D^{\alpha-k}(T_1(x_1,y_1) - T_1(x_2,y_2))(t)\right| \leq$$

$$\left(\frac{t^k}{\Gamma(k+1)}+\frac{\Gamma(n-\gamma)t^{n+k-\alpha-1}}{\Gamma(n+k-\alpha)\Gamma(\alpha-\gamma+1)}\right)$$

$$\times\sup_{s\in J}\sum_{i=1}^{m}\left|\begin{array}{c} f_i\left(\begin{array}{c} s,x_1(s),y_1(s),D^{\alpha-1}x_1(s),D^{\alpha-2}x_1(s),...,D^{\alpha-(n-1)}x_1(s),\\ D^{\beta-1}y_1(s),D^{\beta-2}y_1(s),...,D^{\beta-(n-1)}y_1(s)\end{array}\right)\\ -f_i\left(\begin{array}{c} s,x_2(s),y_2(s),D^{\alpha-1}x_2(s),D^{\alpha-2}x_2(s),...,D^{\alpha-(n-1)}x_2(s),\\ D^{\beta-1}y_2(s),D^{\beta-2}y_2(s),...,D^{\beta-(n-1)}y_2(s)\end{array}\right)\end{array}\right|,$$

Consequently,

$$\left\|D^{\alpha-k}\left(T_1\left(x_1,y_1\right)-T_1\left(x_2,y_2\right)\right)\right\|\leq$$

$$\left(\frac{1}{\Gamma(k+1)}+\frac{\Gamma(n-\gamma)}{\Gamma(n+k-\alpha)\Gamma(\alpha-\gamma+1)}\right)\sum_{i=1}^{m}\left(\eta_1^i+\eta_2^i+...+\eta_{2n}^i\right)$$

$$\times\max\left(\begin{array}{c}\|x_1-x_2\|,\left\|D^{\alpha-1}(x_1-x_2)\right\|,\left\|D^{\alpha-2}(x_1-x_2)\right\|,...,\left\|D^{\alpha-(n-1)}(x_1-x_2)\right\|,\\ \|y_1-y_2\|,\left\|D^{\beta-1}(y_1-y_2)\right\|,\left\|D^{\beta-2}(y_1-y_2)\right\|,...,\left\|D^{\beta-(n-1)}(y_1-y_2)\right\|\end{array}\right).$$

Therefore,

$$\left\|D^{\alpha-k}\left(T_1\left(x_1,y_1\right)-T_1\left(x_2,y_2\right)\right)\right\|\leq A_3^k\left\|(x_1-x_2,y_1-y_2)\right\|_S. \tag{20}$$

Also, we have

$$\left\|D^{\beta-k}\left(T_2\left(x_1,y_1\right)-T_2\left(x_2,y_2\right)\right)\right\|\leq A_4^k\left\|(x_1-x_2,y_1-y_2)\right\|_S. \tag{21}$$

Thanks to (18), (19), (20) and (21), we get

$$\|T(x_1,y_1)-T(x_2,y_2)\|_S\leq$$

$$\max\left(A_1,A_2,A_3^1,A_3^2,...,A_3^{n-1},A_4^1,A_4^2,...,A_4^{n-1}\right)\|(x_1-x_2,y_1-y_2)\|_S. \tag{22}$$

Thanks to (13), we conclude that T is a contractive operator. Therefore, by Banach fixed point theorem, T has a unique fixed point which the solution of the system (1). ∎

For the existence of solution for (1), we prove the following theorem:

Theorem 3.2. *Assume that the hypotheses* (H_1) *and* (H_3) *hold. Then the system* (1) *has at least one solution on J.*

Proof. The operator T is continuous on S in view of the continuity of f_i and g_i (hypothesis (H_1)).

Now, we show that T is completely continuous:

(i) : First, we prove that T maps bounded sets of S into bounded sets of S. Taking $\lambda > 0$, and $(x, y) \in B_\lambda := \{(x, y) \in S;\ \|(x, y)\|_S \leq \lambda\}$, then for each $t \in J$, we have:

$$\begin{aligned} &|T_1(x,y)(t)| \\ &\leq \left(\begin{array}{c} \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha+1-\delta)} \\ + \frac{1}{\Gamma(\alpha+1-\gamma)} \left| \frac{\Gamma(n-\gamma)(\Gamma(n-\delta)+\Gamma(n))}{\Gamma(n)\Gamma(n-\delta)} - \frac{\Gamma(n-\gamma)t^{n-1}}{\Gamma(n)} \right| \end{array} \right) \\ &\times \sup_{s\in J} \sum_{i=1}^m \left| f_i \left(\begin{array}{c} s, x(s), y(s), D^{\alpha-1}x(s), D^{\alpha-2}x(s), ..., D^{\alpha-(n-1)}x(s), \\ D^{\beta-1}y(s), D^{\beta-2}y(s), ..., D^{\beta-(n-1)}y(s) \end{array} \right) \right| \end{aligned} \tag{23}$$

Thanks to (H_3), we can write

$$\|T_1(x,y)\| \leq \sum_{i=1}^m L_i$$

$$\times \left(\frac{2}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha+1-\delta)} + \frac{1}{\Gamma(\alpha+1-\gamma)} \left| \frac{\Gamma(n-\gamma)(\Gamma(n-\delta)+\Gamma(n))}{\Gamma(n)\Gamma(n-\delta)} - \frac{\Gamma(n-\gamma)}{\Gamma(n)} \right| \right).$$

