Electronic Journal of Differential Equations , Vol . 2007 (2007) , No . 1 58 , pp . 1 – 7 .
ISSN : 1 72 - 6691 . URL : http : / / ejde . math . txstate . edu or http : / / ej de . math . unt . edu
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SOLUTION TO NONLINEAR GRADIENT DEPENDENT SYSTEMS WITH A BALANCE LAW

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Abstract . In this paper , we are concerned with the initial boundary value problem (IBVP) and with the Cauchy problem to the reaction - diffusion system

$$u_t - \Delta u = -u^n \mid \nabla v \mid^p,$$
$$v_t - d\Delta v = u^n \mid \nabla v \mid^p,$$

where $1 \leq p \leq 2, d$ and n are positive real numbers . Results on the existence and large - t ime behavior of the solutions are presented .

1 . Introduction

In the first part of this article , we are interested in the existence of global classical nonnegative solutions to the reaction - diffusion equations

$$u_t - \Delta_{v_t} u =_- -u^n d^{|\nabla v|^p}_{\Delta v = u^n |\nabla v} =: -f(u, p\, v), \quad | \; , \tag{1.1}$$

posed on $\mathbb{R}^+ \times \Omega$ with initial data

$$u(0;x) = u_0(x), \quad v(0;x) = v_0(x) \quad \text{in}\Omega \tag{1.2}$$

and boundary conditions (in the case Ω is a bounded domain in $\mathbb{R}^n$)

$$\frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0, \quad \text{on}\mathbb{R}^+ \times \partial\Omega. \tag{1.3}$$

Here Δ is the Laplacian operator , u_0 and v_0 are given bounded nonnegative func - tions , $\Omega \subset \mathbb{R}^n$ is a regular domain , η is the outward normal to $\partial\Omega$. The diffusive coefficient d is a positive real . One of the basic questions for (1 . 1) - (1 . 2) or (1 . 1) -
(1 . 3) is the existence of global solutions . Motivated by extending known results on reaction - diffusion systems with conservation of the total mass but with non linear - ities depending only for the unknowns , Boudiba , Mouley and Pierre succeeded in obtaining L^1 solutions only for the case $u^n \mid \nabla v \mid^p$ with $p < 2$. In this article , we are interested essentially in classical solutions in the case where $p = 2(\Omega$ bounded or $\Omega = \mathbb{R}^n$; in the latter case , there are no boundary conditions) .

2000 *Mathematics Subject Classification* . 35 B 40 , 35 B 50 , 35 K 57 .

Key words and phrases . Reaction - diffusion systems ; global existence ; asymptotic behavior ; maximum principle .

Submitted April 1 5 , 2007 . Published November 2 1 , 2007 .

2 . Results

The existence of a unique classical solution over the whole time interval $[0, T_{\max}[$ can be obtained by a known procedure : a local solution is continued globally by using a priori estimates on $\| u \|_\infty, \| v \|_\infty, \||\nabla u \||_\infty$, and $\||\nabla v\||_\infty$. 2 . 1 . **The Cauchy problem .** *Uniform bounds for u and v.* First , we consider the auxiliary problem

$$L_\lambda \omega := \omega^{-\ \lambda\Delta\omega}_{t_\omega(0\ ,\ x)=} = b\nabla\omega\omega^{,}_{0(x)} \in tL^{>\ 0,}_{\infty,} x \in \mathbb{R}^N \tag{2.1}$$

where $b = (b_1(t,x), ..., b_N(t,x)), b_i(t,x)$ are continuous on $[0,\infty) \times \mathbb{R}^N$, ω is a classical solution of (2 . 1) . **Lemma 2 . 1 .** *Assume that $\omega_t, \nabla\omega, \omega_{x_i x_i}, i = 1, ..., N$ are continuous ,*

$$L_\lambda \omega \le 0, \quad (\ge) \quad (0,\infty) \times \mathbb{R}^N \tag{2.2}$$

and $\omega(t,x)$ satisfies (2 . 1) 2 . Then

$$\omega(t,x) \le C := \sup_{x \in \mathbb{R}^N} \omega_0(x), \quad (0,\infty) \times \mathbb{R}^N.$$

$$\omega(t,x) \ge C := \inf_{x \in \mathbb{R}^N} \omega_0(x), \quad (0,\infty) \times \mathbb{R}^N.$$

