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Coupled Systems of Fractional Integro-Differential Equations Involving Several Functions

Zoubir Dahmani[a,*], Mohamed Amin Abdellaoui[a], Mohamed Houas[b]

[a]*Laboratory LPAM, Faculty SEI, UMAB University, Algeria*
[b]*Laboratory FIMA, Faculty ST, University of Khemis Miliana, Algeria*

Abstract

This paper studies the existence of solutions for a coupled system of nonlinear fractional integro-differential equations involving Riemann-Liouville integrals with several continuous functions. New existence and uniqueness results are established using Banach fixed point theorem, and other existence results are obtained using Schaefer fixed point theorem. Some illustrative examples are also presented.

Keywords: Caputo derivative, fixed point, integro-differential system, existence, uniqueness, Riemann-Liouville integral.
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1. Introduction

The differential equations of fractional order arise in many scientific disciplines, such as physics, chemistry, control theory, signal processing and biophysics. For more details, we refer the reader to (Kilbas & Marzan, 2005; Lakshmikantham & Vatsala, 2008; Su, 2009) and the references therein. Recently, there has been a significant progress in the investigation of these equations, (see (Anber *et al.*, 2013; Bengrine & Dahmani, 2012; Cui *et al.*, 2012; Wang *et al.*, 2010; Zhang, 2006)). On the other hand, the study of coupled systems of fractional differential equations is also of a great importance. Such systems occur in various problems of applied science. For some recent results on the fractional systems, we refer the reader to (Abdellaoui *et al.*, 2013; Bai & Fang, 2004; Gaber & Brikaa, 2012; Gafiychuk *et al.*, 2008).

*Corresponding author
Email addresses: zzdahmani@yahoo.fr (Zoubir Dahmani), abdellaouiamine13@yahoo.fr (Mohamed Amin Abdellaoui), houasmed@yahoo.fr (Mohamed Houas)

In this paper, we discuss the existence and uniqueness of solutions for the following coupled system of fractional integro-differential equations:

$$\begin{cases} D^{\alpha}u(t) = f_1(t,u(t),v(t)) + \sum_{i=1}^{m} \int_0^t \frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)} \varphi_i(s)\, g_i(s,u(s),v(s))\, ds, \\ D^{\beta}v(t) = f_2(t,u(t),v(t)) + \sum_{i=1}^{m} \int_0^t \frac{(t-s)^{\beta_i}}{\Gamma(\beta_i)} \phi_i(s)\, h_i(s,u(s),v(s))\, ds, \\ u(0) = a > 0, v(0) = b > 0, t \in [0,1] \end{cases} \tag{1.1}$$

where D^{α}, D^{β} denote the Caputo fractional derivatives, $0 < \alpha < 1$, $0 < \beta < 1$, α_i,β_i are non negative real numbers, φ_i and ϕ_i are continuous functions, $m \in \mathbb{N}^*$, f_1, f_2 and g_i and $h_i, i = 1, ..., m$, are functions that will be specified later.
The paper is organized as follows: In section 2, we present some preliminaries and lemmas. Section 3 is devoted to existence of solutions of problem (1.1). In the last section, some examples are presented to illustrate our results.

2. Preliminaries

The following notations, definitions and lemmas will be used throughout this paper.

Definition 2.1. The Riemann-Liouville fractional integral operator of order $\alpha > 0$, for $f \in L^1([a,b],\mathbb{R})$ is defined by:

$$I^{\alpha} f(t) = \int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau)\, d\tau, \qquad a \le t \le b, \tag{2.1}$$

where $\Gamma(\alpha) := \int_0^{\infty} e^{-u} u^{\alpha-1} du.$

Definition 2.2. The fractional derivative of $f \in C^n([a,b],\mathbb{R})$, $n \in N^*$, in the sense of Caputo, of order α, $n-1 < \alpha < n$ is defined by:

$$D^{\alpha} f(t) = \int_a^t \frac{(t-\tau)^{n-\alpha-1}}{\Gamma(n-\alpha)} f^{(n)}(\tau)\, d\tau, \quad t \in [a,b]. \tag{2.2}$$

For more details about fractional calculus, we refer the reader to (Mainardi, 1997). The following lemmas give some properties of Riemann-Liouville integrals and Caputo fractional derivatives (Kilbas & Marzan, 2005; Lakshmikantham & Vatsala, 2008):