Thus,

$$\|T_1(x,y)\| \leq \Upsilon_1 \sum_{i=1}^m L_i. \tag{24}$$

As before, we have

$$\|T_2(x,y)\| \leq \Upsilon_2 \sum_{i=1}^m K_i. \tag{25}$$

On the other hand, for all $k = 1, 2, ..., n-1$, we get

$$\begin{aligned} &\left| D^{\alpha-k} T_1(x,y)(t) \right| \\ &\leq \left(\frac{t^k}{\Gamma(k+1)} + \frac{\Gamma(n-\gamma)t^{n+k-\alpha-1}}{\Gamma(n+k-\alpha)\Gamma(\alpha+1-\gamma)} \right) \\ &\times \sup_{s\in J} \sum_{i=1}^m \left| f_i \left(\begin{array}{c} s, x(s), y(s), D^{\alpha-1}x(s), D^{\alpha-2}x(s), ..., D^{\alpha-(n-1)}x(s), \\ D^{\beta-1}y(s), D^{\beta-2}y(s), ..., D^{\beta-(n-1)}y(s) \end{array} \right) \right|. \end{aligned}$$

This implies that

$$\begin{aligned} &\left\| D^{\alpha-k} T_1(x,y) \right\| \\ &\leq \left(\frac{1}{\Gamma(k+1)} + \frac{\Gamma(n-\gamma)}{\Gamma(n+k-\alpha)\Gamma(\alpha+1-\gamma)} \right) \sum_{i=1}^m L_i,\ k = 1, 2, ..., n-1. \end{aligned}$$

Hence,

$$\left\|D^{\alpha-k}T_1(x,y)\right\| \le M_k\sum_{i=1}^{m} L_i,\ k=1,2,...,n-1. \tag{26}$$

Similarly, we have

$$\left\|D^{\beta-k}T_2(x,y)\right\| \le N_k\sum_{i=1}^{m} K_i,\ k=1,2,...,n-1. \tag{27}$$

It follows from (24), (25), (26) and (27) that:

$$\|T(x,y)\|_S \le \max\left(\begin{array}{c}\Upsilon_1\sum_{i=1}^{m}L_i, \Upsilon_2\sum_{i=1}^{m}K_i, M_1\sum_{i=1}^{m}L_i, M_2\sum_{i=1}^{m}L_i, ..., M_{n-1}\sum_{i=1}^{m}L_i,\\ N_1\sum_{i=1}^{m}K_i, N_2\sum_{i=1}^{m}K_i, ..., N_{n-1}\sum_{i=1}^{m}K_i\end{array}\right).$$

Thus,

$$\|T(x,y)\|_S < \infty.$$

(ii) : Second, we prove that T is equi-continuous:
For any $0\le t_1<t_2\le 1$ and $(x,y)\in B_\lambda$, we have

$$|T_1(x,y)(t_2)-T_1(x,y)(t_1)| \le$$

$$\left(\tfrac{1}{\Gamma(\alpha+1)}\left((t_2-t_1)^\alpha+\left(t_2^\alpha-t_1^\alpha\right)\right)+\tfrac{1}{\Gamma(\alpha+1)}(t_2-t_1)^\alpha+\tfrac{\Gamma(n-\gamma)}{\Gamma(n)\Gamma(\alpha+1-\gamma)}\left(t_2^{n-1}-t_1^{n-1}\right)\right)$$

$$\times\sup_{s\in J}\sum_{i=1}^{m}\left|f_i\left(\begin{array}{c}s, x(s), y(s), D^{\alpha-1}x(s), D^{\alpha-2}x(s), ..., D^{\alpha-(n-1)}x(s),\\ D^{\beta-1}y(s), D^{\beta-2}y(s), ..., D^{\beta-(n-1)}y(s)\end{array}\right)\right|.$$

Therefore,

$$\|T_1(x,y)(t_2)-T_1(x,y)(t_1)\| \le$$

$$\sum_{i=1}^{m}L_i\left(\frac{t_2^\alpha-t_1^\alpha}{\Gamma(\alpha+1)}+\frac{2(t_2-t_1)^\alpha}{\Gamma(\alpha+1)}+\frac{\Gamma(n-\gamma)}{\Gamma(n)\Gamma(\alpha+1-\gamma)}\left(t_2^{n-1}-t_1^{n-1}\right)\right). \tag{28}$$

We have also

$$\|T_2(x,y)(t_2)-T_2(x,y)(t_1)\| \le$$

$$\sum_{i=1}^{m}K_i\left(\frac{t_2^\beta-t_1^\beta}{\Gamma(\beta+1)}+\frac{2(t_2-t_1)^\beta}{\Gamma(\beta+1)}+\frac{\Gamma(n-\rho)}{\Gamma(n)\Gamma(\beta+1-\rho)}\left(t_2^{n-1}-t_1^{n-1}\right)\right). \tag{29}$$

On the other hand,

$$\left|D^{\alpha-k}T_1(x,y)(t_2)-D^{\alpha-k}T_1(x,y)(t_1)\right| \le$$

$$\left(\frac{\left(t_2^k-t_1^k\right)}{\Gamma(k+1)}+\frac{2(t_2-t_1)^k}{\Gamma(k+1)}+\frac{\Gamma(n-\gamma)}{\Gamma(n+k-\alpha)\Gamma(\alpha+1-\gamma)}\left(t_2^{n+k-\alpha-1}-t_1^{n+k-\alpha-1}\right)\right)$$

$$\times\sup_{s\in J}\sum_{i=1}^{m}\left|f_i\left(\begin{array}{c} s,x(s),y(s),D^{\alpha-1}x(s),D^{\alpha-2}x(s),...,D^{\alpha-(n-1)}x(s),\\ D^{\beta-1}y(s),D^{\beta-2}y(s),...,D^{\beta-(n-1)}y(s)\end{array}\right)\right|.$$