The proof of the above lemma is elementary and hence is omitted . Now , we consider the problem (1 . 1) - (1 . 2) . It follows by the maximun principle that

$$u, v \ge 0, \quad \text{in} \mathbb{R}^+ \times \mathbb{R}^N.$$

Uniform bounds of u. We have

$$u \le C_1 := \sup_{\mathbb{R}^N} u_0(x),$$

thanks to the maximum principle . Uniform bounds of v. Next , we derive an upper estimate for v. Assume that $1 \le p < 2$. We transform (1 . 1) 2 by the substitution $\omega = e^{\lambda v} - 1$ into

$$\omega_t - \lambda\Delta\omega = \lambda e^{\lambda v}(v_t - d\Delta v - d\lambda \mid \nabla v \mid^2) = \lambda e^{\lambda v}(u^n \mid \nabla v \mid^p - d\lambda \mid \nabla v \mid^2).$$

Let

$$\phi(x) \equiv Cx^p - d\lambda x^2; \quad C > 0, x \ge 0.$$

By elementary computations ,

$$\phi(x) \ge 0 \quad \text{when} x \le (\frac{C}{\lambda d})^{1/(2-p)}.$$

But in this case

$$\mid \nabla v \mid \le (\frac{c}{\lambda d})^{1/(2-p)}.$$

In the case $x \ge (\frac{c}{\lambda d})^{1/(2-p)}$,

$$\phi(x) \leq 0 \tag{2.3}$$

and hence $\omega \leq M$ where

$$M = C(\frac{pC}{2d\lambda})^{p/2-p}(\frac{2-p}{2}). \tag{2.4}$$

Then we have $v \leq C_2$.

 2.1.1. *Uniform bounds for* $|\nabla u|$ *and* $|\nabla v|$. At first, we present the uniform bounds for $|\nabla v|$. We write (1.1)$_2$ in the form

$$L_d v + kv = kv + u^n |\nabla v|^p \tag{2.5}$$

and transform it by the substitutions $\omega = e^{kt} v$ to obtain

$$L_d \omega = e^{kt}(L_d v + kv) = e^{kt}(kv + u^n |\nabla v|^p), \quad t > 0, x \in \mathbb{R}^N$$
$$\omega(0, x) = v_0(x).$$

Now let

$$G_\lambda = G_\lambda(t-\tau; x-\xi) = \frac{1}{[4\pi\lambda(t-\tau)]^{\frac{N}{2}}} \exp(\frac{|x-\xi|^2}{4\lambda(t-\tau)})$$

be the fundamental solution related to the operator L_λ. Then, with $Qt = (0,t)\times\mathbb{R}^N$, we have

$$\omega = e^{kt} v = v^0(t,x) + \int_{Qt} G_d(t-\tau; x-\xi) e^{k\tau}(kv + u^n |\nabla v|^p) d\xi d\tau$$

or

$$v = e^{-kt} v^0 + \int_{Qt} e^{-k(t-\tau)} G_d(t-\tau; x-\xi)(kv + u^n |\nabla v|^p) d\xi d\tau, \tag{2.6}$$

where $v^0(t,x)$ is the solution of the homogeneous problem

$$L_d v^0 = 0, \quad v^0(0,x) = v_0(x).$$

From (2.6) we have

$$\nabla v = e^{-kt} \nabla v^0 + \int_{Qt} e^{-k(t-\tau)} \nabla_x G_d(t-\tau; x-\xi)(kv + u^n |\nabla v|^p) d\xi d\tau. \tag{2.7}$$

Now we set $\nu_1 = \sup |\nabla v|$ and $1_\nu^0 = \sup |\nabla v^0|$, in Qt. From (2.6), and using $v \le C_2$, we have

$$\nu_1 = 1_\nu^0 + (kC_2 + C_1^n \nu_1^p) \int_0^t e^{-k(t-\tau)} (\int_{\mathbb{R}^N} |\nabla_x G_d(t-\tau; x-\xi)| d\xi) d\tau.$$