Lemma 2.1. *Given $f \in L^1([a,b],\mathbb{R})$, then for all $t \in [a,b]$ we have $I^r I^s f(t) = I^{r+s} f(t)$, for $r,s > 0$. $D^s I^s f(t) = f(t)$, for $s > 0$. $D^r I^s f(t) = I^{s-r} f(t)$, for $s > r > 0$.*

To study the coupled system (1.1), we need the following two lemmas (Kilbas & Marzan, 2005):

Lemma 2.2. *For $n-1<\alpha<n$, where $n\in\mathbb{N}^*$, the general solution of the equation $D^{\alpha}x(t)=0$ is given by*

$$x(t)=c_0+c_1t+c_2t^2+...+c_{n-1}t^{n-1}, \tag{2.3}$$

where $c_i\in\mathbb{R}, i=0,1,2,..,n-1$.

Lemma 2.3. *Let $n-1<\alpha<n$, where $n\in\mathbb{N}^*$. Then, for $x\in C^n([0,1],\mathbb{R})$, we have*

$$I^{\alpha}D^{\alpha}x(t)=x(t)+c_0+c_1t+c_2t^2+...+c_{n-1}t^{n-1}, \tag{2.4}$$

for some $c_i\in\mathbb{R}, i=0,1,2,...,n-1, n=[\alpha]+1$.

We prove the following auxiliary lemma:

Lemma 2.4. *Let $f, R_i, K_i\in C([0,1],\mathbb{R}), i=1,...,m$. The solution of the problem*

$$D^{\alpha}x(t)=f(t)+\sum_{i=1}^{m}\int_0^t\frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)}R_i(s)K_i(s)ds, 0<\alpha<1, \quad \alpha_i>0 \tag{2.5}$$

with the condition,$x(0)=x_0^*\in\mathbb{R}_+^*$, is given by

$$x(t)=\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)ds+\sum_{i=1}^{m}\int_0^t\frac{(t-s)^{\alpha+\alpha_i-1}}{\Gamma(\alpha+\alpha_i)}R_i(s)K_i(s)ds+x_0^*. \tag{2.6}$$

Proof. Setting

$$y(t)=x(t)-I^{\alpha}f(t)-\sum_{i=1}^{m}I^{\alpha+\alpha_i}R_i(t)K_i(t). \tag{2.7}$$

Thanks to the linearity of D^{α}, we get

$$D^{\alpha}y(t)=D^{\alpha}x(t)-D^{\alpha}I^{\alpha}f(t)-\sum_{i=1}^{m}D^{\alpha}I^{\alpha+\alpha_i}R_i(t)K_i(t). \tag{2.8}$$

By lemma 2.2, yields

$$D^{\alpha}y(t)=D^{\alpha}x(t)-f(t)-\sum_{i=1}^{m}I^{\alpha_i}R_i(t)K_i(t). \tag{2.9}$$

Thus, (2.5) is equivalent to $D^{\alpha}y(t)=0$.

Finally, thanks to lemma 2.3, we obtain that $y(t)$ is constant, i.e., $y(t)=y(0)=x(0)=x_0^*$, and the proof of lemma 2.4 is achieved. □

3. Main Results

We introduce in this paragraph the following assumptions:

$(H1)$: There exist non negative real numbers $\mu_j, \nu_j; j = 1,2$ and $, l_i, m_i, n_i, k_i, i = 1, ..., m$, such that for all $t \in [0,1]$ and $(u_1, v_1), (u_2, v_2) \in \mathbb{R}^2$, we have

$$\left| f_j(t, u_2, v_2) - f_j(t, u_1, v_1) \right| \le \mu_j |u_2 - u_1| + \nu_j |v_2 - v_1|, j = 1,2$$

$$|g_i(t, u_2, v_2) - g_i(t, u_1, v_1)| \le l_i |u_2 - u_1| + m_i |v_2 - v_1|, i = 1, ..., m,$$

and,

$$|h_i(t, u_2, v_2) - h_i(t, u_1, v_1)| \le n_i |u_2 - u_1| + k_i |v_2 - v_1|, i = 1, ..., m$$

with

$$\overline{L} = max(\mu_1, \mu_2, \nu_1, \nu_2, l_i, m_i, n_i, k_i, i = 1, ..., m.)$$

$(H2)$: The functions f_1, f_2, g_i and $h_i : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous for each $i = 1, ..., m$.

