Theory and Applications of
Mathematics & Computer Science
(ISSN 2067-2764, EISSN 2247-6202)
http://www.uav.ro/applications/se/journal/index.php/tamcs

Theory and Applications of Mathematics & Computer Science 3 (2) (2013) 114–119

New Fractional Integral Results Using Euler Functions

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Abstract

In this paper, we use the the Riemann-Liouville fractional integral to develop some new results related to the Hermite-Hadamard inequality. Our results have some relationships with the paper of M.Z. Sarikaya et al. published in [Int. J. Open Problems Comput. Math., Vol. 5, No. 3, September, 2012]. Some interested inequalities of this paper can be deduced as some special cases.

Keywords: Integral inequalities, fractional integral, log convexity, Euler functions.
2010 MSC: 26D10, 26A33.

1. Introduction

Let us consider the famous Hermite-Hadamard inequality (Hadamard, 1893; Hermite, 1883) :

$$\frac{f(a+b)}{2} \leq \frac{2}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

where f is a convex function on $[a,b]$.
Many researchers have given considerable attention to (1.1) and a number of extensions and generalizations have appeared in the literature, see (Belaidi *et al.*, 2009; Dahmani, 2010; Dragomir & Pearse, 2000; Florea & Niculescu, 2007; Set *et al.*, 2010; Sarikaya *et al.*, 2012).
The aim of this paper is to present new extensions for a Hermite-Hadamard type inequality involving log-convex functions and using Euler Functions. Our results have some relationships with the work of M.Z. Sarikaya et al. (Sarikaya *et al.*, 2012). Some interested results of this reference can be deduced as particular cases.

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2. Preliminaries

We shall introduce the following definitions and properties which are used throughout this paper.

Definition 2.1. The Riemann-Liouville fractional integral operator of order $\alpha > 0$, for a continuous function on $[a, b]$ is defined by:

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau; \quad \alpha > 0, a \leq t \leq b, \tag{2.1}$$

where $\Gamma(\alpha) := \int_0^{\infty} e^{-u} u^{\alpha-1} du.$

We give the semigroup property:

$$J^{\alpha} J^{\beta} f(t) = J^{\alpha+\beta} f(t), \alpha > 0, \beta > 0. \tag{2.2}$$

For more details, one can consult (Gorenflo & Mainardi, 1997).

3. Main Results

Theorem 3.1. *Let f and g be two differentiable positive log-convex functions on I^0 (the interior of the interval I and $a, b \in I^0$.) Then, for $\alpha > 0$, the following inequalities hold.*

$$\begin{aligned} & 2\Gamma^{-2}(\alpha)\Gamma(2\alpha-1)(b-a)J^{2\alpha-1} fg(b) \\ & \geq J^{\alpha}\Big[g(t) exp\Big(bA_b\Big)\Big] exp\Big[\tfrac{-J^{\alpha-1} bf(b) + J^{\alpha} bf'(b)}{J^{\alpha} f(b)}\Big] J^{\alpha} f(b) \\ & + J^{\alpha}\Big[f(b) exp\Big(bD_b\Big)\Big] exp\Big[\tfrac{-J^{\alpha-1} g(b) + J^{\alpha} bg'(b)}{J^{\alpha} g(b)}\Big] J^{\alpha} g(b), \end{aligned} \tag{3.1}$$

where $A_b := \frac{-J^{\alpha-1} f(b) + J^{\alpha} f'(b)}{J^{\alpha} f(b)}, D_b := \frac{-J^{\alpha-1} g(b) + J^{\alpha} g'(b)}{J^{\alpha} g(b)}.$

Proof. Let us consider:
$K(x) := \frac{(t-x)^{\alpha-1}}{\Gamma(\alpha)} f(x), x \in [a, t], a < t \leq b, \alpha > 0.$
We remark immediately that, if $\alpha = 1$, then $K(x) = f(x)$ and hence, we can obtain the first main result of (Sarikaya *et al.*, 2012).
Now, let us take $\alpha \neq 1$. We can write

$$logK(x) - logK(y) \geq \frac{d}{dy}(logK(y))(x-y), x, y \in [a, t]. \tag{3.2}$$

