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The Extended Tanh Method for Solving Some Evolution Equations

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Abstract: In this paper we use the extended tanh method to formally derive traveling wave solutions for some evolution equations of the type $u_t - Du_{xx} = (B_0 + B_1u)u_x + A_3u^3 + A_2u^2 + A_1u + A_0$. The obtained solutions include, also, kink solutions. The extended tanh method presents a wide applicability for handling nonlinear evolution equations.

Key words: Extended tanh method, Kink solutions, Nonlinear evolution equations, Traveling wave solutions.

1 Introduction

Nonlinear evolution equations have a major role in various scientific and engineering fields such as fluid mechanics, plasma physics, optical fibers, solid state physics, chemical kinematics, chemical physics and geochemistry. Seeking analytic approximate solutions for such equations has long been one of the central themes of interest in mathematics and physics.
The pioneering work of Malfiet in [3, 4] introduced the powerful tanh method for a reliable treatment of nonlinear evolution equations. The useful tanh method is widely used in many works [6, 9–11] and the reference therein. Later, the extended tanh method, developed by Wazwaz [7, 8], is a direct and effective algebraic method for handling nonlinear equations [13]. Various extensions of the method were developed as well [1, 2, 12].
Our first interest in the present work is to implement the extended tanh method for some nonlinear equations. The next interest is in the determination of traveling wave solutions for the following equations:
The first equation is

$$u_t - Du_{xx} = B_0u_x + A_3u^3 + A_1u; \quad A_1 > 0, D > 0, A_3 < 0, B_0 \in \mathbb{R}. \tag{1}$$

The second one is of the form

$$u_t - Du_{xx} = B_0u_x + A_2u^2 + A_1u + A_0; \quad D > 0, A_1^2 - 4A_0A_2 \geq, A_2 \neq 0, B_0 \in \mathbb{R}. \tag{2}$$

In addition of these nonlinear evolution equations, we will investigate:

$$u_t - Du_{xx} = (B_0 + B_1u)u_x + A_2u^2 + A_1u + A_0; \quad D > 0, A_1^2 - 4A_0A_2 \geq, A_2 \neq 0, B_1 \neq 0, \tag{3}$$

and

$$u_t - Du_{xx} = (B_0 + B_1u)u_x + A_3u^3 + A_2u^2; \quad D > 0, B_1^2 - 8A_3D \geq 0, B_1 \neq 0, A_3 \neq 0, B_0 \in \mathbb{R}. \tag{4}$$

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It is interesting to note that these equations appear in various areas of applied mathematics, such as modeling of fluid dynamics, turbulence and traffic flow [8]. They can also be encountered in chemical kinetics and population dynamics [8]. They include as particular cases Newell-Whitehead equation [5] (Eq.1 $B_0 = 0, A_3 = -1, A_1 = 1$), Fisher equation [8] (Eq.2 $B_0 = A_0 = 0, A_2 = -1, A_1 = 1$) and Burgers-Fisher equation [10](Eq.3 $B_0 = A_0 = 0, B_1 = A_1 = -A_2 = 1$).

2 The Extended Tanh Method

Wazwaz summarized the main steps for this method as follows [6, 7] . A PDE

$$P(u, u_t, u_x, u_{xx}, ...) = 0 \tag{5}$$

can be converted to an ODE

$$Q(U, U', U'', U''', ...) = 0 \tag{6}$$

upon using a wave variable $\xi = x - ct$. Equation (6) is then integrated as long as all terms contain derivatives where integration constants are considered zeros. Introducing a new independent variable

$$Y = tanh(\mu\xi); \;\; \xi = x - ct, \;\; \mu \text{ is a real constant} \tag{7}$$

leads to change of derivatives:

$$\tfrac{d}{d\xi} = \mu(1 - Y^2)\tfrac{d}{dY}, \quad \tfrac{d^2}{d\xi^2} = \mu^2(1 - Y^2)\Big(- 2Y\tfrac{d}{dY} + (1 - Y^2)\tfrac{d^2}{dY^2}\Big). \tag{8}$$

The extended tanh method admits the use of the finite expansion

$$U(\mu\xi) = S(Y) = \textstyle\sum_{k=0}^{m} a_k Y^k + \sum_{k=1}^{m} b_k Y^{-k}, \tag{9}$$

where m is a positive integer, for this method, to be determined. Expansion (9) reduces to the standard tanh method [6] for $b_k = 0; 1 \leq k \leq m$. The parameter m is usually obtained by balancing the linear terms of the highest order in the resulting equation with the highest order nonlinear terms. Substituting (9) into the ODE results in an algebraic system of equations in powers of Y that will lead to the determination of parameters $a_k,\ b_k\ (1 \leq k \leq m)$, μ and c. Having determined these parameters, we obtain an analytic solution $u(x, t)$ in a closed form.