$(H3)$: There exist positive real numbers $L_1, L_2, L_i', L_i'', i = 1, ..., m$, such that

$$\begin{aligned} |f_1(t,u,v)| &\le L_1, |g_i(t,u,v)| \le L_i', |f_2(t,u,v)| \le L_2, \\ |h_i(t,u,v)| &\le L_i'', t \in [0,1], (u,v) \in \mathbb{R}^2. \end{aligned}$$

Our first result is given by:

Theorem 3.1. *Assume that* $(H1)$ *holds and setting*

$$\begin{aligned} M_1 &:= \frac{1}{\Gamma(\alpha+1)} + \sum_{i=1}^{m} \frac{\|\varphi_i\|_\infty}{\Gamma(\alpha+\alpha_i+1)}, \\ M_2 &:= \frac{1}{\Gamma(\beta+1)} + \sum_{i=1}^{m} \frac{\|\phi_i\|_\infty}{\Gamma(\beta+\beta_i+1)}. \end{aligned}$$

If

$$\overline{L}(M_1 + M_2) < 1, \tag{3.1}$$

then, the system (1.1) *has exactly one solution on* $[0,1]$.

Proof. Setting $X := C([0,1], \mathbb{R})$. This space, equipped with the norm $\|.\|_X = \|.\|_\infty$ defined by $\|f\|_\infty = sup\{|f(x)|, x \in [0,1]\}$, is a Banach space. Also, the product space $(X \times X, \|(u,v)\|_{X\times X})$ is a Banach space, with $\|(u,v)\|_{X\times X} = \|u\|_X + \|v\|_X$.

Consider now the operator $\Psi : X \times X \to X \times X$ defined by

$$\Psi(u,v)(t) = \Big(\Psi_1(u,v)(t), \Psi_2(u,v)(t) \Big), \tag{3.2}$$

where

$$\Psi_1(u,v)(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, u(s), v(s))\,ds + \sum_{i=1}^{m} \int_0^t \frac{(t-s)^{\alpha+\alpha_i-1}}{\Gamma(\alpha+\alpha_i)} \varphi_i(s) g_i(s, u(s), v(s))\,ds + a. \tag{3.3}$$

and

$$\Psi_2(u,v)(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2(s,u(s),v(s))\,ds + \sum_{i=1}^{m} \int_0^t \frac{(t-s)^{\beta+\beta_i-1}}{\Gamma(\beta+\beta_i)} \varphi_i(s)\, g_i(s,u(s),v(s))\,ds + b. \tag{3.4}$$

We shall show that Ψ is contractive: Let $(u_1,v_1),(u_2,v_2) \in X\times X$. Then, for each $t\in[0,1]$, we have

$$|\Psi_1(u_2,v_2)(t) - \Psi_1(u_1,v_1)(t)| \leq$$

$$\begin{aligned}&\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds \times \sup_{0\leq s\leq 1} |f_1(s,u_2(s),v_2(s)) - f_1(s,u_1(s),v_1(s))| \\ &+ \sum_{i=1}^{m} (\sup_{0\leq s\leq 1} |\varphi_i(s)| \int_0^t \frac{(t-s)^{\alpha+\alpha_i-1}}{\Gamma(\alpha+\alpha_i)} ds \\ &\times \sup_{0\leq s\leq 1} |g_i(s,u_2(s),v_2(s)) - g_i(s,u_1(s),v_1(s))|).\end{aligned} \tag{3.5}$$

Therefore,

$$|\Psi_1(u_2,v_2)(t) - \Psi_1(u_1,v_1)(t)| \leq$$

$$\begin{aligned}&\frac{1}{\Gamma(\alpha+1)} \sup_{0\leq s\leq 1} |f_1(s,u_2(s),v_2(s)) - f_1(s,u_1(s),v_1(s))| \\ &+ \sum_{i=1}^{m} \frac{\|\varphi_i\|_\infty}{\Gamma(\alpha+\alpha_i+1)} \sup_{0\leq s\leq 1} |g_i(s,u_2(s),v_2(s)) - g_i(s,u_1(s),v_1(s))|.\end{aligned} \tag{3.6}$$

