

**THE HIGHT ORDER LANE-EMDEN FRACTIONAL  
DIFFERENTIAL SYSTEM: EXISTENCE, UNIQUENESS AND  
ULAM TYPE STABILITIES**

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**ABSTRACT.** In this paper, by considering a more general Lane-Emden system of high order fractional differential equations with two arbitrary orders in each equation, we obtain some results on the existence and uniqueness of solutions using some fixed point theorems. Furthermore, we define and study some types of Ulam stability. Some examples are presented to illustrate the main results.

1. INTRODUCTION AND PRELIMINARIES

In recent years, fractional calculus has attracted great attention. It provides an excellent tool for the description of hereditary properties of various materials and processes. Moreover, the fractional differential equations theory arises in many engineering and scientific disciplines such as mechanics, physics, chemistry, biology, economics, control theory and signal processing, (see [22, 23, 25]). Many authors investigated the existence and uniqueness of solutions for nonlinear fractional differential equations. We refer the reader to [1–5, 10–17, 21, 26, 33] for more information and applications.

On the other hand, the Ulam type stabilities for fractional differential problems are quite significant in realistic problems, numerical analysis, biology and economics. Some results concerning these fractional stabilities have been obtained in [8, 19, 20, 34].

Let us now introduce some other important research papers related to the Lane-Emden model which has inspired our work: we know that modeling of several physical phenomena, such as pattern formation, population evolution and chemical reactions

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gives rise to the Lane-Emden differential problem [9]. Ever after, the Lane-Emden systems and other related systems have exhausted the attention of many authors [6, 7, 18, 27, 29, 30, 35, 36].

Lane-Emden model has the following form:

$$x''(t) + \frac{a}{t}x'(t) + f(t, x(t)) = g(t), \quad t \in [0, 1],$$

with the initial conditions:

$$x(0) = A, \quad x'(0) = B,$$

where  $A$  and  $B$  are constants,  $f$  is a continuous real valued function and  $g \in C([0, 1])$ , (see the paper of J. Serrin and H. Zou [32]).

Recently, S. M. Mechee and N. Senu proposed a numerical study of the Lane-Emden differential problem of fractional order. The imposed Lane-Emden model has a more importance in applied mathematics, mathematical physics and astrophysics. The order appeared in two different fractional order as follows [28]:

$$D^\alpha y(t) + \frac{k}{t^{\alpha-\beta}} D^\beta y(t) + f(t, y(t)) = g(t),$$

where  $t \in [0, 1]$ ,  $k \geq 0$ ,  $1 < \alpha \leq 2$ , and  $0 < \beta \leq 1$ , with the initial conditions  $y(0) = A$ ,  $y'(0) = B$ , where  $A$  and  $B$  are constants,  $f$  is a continuous real valued function and  $g \in C([0, 1])$ .

Very recently, R. W. Ibrahim [19] imposed the Ulam-Hyers stability for the following singular variable Lane-Emden equation:

$$D^\beta \left( D^\alpha + \frac{a}{t} \right) u(t) + f(t, u(t)) = g(t),$$

where  $u(0) = \mu$ ,  $u(1) = \nu$ ,  $0 < \alpha, \beta \leq 1$ ,  $0 \leq t \leq 1$ ,  $a \geq 0$ , and where  $D^\gamma$  denotes the Caputo fractional derivative for  $\gamma > 0$ ,  $f$  is continuous real valued function and  $g \in C([0, 1])$ .

In this paper, we consider a more general and high dimensional Lane-Emden coupled system of fractional differential equations. Then, we discuss the existence, uniqueness and some types of Ulam stabilities for the proposed coupled nonlinear fractional

system. So, let us consider:

$$(1.1) \quad \begin{cases} D^{\beta_1} \left( D^{\alpha_1} + \frac{a_1}{t} \right) x_1(t) + f_1(t, x_1(t), x_2(t), \dots, x_n(t)) = g_1(t), & t \in J, \\ D^{\beta_2} \left( D^{\alpha_2} + \frac{a_2}{t} \right) x_2(t) + f_2(t, x_1(t), x_2(t), \dots, x_n(t)) = g_2(t), & t \in J, \\ \vdots \\ D^{\beta_n} \left( D^{\alpha_n} + \frac{a_n}{t} \right) x_n(t) + f_n(t, x_1(t), x_2(t), \dots, x_n(t)) = g_n(t), & t \in J, \\ \sum_{k=1}^n |x_k(0)| = \sum_{k=1}^n |x'_k(0)| = \dots = \sum_{k=1}^n |x_k^{(l-1)}(0)| = 0, \\ \sum_{k=1}^n |D^{\alpha_k} x_k(0)| = \sum_{k=1}^n |D^{\alpha_{k+1}} x_k(0)| = \dots = \sum_{k=1}^n |D^{\alpha_{k+l-2}} x_k(0)| = 0, \\ D^{\alpha_{k+l-1}} x_k(1) = 0, \quad k = 1, 2, \dots, n, \end{cases}$$

where  $l - 1 < \beta_k < l$ ,  $l - 1 < \alpha_k < l$ ,  $a_k \geq 0$ ,  $l \in \mathbb{N}^* - \{1\}$ ,  $k = 1, 2, \dots, n$ ,  $n \in \mathbb{N}^*$  and  $J := [0, 1]$ . The derivatives  $D^{\beta_k}$  and  $D^{\alpha_k}$ ,  $k = 1, 2, \dots, n$ , are in the sense of Caputo. For each  $k = 1, 2, \dots, n$ , the functions  $f_k : J \times \mathbb{R}^n \rightarrow \mathbb{R}$  and  $g_k : J \rightarrow \mathbb{R}$  will be specified later.

To the best of our knowledge, there are no papers that have developed the Lane-Emden system in multi-variables, with arbitrary orders in each equation.

**Definition 1.1** ([22,31]). The Riemann-Liouville fractional integral operator of order  $\alpha > 0$ , for a continuous function  $f$  on  $[0, \infty[$  is defined as:

$$J^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, & \alpha > 0, \\ f(t), & \alpha = 0, \end{cases}$$

where  $t \geq 0$ , and  $\Gamma(\alpha) := \int_0^\infty e^{-x} x^{\alpha-1} dx$ .

**Definition 1.2** ([22,31]). The Caputo derivative of order  $\alpha$  for a function  $x : [0, \infty) \rightarrow \mathbb{R}$ , which is at least  $l$ -times differentiable can be defined as:

$$D^\alpha x(t) = \frac{1}{\Gamma(l-\alpha)} \int_0^t (t-s)^{l-\alpha-1} x^{(l)}(s) ds = J^{l-\alpha} x^{(l)}(t),$$

for  $l - 1 < \alpha < l$ ,  $l \in \mathbb{N}^*$ .

**Lemma 1.1** ([22]). For  $\alpha > 0$ , the general solution of the fractional differential equation  $D^\alpha x(t) = 0$ , is given by

$$x(t) = \sum_{j=0}^{l-1} c_j t^j,$$

where  $c_j \in \mathbb{R}$ ,  $j = 0, \dots, l - 1$ ,  $l = [\alpha] + 1$ .

**Lemma 1.2** ([22]). *Let  $\alpha > 0$ . Then*

$$J^\alpha D^\alpha x(t) = x(t) + \sum_{j=0}^{l-1} c_j t^j,$$

where  $c_j \in \mathbb{R}$ ,  $j = 0, 1, \dots, l - 1$ ,  $l = [\alpha] + 1$ .

**Lemma 1.3** ([22]). *Let  $q > p > 0$ ,  $g \in L^1([a, b])$ . Then  $D^p J^q g(t) = J^{q-p} g(t)$ ,  $t \in [a, b]$ .*

**Lemma 1.4** ([24], Krasnoselskii). *Let  $M$  be a closed convex and nonempty subset of a Banach space  $X$ . Let  $A$  and  $B$  be the operators such that*

- (i)  $Ax + By \in M$ , whenever  $x, y \in M$ ,
- (ii)  $A$  is a compact and continuous,
- (iii)  $B$  is a contraction mapping.

*Then there exists  $z \in M$  such that  $z = Az + Bz$ .*

The following auxiliary result is important to give the integral solution of (1.1).