Therefore,

$$log\frac{K(x)}{K(y)} \geq \frac{K'(y)}{K(y)}(x-y). \tag{3.3}$$

Hence,

$$\frac{K(x)}{K(y)} \geq exp\Big(\frac{(1-\alpha)(t-y)^{\alpha-2} f(y) + (t-y)^{\alpha-1} f'(y)}{(t-y)^{\alpha-1} f(y)}(x-y)\Big). \tag{3.4}$$

Consequently,

$$\frac{(t-x)^{\alpha-1}f(x)g(x)}{\Gamma(\alpha)} \geq \frac{(t-y)^{\alpha-1}f(y)g(x)}{\Gamma(\alpha)}exp\Big(\frac{(1-\alpha)(t-y)^{\alpha-2}f(y)+(t-y)^{\alpha-1}f'(y)}{(t-y)^{\alpha-1}f(y)}(x-y)\Big). \quad (3.5)$$

Integrating the above inequality with respect to y over $[a,t], a < t \leq b$, yields

$$\frac{(t-a)(t-x)^{\alpha-1}f(x)g(x)}{\Gamma(\alpha)} \geq g(x)\int_a^t \frac{(t-y)^{\alpha-1}f(y)}{\Gamma(\alpha)}exp\Big[\frac{(1-\alpha)(t-y)^{\alpha-2}f(y)+(t-y)^{\alpha-1}f'(y)}{(t-y)^{\alpha-1}f(y)}(x-y)\Big]dy. \quad (3.6)$$

For the right hand side of (3.6) we use Jensen inequality. We obtain

$$\int_a^t \frac{(t-y)^{\alpha-1}f(y)}{\Gamma(\alpha)}exp\Big(\frac{(1-\alpha)(t-y)^{\alpha-2}f(y)+(t-y)^{\alpha-1}f'(y)}{(t-y)^{\alpha-1}f(y)}(x-y)\Big)dy \geq \Big(\int_a^t \frac{(t-y)^{\alpha-1}f(y)}{\Gamma(\alpha)}dy\Big)exp\Big[\frac{\int_a^t \frac{(1-\alpha)(t-y)^{\alpha-2}f(y)+(t-y)^{\alpha-1}f'(y)}{\Gamma(\alpha)}(x-y)dy}{\Big(\int_a^t \frac{(t-y)^{\alpha-1}f(y)}{\Gamma(\alpha)}dy\Big)}\Big]. \quad (3.7)$$

Consequently,

$$\int_a^t \frac{(t-y)^{\alpha-1}f(y)}{\Gamma(\alpha)}exp\Big(\frac{(1-\alpha)(t-y)^{\alpha-2}f(y)+(t-y)^{\alpha-1}f'(y)}{(t-y)^{\alpha-1}f(y)}(x-y)\Big)dy \geq exp\Big[\frac{-J^{\alpha-1}(x-t)f(t)+J^{\alpha}(x-t)f'(t)}{J^{\alpha}f(t)}\Big]J^{\alpha}f(t). \quad (3.8)$$

That is

$$\int_a^t \frac{(t-y)^{\alpha-1}f(y)}{\Gamma(\alpha)}exp\Big(\frac{(1-\alpha)(t-y)^{\alpha-2}f(y)+(t-y)^{\alpha-1}f'(y)}{(t-y)^{\alpha-1}f(y)}(x-y)\Big)dy \geq exp\Big[\frac{J^{\alpha-1}tf(t)-J^{\alpha}tf'(t)}{J^{\alpha}f(t)}\Big]exp\Big[\frac{-J^{\alpha-1}f(t)+J^{\alpha}f'(t)}{J^{\alpha}f(t)}x\Big]J^{\alpha}f(t). \quad (3.9)$$

Thanks to (3.6) and (3.9), we obtain

$$\frac{(t-a)(t-x)^{\alpha-1}f(x)g(x)}{\Gamma(\alpha)} \geq g(x)exp\Big[\frac{J^{\alpha-1}tf(t)-J^{\alpha}tf'(t)}{J^{\alpha}f(t)}\Big]exp\Big[\frac{-J^{\alpha-1}f(t)+J^{\alpha}f'(t)}{J^{\alpha}f(t)}x\Big]J^{\alpha}f(t). \quad (3.10)$$