3 Equation 1

The wave variable $\xi = x - ct$ transforms equation (1) into the ODE

$$DU'' + cU' + B_0U' + A_3U^3 + A_1U = 0. \tag{10}$$

Balancing U'' with U^3 gives $m = 1$. The extended tanh method admits the use of the finite expansion

$$U(\xi) = a_0 + a_1Y + b_1Y^{-1}. \tag{11}$$

Substituting (11) into (10) and equating the coefficients of the powers of Y, we then obtain a system of algebraic equations for a_0, a_1, b_1, c, and μ. Solving this system by Maple gives the following six sets of solutions.

(i) The first set:

$$a_0 = 0, \quad a_1 = \pm\sqrt{\tfrac{A_1}{-A_3}}, \quad b_1 = 0, \quad c = -B_0, \quad \mu = \sqrt{\tfrac{A_1}{2D}}. \tag{12}$$

(ii) The second set:

$$a_0 = \pm\frac{\sqrt{2}B_0\pm 3\sqrt{A_1 D}}{6\sqrt{-A_3 D}}, \quad a_1 = \pm\sqrt{\frac{A_1}{-A_3}}, b_1 = 0, \quad c = B_0 \pm 3\sqrt{\frac{A_1 D}{2}}, \quad \mu = \pm\sqrt{\frac{A_1}{2D}}. \tag{13}$$

(iii) The third set:

$$a_0 = a_1 = 0, \quad b_1 = \pm\sqrt{\frac{A_1}{-A_3}}, \quad c = -B_0, \quad \mu = \sqrt{\frac{A_1}{2D}}. \tag{14}$$

(iv) The forth set:

$$a_0 = \pm 2\sqrt{\frac{-A_1}{A_3}},\ a_1 = 0, \quad b_1 = \pm 0.5\sqrt{\frac{-A_1}{A_3}}, \quad c = B_0 \pm 3\sqrt{\frac{A_1 D}{2}}, \quad \mu = \pm 0.5\sqrt{\frac{A_1}{2D}}. \tag{15}$$

(v) The fifth set:

$$a_0 = 0, \quad a_1 = \pm 0.5\sqrt{\frac{-2A_1}{A_3}},\ b_1 = \pm 0.5\sqrt{\frac{-A_1}{A_3}}, quadc = -B_0,\ \mu = \pm 0.5\sqrt{\frac{A_1}{2D}}. \tag{16}$$

(vi) The sixth set:

$$a_0 = \pm 2\sqrt{\frac{-A_1}{A_3}}, a_1 = \pm 0.5\Big(\frac{-A_1}{A_3}\Big)\sqrt{\frac{-2A_3}{A_1}}, b_1 = \pm 0.5\sqrt{\frac{-A_1}{A_3}}, c = -B_0,\ \mu = \pm 0.5\sqrt{\frac{A_1}{2D}}. \tag{17}$$

In view of this, we obtain the following kink solutions

$$u_1(x,t) = \pm\sqrt{\frac{A_1}{-A_3}}\tanh\Big[\sqrt{\frac{A_1}{2D}}(x + B_0 t)\Big], \tag{18}$$

$$u_2(x,t) = \pm\frac{\sqrt{2}B_0\pm 3\sqrt{A_1 D}}{6\sqrt{-A_3 D}} \pm \sqrt{\frac{A_1}{-A_3}}\tanh\Big[\pm 0.5\sqrt{\frac{A_1}{2D}}(x - \frac{4B_0\pm\sqrt{72A_1 D}}{4}t)\Big]. \tag{19}$$