Consequently, for all $k=1,2,...,n-1$, we obtain

$$\left\|D^{\alpha-k}T_1\left(x,y\right)\left(t_2\right)-D^{\alpha-k}T_1\left(x,y\right)\left(t_1\right)\right\|$$

$$\leq\left(\begin{array}{c}\frac{\left(t_2^k-t_1^k\right)}{\Gamma(k+1)}+\frac{2(t_2-t_1)^k}{\Gamma(k+1)}\\ +\frac{\Gamma(n-\gamma)}{\Gamma(n+k-\alpha)\Gamma(\alpha+1-\gamma)}\left(t_2^{n+k-\alpha-1}-t_1^{n+k-\alpha-1}\right)\end{array}\right)\sum_{i=1}^{m}L_i. \tag{30}$$

Similarly,

$$\left\|D^{\beta-k}T_2\left(x,y\right)\left(t_2\right)-D^{\beta-k}T_2\left(x,y\right)\left(t_1\right)\right\|$$

$$\leq\left(\begin{array}{c}\frac{\left(t_2^k-t_1^k\right)}{\Gamma(k+1)}+\frac{2(t_2-t_1)^k}{\Gamma(k+1)}\\ +\frac{\Gamma(n-\rho)}{\Gamma(n+k-\beta)\Gamma(\beta+1-\rho)}\left(t_2^{n+k-\beta-1}-t_1^{n+k-\beta-1}\right)\end{array}\right)\sum_{i=1}^{m}K_i, \tag{31}$$

where $k=1,2,...,n-1$. Using (28), (29), (30) and (31), we deduce that

$$\left\|T\left(x,y\right)\left(t_2\right)-T\left(x,y\right)\left(t_1\right)\right\|_S\to 0$$

as $t_2\to t_1$.
Combining (i) and (ii), we conclude that T is completely continuous.

(iii) : Finally, we shall prove that the set F defined by

$$F:=\left\{(x,y)\in S,\ \ (x,y)=\omega T\left(x,y\right),\ 0<\omega<1\ \right\}$$

is bounded.
Let $(x,y)\in F$, then $(x,y)=\omega T\left(x,y\right)$, for some $0<\omega<1$. Thus, for each $t\in J$, we have:

$$x\left(t\right)=\omega T_1\left(x,y\right)\left(t\right),\ y\left(t\right)=\omega T_2\left(x,y\right)\left(t\right). \tag{32}$$

Thanks to (H_3) and using (24) and (25), we deduce that

$$\|x\|\leq\omega\Upsilon_1\sum_{i=1}^{m}L_i,\ \ \|y\|\leq\omega\Upsilon_2\sum_{i=1}^{m}K_i. \tag{33}$$

Using (26) and (27), it yields that

$$\left\|D^{\alpha-k}x\right\|\leq\omega M_k\sum_{i=1}^{m}L_i,\ \ \left\|D^{\beta-k}y\right\|\leq\omega N_k\sum_{i=1}^{m}K_i, \tag{34}$$

where $k=1,2,...,n-1$.

It follows from (33) and (34) that

$$\|(x,y)\|_S \leq \omega \max\left(\begin{array}{c} \Upsilon_1 \sum_{i=1}^{m} L_i, \Upsilon_2 \sum_{i=1}^{m} K_i, M_1 \sum_{i=1}^{m} L_i, M_2 \sum_{i=1}^{m} L_i, ..., M_{n-1} \sum_{i=1}^{m} L_i, \\ N_1 \sum_{i=1}^{m} K_i, N_2 \sum_{i=1}^{m} K_i, ..., N_{n-1} \sum_{i=1}^{m} K_i \end{array}\right). \quad (35)$$

Consequently,

$$\|(x,y)\|_S < \infty. \quad (36)$$

This shows that F is bounded.

By lemma 2.5, we deduce that T has a fixed point, which is a solution of (1). ∎

The third main result is based on Krasnoselskii theorem [15]. We prove the following theorem:

Theorem 3.3. *Assume that* (H_1), (H_2) *and* (H_3) *hold. If*

$$\max\left(\frac{\Sigma_1}{\Gamma(\alpha+1)}, \frac{\Sigma_2}{\Gamma(\beta+1)}, \Sigma_1, \frac{\Sigma_1}{2!}, \frac{\Sigma_1}{3!}, ..., \frac{\Sigma_1}{(n-1)!}, \Sigma_2, \frac{\Sigma_2}{2!}, \frac{\Sigma_2}{3!}, ..., \frac{\Sigma_2}{(n-1)!}\right) < 1, \quad (37)$$

then the coupled system (1) *has at least one solution on* J.

Proof. Let us fixe:

$$\theta \geq \max\left(\begin{array}{c} \Upsilon_1 \sum_{i=1}^{m} L_i, \Upsilon_2 \sum_{i=1}^{m} K_i, M_1 \sum_{i=1}^{m} L_i, M_2 \sum_{i=1}^{m} L_i, ..., M_{n-1} \sum_{i=1}^{m} L_i, \\ N_1 \sum_{i=1}^{m} K_i, N_2 \sum_{i=1}^{m} K_i, ..., N_{n-1} \sum_{i=1}^{m} K_i \end{array}\right).$$