Then,

$$\Gamma^{-2}(\alpha)\Gamma(2\alpha-1)(t-a)J^{2\alpha-1}fg(t) \geq J^{\alpha}\Big[g(t)exp\Big(tA_t\Big)\Big]exp\Big[\frac{-J^{\alpha-1}tf(t)+J^{\alpha}tf'(t)}{J^{\alpha}f(t)}\Big]J^{\alpha}f(t), \quad (3.11)$$

where
$$A_t := \frac{-J^{\alpha-1}f(t) + J^{\alpha}f'(t)}{J^{\alpha}f(t)}.$$
With the same arguments, we obtain:
$$\Gamma^{-2}(\alpha)\Gamma(2\alpha-1)(t-a)J^{2\alpha-1}fg(t) \geq J^{\alpha}\Big[f(t)exp\Big(tB_t\Big)\Big]exp\Big[\frac{-J^{\alpha-1}tg(t) + J^{\alpha}tg'(t)}{J^{\alpha}g(t)}\Big]J^{\alpha}g(t), \quad (3.12)$$
where
$$D_t := \frac{-J^{\alpha-1}g(t) + J^{\alpha}g'(t)}{J^{\alpha}g(t)}.$$
Adding (3.11) and (3.12), yields
$$2\Gamma^{-2}(\alpha)\Gamma(2\alpha-1)(t-a)J^{2\alpha-1}fg(t) \geq J^{\alpha}\Big[g(t)exp\Big(tA_t\Big)\Big]exp\Big[\frac{-J^{\alpha-1}tf(t) + J^{\alpha}tf'(t)}{J^{\alpha}f(t)}\Big]J^{\alpha}f(t)$$
$$+J^{\alpha}\Big[f(t)exp\Big(tD_t\Big)\Big]exp\Big[\frac{-J^{\alpha-1}tg(t)+J^{\alpha}tg'(t)}{J^{\alpha}g(t)}\Big]J^{\alpha}g(t). \quad (3.13)$$
Taking $t = b$, we obtain the desired inequality (3.1). □

Theorem 3.2. *Let f and g be two differentiable positive log-convex functions on I^0 and $a, b \in I^0$. Then, for $\alpha > 0, \beta > 0, \alpha + \beta \neq 1$, we have:*
$$2\Gamma(2\alpha+2\beta-3)(b-a)\frac{J^{2\alpha+2\beta-3}fg(b)}{\Gamma^2(\alpha)\Gamma^2(\beta)}$$
$$\geq exp[\frac{-J^{\alpha+\beta-2}bf(b) + J^{\alpha+\beta}bf'(b)}{J^{\alpha+\beta-1}f(b)}]\frac{J^{\alpha+\beta-1}(g(b)exp[bE_b])}{(\alpha+\beta-1)B(\alpha,\beta)}\frac{J^{\alpha+\beta-1}f(b)}{(\alpha+\beta-1)B(\alpha,\beta)}$$
$$+exp[\frac{-J^{\alpha+\beta-2}bg(b) + J^{\alpha+\beta}bg'(b)}{J^{\alpha+\beta-1}g(b)}]\frac{J^{\alpha+\beta-1}(f(b)exp[bL_b])}{(\alpha+\beta-1)B(\alpha,\beta)}\frac{J^{\alpha+\beta-1}g(b)}{(\alpha+\beta-1)B(\alpha,\beta)}, \quad (3.14)$$
where
$$E_b := \frac{-J^{\alpha+\beta-2}f(b) + J^{\alpha+\beta-1}f'(b)}{J^{\alpha+\beta-1}f(b)}, L_b := \frac{-J^{\alpha+\beta-2}g(b) + J^{\alpha+\beta-1}g'(b)}{J^{\alpha+\beta-1}g(b)}.$$
Proof. We consider: $K(x) := \frac{(t-x)^{\alpha-1}(t-x)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}f(x), x \in [a,t], a < t \leq b, \alpha > 0, \beta > 0.$

We remark immediately that if $\alpha = 1, \beta = 1$, then we obtain the first main result in (Sarikaya *et al.*, 2012).