**Lemma 1.5.** *Suppose that  $(G_k)_{k=1}^n \in C(J, \mathbb{R})$ ,  $J = [0, 1]$  and consider the nonlinear system*

$$(1.2) \quad \begin{cases} D^{\beta_1} \left( D^{\alpha_1} + \frac{a_1}{t} \right) x_1(t) = G_1(t), & t \in J, \\ D^{\beta_2} \left( D^{\alpha_2} + \frac{a_2}{t} \right) x_2(t) = G_2(t), & t \in J, \\ \vdots \\ D^{\beta_n} \left( D^{\alpha_n} + \frac{a_n}{t} \right) x_n(t) = G_n(t), & t \in J, \end{cases}$$

where  $l - 1 < \beta_k, \alpha_k < l$ ,  $a_k \geq 0, k = 1, 2, \dots, n$ ,  $l \in \mathbb{N}^* - \{1\}$ , with the conditions:

$$(1.3) \quad \begin{aligned} \sum_{k=1}^n |x_k(0)| &= \sum_{k=1}^n |x'_k(0)| = \dots = \sum_{k=1}^n |x_k^{(l-1)}(0)| = 0, \\ \sum_{k=1}^n |D^{\alpha_k} x_k(0)| &= \sum_{k=1}^n |D^{\alpha_{k+1}} x_k(0)| = \dots = \sum_{k=1}^n |D^{\alpha_{k+l-2}} x_k(0)| = 0, \end{aligned}$$

where  $D^{\alpha_{k+l-1}} x_k(1) = 0, k = 1, 2, \dots, n, l \in \mathbb{N}^* - \{1\}$ . Then, (1.2)–(1.3) has a unique solution given by  $(x_1, x_2, \dots, x_n)(t)$ , where

$$(1.4) \quad \begin{aligned} x_k(t) &= \int_0^t \frac{(t-\tau)^{\alpha_k-1}}{\Gamma(\alpha_k)} \left( \int_0^\tau \frac{(\tau-s)^{\beta_k-1}}{\Gamma(\beta_k)} G_k(s) ds - \frac{a_k}{\tau} x_k(\tau) \right) d\tau \\ &\quad - \frac{t^{\alpha_k+l-1}}{\Gamma(\alpha_k+l)} \int_0^1 \frac{(1-\tau)^{\alpha_k-1}}{\Gamma(\alpha_k)} \left( \int_0^\tau \frac{(\tau-s)^{\beta_k-1}}{\Gamma(\beta_k)} G_k(s) ds - \frac{a_k}{\tau} x_k(\tau) \right) d\tau, \end{aligned}$$

and  $k = 1, 2, \dots, n$ .

*Proof.* By Applying Lemma 1.2 to the problem (1.2), we get:

$$\left(D^{\alpha_k} + \frac{a_k}{\tau}\right) x_k(\tau) = \int_0^\tau \frac{(\tau - s)^{\beta_k - 1}}{\Gamma(\beta_k)} G_k(s) ds - c_0^k - c_1^k \tau - \dots - c_{l-1}^k \tau^{l-1},$$

where  $k = 1, 2, \dots, n$ ,  $(c_j^k)_{j=0, \dots, l-1} \in \mathbb{R}$ , and  $l - 1 < \beta_k < l, l \in \mathbb{N}^* - \{1\}$ .

Now, applying Lemma 1.2 to the last assertion, yields

$$\begin{aligned} (1.5) \quad x_k(t) &= \int_0^t \frac{(t - \tau)^{\alpha_k - 1}}{\Gamma(\alpha_k)} \left( \int_0^\tau \frac{(\tau - s)^{\beta_k - 1}}{\Gamma(\beta_k)} G_k(s) ds - \frac{a_k}{\tau} x_k(\tau) \right) d\tau \\ &\quad - \frac{c_0^k t^{\alpha_k}}{\Gamma(\alpha_k + 1)} - \frac{c_1^k t^{\alpha_k + 1}}{\Gamma(\alpha_k + 2)} - \dots - \frac{(l - 1)! c_{l-1}^k t^{\alpha_k + l - 1}}{\Gamma(\alpha_k + l)} \\ &\quad - c_0'^k - c_1'^k t - \dots - c_{l-1}'^k t^{l-1}, \end{aligned}$$

where  $k = 1, 2, \dots, n$  and  $(c_j'^k)_{j=0, \dots, l-1} \in \mathbb{R}$ ,  $l - 1 < \alpha_k < l, l \in \mathbb{N}^* - \{1\}$ .

Using Lemma 1.3 and applying the conditions (1.3), we obtain the values of  $c_j^k$  and  $c_j'^k$ . Substituting the last condition in (1.5), we obtain (1.4). The proof of Lemma 1.5 is thus completed.  $\square$

Now, let us introduce the Banach space:

$$S := \{(x_1, x_2, \dots, x_n) : x_k \in C(J, \mathbb{R}), \quad k = 1, 2, \dots, n\},$$

endowed with the norm:

$$\begin{aligned} \|(x_1, x_2, \dots, x_n)\|_S &= \max(\|x_1\|, \|x_2\|, \dots, \|x_n\|), \\ \|x_k\| &= \sup_{t \in J} |x_k(t)|, \end{aligned}$$

where  $k = 1, 2, \dots, n$ .

## 2. MAIN RESULTS

In this section, we will formulate and establish sufficient conditions for the existence and uniqueness of solutions to (1.1). Then, we continue our study by imposing some types of Ulam stability: Ulam-Hyers stability, generalized Ulam-Hyers stability and Ulam-Hyers-Rassias stability for the problem (1.1).

We begin by list the following hypotheses:

( $H_1$ ): There exist nonnegative constants  $(\mu_k)_j, j, k = 1, 2, \dots, n$ , such that for all  $t \in [0, 1]$  and all  $(x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n) \in S$ , we have

$$|f_k(t, x_1, x_2, \dots, x_n) - f_k(t, y_1, y_2, \dots, y_n)| \leq \sum_{j=1}^n (\mu_k)_j |x_j - y_j|.$$

( $H_2$ ): The functions  $f_k : [0, 1] \times \mathbb{R}^n \rightarrow \mathbb{R}$  and  $g_k : [0, 1] \rightarrow \mathbb{R}$  are continuous for each  $k = 1, 2, \dots, n, n \in \mathbb{N}^*$ .

(H<sub>3</sub>): For all  $k = 1, 2, \dots, n$ ,  $n \in \mathbb{N}^*$ , the function  $f_k$  maps bounded subsets of  $J \times \mathbb{R}^n$  into relatively compact subsets of  $\mathbb{R}$ .

(H<sub>4</sub>): There exist nonnegative constants  $(L_k)_{k=1,2,\dots,n}$ , such that, for each  $t \in J$  and all  $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ ,

$$|f_k(t, x_1, x_2, \dots, x_n)| \leq L_k, \quad k = 1, 2, \dots, n.$$

(H<sub>5</sub>): There exist nonnegative constants  $(M_k)_{k=1,2,\dots,n}$ , such that, for each  $t \in J$ ,  $|g_k(t)| \leq M_k$ ,  $k = 1, 2, \dots, n$ .

Then, we set the following quantities:

$$(2.1) \quad F_k = \frac{\Gamma(\alpha_k + l) + 1}{\Gamma(\alpha_k + l)\Gamma(\alpha_k + \beta_k + 1)}, \quad \Lambda_k = \frac{\Gamma(\alpha_k + l) + 1}{(\alpha_k + l - 1)\Gamma(2\alpha_k + l - 1)},$$

and

$$\Sigma_k = \sum_{j=1}^n (\mu_k)_j, \quad k = 1, 2, \dots, n.$$

**2.1. Existence and Uniqueness of Solutions.** The first result is based on Banach contraction principle. We have:

**Theorem 2.1.** *Assume that hypotheses (H<sub>1</sub>), (H<sub>4</sub>) and (H<sub>5</sub>) are satisfied. Then, the system (1.1) has a unique solution on J provided that*

$$(2.2) \quad \lambda_k =: \Sigma_k F_k + a_k \Lambda_k < 1, \quad k = 1, 2, \dots, n.$$

*Proof.* Define the nonlinear operator  $T : S \rightarrow S$  by

$$T(x_1, x_2, \dots, x_n)(t) := (T_1(x_1, x_2, \dots, x_n)(t), T_2(x_1, x_2, \dots, x_n)(t), \dots, T_n(x_1, x_2, \dots, x_n)(t)), \quad t \in J,$$

where, for all  $k = 1, 2, \dots, n$ ,

$$\begin{aligned} T_k(x_1, \dots, x_n)(t) &= \int_0^t \frac{(t-\tau)^{\alpha_k-1}}{\Gamma(\alpha_k)} \left( \int_0^\tau \frac{(\tau-s)^{\beta_k-1}}{\Gamma(\beta_k)} \right. \\ &\quad \left. \times (g_k(s) - f_k(s, x_1(s), \dots, x_n(s))) ds - \frac{a_k}{\tau} x_k(\tau) \right) d\tau \\ &\quad - \frac{t^{\alpha_k+l-1}}{\Gamma(\alpha_k+l)} \int_0^1 \frac{(1-\tau)^{\alpha_k-1}}{\Gamma(\alpha_k)} \left( \int_0^\tau \frac{(\tau-s)^{\beta_k-1}}{\Gamma(\beta_k)} \right. \\ &\quad \left. \times (g_k(s) - f_k(s, x_1(s), \dots, x_n(s))) ds - \frac{a_k}{\tau} x_k(\tau) \right) d\tau. \end{aligned}$$

For

$$(2.3) \quad r \geq \frac{(M_k + L_k) F_k}{1 - \sigma}, \quad \lambda_k \leq \sigma < 1,$$

we consider the set  $B_r := \{(x_1, x_2, \dots, x_n) \in S; \|(x_1, x_2, \dots, x_n)\|_S \leq r\}$  and we will show that  $TB_r \subset B_r$ .