We also have

$$\int_{\mathbb{R}^N} |\nabla_x G_d(t-\tau; x-\xi)| d\xi = \int_{\mathbb{R}^N} \frac{|x-\xi|}{2d(t-\tau)} |G_d(t-\tau,; x-\xi)| d\xi$$

which is transformed by the substitution $\rho = 2\sqrt{d(t-\tau)\nu}$ into

$$\int_{\mathbb{R}^N} |\nabla_x G_d| d\rho = \frac{w_N}{\pi^{N/2}} \int_0^\infty e^{-\nu^2} d\nu = \frac{\chi}{\sqrt{d(t-\tau)}}$$

where $\chi = line-slash^w_{2\pi} N^N 2\Gamma(\frac{N+1}{2}) = \frac{\Gamma(\frac{N+1}{2})}{\Gamma(\frac{N}{2})}$. It follows that

$$\nu_1 = 1_\nu^0 + (kC_2 + C_1^n \nu_1^p) \frac{\chi}{\sqrt{d}} \int_0^t e^{-k(t-\tau)} \frac{d\tau}{\sqrt{t-\tau}}. \tag{2.8}$$

Recall that

$$\int_0^t e^{-k(t-\tau)} \frac{d\tau}{\sqrt{t-\tau}} = \frac{2}{\sqrt{k}} \int_0^t e^{-z^2} dz < \sqrt{\frac{\pi}{k}}.$$

If we set $s = \sqrt{k}$ in (2 . 8) then we have

$$\nu_1 \leq 1_\nu^0 + (sC_2 + \frac{C_1^n}{s} \nu_1^p) \chi \sqrt{\frac{\pi}{d}}. \tag{2.9}$$

 Now we minimize the right hand side of (2 . 9) with respect to s to obtain

$$\nu_1 \le 1_\nu^0 + \frac{2\chi\sqrt{\pi}}{d}(C_2C_1^n p_{\nu_1})1/p. \quad (2.10)$$

Notethat$1_\nu^0 = C_2$.

We have two cases : Case (i)$1 \le p < 2$. In this case (2 . 1 0) implies

$$| \nabla v | \le \nu_1 \le \_\_\_\_\_\__\nu(p) = D, \quad \text{in} Qt, \tag{2.11}$$

where D is a positive constant .

Case (i i)$p = 2$. In this case (2 . 1 0) holds under the additional condition

$$C_2C_1^n \le \frac{d}{4\pi\chi_.} \tag{2.12}$$

Similarly we obtain from (1 . 1) 1 ,

$$U_1 := \sup_{QT} | \nabla u | \le C_1 + C_1\frac{2\sqrt{\pi}\chi}{\sqrt{d}}1_\nu^{p/2} \le Constant. \tag{2.13}$$

The estimates (2 . 1 0) and (2 . 1 3) are independent of t, hence $T_{\max} = +\infty$.

Finally , we have the main result .

Theorem 2 . 2 . *Let* $p =$ 2 *and* (u_0, v_0) *be bounded such that* (2 . 1 2) *holds , then system* (1 . 1) - (1 . 2) *admits a glo bal s o lution .*

2 . 2 . **The Neumann Problem .** In this section , we are concerned with the Neu - mann problem

$$u_{v_t^t} -_- \Delta_{d\Delta v} u =_= -u^n u^{|\nabla v|^2}_{n|\nabla v|^2} \tag{2.14}$$

where Ω be a bounded domain in $\mathbb{R}^N$, with the homogeneous Neumann boundary condition

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \quad \text{on}\mathbb{R}^+ \times \partial\Omega \tag{2.15}$$

subject to the initial conditions

$$u(0;x) = u_0(x); \quad v(0;x) = v_0(x) \quad \text{in}\Omega. \tag{2.16}$$

The initial nonnegative functions u_0, v_0 are assumed to belong to the Holder space

$$C^{2,\alpha}(\Omega).$$

Uniform bounds for u *and* v. In this section a priori estimates on $\| u \|_\infty$ and $\| v \|_\infty$ are presented . **Lemma 2 . 3 .** *For each* $0 < t < T_{\max}$ *we have*

$$0 \le u(t,x) \le M, \quad 0 \le v(t,x) \le M, \quad (2.17)$$

$$for any x \in \Omega.$$

Proof . Since $u_0(x) \geq 0$ and $f(0, v) = 0$, we first obtain $u \geq 0$ and then $v \geq 0$ as $v_0(x) \geq 0$. Using the maximum principle , we conclude that