and the traveling wave solutions

$$u_3(x,t) = \pm\sqrt{\frac{A_1}{-A_3}}\coth\Big[\sqrt{\frac{A_1}{2D}}(x + B_0 t)\Big], \tag{20}$$

$$u_4(x,t) = \pm 2\sqrt{\frac{A_1}{-A_3}} \pm \sqrt{\frac{A_1}{-A_3}}\coth\Big[\pm 0.5\sqrt{\frac{A_1}{2D}}(x - \frac{4B_0\pm\sqrt{72A_1 D}}{4}t)\Big], \tag{21}$$

$$u_5(x,t) = \pm 0.5\sqrt{\frac{-2A_1}{A_3}}\tanh\Big[\sqrt{\frac{A_1}{2D}}(x + B_0 t)\Big] \pm \sqrt{\frac{A_1}{-A_3}}\coth\Big[\sqrt{\frac{A_1}{2D}}(x + B_0 t)\Big], \tag{22}$$

$$u_6(x,t) = \pm 2\sqrt{\frac{-A_1}{A_3}} \pm 0.5\Big(\frac{-A_1}{A_3}\Big)\sqrt{\frac{-2A_3}{A_1}}\tanh\Big[\sqrt{\frac{A_1}{2D}}(x + B_0 t)\Big] \pm \sqrt{\frac{A_1}{-A_3}}\coth\Big[\sqrt{\frac{A_1}{2D}}(x + B_0 t)\Big]. \tag{23}$$

4 Equation 2

The wave variable $\xi = x - ct$ converts equation (2) into the ODE

$$DU'' + cU' + B_0 U' + A_2 U^2 + A_1 U + A_0 = 0. \tag{24}$$

Balancing U'' with U^2 gives $m = 2$. The extended tanh method admits the use of the finite expansion

$$U(\xi) = a_0 + a_1 Y + a_2 Y^2 + b_1 Y^{-1} + b_2 Y^{-2}. \tag{25}$$

Substituting (25) into (24) and equating the coefficients of the powers of Y, we then obtain a system of algebraic equations for $a_0, a_1, a_2, b_1, b_2, c$, and μ, which solved gives the following set of nine solutions

(i) The first set:

$$a_0 = \frac{2\sqrt{A_1^2 - 4A_0 A_2} - A_1}{2A_2}, b_1 = b_2 = a_1 = 0, c = -B_0, a_2 = -\frac{3\sqrt{A_1^2 - 4A_0 A_2}}{2A_2}, \mu = 0.5\sqrt[4]{\frac{A_1^2 - 4A_0 A_2}{D^2}}. \tag{26}$$

(ii) The second set:

$$\begin{aligned} a_0 = \tfrac{0.5\sqrt{A_1^2-4A_0A_2}-A_1}{2A_2}, \quad b_1 = b_2 = 0, \quad c = -B_0 + 0.5D\sqrt[4]{\tfrac{A_1^2-4A_0A_2}{36D^2}}, \\ a_2 = -\tfrac{\sqrt{A_1^2-4A_0A_2}}{4A_2}, \quad \mu = 0.5\sqrt[4]{\tfrac{A_1^2-4A_0A_2}{36D^2}}, \quad a_1 = \tfrac{\sqrt{A_1^2-4A_0A_2}}{2A_2}. \end{aligned} \tag{27}$$

(iii) The third set:

$$\begin{aligned} a_0 = \tfrac{0.5\sqrt{A_1^2-4A_0A_2}-A_1}{2A_2}, \quad b_1 = b_2 = 0, \quad c = -B_0 - 5D\sqrt[4]{\tfrac{A_1^2-4A_0A_2}{36D^2}}, \\ a_2 = -\tfrac{\sqrt{A_1^2-4A_0A_2}}{4A_2}, \quad \mu = 0.5\sqrt[4]{\tfrac{A_1^2-4A_0A_2}{36D^2}}, \quad a_1 = -\tfrac{\sqrt{A_1^2-4A_0A_2}}{2A_2}. \end{aligned} \tag{28}$$