For  $(x_1, x_2, \dots, x_n) \in B_r$  and each  $k = 1, 2, \dots, n$ , we have

$$\begin{aligned} & \|T_k(x_1, x_2, \dots, x_n)\| \\ & \leq \sup_{t \in J} \int_0^t \frac{(t-\tau)^{\alpha_k-1}}{\Gamma(\alpha_k)} \left( \int_0^\tau \frac{(\tau-s)^{\beta_k-1}}{\Gamma(\beta_k)} \right. \\ & \quad \times |g_k(s) - f_k(s, x_1(s), \dots, x_n(s)) - f_k(s, 0, \dots, 0) + f_k(s, 0, \dots, 0)| ds \\ & \quad \left. + \left| \frac{a_k}{\tau} x_k(\tau) \right| \right) d\tau \\ & + \sup_{t \in J} \frac{t^{\alpha_k+l-1}}{\Gamma(\alpha_k+l)} \int_0^1 \frac{(1-\tau)^{\alpha_k-1}}{\Gamma(\alpha_k)} \left( \int_0^\tau \frac{(\tau-s)^{\beta_k-1}}{\Gamma(\beta_k)} \right. \\ & \quad \times |g_k(s) - f_k(s, x_1(s), \dots, x_n(s)) - f_k(s, 0, \dots, 0) + f_k(s, 0, \dots, 0)| ds \\ & \quad \left. + \left| \frac{a_k}{\tau} x_k(\tau) \right| \right) d\tau. \end{aligned}$$

The hypotheses  $(H_4)$  and  $(H_5)$  allow us to write

$$\begin{aligned} & \|T_k(x_1, x_2, \dots, x_n)\| \\ & \leq \sup_{t \in J} \int_0^t \frac{(t-\tau)^{\alpha_k-1}}{\Gamma(\alpha_k)} \left( \int_0^\tau \frac{(\tau-s)^{\beta_k-1}}{\Gamma(\beta_k)} ds \right) d\tau (M_k + L_k + \Sigma_k \|(x_1, x_2, \dots, x_n)\|_S) \\ & + \sup_{t \in J} \int_0^t \frac{(t-\tau)^{\alpha_k-1}}{\Gamma(\alpha_k)} \frac{1}{\tau} d\tau a_k \|(x_1, x_2, \dots, x_n)\|_S \\ & + \sup_{t \in J} \frac{t^{\alpha_k+l-1}}{\Gamma(\alpha_k+l)} \int_0^1 \frac{(1-\tau)^{\alpha_k-1}}{\Gamma(\alpha_k)} \left( \int_0^\tau \frac{(\tau-s)^{\beta_k-1}}{\Gamma(\beta_k)} ds \right) d\tau \\ & \quad \times (M_k + L_k + \Sigma_k \|(x_1, x_2, \dots, x_n)\|_S) \\ & + \sup_{t \in J} \frac{t^{\alpha_k+l-1}}{\Gamma(\alpha_k+l)} \int_0^1 \frac{(1-\tau)^{\alpha_k-1}}{\Gamma(\alpha_k)} \frac{1}{\tau} d\tau a_k \|(x_1, x_2, \dots, x_n)\|_S. \end{aligned}$$

Therefore, by  $(H_1)$

$$\begin{aligned} & \|T_k(x_1, x_2, \dots, x_n)\| \\ & \leq (M_k + L_k + \Sigma_k \|(x_1, x_2, \dots, x_n)\|_S) \\ & \quad \times \left(1 + \frac{1}{\Gamma(\alpha_k + l)}\right) \int_0^1 \frac{(1 - \tau)^{\alpha_k - 1}}{\Gamma(\alpha_k)} \left(\int_0^\tau \frac{(\tau - s)^{\beta_k - 1}}{\Gamma(\beta_k)} ds\right) d\tau \\ & \quad + a_k \left(1 + \frac{1}{\Gamma(\alpha_k + l)}\right) \|(x_1, x_2, \dots, x_n)\|_S \int_0^1 \frac{(1 - \tau)^{\alpha_k - 1}}{\Gamma(\alpha_k)} \tau^{\alpha_k + l - 2} d\tau. \end{aligned}$$

Hence,

$$\begin{aligned} & \|T_k(x_1, x_2, \dots, x_n)\| \\ & \leq (M_k + L_k + \Sigma_k r) \left(\frac{1}{\Gamma(\alpha_k) \Gamma(\beta_k + 1)} + \frac{1}{\Gamma(\alpha_k) \Gamma(\beta_k + 1) \Gamma(\alpha_k + l)}\right) \\ & \quad \times \int_0^1 (1 - \tau)^{\alpha_k - 1} \tau^{\beta_k} d\tau \\ & \quad + a_k r \left(\frac{1}{\Gamma(\alpha_k)} + \frac{1}{\Gamma(\alpha_k) \Gamma(\alpha_k + l)}\right) \int_0^1 (1 - \tau)^{\alpha_k - 1} \tau^{\alpha_k + l - 2} d\tau. \end{aligned}$$

Using the Beta function, we get

$$\begin{aligned} & \|T_k(x_1, x_2, \dots, x_n)\| \\ & \leq (M_k + L_k + \Sigma_k r) \left(\frac{1}{\Gamma(\alpha_k) \Gamma(\beta_k + 1)} + \frac{1}{\Gamma(\alpha_k + l) \Gamma(\alpha_k) \Gamma(\beta_k + 1)}\right) \\ & \quad \times \beta(\alpha_k, \beta_k + 1) + a_k r \left(\frac{1}{\Gamma(\alpha_k)} + \frac{1}{\Gamma(\alpha_k) \Gamma(\alpha_k + l)}\right) \beta(\alpha_k, \alpha_k + l - 1). \end{aligned}$$

By (2.3), we obtain

$$\|T_k(x_1, x_2, \dots, x_n)\| \leq (1 - \sigma) r + \lambda_k r = (1 - \sigma + \lambda_k) r$$

and

$$\|T_k(x_1, x_2, \dots, x_n)\| \leq r,$$

where  $k = 1, 2, \dots, n$ .

Thus,

$$\|T(x_1, x_2, \dots, x_n)\|_S \leq r.$$

We need to prove that  $T$  is a contractive.



For  $(x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n) \in S$  and for all  $t \in J$ , we have:

$$\begin{aligned} & \|T_k(x_1, x_2, \dots, x_n) - T_k(y_1, y_2, \dots, y_n)\| \\ & \leq \sup_{t \in J} \int_0^t \frac{(t-\tau)^{\alpha_k-1}}{\Gamma(\alpha_k)} \left( \int_0^\tau \frac{(\tau-s)^{\beta_k-1}}{\Gamma(\beta_k)} \right. \\ & \quad \times |f_k(s, x_1(s), \dots, x_n(s)) - f_k(s, y_1(s), \dots, y_n(s))| ds \\ & \quad \left. + \frac{a_k}{\tau} |x_k(\tau) - y_k(\tau)| \right) d\tau \\ & \quad + \sup_{t \in J} \frac{t^{\alpha_k+l-1}}{\Gamma(\alpha_k+l)} \int_0^1 \frac{(1-\tau)^{\alpha_k-1}}{\Gamma(\alpha_k)} \left( \int_0^\tau \frac{(\tau-s)^{\beta_k-1}}{\Gamma(\beta_k)} \right. \\ & \quad \times |f_k(s, x_1(s), \dots, x_n(s)) - f_k(s, y_1(s), \dots, y_n(s))| ds \\ & \quad \left. + \frac{a_k}{\tau} |x_k(\tau) - y_k(\tau)| \right) d\tau. \end{aligned}$$