Then, we consider $B_\theta := \{(x,y) \in S : \|(x,y)\|_S \leq \theta\}$. On B_θ, we define the operators P and Q as follows:

$$P(x,y)(t) := (P_1(x,y)(t), P_2(x,y)(t)),$$

$$Q(x,y)(t) := (Q_1(x,y)(t), Q_2(x,y)(t)),$$

where,

$$\begin{aligned} P_1(x,y)(t) &= \sum_{i=1}^{m} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \varphi_i(s)\,ds - \sum_{i=1}^{m} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \varphi_i(s)\,ds \\ &- \sum_{i=1}^{m} \int_0^1 \frac{(1-s)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)} \varphi_i(s)\,ds + \frac{\Gamma(n-\gamma)(\Gamma(n-\delta)+\Gamma(n))}{\Gamma(n)\Gamma(n-\delta)} \sum_{i=1}^{m} \int_0^1 \frac{(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} \varphi_i(s)\,ds, \end{aligned} \quad (38)$$

$$\begin{aligned} P_2(x,y)(t) &= \sum_{i=1}^{m} \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \psi_i(s)\,ds - \sum_{i=1}^{m} \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \psi_i(s)\,ds \\ &- \sum_{i=1}^{m} \int_0^1 \frac{(1-s)^{\beta-\sigma-1}}{\Gamma(\beta-\sigma)} \psi_i(s)\,ds + \frac{\Gamma(n-\rho)(\Gamma(n-\sigma)+\Gamma(n))}{\Gamma(n)\Gamma(n-\sigma)} \sum_{i=1}^{m} \int_0^1 \frac{(1-s)^{\beta-\rho-1}}{\Gamma(\beta-\rho)} \psi_i(s)\,ds, \end{aligned} \quad (39)$$

$$Q_1(x,y)(t) = -\frac{\Gamma(n-\gamma)\,t^{n-1}}{\Gamma(n)} \sum_{i=1}^{m} \int_0^1 \frac{(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} \varphi_i(s)\,ds \tag{40}$$

and

$$Q_2(x,y)(t) = -\frac{\Gamma(n-\rho)\,t^{n-1}}{\Gamma(n)} \sum_{i=1}^{m} \int_0^1 \frac{(1-s)^{\beta-\rho-1}}{\Gamma(\beta-\rho)} \psi_i(s)\,ds. \tag{41}$$

Let $(x_1,y_1), (x_2,y_2) \in B_\theta$, $t \in [0,1]$. Then, thanks to (H_3), we get

$$\|P_1(x_1,y_1) + Q_1(x_2,y_2)\| \le \Upsilon_1 \sum_{i=1}^{m} L_i, \tag{42}$$

$$\|P_2(x_1,y_1) + Q_2(x_2,y_2)\| \le \Upsilon_2 \sum_{i=1}^{m} K_i. \tag{43}$$

On the other hand for all $k = 1, 2, ..., n-1$, we have

$$\left\|D^{\alpha-k}\left(P_1(x_1,y_1) + Q_1(x_2,y_2)\right)\right\| \le M_k \sum_{i=1}^{m} L_i, \tag{44}$$

$$\left\|D^{\beta-k}\left(P_2(x_1,y_1) + Q_2(x_2,y_2)\right)\right\| \le N_k \sum_{i=1}^{m} K_i. \tag{45}$$

Thanks to (42), (43), (44) and (45), we have

$$\|P(x_1,y_1) + Q(x_2,y_2)\|_S \le$$

$$\max\begin{pmatrix} \Upsilon_1 \sum_{i=1}^{m} L_i, \Upsilon_2 \sum_{i=1}^{m} K_i, M_1 \sum_{i=1}^{m} L_i, M_2 \sum_{i=1}^{m} L_i, ..., M_{n-1} \sum_{i=1}^{m} L_i, \\ N_1 \sum_{i=1}^{m} K_i, N_2 \sum_{i=1}^{m} K_i, ..., N_{n-1} \sum_{i=1}^{m} K_i \end{pmatrix}. \tag{46}$$

Therefore,

$$P(x_1,y_1) + Q(x_2,y_2) \in B_\theta. \tag{47}$$

Now, we shall prove that the operator P is contractive. Using the hypothesis (H_2), we can write

$$\|P_1(x_1,y_1) - P_1(x_2,y_2)\|$$

$$\le \frac{1}{\Gamma(\alpha+1)} \sum_{i=1}^{m} \left(\eta_1^i + \eta_2^i + ... + \eta_{2n}^i\right) \|(x_1 - x_2, y_1 - y_2)\|_S\,, \tag{48}$$

$$\|P_2(x_1,y_1) - P_2(x_2,y_2)\|$$

$$\leq \frac{1}{\Gamma(\beta+1)} \sum_{i=1}^{m} \left(\upsilon_1^i + \upsilon_2^i + ... + \upsilon_{2n}^i\right) \|(x_1 - x_2, y_1 - y_2)\|_S , \tag{49}$$

and for $k = 1, 2, ..., n-1$, we have

$$\left\| D^{\alpha-k} \left(P_1 (x_1, y_1) - P_1 (x_2, y_2)\right) \right\|$$

$$\leq \frac{1}{\Gamma(k+1)} \sum_{i=1}^{m} \left(\eta_1^i + \eta_2^i + ... + \eta_{2n}^i\right) \|(x_1 - x_2, y_1 - y_2)\|_S , \tag{50}$$

$$\left\| D^{\beta-k} \left(P_2 (x_1, y_1) - P_2 (x_2, y_2)\right) \right\|$$

$$\leq \frac{1}{\Gamma(k+1)} \left(\upsilon_1^i + \upsilon_2^i + ... + \upsilon_{2n}^i\right) \|(x_1 - x_2, y_1 - y_2)\|_S . \tag{51}$$