$$0 \leq u(t, x) \leq M, \quad \text{on} QT$$

where

$$M \geq M_1 := \max_{x \in \Omega} u_0(x).$$

 Using $\omega = e^{\lambda v} - 1$, with $d\lambda \geq M_1^n$, from (2 . 1 4) , we obtain

$$\omega_t - d\Delta\omega = \lambda \mid \nabla v \mid^2 (u^n - d\lambda)e^{\lambda v}, \quad \text{on} QT$$

$$\frac{\partial u}{\partial v} = 0 \quad \text{on} \partial S_T.$$

Consequently as $d\lambda > \max_\Omega u^n$, we deduce from the maximum principle that

$$0 \leq \omega(t,x) \leq \exp(\lambda \mid v_0 \mid_\infty) - 1.$$

Hence

$$v(x,t) \leq \frac{1}{\lambda} \ln(\mid \omega \mid_\infty +1) \leq Constant < \infty.$$

□

Uniform bounds for $\mid \nabla v \mid$ *and* $\mid \nabla u \mid$ *.* To obtain uniform a priori estimates for $\mid \nabla v \mid$, we make use of some techniques already used by Tomi [8] and von Wahl [9]
Lemma 2 . 4 . *Let* (u,v) *be a s o lutio n to (2 . 1 0) - (2 . 1 2) in its maximal interval of existence* $[0, T_{\max}[$*. Then there exist a constant* C *such that*

$$\| u \| L\infty([0,T[, W^2, q(\Omega)) \leq C \quad and \quad \| v \| L\infty([0,T[, W^2, q(\Omega)) \leq C.$$

Proof . Let us introduce the function

$$f_{\sigma,\epsilon}(t,x,u,\nabla v) = \sigma u^n(t,x) \frac{\epsilon + \mid \nabla v \mid^2}{1 + \epsilon \mid \nabla v \mid^2}.$$

It is clear that $\mid f_{\sigma,\epsilon}(t,x,u,\nabla v) \mid \leq C(1 + \mid \nabla v \mid^2)$ and a global solution $v_{\sigma,\epsilon}$ differen - tiable in σ for the equation

$$v_t - d\Delta v = f_{\sigma,\epsilon}(t,x,u,\nabla v)$$

exists . Moreover , $v_{\sigma,\epsilon} \to v$ as $\sigma \to 1$ and $\epsilon \to 0$, uniformly on every compact of

$$[0, T_{\max}[.$$

The function $\omega_\sigma := \frac{\partial v_{\sigma,\epsilon}}{\partial \sigma}$ satisfies

$$\partial_t \omega_\sigma - d\Delta\omega_\sigma = u^n(t,x) \frac{\epsilon + \mid \nabla v_\sigma \mid^2}{1 + \epsilon \mid \nabla v\sigma \mid^2} - 2\sigma u^{n(\epsilon} \frac{2 \quad -1)\nabla v_\sigma . \nabla\omega_\sigma}{(1+\epsilon|\nabla v_\sigma|^2)^2}. \tag{2.18}$$

Hereafter , we derive uniform estimates in σ and ϵ. Using Solonnikov ' s estimates for parabolic equation [5] we have

$$\| \omega_\sigma \| L\infty([0, T(u_0,v_0)[, W^2, p(\Omega)) \leq C[\| \nabla v_\sigma \|^2_{Lp(\Omega)} + \| \nabla v_\sigma . \nabla\omega_\sigma \|^2_{Lp(\Omega)}].$$

The Gagliardo - Nirenberg inequality [5] in the in the form

$$\| u \| W^1, 2p(\Omega) \leq C \| u \|^{1/2}_{L\infty(\Omega)} C \| u \|^{1/2}_{W^2,p(\Omega)}$$

and the $\delta-$ Young inequality (where $\delta > 0$)

$$\alpha\beta \leq \frac{1}{2}(\delta\alpha^2 + \frac{\beta^2}{\delta}),$$

allows one to obtain the estimate

$$\parallel \omega_\sigma \parallel L\infty([0, T(u_0, v_0)[, W^2, p(\Omega)) \leq C(1+ \parallel \omega_\sigma \parallel W^2, p(\Omega)).$$