Thanks to  $(H_1)$ , we can write

$$\begin{aligned} & \|T_k(x_1, x_2, \dots, x_n) - T_k(y_1, y_2, \dots, y_n)\| \\ & \leq \sup_{t \in J} \int_0^t \frac{(t-\tau)^{\alpha_k-1}}{\Gamma(\alpha_k)} \left( \int_0^\tau \frac{(\tau-s)^{\beta_k-1}}{\Gamma(\beta_k)} ds \right) d\tau \Sigma_k \|(x_1 - y_1, x_2 - y_2, \dots, x_n - y_n)\|_S \\ & \quad + \sup_{t \in J} \int_0^t \frac{(t-\tau)^{\alpha_k-1}}{\Gamma(\alpha_k)} \frac{1}{\tau} d\tau a_k \|(x_1 - y_1, x_2 - y_2, \dots, x_n - y_n)\|_S + \sup_{t \in J} \frac{t^{\alpha_k+l-1}}{\Gamma(\alpha_k+l)} \\ & \quad \times \int_0^1 \frac{(1-\tau)^{\alpha_k-1}}{\Gamma(\alpha_k)} \left( \int_0^\tau \frac{(\tau-s)^{\beta_k-1}}{\Gamma(\beta_k)} ds \right) d\tau \Sigma_k \|(x_1 - y_1, x_2 - y_2, \dots, x_n - y_n)\|_S \\ & \quad + \sup_{t \in J} \frac{t^{\alpha_k+l-1}}{\Gamma(\alpha_k+l)} \int_0^1 \frac{(1-\tau)^{\alpha_k-1}}{\Gamma(\alpha_k)} \frac{1}{\tau} d\tau a_k \|(x_1 - y_1, x_2 - y_2, \dots, x_n - y_n)\|_S. \end{aligned}$$

Consequently,

$$\begin{aligned} & \|T_k(x_1, x_2, \dots, x_n) - T_k(y_1, y_2, \dots, y_n)\| \\ & \leq \Sigma_k \frac{\Gamma(\alpha_k+l)+1}{\Gamma(\alpha_k+l)} \|(x_1 - y_1, x_2 - y_2, \dots, x_n - y_n)\|_S \end{aligned}$$

$$\begin{aligned}
 & \times \int_0^1 \frac{(1-\tau)^{\alpha_k-1}}{\Gamma(\alpha_k)} \left( \int_0^\tau \frac{(\tau-s)^{\beta_k-1}}{\Gamma(\beta_k)} ds \right) d\tau \\
 (2.4) \quad & + a_k \frac{\Gamma(\alpha_k+l)+1}{\Gamma(\alpha_k+l)} \|(x_1-y_1, x_2-y_2, \dots, x_n-y_n)\|_S \\
 & \times \int_0^1 \frac{(1-\tau)^{\alpha_k-1}}{\Gamma(\alpha_k)} \tau^{\alpha_k+l-2} d\tau \\
 & \leq \lambda_k \|(x_1-y_1, x_2-y_2, \dots, x_n-y_n)\|_S,
 \end{aligned}$$

which implies that

$$\begin{aligned}
 & \|T(x_1, x_2, \dots, x_n) - T(y_1, y_2, \dots, y_n)\|_S \\
 & \leq \max(\lambda_1, \lambda_2, \dots, \lambda_n) \|(x_1-y_1, x_2-y_2, \dots, x_n-y_n)\|_S,
 \end{aligned}$$

where,  $k = 1, 2, \dots, n$ .

Then by (2.2),  $T$  is contractive. □

**Theorem 2.2.** *Assume that the hypotheses  $(H_i)_{i=1,2,\dots,5}$  and the inequalities*

$$(2.5) \quad C_k := \frac{\lambda_k}{\Gamma(\alpha_k+l)+1} < 1, \quad k = 1, 2, \dots, n,$$

are satisfied. Then system (1.1) has at least one solution on  $J$ .

*Proof.* On  $B_R$ , such that

$$R \geq \frac{(M_k + L_k) F_k}{1 - a_k \Lambda_k}, \quad a_k \Lambda_k \neq 1,$$

we define the operators  $P$  and  $Q$  as follows:

$$\begin{aligned}
 P(x_1, x_2, \dots, x_n)(t) & := \\
 & (P_1(x_1, x_2, \dots, x_n)(t), P_2(x_1, x_2, \dots, x_n)(t), \dots, P_n(x_1, x_2, \dots, x_n)(t)), \\
 Q(x_1, x_2, \dots, x_n)(t) & := \\
 & (Q_1(x_1, x_2, \dots, x_n)(t), Q_2(x_1, x_2, \dots, x_n)(t), \dots, Q_n(x_1, x_2, \dots, x_n)(t)).
 \end{aligned}$$

For each  $k = 1, 2, \dots, n$ ,

$$\begin{aligned}
 P_k(x_1, x_2, \dots, x_n)(t) & := \\
 & \int_0^t \frac{(t-\tau)^{\alpha_k-1}}{\Gamma(\alpha_k)} \left( \int_0^\tau \frac{(\tau-s)^{\beta_k-1}}{\Gamma(\beta_k)} (g_k(s) - f_k(s, x_1(s), \dots, x_n(s))) ds - \frac{a_k}{\tau} x_k(\tau) \right) d\tau,
 \end{aligned}$$

and

$$Q_k(x_1, x_2, \dots, x_n)(t) := -\frac{t^{\alpha_k+l-1}}{\Gamma(\alpha_k+l)} \int_0^1 \frac{(1-\tau)^{\alpha_k-1}}{\Gamma(\alpha_k)} \left( \int_0^\tau \frac{(\tau-s)^{\beta_k-1}}{\Gamma(\beta_k)} \right. \\ \left. \times (g_k(s) - f_k(s, x_1(s), \dots, x_n(s))) ds - \frac{a_k}{\tau} x_k(\tau) \right) d\tau.$$

For  $(x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n) \in B_R$  and for each  $k = 1, 2, \dots, n$ , we can write

$$\|P_k(x_1, x_2, \dots, x_n) + Q_k(y_1, y_2, \dots, y_n)\| \\ \leq \sup_{t \in J} \int_0^t \frac{(t-\tau)^{\alpha_k-1}}{\Gamma(\alpha_k)} \left( \int_0^\tau \frac{(\tau-s)^{\beta_k-1}}{\Gamma(\beta_k)} \right. \\ \left. \times (|g_k(s)| + |f_k(s, x_1(s), x_2(s), \dots, x_n(s))|) ds + \left| \frac{a_k}{\tau} x_k(\tau) \right| \right) d\tau \\ + \sup_{t \in J} \frac{t^{\alpha_k+l-1}}{\Gamma(\alpha_k+l)} \int_0^1 \frac{(1-\tau)^{\alpha_k-1}}{\Gamma(\alpha_k)} \left( \int_0^\tau \frac{(\tau-s)^{\beta_k-1}}{\Gamma(\beta_k)} \right. \\ \left. \times (|g_k(s)| + |f_k(s, y_1(s), y_2(s), \dots, y_n(s))|) ds + \left| \frac{a_k}{\tau} y_k(\tau) \right| \right) d\tau.$$

Using  $(H_4)$  and  $(H_5)$ , we obtain

$$\|P_k(x_1, x_2, \dots, x_n) + Q_k(y_1, y_2, \dots, y_n)\| \\ \leq \sup_{t \in J} \int_0^t \frac{(t-\tau)^{\alpha_k-1}}{\Gamma(\alpha_k)} \left( \int_0^\tau \frac{(\tau-s)^{\beta_k-1}}{\Gamma(\beta_k)} ds \right) d\tau (M_k + L_k) \\ + \sup_{t \in J} \int_0^t \frac{(t-\tau)^{\alpha_k-1}}{\Gamma(\alpha_k)} \frac{1}{\tau} d\tau a_k \| (x_1, x_2, \dots, x_n) \|_S \\ + \sup_{t \in J} \frac{t^{\alpha_k+l-1}}{\Gamma(\alpha_k+l)} \int_0^1 \frac{(1-\tau)^{\alpha_k-1}}{\Gamma(\alpha_k)} \left( \int_0^\tau \frac{(\tau-s)^{\beta_k-1}}{\Gamma(\beta_k)} ds \right) d\tau (M_k + L_k) \\ + \sup_{t \in J} \frac{t^{\alpha_k+l-1}}{\Gamma(\alpha_k+l)} \int_0^1 \frac{(1-\tau)^{\alpha_k-1}}{\Gamma(\alpha_k)} \frac{1}{\tau} d\tau a_k \| (y_1, y_2, \dots, y_n) \|_S.$$