From (48), (49), (50) and (51), we deduce that

$$\|P (x_1, y_1) - P (x_2, y_2)\|_S$$

$$\begin{aligned} \leq \quad & \max\left(\frac{\Sigma_1}{\Gamma(\alpha+1)}, \frac{\Sigma_2}{\Gamma(\beta+1)}, \Sigma_1, \frac{\Sigma_1}{2!}, ..., \frac{\Sigma_1}{(n-1)!}, \Sigma_2, \frac{\Sigma_2}{2!}, ..., \frac{\Sigma_2}{(n-1)!}\right) \\ & \times \|(x_1 - x_2, y_1 - y_2)\|_S . \end{aligned} \tag{51}$$

Using (37), we conclude that P is a contractive operator.
For $i = 1, 2, ..., m$, the operator Q is continuous in view of the continuity of f_i and g_i, (see (H_1)). Now, we prove the compactness of the operator Q :

Let $t_1, t_2 \in [0, 1]$, $t_1 < t_2$ and $(x, y) \in B_\theta$. We have

$$\|Q_1 (x, y) (t_2) - Q_1 (x, y) (t_1)\| \leq \frac{\Gamma(n-\gamma)}{\Gamma(n)\Gamma(\alpha+1-\gamma)} \left(t_2^{n-1} - t_1^{n-1}\right) \tag{53}$$

and

$$\|Q_2 (x, y) (t_2) - Q_2 (x, y) (t_1)\| \leq \frac{\Gamma(n-\rho)}{\Gamma(n)\Gamma(\beta+1-\rho)} \left(t_2^{n-1} - t_1^{n-1}\right). \tag{54}$$

On the other hand for $k = 1, 2, ..., n-1$, we get

$$\left\| D^{\alpha-k} Q_1 (x, y) (t_2) - D^{\alpha-k} Q_1 (x, y) (t_1) \right\|$$

$$\leq \frac{\Gamma(n-\gamma)}{\Gamma(n+k-\alpha)\Gamma(\alpha+1-\gamma)} \left(t_2^{n+k-\alpha-1} - t_1^{n+k-\alpha-1}\right), \tag{55}$$

$$\left\| D^{\beta-k} \left(Q_2 (x, y) (t_2) - D^{\beta-k} Q_2 (x, y) (t_1)\right) \right\|$$

$$\leq \frac{\Gamma(n-\rho)}{\Gamma(n+k-\beta)\Gamma(\beta+1-\rho)} \left(t_2^{n+k-\beta-1} - t_1^{n+k-\beta-1}\right). \tag{56}$$

The right hand sides of (53) , (54) , (55) and (56) are independent of (x, y) and tend to zero as $t_1 \to t_2$, so Q is relatively compact on B_θ. Then by Ascolli-Arzela Theorem, the operator Q is compact. Therefore by Krasnoselskii Theorem [15], we conclude that system (1) has a solution. Theorem 3.3 is thus proved. ∎

Corolar 3.1. *Assume that* (H_1) , (H_2) *and* (H_3) *hold and*

$$\max(\Sigma_1, \Sigma_2) < 1, \tag{57}$$

then the coupled system (1) *has at least one solution on J.*

4. EXAMPLES

We present two examples to illustrate the main results.

Example 4.1. *Consider the following system:*

$$\begin{cases} D^{\frac{13}{4}} x(t) = \\ \frac{|x(t)|+|y(t)|+\left|D^{\frac{9}{4}} x(t)\right|+\left|D^{\frac{5}{4}} x(t)\right|+\left|D^{\frac{1}{4}} x(t)\right|+\left|D^{\frac{8}{3}} y(t)\right|+\left|D^{\frac{5}{3}} y(t)\right|+\left|D^{\frac{2}{3}} y(t)\right|}{16\pi^2\left(t^2+1+|x(t)|+|y(t)|+\left|D^{\frac{9}{4}} x(t)\right|+\left|D^{\frac{5}{4}} x(t)\right|+\left|D^{\frac{1}{4}} x(t)\right|+\left|D^{\frac{8}{3}} y(t)\right|+\left|D^{\frac{5}{3}} y(t)\right|+\left|D^{\frac{2}{3}} y(t)\right|\right)} \\ +\frac{1}{32\pi^2 e}\left(\begin{array}{c} \sin x(t) + \sin y(t) + \frac{\cos D^{\frac{9}{4}} x(t)+\cos D^{\frac{5}{4}} x(t)+\cos D^{\frac{1}{4}} x(t)}{2\pi} \\ +\frac{\cos D^{\frac{8}{3}} y(t)+\cos D^{\frac{5}{3}} y(t)+\cos D^{\frac{2}{3}} y(t)}{e^{2t}} \end{array}\right), \\ t \in [0,1], \\ D^{\frac{11}{3}} y(t) = \\ \frac{1}{8\pi^3 e^{t+2}(t^2+1)}\left(\frac{\left|x(t)+D^{\frac{9}{4}} x(t)+D^{\frac{5}{4}} x(t)+D^{\frac{1}{4}} x(t)\right|}{1+\left|x(t)+D^{\frac{9}{4}} x(t)+D^{\frac{5}{4}} x(t)+D^{\frac{1}{4}} x(t)\right|} + \frac{\left|y(t)+D^{\frac{8}{3}} y(t)+D^{\frac{5}{3}} y(t)+D^{\frac{2}{3}} y(t)\right|}{1+\left|y(t)+D^{\frac{8}{3}} y(t)+D^{\frac{5}{3}} y(t)+D^{\frac{2}{3}} y(t)\right|}\right) \\ +\frac{t^2}{16\pi^2 e^{t+1}}\left(\begin{array}{c} \frac{\sin x(t)+\sin y(t)+\sin\left(D^{\frac{9}{4}} x(t)\right)-\sin\left(D^{\frac{5}{4}} x(t)\right)}{1+\left|\sin x(t)+\sin y(t)+\sin\left(D^{\frac{9}{4}} x(t)\right)-\sin\left(D^{\frac{5}{4}} x(t)\right)\right|} \\ +\frac{\sin\left(D^{\frac{1}{4}} x(t)\right)+\cos\left(D^{\frac{8}{3}} y(t)\right)+\cos\left(D^{\frac{5}{3}} y(t)\right)-\cos D^{\frac{2}{3}} y(t)}{1+\left|\sin\left(D^{\frac{1}{4}} x(t)\right)+\cos\left(D^{\frac{8}{3}} y(t)\right)+\cos\left(D^{\frac{5}{3}} y(t)\right)-\cos D^{\frac{2}{3}} y(t)\right|} \end{array}\right), \\ t \in [0,1], \\ D^{\frac{1}{4}} x(0) + D^{\frac{1}{4}} x(1) = D^{\frac{5}{7}} y(0) + D^{\frac{5}{7}} y(1) = 0, \\ \left|x'(0)\right| + \left|x''(0)\right| = \left|y'(0)\right| + \left|y''(0)\right| = 0, \\ x(1) + D^{\frac{8}{3}} x(1) = y(1) + D^{\frac{10}{4}} y(1) = 0. \end{cases} \tag{58}$$