But $\omega_\sigma = \frac{\partial v_\sigma}{\partial \sigma}$, hence by Gronwall ' s inequality we have

$$\parallel v_\sigma \parallel L\infty([0, T[, W^2, p(\Omega)) \leq Ce^{C\sigma}.$$

 Letting $\sigma \to 1$ and $\epsilon \to 0$, we obtain

$$\| v \| L\infty([0,T[,W^2,p(\Omega)) \leq C.$$

On the other hand , the Sobolev inj ection theorem allows to assert that $u \in C^{1,\alpha}(\Omega)$. Hence in particular $| \nabla u | \in C^{0,\alpha}(\Omega)$. Since $| \nabla v |$ is uniformly bounded , it is easy then to bound $| \nabla u |$ in $L^\infty(\Omega)$. As a consequence , one can affirm that the solution (u,v) to problem (2 . 1 4) - (2 . 1 6) is global ; that is $T_{\max} = \infty$. □

2 . 3 . **Larg e – hyphen t ime behavior .** In this section , the large time behavior of the global solutions to (2 . 1 4) - (2 . 1 6) is briefly presented .

Theorem 2 . 5 . *Let* $(u_0, v_0) \in C^{2,\epsilon}(\Omega) \times C^{2,\epsilon}(\Omega)$ *for s ome* $0 < \epsilon < 1$. *The system (2 . 14) - (2 . 1 6) has a glo bal classical s o lution . Moreover , as* $t \to \infty$, $u \to k_1$ *and* $v \to k_2$ *uniformly in* x, *and*

$$k_1 + k_2 = \frac{1}{| \Omega |} \int_\Omega [u_0(x) + v_0(x)]dx.$$

Proof . The proof of the first part of the Theorem is presented above . Concerning the large time behavior , observe first that for any $t \geq 0$,

$$\int_\Omega [u(t,x) + v(t,x)]dx = \int_\Omega [u_0(x) + v_0(x)]dx.$$

Then , the function $t \to \int_\Omega u(x)dx$ is bounded ; as it is decreasing , we have

$$\int_\Omega u(x)dx \to k_1 \quad \text{ast} \to \infty;$$

the function $t \to \int_\Omega v(x)dx$ is increasing and bounded , hence admits a finite limit k_2 as $t \to \infty$. As $\bigcup_{t\geq 0\{(} u(t), v(t))\}$ is relatively compact in $C(\overline{\quad\quad}\Omega) \times C(\overline{\quad\quad}\Omega)$,

$$u(\tau_n) \to \widetilde{u}_, \quad v(\tau_n) \to \widetilde{v} \quad \text{in} C(\overline{\quad\quad}\Omega),$$

through a sequence $\tau_n \to \infty$. It is not difficult to show that in fact ($\widetilde{u}_,$parenright − tildewide − v is the stationary solution to (2 . 1 4) - (2 . 1 6) (see [3]) .

As the stationary solution (u_s, v_s) to (2 . 1 4) - (2 . 1 6) satisfies

$$-\Delta u_s = -u_s^n \mid \nabla v_s \mid^2, \quad \text{in}\Omega,$$
$$-d\Delta v_s = u_s^n \mid \nabla v_s \mid^2, \quad \text{in}\Omega, \frac{\partial u_s}{\partial \nu} = \frac{\partial v_s}{\partial \nu} = 0, \quad \text{on}\partial\Omega,$$

we have

$$-\int_\Omega \Delta u_s . u_s dx = -\int_\Omega u_s^{n+1} \mid \nabla v_s \mid^2 dx$$

which in the light of the Green formula can be written

$$\int_\Omega \mid \nabla u_s \mid^2 dx = -\int_\Omega u_s^{n+1} \mid \nabla v_s \mid^2 dx$$

hence $| \nabla u_s |=| \nabla v_s |= 0$ implies $u_s = k_1$ and $v_s = k_2$. □

Remarks . (1) It is very interesting to address the question of existence global solutions of the system (2 . 1 4) - (2 . 1 6) with a genuine nonlinearity of the form $u^n \mid \nabla v \mid^p$ with $p \geq 2$.

(2) It is possible to extend the results presented here for systems with nonlinear boundary conditions satisfying reasonable growth restrictions .

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