And then,

$$\begin{aligned}
 & \|P_k(x_1, x_2, \dots, x_n) + Q_k(y_1, y_2, \dots, y_n)\| \\
 & \leq (M_k + L_k) \left(1 + \frac{1}{\Gamma(\alpha_k + l)}\right) \int_0^1 \frac{(1 - \tau)^{\alpha_k - 1}}{\Gamma(\alpha_k)} \left(\int_0^\tau \frac{(\tau - s)^{\beta_k - 1}}{\Gamma(\beta_k)} ds\right) d\tau \\
 & \quad + a_k R \left(1 + \frac{1}{\Gamma(\alpha_k + l)}\right) \int_0^1 \frac{(1 - \tau)^{\alpha_k - 1}}{\Gamma(\alpha_k)} \tau^{\alpha_k + l - 2} d\tau \\
 & \leq \left(\frac{M_k + L_k}{\Gamma(\alpha_k) \Gamma(\beta_k + 1)} + \frac{M_k + L_k}{\Gamma(\alpha_k) \Gamma(\beta_k + 1) \Gamma(\alpha_k + l)}\right) \int_0^1 (1 - \tau)^{\alpha_k - 1} \tau^{\beta_k} d\tau \\
 & \quad + a_k R \left(\frac{1}{\Gamma(\alpha_k)} + \frac{1}{\Gamma(\alpha_k) \Gamma(\alpha_k + l)}\right) \int_0^1 (1 - \tau)^{\alpha_k - 1} \tau^{\alpha_k + l - 2} d\tau.
 \end{aligned}$$

Such that  $R \geq (M_k + L_k)F_k/(1 - \alpha_k\Lambda_k)$ ,  $\alpha_k\Lambda_k \neq 1$ , we have

$$\begin{aligned}
 \|P_k(x_1, x_2, \dots, x_n) + Q_k(y_1, y_2, \dots, y_n)\| & \leq (M_k + L_k) \frac{\Gamma(\alpha_k + l) + 1}{\Gamma(\alpha_k + l) \Gamma(\alpha_k + \beta_k + 1)} \\
 & \quad + a_k R \frac{(\Gamma(\alpha_k + l) + 1)}{(\alpha_k + l - 1) \Gamma(2\alpha_k + l - 1)} \\
 & = (M_k + L_k) F_k + a_k \Lambda_k R \leq R.
 \end{aligned}$$

Consequently,

$$\|P(x_1, x_2, \dots, x_n) + Q(y_1, y_2, \dots, y_n)\|_S \leq R.$$

Thus,  $P(x_1, x_2, \dots, x_n) + Q(y_1, y_2, \dots, y_n) \in B_R$ .

We proceed to prove that  $Q$  is a contraction mapping in  $B_R$ . For  $(x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n) \in B_R$  and for each  $k = 1, 2, \dots, n$ , we obtain:

$$\begin{aligned}
 & \|Q_k(x_1, x_2, \dots, x_n) - Q_k(y_1, y_2, \dots, y_n)\| \\
 & \leq \sup_{t \in J} \frac{t^{\alpha_k + l - 1}}{\Gamma(\alpha_k + l)} \int_0^1 \frac{(1 - \tau)^{\alpha_k - 1}}{\Gamma(\alpha_k)} \int_0^\tau \frac{(\tau - s)^{\beta_k - 1}}{\Gamma(\beta_k)} \\
 & \quad \times |f_k(s, x_1(s), \dots, x_n(s)) - f_k(s, y_1(s), \dots, y_n(s))| ds \\
 & \quad + \frac{a_k}{\tau} |x_k(\tau) - y_k(\tau)|
 \end{aligned}$$

Using  $(H_1)$ , we get

$$\begin{aligned}
 & \|Q_k(x_1, x_2, \dots, x_n) - Q_k(y_1, y_2, \dots, y_n)\| \\
 & \leq \frac{\Sigma_k}{\Gamma(\alpha_k + l)} \|(x_1 - y_1, x_2 - y_2, \dots, x_n - y_n)\|_S
 \end{aligned}$$

$$\begin{aligned}
& \times \int_0^1 \frac{(1-\tau)^{\alpha_k-1}}{\Gamma(\alpha_k)} \left( \int_0^\tau \frac{(\tau-s)^{\beta_k-1}}{\Gamma(\beta_k)} ds \right) d\tau \\
& + \frac{a_k}{\Gamma(\alpha_k+l)} \|(x_1-y_1, x_2-y_2, \dots, x_n-y_n)\|_S \int_0^1 \frac{(1-\tau)^{\alpha_k-1}}{\Gamma(\alpha_k)} \tau^{\alpha_k+l-2} d\tau \\
\leq & \frac{\Sigma_k \|(x_1-y_1, x_2-y_2, \dots, x_n-y_n)\|_S}{\Gamma(\alpha_k+l) \Gamma(\alpha_k) \Gamma(\beta_k+1)} \int_0^1 (1-\tau)^{\alpha_k-1} \tau^{\beta_k} d\tau \\
& + \frac{a_k \|(x_1-y_1, x_2-y_2, \dots, x_n-y_n)\|_S}{\Gamma(\alpha_k+l) \Gamma(\alpha_k)} \int_0^1 (1-\tau)^{\alpha_k-1} \tau^{\alpha_k+l-2} d\tau \\
\leq & \frac{\Sigma_k \|(x_1-y_1, x_2-y_2, \dots, x_n-y_n)\|_S}{\Gamma(\alpha_k+l) \Gamma(\alpha_k) \Gamma(\beta_k+1)} \beta(\alpha_k, \beta_k+1) \\
& + \frac{a_k \|(x_1-y_1, x_2-y_2, \dots, x_n-y_n)\|_S}{\Gamma(\alpha_k+l) \Gamma(\alpha_k)} \beta(\alpha_k, \alpha_k+l-1).
\end{aligned}$$

Then,

$$\begin{aligned}
& \|Q_k(x_1, x_2, \dots, x_n) - Q_k(y_1, y_2, \dots, y_n)\| \\
\leq & \frac{\Sigma_k \|(x_1-y_1, x_2-y_2, \dots, x_n-y_n)\|_S}{\Gamma(\alpha_k+l) \Gamma(\alpha_k+\beta_k+1)} \\
& + \frac{a_k \Gamma(\alpha_k+l-1) \|(x_1-y_1, x_2-y_2, \dots, x_n-y_n)\|_S}{\Gamma(\alpha_k+l) \Gamma(2\alpha_k+l-1)} \\
\leq & \left( \frac{\Sigma_k}{\Gamma(\alpha_k+l) \Gamma(\alpha_k+\beta_k+1)} + \frac{a_k}{(\alpha_k+l-1) \Gamma(2\alpha_k+l-1)} \right) \\
& \times \|(x_1-y_1, x_2-y_2, \dots, x_n-y_n)\|_S.
\end{aligned}$$

Thus,

$$\begin{aligned}
& \|Q(x_1, x_2, \dots, x_n) - Q(y_1, y_2, \dots, y_n)\|_S \\
\leq & \max_{1 \leq k \leq n} C_k \|(x_1-y_1, x_2-y_2, \dots, x_n-y_n)\|_S.
\end{aligned}$$

Thanks to (2.2) and (2.5), we conclude that  $Q$  is a contractive.

The hypothesis  $(H_2)$  implies that  $P$  is continuous. Then, for all  $(x_1, x_2, \dots, x_n) \in B_R$  and each  $t \in J$ , we obtain

$$\begin{aligned}
& \|P_k(x_1, x_2, \dots, x_n)\| \\
\leq & \sup_{t \in J} \int_0^t \frac{(t-\tau)^{\alpha_k-1}}{\Gamma(\alpha_k)} \left( \int_0^\tau \frac{(\tau-s)^{\beta_k-1}}{\Gamma(\beta_k)} \right)
\end{aligned}$$

$$\begin{aligned}
 & \times \left( |g_k(s)| + |f_k(s, x_1(s), x_2(s), \dots, x_n(s))| \right) ds + \left| \frac{a_k}{\tau} x_k(\tau) \right| d\tau \\
 (2.6) \quad & \leq \sup_{t \in J} \int_0^t \frac{(t-\tau)^{\alpha_k-1}}{\Gamma(\alpha_k)} \left( \int_0^\tau \frac{(\tau-s)^{\beta_k-1}}{\Gamma(\beta_k)} ds \right) d\tau (M_k + L_k) \\
 & \quad + \sup_{t \in J} \int_0^t \frac{(t-\tau)^{\alpha_k-1}}{\Gamma(\alpha_k)} \tau^{\alpha_k-l-2} d\tau a_k \| (x_1, x_2, \dots, x_n) \|_S \\
 & \leq \frac{M_k + L_k}{\Gamma(\alpha_k)\Gamma(\beta_k + 1)} \int_0^1 (1-\tau)^{\alpha_k-1} \tau^{\beta_k} d\tau + a_k R \frac{1}{\Gamma(\alpha_k)} \int_0^1 (1-\tau)^{\alpha_k-1} \tau^{\alpha_k-l-2} d\tau \\
 & \leq \frac{M_k + L_k}{\Gamma(\alpha_k + \beta_k + 1)} + \frac{\Gamma(\alpha_k + l - 1)}{\Gamma(2\alpha_k + l - 1)} a_k R.
 \end{aligned}$$

And by (2.6), we get

$$\|P(x_1, x_2, \dots, x_n)\|_S \leq R.$$

Therefore,  $P$  is uniformly bounded on  $B_R$ . Furthermore, we show that  $P$  is a compact operator in  $B_R$ .