Using $(H1)$, we can write:

$$|\Psi_1(u_2,v_2)(t) - \Psi_1(u_1,v_1)(t)| \leq$$

$$\begin{aligned}&\frac{\overline{L}}{\Gamma(\alpha+1)} \left(\sup_{0\leq t\leq 1} |u_2(t) - u_1(t)| + \sup_{0\leq t\leq 1} |v_2(t) - v_1(t)| \right) \\ &+ \sum_{i=1}^{m} \frac{\|\varphi_i\|_\infty \overline{L}}{\Gamma(\alpha+\alpha_i+1)} \left(\sup_{0\leq t\leq 1} |u_2(t) - u_1(t)| + \sup_{0\leq t\leq 1} |v_2(t) - v_1(t)| \right).\end{aligned} \tag{3.7}$$

This implies that

$$|\Psi_1(u_2,v_2)(t) - \Psi_1(u_1,v_1)(t)| \leq M_1 \overline{L} (\|u_2-u_1\|_X + \|v_2-v_1\|_X). \tag{3.8}$$

And consequently,

$$\|\Psi_1(u_2,v_2) - \Psi_1(u_1,v_1)\|_X \leq M_1 \overline{L} \|(u_2-u_1, v_2-v_1)\|_{X\times X}. \tag{3.9}$$

With the same arguments as before, we can write

$$\|\Psi_2(u_2,v_2)-\Psi_2(u_1,v_1)\|_X \le M_2\overline{L}\,\|(u_2-u_1,v_2-v_1)\|_{X\times X}\,. \tag{3.10}$$

Finally, using (3.9) and (3.10), we deduce that

$$\|\Psi(u_2,v_2)-\Psi(u_1,v_1)\|_{X\times X} \le \overline{L}\,(M_1+M_2)\,\|(u_2-u_1,v_2-v_1)\|_{X\times X}\,. \tag{3.11}$$

Thanks to (3.1), we conclude that Ψ is a contraction mapping. Hence, by Banach fixed point theorem, there exists a unique fixed point which is a solution of (1.1). □

The second result is the following:

Theorem 3.2. *Assume that* $(H2)$ *and* $(H3)$ *are satisfied and* $L_i' \le L_1, L_i'' \le L_2, i=1,...,m$. *Then problem* (1.1) *has at least one solution on* $[0,1]$.

Proof. First of all, we show that the operator T is completely continuous.

Step 1: Let us take $\gamma>0$ and $B_\gamma := \{(u,v)\in X\times X; \|(u,v)\|_{X\times X}\le\gamma\}$. Now, assume that $(H3)$ holds, and $L_i'\le L_1, L_i''\le L_2$. Then for $(u,v)\in B_\gamma$, we have

$$\begin{aligned}|\Psi_1(u,v)(t)| &\le a+\tfrac{t^\alpha}{\Gamma(\alpha+1)}\sup_{0\le t\le 1}|f_1(t,u(t),v(t))|\\ &+\sum_{i=1}^{m}\tfrac{\|\varphi_i\|_\infty t^{\alpha+\alpha_i}}{\Gamma(\alpha+\alpha_i+1)}\sup_{0\le t\le 1}|g_i(t,u(t),v(t))|\,,\ t\in[0,1].\end{aligned} \tag{3.12}$$

Hence, we obtain

$$\|\Psi_1(u,v)\|_X \le L_1M_1+a<+\infty. \tag{3.13}$$

With the same arguments, we have

$$\|\Psi_2(u,v)\|_X \le L_2M_2+b<+\infty. \tag{3.14}$$

Then, by (23) and (24), we can state that $\|T(u,v)\|_{X\times X}$ is bounded by C, where

$$C := L_1M_1+L_2M_2+a+b. \tag{3.15}$$

Step 2: Let $t_1,t_2\in[0,1], t_1<t_2$ and $(u,v)\in B_\gamma$. We have

$$\begin{aligned}|\Psi_1(u,v)(t_2)-\Psi_1(u,v)(t_1)| \le &\left|\int_0^{t_2}\tfrac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}f_1(s,u(s),v(s))\,ds-\int_0^{t_1}\tfrac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}f_1(s,u(s),v(s))\,ds\right|\\ +&\left|\sum_{i=1}^{m}\int_0^{t_2}\tfrac{(t_2-s)^{\alpha+\alpha_i-1}}{\Gamma(\alpha+\alpha_i)}\varphi_i(s)\,g_i(s,u(s),v(s))\,ds-\sum_{i=1}^{m}\int_0^{t_1}\tfrac{(t_1-s)^{\alpha+\alpha_i-1}}{\Gamma(\alpha+\alpha_i)}\varphi_i(s)\,g_i(s,u(s),v(s))\,ds\right|.\end{aligned} \tag{3.16}$$