We have:
$n = 4, m = 2, \alpha = \frac{13}{4}, \beta = \frac{11}{3}, \gamma = \frac{1}{4}, \rho = \frac{5}{7}, \delta = \frac{8}{3}, \sigma = \frac{10}{4}, J = [0,1]$.

Also,

$$f_1(t,x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8) = \frac{|x_1|+|x_2|+|x_3|+|x_4|+|x_5|+|x_6|+|x_7|+|x_8|}{16\pi^2(t^2+1+|x_1|+|x_2|+|x_3|+|x_4|+|x_5|+|x_6|+|x_7|+|x_8|)}, \tag{59}$$

$$f_2(t,x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8) = \frac{1}{32\pi^2 e}\left(\sin x_1+\sin x_2+\frac{\cos x_3+\cos x_4+\cos x_5}{2\pi}+\frac{\cos x_6+\cos x_7+\cos x_8}{e^{2t}}\right). \tag{60}$$

For $t\in[0,1]$ *and* $(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8),(y_1,y_2,y_3,y_4,y_5,y_6,y_7,y_8)\in\mathbb{R}^8$, *we have:*

$$\begin{aligned}|f_1(t,x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8)-f_1(t,y_1,y_2,y_3,y_4,y_5,y_6,y_7,y_8)| \le \\ \tfrac{1}{16\pi^2}|x_1-y_1|+\tfrac{1}{16\pi^2}|x_2-y_2|+\tfrac{1}{16\pi^2}|x_3-y_3|+\tfrac{1}{16\pi^2}|x_4-y_4| \\ +\tfrac{1}{16\pi^2}|x_5-y_5|+\tfrac{1}{16\pi^2}|x_6-y_6|+\tfrac{1}{16\pi^2}|x_7-y_7|+\tfrac{1}{16\pi^2}|x_8-y_8|\end{aligned} \tag{61}$$

and

$$\begin{aligned}|f_2(t,x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8)-f_2(t,y_1,y_2,y_3,y_4,y_5,y_6,y_7,y_8)| \le \\ \tfrac{1}{32\pi^2 e}|x_1-y_1|+\tfrac{1}{32\pi^2 e}|x_2-y_2|+\tfrac{1}{64\pi^3 e}|x_3-y_3|+\tfrac{1}{64\pi^3 e}|x_4-y_4| \\ +\tfrac{1}{64\pi^3 e}|x_5-y_5|+\tfrac{1}{32\pi^2 e}|x_6-y_6|+\tfrac{1}{32\pi^2 e}|x_7-y_7|+\tfrac{1}{32\pi^2 e}|x_8-y_8|.\end{aligned} \tag{62}$$

So, we can take:

$$\eta_1^1=\eta_2^1=\eta_3^1=\eta_4^1=\eta_5^1=\eta_6^1=\eta_7^1=\eta_8^1=\frac{1}{8\pi^2}, \tag{63}$$

$$\eta_1^2=\eta_2^2=\eta_6^2=\eta_7^2=\eta_8^2=\frac{1}{32\pi^2 e},\ \eta_3^2=\eta_4^2=\eta_5^2=\frac{1}{64\pi^3 e}. \tag{64}$$

We have also

$$g_1(t,x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8) = \frac{1}{8\pi^3 e^{t+2}(t^2+1)}\left(\frac{|x_1+x_3+x_4+x_5|}{1+|x_1+x_3+x_4+x_5|}+\frac{|x_2+x_6,x_7,x_8|}{1+|x_2+x_6,x_7,x_8|}\right) \tag{65}$$

and

$$g_2(t,x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8) = \frac{t^2}{16\pi^2 e^{t+1}} \times\left(\frac{\sin x_1+\sin x_2+\sin x_3-\sin x_4}{1+|\sin x_1+\sin x_2+\sin x_3-\sin x_4|}+\frac{\sin x_5+\cos x_6+\cos x_7-\cos x_8}{1+|\sin x_5+\cos x_6+\cos x_7-\cos x_8|}\right). \tag{66}$$