Let  $0 \leq t_1 < t_2 \leq 1$  and  $(x_1, x_2, \dots, x_n) \in B_R$ , then

$$\begin{aligned}
 & \|P_k(x_1, x_2, \dots, x_n)(t_2) - P_k(x_1, x_2, \dots, x_n)(t_1)\| \\
 & \leq \sup_{t \in J} \left| \int_0^{t_2} (t_1 - \tau)^{\alpha_k-1} \tau^{\beta_k} d\tau - \int_0^{t_1} (t_2 - \tau)^{\alpha_k-1} \tau^{\beta_k} d\tau \right| \frac{M_k + L_k}{\Gamma(\alpha_k)\Gamma(\beta_k + 1)} \\
 & \quad + \frac{a_k R}{\Gamma(\alpha_k)} \sup_{t \in J} \left| \int_0^{t_2} (t_2 - \tau)^{\alpha_k-1} \tau^{\alpha_k-l-2} d\tau - \int_0^{t_1} (t_1 - \tau)^{\alpha_k-1} \tau^{\alpha_k-l-2} d\tau \right| \\
 & = \frac{M_k + L_k}{\Gamma(\alpha_k + \beta_k + 1)} \left( t_2^{\alpha_k+\beta_k} - t_1^{\alpha_k+\beta_k} \right) + \frac{a_k R \Gamma(\alpha_k + l - 1)}{\Gamma(2\alpha_k + l - 1)} \left( t_2^{2\alpha_k+l-2} - t_1^{2\alpha_k+l-2} \right).
 \end{aligned}$$

Thus,

$$\begin{aligned}
 & \|P(x_1, x_2, \dots, x_n)(t_2) - P(x_1, x_2, \dots, x_n)(t_1)\|_S \\
 & \leq \max_{1 \leq k \leq n} \frac{M_k + L_k}{\Gamma(\alpha_k + \beta_k + 1)} \left( t_2^{\alpha_k+\beta_k} - t_1^{\alpha_k+\beta_k} \right) \\
 (2.7) \quad & \quad + \max_{1 \leq k \leq n} \frac{a_k R \Gamma(\alpha_k + l - 1)}{\Gamma(2\alpha_k + l - 1)} \left( t_2^{2\alpha_k+l-2} - t_1^{2\alpha_k+l-2} \right).
 \end{aligned}$$

The right-hand side of the above inequalities (2.7) is independent of  $(x_1, x_2, \dots, x_n)$  and tend to zero as  $t_2 \rightarrow t_1$ . Therefore,  $P$  is equicontinuous. Furthermore, by the hypothesis  $(H_3)$ ,  $P$  is relatively compact on  $B_R$ . By Arzela-Ascoli Theorem,  $P$  is a

compact operator on  $B_R$ . Then thanks to Lemma 1.6, we can state that (1.1) has at least one solution on  $J$ . Theorem 2.2 is thus proved.  $\square$

**2.2. Ulam Stabilities.** We introduce the following definitions.

**Definition 2.1.** The Lane-Emden fractional system (1.1) has the Ulam-Hyers stability if there exists a positive constant  $K$  with the following property:

For every  $\epsilon > 0$ , for any  $t \in J$ , and  $(x_1, x_2, \dots, x_n) \in S$  of (1.1), with

$$\left| D^{\beta_k} \left( D^{\alpha_k} + \frac{a_k}{t} \right) x_k(t) + f_k(t, x_1(t), x_2(t), \dots, x_n(t)) - g_k(t) \right| < \epsilon,$$

and  $k = 1, \dots, n$ , then, there exists a solution  $(y_1, y_2, \dots, y_n) \in S$  that satisfies

$$(2.8) \quad D^{\beta_k} \left( D^{\alpha_k} + \frac{a_k}{t} \right) y_k(t) + f_k(t, y_1(t), y_2(t), \dots, y_n(t)) = g_k(t), \quad k = 1, \dots, n,$$

and

$$\begin{aligned} \sum_{k=1}^n |y_k(0)| &= \sum_{k=1}^n |y'_k(0)| = \dots = \sum_{k=1}^n \left| y_k^{(l-1)}(0) \right| \\ &= \sum_{k=1}^n |D^{\alpha_k} y_k(0)| = \sum_{k=1}^n |D^{\alpha_{k+1}} y_k(0)| = \dots = \sum_{k=1}^n |D^{\alpha_{k+l-2}} y_k(0)| = 0, \end{aligned}$$

where  $D^{\alpha_{k+l-1}} y_k(1) = 0$ , such that,

$$\|(x_1 - y_1, x_2 - y_2, \dots, x_n - y_n)\|_S < K\epsilon.$$

**Definition 2.2.** The system (1.1) is Ulam-Hyers stable in the generalized sense if there exists  $\varphi \in C(\mathbb{R}^+, \mathbb{R}^+)$ , such that  $\varphi(0) = 0$  and for each  $\epsilon > 0$ , and for each solution  $(x_1, x_2, \dots, x_n) \in S$  of (1.1), with  $|D^{\beta_k} (D^{\alpha_k} + \frac{a_k}{t}) y_k(t) + f_k(t, x_1(t), x_2(t), \dots, x_n(t)) - g_k(t)| < \epsilon$ , and  $k = 1, \dots, n$ , there exists  $(y_1, y_2, \dots, y_n) \in S$  of (2.8), with

$$\|(x_1 - y_1, x_2 - y_2, \dots, x_n - y_n)\|_S < \varphi(\epsilon).$$

**Definition 2.3.** The system (1.1) is stable in the sense of Ulam-Hyers-Rassias if there exist  $\Psi \in C(J, \mathbb{R}^+)$  and some positive  $\rho_i$ , such that for each  $\epsilon_i > 0$ ,  $i = 1, \dots, n$ , and for all solution  $(x_1, x_2, \dots, x_n) \in S$  of the inequalities

$$(2.9) \quad \begin{cases} \left| D^{\beta_1} \left( D^{\alpha_1} + \frac{a_1}{t} \right) x_1(t) + f_1(t, x_1(t), x_2(t), \dots, x_n(t)) - g_1(t) \right| \leq \epsilon_1 \Psi(t), \\ \left| D^{\beta_2} \left( D^{\alpha_2} + \frac{a_2}{t} \right) x_2(t) + f_2(t, x_1(t), x_2(t), \dots, x_n(t)) - g_2(t) \right| \leq \epsilon_2 \Psi(t), \\ \vdots \\ \left| D^{\beta_n} \left( D^{\alpha_n} + \frac{a_n}{t} \right) x_n(t) + f_n(t, x_1(t), x_2(t), \dots, x_n(t)) - g_n(t) \right| \leq \epsilon_n \Psi(t), \end{cases}$$

there exists  $(y_1, y_2, \dots, y_n) \in S$  of (2.8) that satisfies

$$\|(x_1 - y_1, x_2 - y_2, \dots, x_n - y_n)\|_S \leq \rho \epsilon \Psi(t), \quad \rho := \max_{1 \leq k \leq n} \rho_k, \quad \epsilon := \max_{1 \leq k \leq n} \epsilon_k.$$

*Remark 2.1.* An element  $(x_1, x_2, \dots, x_n)$  of  $S$  is a solution of (1.1) if and only if there exists  $h_k : [0, 1] \rightarrow \mathbb{R}, k = 1, 2, \dots, n$ , such that

- (i)  $|h_k(t)| < \epsilon_k, t \in [0, 1], (\epsilon_k > 0, \text{ and is sufficient small}),$  and
- (ii)  $D^{\beta_k} \left( D^{\alpha_k} + \frac{a_k}{t} \right) x_k(t) + f_k(t, x_1(t), x_2(t), \dots, x_n(t)) = g_k(t) + h_k(t), t \in [0, 1].$

**Theorem 2.3.** *Under the assumptions of Theorem 2.1, if the inequalities*

$$(2.10) \quad \sup_{t \in J} \left| D^{\beta_k} \left( D^{\alpha_k} + \frac{a_k}{t} \right) x_k(t) \right| \geq (M_k + L_k) F_k + (\Sigma_k F_k + a_k \Lambda_k) r, \quad \Sigma_k < 1,$$

*are satisfied, then the Lane-Emden problem (1.1) is Ulam-Hyers stable in  $S$ .*

*Proof.* Thanks to Theorem 2.1, we can state that the problem (1.1) has a solution  $(y_1, y_2, \dots, y_n) \in S$  that satisfies (2.8).