To prove Theorem 3.2, we need to take $\alpha + \beta \neq 1$. We have
$$\frac{K(x)}{K(y)} \geq exp\Big(\frac{(2-\alpha-\beta)(t-y)^{\alpha+\beta-3}f(y) + (t-y)^{\alpha+\beta-2}f'(y)}{(t-y)^{\alpha+\beta-2}f(y)}(x-y)\Big). \quad (3.15)$$
Then,
$$\frac{(t-x)^{\alpha-1}(t-x)^{\beta-1}f(x)g(x)}{\Gamma(\alpha)\Gamma(\beta)}$$
$$\geq \frac{(t-y)^{\alpha+\beta-2}f(y)g(x)}{\Gamma(\alpha)\Gamma(\beta)}exp\Big(\frac{(2-\alpha-\beta)(t-y)^{\alpha+\beta-3}f(y) + (t-y)^{\alpha+\beta-2}f'(y)}{(t-y)^{\alpha+\beta-2}f(y)}(x-y)\Big). \quad (3.16)$$

Integrating the above inequality with respect to y over $[a,t], a < t \leq b$, yields

$$\frac{(t-a)(t-x)^{\alpha+\beta-2}f(x)g(x)}{\Gamma(\alpha)\Gamma(\beta)}$$

$$\geq g(x)\int_a^t \frac{(t-y)^{\alpha+\beta-2}f(y)}{\Gamma(\alpha)\Gamma(\beta)} exp\Big[\frac{(2-\alpha-\beta)(t-y)^{\alpha+\beta-3}f(y)+(t-y)^{\alpha+\beta-2}f'(y)}{(t-y)^{\alpha+\beta-2}f(y)}(x-y)\Big]dy. \quad (3.17)$$

Thanks to Jensen inequality, we can write

$$\int_a^t \frac{(t-y)^{\alpha+\beta-2}f(y)}{\Gamma(\alpha)\Gamma(\beta)} exp\Big(\frac{(2-\alpha+\beta)(t-y)^{\alpha+\beta-3}f(y)+(t-y)^{\alpha+\beta-2}f'(y)}{(t-y)^{\alpha+\beta-2}f(y)}(x-y)\Big)dy$$

$$\geq \Big(\int_a^t \frac{(t-y)^{\alpha+\beta-2}f(y)}{\Gamma(\alpha)\Gamma(\beta)}dy\Big) exp\Big[\frac{\int_a^t \frac{(2-\alpha-\beta)(t-y)^{\alpha+\beta-3}f(y)+(t-y)^{\alpha+\beta-2}f'(y)}{\Gamma(\alpha)\Gamma(\beta)}(x-y)dy}{\Big(\int_a^t \frac{(t-y)^{\alpha+\beta-2}f(y)}{\Gamma(\alpha)\Gamma(\beta)}dy\Big)}\Big]. \quad (3.18)$$

By simple calculation, we can state that

$$\int_a^t \frac{(t-y)^{\alpha+\beta-2}f(y)}{\Gamma(\alpha)\Gamma(\beta)} exp\Big(\frac{(2-\alpha-\beta)(t-y)^{\alpha+\beta-3}f(y)+(t-y)^{\alpha+\beta-2}f'(y)}{(t-y)^{\alpha+\beta-2}f(y)}(x-y)\Big)dy$$

$$\geq exp\Big[\frac{-J^{\alpha+\beta-2}tf(t)+J^{\alpha+\beta-1}tf'(t)}{J^{\alpha+\beta-1}f(t)}\Big] exp\Big[\frac{-J^{\alpha+\beta-2}f(t)+J^{\alpha+\beta-1}f'(t)}{J^{\alpha+\beta-1}f(t)}x\Big]\frac{J^{\alpha+\beta-1}f(t)}{(\alpha+\beta-1)B(\alpha,\beta)}, \quad (3.19)$$

where $B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$.