(iv) The forth set:

$$a_0 = \tfrac{2\sqrt{A_1^2-4A_0A_2}-A_1}{2A_2}, b_1 = a_1 = a_2 = 0, c = -B_0, b_2 = -\tfrac{3\sqrt{A_1^2-4A_0A_2}}{2A_2}, \mu = 0.5\sqrt[4]{\tfrac{A_1^2-4A_0A_2}{D^2}}. \tag{29}$$

(v) The fifth set:

$$\begin{aligned} a_0 = \tfrac{0.5\sqrt{A_1^2-4A_0A_2}-A_1}{2A_2}, \quad a_1 = a_2 = 0, \quad c = -B_0 + 5D\sqrt[4]{\tfrac{A_1^2-4A_0A_2}{36D^2}}, \\ b_1 = \tfrac{\sqrt{A_1^2-4A_0A_2}}{2A_2}, \quad \mu = 0.5\sqrt[4]{\tfrac{A_1^2-4A_0A_2}{36D^2}}, \quad b_2 = -\tfrac{\sqrt{A_1^2-4A_0A_2}}{4A_2}. \end{aligned} \tag{30}$$

(vi) The sixth set:

$$\begin{aligned} a_0 = \tfrac{0.5\sqrt{A_1^2-4A_0A_2}-A_1}{2A_2}, \quad a_1 = a_2 = 0, \quad c = -B_0 - 5D\sqrt[4]{\tfrac{A_1^2-4A_0A_2}{36D^2}}, \\ b_1 = -\tfrac{\sqrt{A_1^2-4A_0A_2}}{2A_2}, \quad \mu = 0.5\sqrt[4]{\tfrac{A_1^2-4A_0A_2}{36D^2}}, \quad 2b_2 = b_1. \end{aligned} \tag{31}$$

(vii) The seventh set:

$$\begin{aligned} a_0 = \tfrac{\sqrt{A_1^2-4A_0A_2}-2A_1}{4A_2}, \quad a_1 = b_1 = 0, \quad c = -B_0, \\ a_2 = b_2 = -\tfrac{3\sqrt{A_1^2-4A_0A_2}}{8A_2}, \quad \mu = 0.25\sqrt[4]{\tfrac{A_1^2-4A_0A_2}{D^2}}. \end{aligned} \tag{32}$$

(viii) The eighth set:

$$\begin{aligned} a_0 = \tfrac{0.5\sqrt{A_1^2-4A_0A_2}-2A_1}{4A_2}, \quad a_1 = b_1 = \tfrac{\sqrt{A_1^2-4A_0A_2}}{4A_2}, \\ c = -B_0 + 5D\sqrt[4]{\tfrac{A_1^2-4A_0A_2}{36D^2}}, \quad \mu = 0.25\sqrt[4]{\tfrac{A_1^2-4A_0A_2}{36D^2}}, \quad a_2 = b_2 = -\tfrac{0.5\sqrt{A_1^2-4A_0A_2}}{8A_2}. \end{aligned} \tag{33}$$

(ix) The ninth set:

$$\begin{aligned} a_0 = \tfrac{0.5\sqrt{A_1^2-4A_0A_2}-2A_1}{4A_2}, \quad a_1 = b_1 = -\tfrac{\sqrt{A_1^2-4A_0A_2}}{4A_2}, \\ c = -B_0 - 5D\sqrt[4]{\tfrac{A_1^2-4A_0A_2}{36D^2}}, \quad \mu = 0.25\sqrt[4]{\tfrac{A_1^2-4A_0A_2}{36D^2}}, \quad a_2 = b_2 = -\tfrac{0.5\sqrt{A_1^2-4A_0A_2}}{8A_2}. \end{aligned} \tag{34}$$

In view of this, we obtain the following kink and travelling wave solutions

$$u_1(x,t) = \tfrac{2\sqrt{A_1^2-4A_0A_2}-A_1}{2A_2} - \tfrac{3\sqrt{A_1^2-4A_0A_2}}{2A_2}\tanh^2\left[0.5\sqrt[4]{\tfrac{A_1^2-4A_0A_2}{D^2}}(x+B_0t)\right], \tag{35}$$