Thus, we get

$$|\Psi_1(u,v)(t_2)-\Psi_1(u,v)(t_1)| \le \frac{L_1\left(t_2^\alpha-t_1^\alpha+(t_2-t_1)^\alpha\right)}{\Gamma(\alpha+1)}+\sum_{i=1}^{m}\frac{L_1\|\varphi_i\|_\infty\left(t_2^{\alpha+\alpha_i}-t_1^{\alpha+\alpha_i}+(t_2-t_1)^{\alpha+\alpha_i}\right)}{\Gamma(\alpha+\alpha_i+1)}. \tag{3.17}$$

Analogously, we can obtain

$$|\Psi_2(u,v)(t_2) - \Psi_2(u,v)(t_1)| \le \frac{L_2\left(t_2^{\beta} - t_1^{\beta} + (t_2 - t_1)^{\beta}\right)}{\Gamma(\beta+1)} + \sum_{i=1}^{m} \frac{L_2\,\|\varphi_i\|_\infty\left(t_2^{\beta+\beta_i} - t_1^{\beta+\beta_i} + (t_2 - t_1)^{\beta+\beta_i}\right)}{\Gamma(\beta+\beta_i+1)}. \tag{3.18}$$

Therefore,

$$|\Psi(u,v)(t_2) - \Psi(u,v)(t_1)| \le \frac{L_1\left(t_2^{\alpha} - t_1^{\alpha} + (t_2 - t_1)^{\alpha}\right)}{\Gamma(\alpha+1)} + \sum_{i=1}^{m} \frac{L_1\,\|\varphi_i\|_\infty\left(t_2^{\alpha+\alpha_i} - t_1^{\alpha+\alpha_i} + (t_2 - t_1)^{\alpha+\alpha_i}\right)}{\Gamma(\alpha+\alpha_i+1)}$$
$$+\frac{L_2\left(t_2^{\beta} - t_1^{\beta} + (t_2 - t_1)^{\beta}\right)}{\Gamma(\beta+1)} + \sum_{i=1}^{m} \frac{L_2\,\|\varphi_i\|_\infty\left(t_2^{\beta+\beta_i} - t_1^{\beta+\beta_i} + (t_2 - t_1)^{\beta+\beta_i}\right)}{\Gamma(\beta+\beta_i+1)}. \tag{3.19}$$

As $t_2 \to t_1$, the right-hand side of (3.19) tends to zero. Then, as a consequence of Steps 1, 2, and by Arzela-Ascoli theorem, we conclude that Ψ is completely continuous.
Next, we consider the set:

$$\Omega = \{(u,v) \in X \times X; (u,v) = \lambda T(u,v), 0 < \lambda < 1\}. \tag{3.20}$$

We shall show that Ω is bounded:
Let $(u,v) \in \Omega$, then $(u,v) = \lambda\Psi(u,v)$, for some $0 < \lambda < 1$. Hence, for $t \in [0,1]$, we have:

$$u(t) = \lambda\Psi_1(u,v)(t), v(t) = \lambda\Psi_2(u,v)(t). \tag{3.21}$$

Thus,

$$\|(u,v)\|_{X\times X} = \lambda\,\|\Psi(u,v)\|_{X\times X}. \tag{3.22}$$

Thanks to (H_3),

$$\|(u,v)\|_{X\times X} \le \lambda C, \tag{3.23}$$

where C is defined by (3.15). Therefore, Ω is bounded.