For $t\in[0,1]$ *and* $(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8),(y_1,y_2,y_3,y_4,y_5,y_6,y_7,y_8)\in\mathbb{R}^8$, *we can write*

$$|g_1(t,x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8)-g_1(t,y_1,y_2,y_3,y_4,y_5,y_6,y_7,y_8)|$$

$$\leq \tfrac{1}{8\pi^3e^2}|x_1-y_1| + \tfrac{1}{8\pi^3e^2}|x_2-y_2| + \tfrac{1}{8\pi^3e^2}|x_3-y_3| + \tfrac{1}{8\pi^3e^2}|x_4-y_4| + \tfrac{1}{8\pi^3e^2}|x_5-y_5| + \tfrac{1}{8\pi^3e^2}|x_6-y_6| + \tfrac{1}{8\pi^3e^2}|x_7-y_7| + \tfrac{1}{8\pi^3e^2}|x_8-y_8|, \tag{67}$$

$$|g_2(t,x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8) - g_2(t,y_1,y_2,y_3,y_4,y_5,y_6,y_7,y_8)|$$
$$\leq \tfrac{1}{16\pi^2e}|x_1-y_1| + \tfrac{1}{16\pi^2e}|x_2-y_2| + \tfrac{1}{16\pi^2e}|x_3-y_3| + \tfrac{1}{16\pi^2e}|x_4-y_4| + \tfrac{1}{16\pi^2e}|x_5-y_5| + \tfrac{1}{16\pi^2e}|x_6-y_6| + \tfrac{1}{16\pi^2e}|x_7-y_7| + \tfrac{1}{16\pi^2e}|x_8-y_8|. \tag{68}$$

Hence,

$$\upsilon_1^1 = \upsilon_2^1 = \upsilon_3^1 = \upsilon_4^1 = \upsilon_5^1 = \upsilon_6^1 = \upsilon_7^1 = \upsilon_8^1 = \frac{1}{8\pi^3e^2}, \tag{69}$$

$$\upsilon_1^2 = \upsilon_2^2 = \upsilon_3^2 = \upsilon_4^2 = \upsilon_5^2 = \upsilon_6^2 = \upsilon_7^2 = \upsilon_8^2 = \frac{1}{16\pi^2e}. \tag{70}$$

Therefore,

$$\Sigma_1 = 0.5066,\ \Sigma_2 = 0.00638, \tag{71}$$

$$A_1 = 0.006135,\ A_2 = 0.000432, \tag{72}$$

$$A_3^1 = 0.091296,\ A_3^2 = 0.048554,\ A_3^3 = 0.383993, \tag{73}$$

$$A_4^1 = 0.009723,\ A_4^2 = 0.005699,\ A_4^3 = 0.002138. \tag{74}$$

Thus,

$$\max\left(A_1, A_2, A_3^1, A_3^2, A_3^3, A_4^1, A_4^2, A_4^3\right) < 1. \tag{75}$$

And by Theorem 3.1, we conclude that the system (58) *has a unique solution on* $[0,1]$.

Example 4.2. *We consider the following system:*

$$\begin{cases} D^{\frac{5}{2}}x(t) = \dfrac{e^t}{2\pi+\sin(x(t)+y(t))+\cos\left(D^{\frac{3}{2}}x(t)+D^{\frac{1}{2}}x(t)\right)+\cos\left(D^{\frac{3}{2}}y(t)+D^{\frac{1}{2}}y(t)\right)} \\ \qquad + \dfrac{e^t\sin(x(t)+y(t))}{5+\cos\left(D^{\frac{3}{2}}x(t)+D^{\frac{1}{2}}x(t)\right)+\cos\left(D^{\frac{3}{2}}y(t)+D^{\frac{1}{2}}y(t)\right)},\ t\in[0,1], \\ D^{\frac{5}{2}}y(t) = \dfrac{(t+1)\sin\left(x(t)+D^{\frac{3}{2}}x(t)+D^{\frac{1}{2}}x(t)\right)}{2\pi+\cos\left(y(t)+D^{\frac{3}{2}}y(t)+D^{\frac{1}{2}}y(t)\right)} + \dfrac{3t^2\cos\left(D^{\frac{3}{2}}y(t)+D^{\frac{1}{2}}y(t)\right)\cos(x(t)+y(t))}{e-\sin\left(D^{\frac{3}{2}}x(t)+D^{\frac{1}{2}}x(t)\right)}, \\ \qquad t\in[0,1], \\ D^{\frac{3}{4}}x(0) + D^{\frac{3}{4}}x(1) = 0,\ D^{\frac{5}{6}}y(0) + D^{\frac{5}{6}}y(1) = 0, \\ \left|x'(0)\right| = \left|y'(0)\right| = 0, \\ x(1) + D^{\frac{8}{5}}x(1) = 0,\ y(1) + D^{\frac{3}{2}}y(1) = 0. \end{cases} \tag{76}$$

We have: $n = 3, m = 2, \alpha = \beta = \frac{5}{2}, \gamma = \frac{3}{4}, \rho = \frac{5}{6}, \delta = \frac{8}{5}, \sigma = \frac{3}{2}, J = [0,1]$, *and then*

$$f_1(t, x_1, x_2, x_3, x_4, x_5, x_6) = \frac{e^t}{2\pi + \sin(x_1 + x_2) + \cos(x_3 + x_4) + \cos(x_5 + x_6)}, \quad (77)$$

$$f_2(t, x_1, x_2, x_3, x_4, x_5, x_6) = \frac{e^t \sin(x_1 + x_2)}{5 + \cos(x_3 + x_4) + \cos(x_5 + x_6)}, \quad (78)$$

$$g_1(t, x_1, x_2, x_3, x_4, x_5, x_6) = \frac{(t+1)\sin(x_1 + x_3 + x_4)}{2\pi + \cos(x_2 + x_5 + x_6)}, \quad (79)$$

$$g_2(t, x_1, x_2, x_3, x_4, x_5, x_6) = \frac{3t^2 \cos(x_5 + x_6)\cos(x_1 + x_2)}{e - \sin(x_3 + x_4)}. \quad (80)$$