$$\begin{aligned} u_2(x,t) = \tfrac{0.5\sqrt{A_1^2-4A_0A_2}-A_1}{2A_2} + \tfrac{\sqrt{A_1^2-4A_0A_2}}{2A_2}\tanh\left[0.5\sqrt[4]{\tfrac{A_1^2-4A_0A_2}{D^2}}(x+B_0-5D\sqrt[4]{\tfrac{A_1^2-4A_0A_2}{36D^2}}t)\right] \\ -\tfrac{\sqrt{A_1^2-4A_0A_2}}{4A_2}\tanh^2\left[0.5\sqrt[4]{\tfrac{A_1^2-4A_0A_2}{D^2}}(x+B_0-5D\sqrt[4]{\tfrac{A_1^2-4A_0A_2}{36D^2}}t)\right], \end{aligned} \tag{36}$$

$$u_3(x,t)=\frac{0.5\sqrt{A_1^2-4A_0A_2}-A_1}{2A_2}-\frac{\sqrt{A_1^2-4A_0A_2}}{2A_2}\tanh\left[0.5\sqrt[4]{\frac{A_1^2-4A_0A_2}{D^2}}(x-B_0-5D\sqrt[4]{\frac{A_1^2-4A_0A_2}{36D^2}}t)\right]-\frac{\sqrt{A_1^2-4A_0A_2}}{4A_2}\tanh^2\left[0.5\sqrt[4]{\frac{A_1^2-4A_0A_2}{D^2}}(x-B_0+5D\sqrt[4]{\frac{A_1^2-4A_0A_2}{36D^2}}t)\right], \tag{37}$$

$$u_4(x,t)=\frac{2\sqrt{A_1^2-4A_0A_2}-A_1}{2A_2}-\frac{3\sqrt{A_1^2-4A_0A_2}}{2A_2}\coth^2\left[0.5\sqrt[4]{\frac{A_1^2-4A_0A_2}{D^2}}(x+B_0t)\right], \tag{38}$$

$$u_5(x,t)=\frac{0.5\sqrt{A_1^2-4A_0A_2}-A_1}{2A_2}+\frac{\sqrt{A_1^2-4A_0A_2}}{2A_2}\coth\left[0.5\sqrt[4]{\frac{A_1^2-4A_0A_2}{D^2}}(x+B_0+5D\sqrt[4]{\frac{A_1^2-4A_0A_2}{36D^2}}t)\right]-\frac{\sqrt{A_1^2-4A_0A_2}}{4A_2}\coth^2\left[0.5\sqrt[4]{\frac{A_1^2-4A_0A_2}{D^2}}(x+B_0+5D\sqrt[4]{\frac{A_1^2-4A_0A_2}{36D^2}}t)\right], \tag{39}$$

$$u_6(x,t)=\frac{0.5\sqrt{A_1^2-4A_0A_2}-A_1}{2A_2}-\frac{\sqrt{A_1^2-4A_0A_2}}{2A_2}\coth\left[0.5\sqrt[4]{\frac{A_1^2-4A_0A_2}{D^2}}(x+B_0-5D\sqrt[4]{\frac{A_1^2-4A_0A_2}{36D^2}}t)\right]--\frac{\sqrt{A_1^2-4A_0A_2}}{4A_2}\coth^2\left[0.5\sqrt[4]{\frac{A_1^2-4A_0A_2}{D^2}}(x+B_0-5D\sqrt[4]{\frac{A_1^2-4A_0A_2}{36D^2}}t)\right], \tag{40}$$

$$u_7(x,t)=\frac{\sqrt{A_1^2-4A_0A_2}-2A_1}{4A_2}-\frac{3\sqrt{A_1^2-4A_0A_2}}{8A_2}\tanh^2\left[0.5\sqrt[4]{\frac{A_1^2-4A_0A_2}{D^2}}(x+B_0t)\right]-\frac{3\sqrt{A_1^2-4A_0A_2}}{8A_2}\coth^2\left[0.5\sqrt[4]{\frac{A_1^2-4A_0A_2}{D^2}}(x+B_0t)\right], \tag{41}$$