As a conclusion of Schaefer fixed point theorem, we deduce that Ψ has at least one fixed point, which is a solution of (1.1). □

4. Examples

Example 4.1. Consider the following fractional system:

$$\begin{cases} D^{\frac{1}{2}}u(t) = \left(\frac{\sin(u(t)+v(t))}{18(\ln(t+1)+1)} + 6\right) + \int_0^t \frac{(t-s)^{\frac{1}{2}}}{\Gamma\left(\frac{3}{2}\right)}\left(\frac{\exp(-s)}{18(s+1)}\frac{\sin(u(s)+v(s))}{18(s+5)}\right)ds, t \in [0,1], \\ D^{\frac{1}{2}}v(t) = \frac{\sin u(s)+\sin v(s)}{16(t\exp(t^2)+1)} + \int_0^t \frac{(t-s)^{\frac{3}{2}}}{\Gamma\left(\frac{5}{2}\right)}\left(\frac{\exp\left(-s^2\right)}{32\sqrt{1+s^2}}\frac{\sin u(s)+\sin v(s)}{16(s\exp(s^2)+1)}\right)ds, t \in [0,1], \\ u(0) = \sqrt{3}, v(0) = \sqrt{2}, \end{cases} \tag{4.1}$$

We have $\alpha = \beta = \frac{1}{2}, \alpha_1 = \frac{3}{2}, \beta_1 = \frac{5}{2}, a = \sqrt{3}, b = \sqrt{2}, f_1(t,u,v) = \frac{\sin(u+v)}{18(\ln(t+1)+1)} + 6, f_2(t,u,v) = \frac{\sin u + \sin v}{16(t\exp(t^2)+6)}, \varphi_1(t) = \frac{\exp(-t)}{18(t+1)}$ and $\phi_1(t) = \frac{\exp(-t^2)}{32\sqrt{1+t^2}}$. Also, for $(u_1, v_1), (u_2, v_2) \in \mathbb{R}^2, t \in [0,1]$, we have

$$\begin{aligned}
|f_1(t,u_2,v_2) - f_1(t,u_1,v_1)| &\leq \frac{1}{18}(|u_2-u_1| + |v_2-v_1|),\\
|f_2(t,u_2,v_2) - f_2(t,u_1,v_1)| &\leq \frac{1}{16}(|u_2-u_1| + |v_2-v_1|),\\
|g_1(t,u_2,v_2) - g_1(t,u_1,v_1)| &\leq \frac{1}{18}(|u_2-u_1| + |v_2-v_1|),\\
|h_1(t,u_2,v_2) - h_1(t,u_1,v_1)| &\leq \frac{1}{16}(|u_2-u_1| + |v_2-v_1|).
\end{aligned}$$

Hence, $M_1 = 2.271, M_2 = 2.261, \mu_1 = \nu_1 = \frac{1}{18}, \mu_2 = \nu_2 = \frac{1}{16}, l_1 = m_1 = \frac{1}{18}, n_1 = k_1 = \frac{1}{16}$. Thus, we obtain $\overline{L} = \frac{1}{16}, \overline{L}(M_1 + M_2) = 0.283$. The conditions of the Theorem 3.1 hold. Therefore, the problem (4.1) has a unique solution on $[0,1]$.

Example 4.2. Consider the following problem:

$$\begin{cases}
D^{\frac{3}{4}}u(t) = e^t \cos(u(t)v(t)) + \ln(t+4)\\
+\int_0^t \frac{(t-s)^{\sqrt{11}-1}}{\Gamma(\sqrt{11})}\left[\frac{s}{e^s}\cos(su(s)v(s))\right]ds, t \in [0,1],\\
D^{\frac{5}{7}}v(t) = \sinh\left(-\pi t^2 |u(t)v(t)|\right)\\
+\int_0^t \frac{(t-s)^{\sqrt{7}-1}}{\Gamma(\sqrt{7})}\left[\sqrt{s}\exp(-|u(s)| - |v(s)|)\right]ds, t \in [0,1],\\
u(0) = \sqrt{2}, v(0) = \sqrt{5}.
\end{cases} \tag{4.2}$$

For this example, we have $\alpha = \frac{3}{4}, \beta = \frac{5}{7}, a = \sqrt{2}, b = \sqrt{5}$, and for all $t \in [0,1], \varphi_1(t) = \frac{t}{e^t}, \phi_1(t) = \sqrt{t}$, and for each $(u,v) \in \mathbb{R}^2$,

$$\begin{aligned}
f_1(t,u,v) &= e^t \cos(uv) + \ln(t+4),\\
f_2(t,u,v) &= \sinh\left(-\pi t^2 |uv|\right).
\end{aligned}$$

The conditions of Theorem 3.2 hold. Then (4.2) has at least one solution on $[0,1]$.

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