It is clear that

$$\begin{aligned} |f_1(t, x_1, x_2, x_3, x_4, x_5, x_6)| &\leq \frac{e}{2\pi - 3}, \quad (80)\\ |f_2(t, x_1, x_2, x_3, x_4, x_5, x_6)| &\leq \frac{e}{3}, \\ |g_1(t, x_1, x_2, x_3, x_4, x_5, x_6)| &\leq \frac{2}{2\pi - 1}, \\ |g_2(t, x_1, x_2, x_3, x_4, x_5, x_6)| &\leq \frac{3}{e-1}. \end{aligned}$$

The functions f_1, f_2, g_1 *and* g_2 *are continuous and bounded on* $[0,1] \times \mathbb{R}^6$. *So, by Theorem 3.2, the system* (76) *has at least one solution on* $[0,1]$.

References

[1] A. Anber, S. Belarbi, Z. Dahmani: *New Existence And Uniqueness Results For Fractional Differential Equations*, An. St. Univ. Ovidius Constant., (2013), pp. 33-41.

[2] S. Belarbi, Z. Dahmani: *Existence Of Solutions For Multi-Points Fractional Evolution Equations,* Appl. Math., Vol. 9, (2014), pp. 416 – 427.

[3] M. Benchohra, S. Hamani, S.K. Ntouyas: *Boundary Value Problems For Differential Equations With Fractional Order And Nonlocal Conditions*, Nonlinear Anal., 71, (2009), pp. 2391-2396.

[4] M.E. Bengrine, Z. Dahmani: *Boundary Value Problem For Fractional Differential Equations*, Int. J. Open Problems Compt. Math., 5(4), (2012), pp.7-15.

[5] T.A. Burton: *A Fixed-Point Theorem Of Krasnoselskii,* App. Math. Lett., Vol. 11, No. 1, (1998), pp. 85-88.

[6] Z. Cui, P. Yu, Z. Mao: *Existence Of Solutions For Nonlocal Boundary Value Problems Of Nonlinear Fractional Differential Equations*, Advances In Dynamical Systems And Applications., 7(1), (2012), pp. 31-40.

[7] Z. Dahmani, L. Tabharit: *Fractional Order Differential Equations Involving Caputo Derivative*, Theory And Applications of Mathematics & Computer Science., 4 (1), (2014), pp. 40–55.

[8] R. Gorenflo, F. Mainardi: *Fractional Calculus: Integral And Differential Equations Of Fractional Order*, Fractals And Fractional Calculus In Continuum Mechanics, Springer-Verlag Vienna, (1997), pp. 291-348.

[9] M. Houas, Z. Dahmani: *New Results For A Coupled System Of Fractional Differential Equations,* Ser. Math. Inform., Vol. 28, No. 2, (2013), pp. 133-150.

[10] M. Houas, Z. Dahmani: *New Fractional Results For A Boundary Value Problem With Caputo Derivative*, Int. J. Open Problems Compt. Math., 6(2), (2013), pp. 30-42.

[11] M. Houas, Z. Dahmani, M. Benbachir: *New Results For A Boundary Value Problem For Differential Equations Of Arbitrary Order*, International Journal Of Modern Mathematical Sciences., 7(2), (2013), pp. 195-211.

[12] M. Houas, Z. Dahmani: *New Results For A Caputo Boundary Value Problem*, American Journal Of Computational And Applied Mathematics., 3(3), (2013), pp. 143-161.

[13] A.A. Kilbas, S.A. Marzan: *Nonlinear Differential Equation With The Caputo Fractional Derivative In The Space Of Continuously Differentiable Functions*, Differ. Equ., 41(1), (2005), pp. 84-89.

[14] M.A. Krasnoselskii, Amer. Math. Soc. Transl. 10 (2), (1958), pp. 345-409.

[15] R. Lakshmikantham, A.S. Vatsala: *Basic Theory Of Fractional Defferential Equations*, Nonlinear Anal., (2008), pp. 2677-2682.

[16] M. Li, Y. Liu: *Existence And Uniqueness Of Positive Solutions For A Coupled System Of Nonlinear Fractional Differential Equations*, Open Journal Of Applied Sciences., (3), (2013), pp. 53-61.

[17] F. Mainardi: *Fractional Calculus: Some Basic Problems In Continum And statistical mechanics*, Fractals And Fractional Calculus In Continuum Mechanics., (1999).

[18] J. Nieto, J. Pimente: *Positive Solutions Of A Fractional Thermostat Model*, Boundary Value Problems., 1(5), (2013), pp. 1-11.

[19] S.K. Ntouyas: *Existence Results For First Order Boundary Value Problems For Fractional Differential Equations And Inclusions With Fractional Integral Boundary Conditions*, Journal Of Fractional Calculus And Applications., 3(9), (2012), pp. 1-14.

[20] N. Octavia: *Nonlocal Initial Value Problems For First Order Differential Systems*, Fixed Point Theory, Vol. 22, No. 1, (2009), pp. 64-69.

[21] L. Podlubny: *Fractional Differential Equations*, Academic Press, New York, (1999).

[22] D.R. Smart: *Fixed point Theorems*, Cambridge University Press, (1980).

[23] W. Yang: *Positive Solutions For A Coupled System Of Nonlinear Fractional Differential Equations With Integral Boundary Conditions*, Comput. Math. Appl., Vol. 63, (2012), pp. 288-297.