$$u_8(x,t)=\frac{0.5\sqrt{A_1^2-4A_0A_2}-2A_1}{4A_2}+\frac{\sqrt{A_1^2-4A_0A_2}}{4A_2}\tanh\left[0.5\sqrt[4]{\frac{A_1^2-4A_0A_2}{D^2}}(x+B_0-5D\sqrt[4]{\frac{A_1^2-4A_0A_2}{36D^2}}t)\right]-\frac{0.5\sqrt{A_1^2-4A_0A_2}}{8A_2}\tanh^2\left[0.5\sqrt[4]{\frac{A_1^2-4A_0A_2}{D^2}}(x+B_0-5D\sqrt[4]{\frac{A_1^2-4A_0A_2}{36D^2}}t)\right]+\frac{\sqrt{A_1^2-4A_0A_2}}{4A_2}\coth\left[0.5\sqrt[4]{\frac{A_1^2-4A_0A_2}{D^2}}(x+B_0-5D\sqrt[4]{\frac{A_1^2-4A_0A_2}{36D^2}}t)\right]-\frac{0.5\sqrt{A_1^2-4A_0A_2}}{8A_2}\coth^2\left[0.5\sqrt[4]{\frac{A_1^2-4A_0A_2}{D^2}}(x+B_0-5D\sqrt[4]{\frac{A_1^2-4A_0A_2}{36D^2}}t)\right]. \tag{42}$$

$$u_9(x,t)=\frac{0.5\sqrt{A_1^2-4A_0A_2}-2A_1}{4A_2}-\frac{\sqrt{A_1^2-4A_0A_2}}{4A_2}\tanh\left[0.5\sqrt[4]{\frac{A_1^2-4A_0A_2}{D^2}}(x+B_0+5D\sqrt[4]{\frac{A_1^2-4A_0A_2}{36D^2}}t)\right]-\frac{0.5\sqrt{A_1^2-4A_0A_2}}{8A_2}\tanh^2\left[0.5\sqrt[4]{\frac{A_1^2-4A_0A_2}{D^2}}(x+B_0+5D\sqrt[4]{\frac{A_1^2-4A_0A_2}{36D^2}}t)\right]-\frac{\sqrt{A_1^2-4A_0A_2}}{4A_2}\coth\left[0.5\sqrt[4]{\frac{A_1^2-4A_0A_2}{D^2}}(x+B_0+5D\sqrt[4]{\frac{A_1^2-4A_0A_2}{36D^2}}t)\right]-\frac{0.5\sqrt{A_1^2-4A_0A_2}}{8A_2}\coth^2\left[0.5\sqrt[4]{\frac{A_1^2-4A_0A_2}{D^2}}(x+B_0+5D\sqrt[4]{\frac{A_1^2-4A_0A_2}{36D^2}}t)\right]. \tag{43}$$

5 Equation 3

The wave variable $\xi = x - ct$ converts the equation (3) to the ODE

$$DU'' + cU' + (B_0 + B_1U)U' + A_2U^2 + A_1U + A_0 = \quad 0. \tag{44}$$

Balancing U'' with UU' gives $m = 1$. The extended tanh method admits the use of the

$$U(\xi) = \quad a_0 + a_1Y + b_1Y^{-1}. \tag{45}$$

Substituting (45) into (44) and equating the coefficients of the powers of Y, we then obtain a system of algebraic equations for a_0, a_1, b_1, c, and μ, which solved gives the following three sets of solutions

(i) The first set:

$$a_0=-\frac{A_1}{2A_2},\quad b_1=0,\quad a_1=\pm\frac{\sqrt{A_1^2-4A_0A_2}}{2A_2},$$
$$\mu=\pm B_1\frac{\sqrt{A_1^2-4A_0A_2}}{4DA_2},\quad c=\frac{4DA_2^2+B_1^2A_1-2B_0B_1A_2}{2B_1A_2}. \tag{46}$$

(ii) The second set:

$$a_0=-\frac{A_1}{2A_2},\quad a_1=0,\quad b_1=\pm\frac{\sqrt{A_1^2-4A_0A_2}}{2A_2},$$
$$\mu=\pm B_1\frac{\sqrt{A_1^2-4A_0A_2}}{4DA_2},\quad c=\frac{4DA_2^2+B_1^2A_1-2B_0B_1A_2}{2B_1A_2}. \tag{47}$$