Thanks to (3.17) and (3.19), we obtain

$$\frac{(t-a)(t-x)^{\alpha+\beta-2}f(x)g(x)}{\Gamma(\alpha)\Gamma(\beta)}$$

$$\geq exp\Big[\frac{-J^{\alpha+\beta-2}tf(t)+J^{\alpha+\beta-1}tf'(t)}{J^{\alpha+\beta-1}f(t)}\Big] exp\Big[\frac{-J^{\alpha+\beta-2}f(t)+J^{\alpha+\beta-1}f'(t)}{J^{\alpha+\beta-1}f(t)}x\Big]g(x)\frac{J^{\alpha+\beta-1}f(t)}{(\alpha+\beta-1)B(\alpha,\beta)}. \quad (3.20)$$

Then,

$$\Gamma(2\alpha+2\beta-3)(t-a)\frac{J^{2\alpha+2\beta-3}fg(t)}{\Gamma^2(\alpha)\Gamma^2(\beta)}$$

$$\geq exp\Big[\frac{-J^{\alpha+\beta-2}tf(t)+J^{\alpha+\beta}tf'(t)}{J^{\alpha+\beta-1}f(t)}\Big]\frac{J^{\alpha+\beta-1}\Big(g(t)exp\Big[\frac{-J^{\alpha+\beta-2}f(t)+J^{\alpha+\beta-1}f'(t)}{J^{\alpha+\beta-1}f(t)}t\Big]\Big)}{(\alpha+\beta-1)B(\alpha,\beta)}\frac{J^{\alpha+\beta-1}f(t)}{(\alpha+\beta-1)B(\alpha,\beta)}. \quad (3.21)$$

With the same arguments, we obtain

$$\Gamma(2\alpha+2\beta-3)(t-a)\frac{J^{2\alpha+2\beta-3}fg(t)}{\Gamma^2(\alpha)\Gamma^2(\beta)}$$
$$\geq exp\Big[\frac{-J^{\alpha+\beta-2}tg(t)+J^{\alpha+\beta}tg'(t)}{J^{\alpha+\beta-1}g(t)}\Big]\frac{J^{\alpha+\beta-1}\Big(f(t)exp\Big[\frac{-J^{\alpha+\beta-2}g(t)+J^{\alpha+\beta-1}g'(t)}{J^{\alpha+\beta-1}g(t)}t\Big]\Big)}{(\alpha+\beta-1)B(\alpha,\beta)}\frac{J^{\alpha+\beta-1}g(t)}{(\alpha+\beta-1)B(\alpha,\beta)}. \tag{3.22}$$

Adding (3.21) and (3.22), yields

$$2\Gamma(2\alpha+2\beta-3)(t-a)\frac{J^{2\alpha+2\beta-3}fg(t)}{\Gamma^2(\alpha)\Gamma^2(\beta)}$$
$$\geq exp\Big[\frac{-J^{\alpha+\beta-2}tf(t)+J^{\alpha+\beta}tf'(t)}{J^{\alpha+\beta-1}f(t)}\Big]\frac{J^{\alpha+\beta-1}\Big(g(t)exp\Big[\frac{-J^{\alpha+\beta-2}f(t)+J^{\alpha+\beta-1}f'(t)}{J^{\alpha+\beta-1}f(t)}t\Big]\Big)}{(\alpha+\beta-1)B(\alpha,\beta)}\frac{J^{\alpha+\beta-1}f(t)}{(\alpha+\beta-1)B(\alpha,\beta)}$$
$$+exp\Big[\frac{-J^{\alpha+\beta-2}tg(t)+J^{\alpha+\beta}tg'(t)}{J^{\alpha+\beta-1}g(t)}\Big]\frac{J^{\alpha+\beta-1}\Big(f(t)exp\Big[\frac{-J^{\alpha+\beta-2}g(t)+J^{\alpha+\beta-1}g'(t)}{J^{\alpha+\beta-1}g(t)}t\Big]\Big)}{(\alpha+\beta-1)B(\alpha,\beta)}\frac{J^{\alpha+\beta-1}g(t)}{(\alpha+\beta-1)B(\alpha,\beta)}. \tag{3.23}$$

Taking $t = b$, we obtain (3.14). Theorem 3.2 is thus proved. □

Remark. Applying Theorem 3.2 for $\alpha = 1, \beta \neq 1$ or $\beta = 1, \alpha \neq 1$, we obtain Theorem 3.1.

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