Now, suppose  $(x_1, x_2, \dots, x_n) \in S$  is a solution of (1.1), where

$$(2.11) \quad \left| D^{\beta_k} \left( D^{\alpha_k} + \frac{a_k}{t} \right) x_k(t) + f_k(t, x_1(t), x_2(t), \dots, x_n(t)) - g_k(t) \right| < \epsilon_k.$$

According to assumptions of Theorem 2.1, we have

$$(2.12) \quad |x_k(t)| \leq (M_k + L_k) F_k + (\Sigma_k F_k + a_k \Lambda_k) r, \quad k = 1, 2, \dots, n,$$

where  $\epsilon_k > 0$ .

Then, by (2.10) and (2.12), we obtain

$$(2.13) \quad \sup_{t \in J} |x_k(t)| \leq \sup_{t \in J} \left| D^{\beta_k} \left( D^{\alpha_k} + \frac{a_k}{t} \right) x_k(t) \right|.$$

Using (2.13), we get

$$\begin{aligned} & \sup_{t \in J} |x_k(t) - y_k(t)| \\ & \leq \sup_{t \in J} \left| D^{\beta_k} \left( D^{\alpha_k} + \frac{a_k}{t} \right) (x_k(t) - y_k(t)) \right| \\ & \leq \sup_{t \in J} \left| D^{\beta_k} \left( D^{\alpha_k} + \frac{a_k}{t} \right) x_k(t) - g_k(t) + f_k(t, x_1(t), x_2(t), \dots, x_n(t)) \right. \\ & \quad \left. - \left( D^{\beta_k} \left( D^{\alpha_k} + \frac{a_k}{t} \right) y_k(t) - g_k(t) + f_k(t, y_1(t), y_2(t), \dots, y_n(t)) \right) \right. \\ & \quad \left. - f_k(t, x_1(t), x_2(t), \dots, x_n(t)) + f_k(t, y_1(t), y_2(t), \dots, y_n(t)) \right| \end{aligned}$$

Using (2.8) and (2.11), we obtain



$$\begin{aligned} \sup_{t \in J} |x_k(t) - y_k(t)| &\leq \epsilon_k + (\mu_k)_1 |x_1(t) - y_1(t)| + \dots + (\mu_k)_n |x_n(t) - y_n(t)| \\ &\leq \epsilon_k + \sum_{j=1}^n (\mu_k)_j \max_{1 \leq j \leq n} |x_j(t) - y_j(t)| \\ &\leq \epsilon_k + \sum_{j=1}^n (\mu_k)_j \max_{1 \leq k \leq n} |x_k(t) - y_k(t)|. \end{aligned}$$

Hence,

$$\max_{1 \leq k \leq n} \|x_k - y_k\|_\infty \leq \max_{1 \leq k \leq n} \omega_k \epsilon_k, \quad \omega_k := \frac{1}{1 - \Sigma_k}.$$

Therefore,

$$\|(x_1 - y_1, x_2 - y_2, \dots, x_n - y_n)\|_S \leq \max_{1 \leq k \leq n} \omega_k \epsilon_k := K\epsilon, \quad \epsilon = \max_{1 \leq k \leq n} \epsilon_k.$$

Since  $\Sigma_k < 1$ , we get  $\omega_k > 0$ , that is  $K$  is a positive constant.

From (1.4), we see that

$$\begin{aligned} \sum_{k=1}^n |y_k(0)| &= \sum_{k=1}^n |y'_k(0)| = \dots = \sum_{k=1}^n |y_k^{(l-1)}(0)| = \sum_{k=1}^n |D^{\alpha_k} y_k(0)| \\ &= \sum_{k=1}^n |D^{\alpha_{k+1}} y_k(0)| = \dots = \sum_{k=1}^n |D^{\alpha_{k+l-2}} y_k(0)| = D^{\alpha_{k+l-1}} y_k(1) = 0. \end{aligned}$$

Then,

$$\|(x_1 - y_1, x_2 - y_2, \dots, x_n - y_n)\|_S \leq K\epsilon.$$

Therefore, the Lane-Emden fractional system (1.1) is Ulam-Hyers stable. □

*Remark 2.2.* Taking  $\varphi(\epsilon) = K\epsilon$ , we conclude that the problem (1.1) is generalized Ulam-Hyers stable.

**Theorem 2.4.** *Assume that:*

- (i) *The assumptions and the condition (2.2) of Theorem 2.1 are satisfied.*
- (ii) *There exists a function  $\Psi \in C([0, 1], \mathbb{R}_+)$  that satisfies (2.9).*

*Then, the fractional system (1.1) is Ulam-Hyers-Rassias stable in  $S$ .*

*Proof.* Let  $(x_1, x_2, \dots, x_n) \in S$  be a solution of (1.1). In virtue of Remark 2.1 and by some easy calculations, we get:

$$\max_{1 \leq k \leq n} \|x_k - y_k\| \leq \max_{1 \leq k \leq n} \frac{\epsilon_k \Psi(t)}{1 - \lambda_k} := \max_{1 \leq k \leq n} \rho_k \epsilon_k \Psi(t).$$

Thus,

$$\|(x_1 - y_1, x_2 - y_2, \dots, x_n - y_n)\|_S = \max_{1 \leq k \leq n} \|x_k - y_k\| \leq \rho \epsilon \Psi(t),$$

where,

$$\rho = \max_{1 \leq k \leq n} \rho_k, \quad \epsilon = \max_{1 \leq k \leq n} \epsilon_k.$$

The Condition (2.2) implies that  $\rho_k = 1/(1 - \lambda_k) > 0, k = 1, 2, \dots, n$ . Hence, (1.1) is Ulam-Hyers-Rassias stable. □

### 3. APPLICATIONS

In this section, we present some examples to illustrate some applications of the main results.

*Example 3.1.* Consider the following system:

$$(3.1) \left\{ \begin{array}{l}
 D^{\frac{9}{4}} \left( D^{\frac{5}{2}} + \frac{5}{t} \right) x_1(t) \\
 + \frac{|x_1(t) + x_2(t) + x_3(t) + x_4(t)|}{9\pi^2 e^t (1 + |x_1(t) + x_2(t) + x_3(t) + x_4(t)|)} = \frac{t}{2}, \quad t \in [0, 1], \\
 D^{\frac{12}{5}} \left( D^{\frac{8}{3}} + \frac{10.5}{t} \right) x_2(t) + \frac{1}{32\pi(t+1)} \\
 \times \left( \frac{\sin(x_1(t)) + \sin(x_2(t)) + \cos(x_3(t))}{2e^{t+1}} + \sin(x_4(t)) \right) = t, \quad t \in [0, 1], \\
 D^{\frac{5}{2}} \left( D^{\frac{11}{4}} + \frac{15.2}{t} \right) x_3(t) + \frac{1}{t^2 + 1} \\
 \times \frac{\cos(x_1(t)) + \cos(x_2(t)) + \cos(x_3(t)) + \cos(x_4(t))}{4\pi e^2} = \frac{3t}{2}, \quad t \in [0, 1], \\
 D^{\frac{7}{3}} \left( D^{\frac{11}{5}} + \frac{3 \times 10^8}{t} \right) x_4(t) + \frac{t^2}{4\pi^2 e^{t^2+1}} \\
 \times \left( \sin x_1(t) + \frac{|x_2(t) + x_3(t) + x_4(t)|}{2\pi^3 (1 + |x_2(t) + x_3(t) + x_4(t)|)} \right) = \frac{t}{4}, \quad t \in [0, 1], \\
 |x_1(0)| + |x_2(0)| + |x_3(0)| + |x_4(0)| = 0, \\
 |x'_1(0)| + |x'_2(0)| + |x'_3(0)| = |x'_4(0)| = 0, \\
 |x''_1(0)| + |x''_2(0)| + |x''_3(0)| + |x''_4(0)| = 0, \\
 \left| D^{\frac{5}{2}} x_1(0) \right| + \left| D^{\frac{8}{3}} x_2(0) \right| + \left| D^{\frac{11}{4}} x_3(0) \right| + \left| D^{\frac{11}{5}} x_4(0) \right| = 0, \\
 \left| D^{\frac{7}{2}} x_1(0) \right| + \left| D^{\frac{11}{3}} x_2(0) \right| + \left| D^{\frac{15}{4}} x_3(0) \right| + \left| D^{\frac{16}{5}} x_4(0) \right| = 0, \\
 D^{\frac{9}{2}} x_1(1) = D^{\frac{14}{3}} x_2(1) = D^{\frac{19}{4}} x_3(1) = D^{\frac{21}{5}} x_4(1) = 0,
 \end{array} \right.$$

In this example, we have:  $n = 4, l = 3, \beta_1 = 9/4, \beta_2 = 12/5, \beta_3 = 5/2, \beta_4 = 7/3, \alpha_1 = 5/2, \alpha_2 = 8/3, \alpha_3 = 11/4, \alpha_4 = 11/5, a_1 = 5, a_2 = 10.5, a_3 = 15.2, a_4 = 3 \times 10^8, J = [0, 1]$ .

It is clear that, for all  $t \in [0, 1]$  and  $(x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4) \in \mathbb{R}^4$ , we have:

$$\begin{aligned}
& |f_1(t, x_1, x_2, x_3, x_4) - f_1(t, y_1, y_2, y_3, y_4)| \\
& \leq \frac{1}{9\pi^2} |x_1 - y_1| + \frac{1}{9\pi^2} |x_2 - y_2| + \frac{1}{9\pi^2} |x_3 - y_3| + \frac{1}{9\pi^2} |x_4 - y_4|, \\
& |f_2(t, x_1, x_2, x_3, x_4) - f_2(t, y_1, y_2, y_3, y_4)| \\
& \leq \frac{1}{64\pi e} |x_1 - y_1| + \frac{1}{64\pi e} |x_2 - y_2| + \frac{1}{64\pi e} |x_3 - y_3| + \frac{1}{32\pi} |x_4 - y_4|, \\
& |f_3(t, x_1, x_2, x_3, x_4) - f_3(t, y_1, y_2, y_3, y_4)| \\
& \leq \frac{1}{4\pi e^2} |x_1 - y_1| + \frac{1}{4\pi e^2} |x_2 - y_2| + \frac{1}{4\pi e^2} |x_3 - y_3| + \frac{1}{4\pi e^2} |x_4 - y_4|, \\
& |f_4(t, x_1, x_2, x_3, x_4) - f_4(t, y_1, y_2, y_3, y_4)| \\
& \leq \frac{1}{4\pi^2 e} |x_1 - y_1| + \frac{1}{8\pi^5 e} |x_2 - y_2| + \frac{1}{8\pi^5 e} |x_3 - y_3| + \frac{1}{8\pi^5 e} |x_4 - y_4|,
\end{aligned}$$

and

$$\begin{aligned}
\sup_{t \in J} |g_1(t)| &= \frac{1}{2}, & \sup_{t \in J} |g_2(t)| &= 1, & \sup_{t \in J} |g_3(t)| &= \frac{3}{2}, & \sup_{t \in J} |g_4(t)| &= \frac{1}{4}, \\
\sup_{t \in J} |f_1(t, x_1, x_2, x_3, x_4)| &= \frac{1}{9\pi^2}, & \sup_{t \in J} |f_2(t, x_1, x_2, x_3, x_4)| &= \frac{2e+1}{64\pi e}, \\
\sup_{t \in J} |f_3(t, x_1, x_2, x_3, x_4)| &= \frac{1}{4\pi e^2}, & \sup_{t \in J} |f_4(t, x_1, x_2, x_3, x_4)| &= \frac{2\pi^3+1}{8\pi^5 e}.
\end{aligned}$$

We can take

$$\begin{aligned}
(\mu_1)_1 &= (\mu_1)_2 = (\mu_1)_3 = (\mu_1)_4 = \frac{1}{9\pi^2}, \\
(\mu_2)_1 &= (\mu_2)_2 = (\mu_2)_3 = \frac{1}{64\pi e}, \\
(\mu_2)_4 &= \frac{1}{32\pi}, \\
(\mu_3)_1 &= (\mu_3)_2 = (\mu_3)_3 = (\mu_3)_4 = \frac{1}{4\pi e^2}, \\
(\mu_4)_1 &= \frac{1}{4\pi^2 e} = 2(\mu_4)_i,
\end{aligned}$$

where  $i = 2, 3, 4$ . In addition

$$\begin{aligned}
\Sigma_1 &= 0.045032, & \Sigma_2 &= 0.015436, & \Sigma_3 &= 0.043079, & \Sigma_4 &= 0.009769, \\
F_1 &= 0.012936, & F_2 &= 0.007538, & F_3 &= 0.005478, & F_4 &= 0.018669, \\
\Lambda_1 &= 0.016461, & \Lambda_2 &= 0.011025, & \Lambda_3 &= 0.008978, & \Lambda_4 &= 2.03516 \times 10^{-12}.
\end{aligned}$$

Furthermore, we have:

$$\lambda_1 = 0.082888 < 1, \quad \lambda_2 = 0.115879 < 1, \quad \lambda_3 = 0.136702 < 1, \quad \lambda_4 = 0.000793 < 1.$$

Then, by Theorem 2.1, the fractional coupled system (3.1) has a unique solution on  $[0, 1]$ .

To illustrate the second main result, we consider the following example:

*Example 3.2.*

$$(3.2) \quad \left\{ \begin{array}{l} D^{\frac{10}{3}} \left( D^{\frac{7}{2}} + \frac{10^2}{t} \right) x_1(t) + \frac{|x_1(t) + x_2(t)|}{e^{6(t+1)} (1 + |x_1(t) + x_2(t)|)} = \frac{e^t}{2} + t, \quad t \in [0, 1], \\ D^{\frac{22}{7}} \left( D^{\frac{15}{4}} + \frac{5 \times 10^3}{t} \right) x_2(t) \\ \quad + \frac{1}{12\pi^2 (t^2 + 1)} (\cos(x_1(t)) + \cos(x_2(t))) = t^2, \quad t \in [0, 1], \\ |x_1(0)| + |x_2(0)| = |x_1'(0)| + |x_2'(0)| = 0, \\ |x_1''(0)| + |x_2''(0)| = |x_1'''(0)| + |x_2'''(0)| = 0, \\ \left| D^{\frac{7}{2}} x_1(0) \right| + \left| D^{\frac{15}{4}} x_2(0) \right| = \left| D^{\frac{9}{2}} x_1(0) \right| + \left| D^{\frac{19}{4}} x_2(0) \right| = 0, \\ \left| D^{\frac{11}{2}} x_1(0) \right| + \left| D^{\frac{23}{4}} x_2(0) \right| = 0, \\ D^{\frac{13}{2}} x_1(1) = D^{\frac{27}{4}} x_2(1) = 0. \end{array} \right.$$

For this example, we have:  $n = 2, l = 4, \beta_1 = 10/3, \beta_2 = 22/7, \alpha_1 = 7/2, \alpha_2 = 15/4, a_1 = 10^2, a_2 = 5 \times 10^3, J = [0, 1]$ .

We see that, for all  $t \in [0, 1]$  and  $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$ , we get:

$$\begin{aligned} |f_1(t, x_1, x_2) - f_1(t, y_1, y_2)| &\leq \frac{1}{e^6} |x_1 - y_1| + \frac{1}{e^6} |x_2 - y_2|, \\ |f_2(t, x_1, x_2) - f_2(t, y_1, y_2)| &\leq \frac{1}{12\pi^2} |x_1 - y_1| + \frac{1}{12\pi^2} |x_2 - y_2|, \end{aligned}$$

and

$$\begin{aligned} \sup_{t \in J} |g_1(t)| &= \frac{e + 2}{2}, & \sup_{t \in J} |g_2(t)| &= 1, \\ \sup_{t \in J} |f_1(t)| &= \frac{1}{e^6}, & \sup_{t \in J} |f_2(t)| &= \frac{1}{12\pi^2}. \end{aligned}$$

We can take:

$$(\mu_1)_1 = (\mu_1)_2 = \frac{1}{e^6}, \quad (\mu_2)_1 = (\mu_2)_2 = \frac{1}{12\pi^2}.$$

In addition:

$$\Sigma_1 = 0.004958, \quad \Sigma_2 = 0.016887.$$

Moreover, we obtain:

$$C_1 = 0.000042 < 1, \quad C_2 = 0.000654 < 1.$$

The functions  $f_k : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$  and  $g_k : [0, 1] \rightarrow \mathbb{R}$  are also continuous. So, by Theorem 2.2, the system (3.2) has at least one solution on  $[0, 1]$ .

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