(iii) The third set:

$$a_0=-\frac{A_1}{2A_2},\quad a_1=b_1=\pm\frac{\sqrt{A_1^2-4A_0A_2}}{4A_2},$$
$$\mu=\pm B_1\frac{\sqrt{A_1^2-4A_0A_2}}{8DA_2},\quad c=\frac{4DA_2^2+B_1^2A_1-2B_0B_1A_2}{2B_1A_2}. \tag{48}$$

This in turn gives the kink solution

$$u_1(x,t)=-\frac{A_1}{2A_2}\pm\frac{\sqrt{A_1^2-4A_0A_2}}{4A_2}$$
$$\tanh\Big[\pm B_1\frac{\sqrt{A_1^2-4A_0A_2}}{8DA_2}(x-\frac{4DA_2^2+B_1^2A_1-2B_0B_1A_2}{2B_1A_2}t)\Big], \tag{49}$$

and the traveling wave solutions

$$u_2(x,t)=-\frac{A_1}{2A_2}\pm\frac{\sqrt{A_1^2-4A_0A_2}}{4A_2}$$
$$\coth\Big[\pm B_1\frac{\sqrt{A_1^2-4A_0A_2}}{8DA_2}(x-\frac{4DA_2^2+B_1^2A_1-2B_0B_1A_2}{2B_1A_2}t)\Big], \tag{50}$$

$$u_3(x,t)=-\frac{A_1}{2A_2}\pm\frac{\sqrt{A_1^2-4A_0A_2}}{4A_2}$$
$$\tanh\Big[\pm B_1\frac{\sqrt{A_1^2-4A_0A_2}}{8DA_2}(x-\frac{4DA_2^2+B_1^2A_1-2B_0B_1A_2}{2B_1A_2}t)\Big]$$
$$\pm\frac{\sqrt{A_1^2-4A_0A_2}}{4A_2}\coth\Big[\pm B_1\frac{\sqrt{A_1^2-4A_0A_2}}{8DA_2}(x-\frac{4DA_2^2+B_1^2A_1-2B_0B_1A_2}{2B_1A_2}t)\Big]. \tag{51}$$

6 Equation 4

Proceeding as before, the equation (4) can be converted to the ODE

$$DU''+cU'+(B_0+B_1U)U'+A_3U^3+A_2U^2=\ 0, \tag{52}$$

upon using the wave variable $\xi=x-ct$. Balancing U'' with UU' gives $m=1$. This allows us to set

$$U(\xi)=\ a_0+a_1Y+b_1Y^{-1}. \tag{53}$$

Substituting the equation(53) into (52), solving the resulting system and putting $R_\pm=\frac{-B_1\pm\sqrt{B_1^2-8A_3D}}{4DA_3}$, $Z_\pm=\frac{B_1\pm\sqrt{B_1^2-8A_3D}}{4DA_3}$, we then have the sets of solutions

(i) The first set

$$a_0=-\frac{A_2}{2A_3},\quad b_1=0,\quad a_1=\frac{A_2(1+2DA_3R_\pm^2)}{2A_3B_1R_\pm},\quad \mu=\frac{A_2R_\pm}{2},$$
$$c=\frac{(-B_0+DA_2B_1R_\pm^2)-2B_0DA_3R_\pm^2}{1+2DA_3R_\pm^2}. \tag{54}$$

(ii) The second set

$$a_0=-\frac{A_2}{2A_3},\quad b_1=0,\quad a_1=\frac{A_2(1+2DA_3Z_\pm^2)}{2A_3B_1Z_\pm},\quad \mu=\frac{A_2Z_\pm}{2},$$
$$c=\frac{(-B_0+DA_2B_1Z_\pm^2)-2B_0DA_3Z_\pm^2}{1+2DA_3Z_\pm^2}. \tag{55}$$

(iii) The third set

$$a_0 = -\frac{A_2}{2A_3}, \quad a1 = 0, \quad b_1 = \frac{A_2(1+2DA_3R_\pm^2)}{2A_3B_1R_\pm}, \quad \mu = \frac{A_2R_\pm}{2}, \\ c = \frac{(-B_0+DA_2B_1R_\pm^2)-2B_0DA_3R_\pm^2}{1+2DA_3R_\pm^2}. \tag{56}$$

(iv) The forth set

$$a_0 = -\frac{A_2}{2A_3}, \quad a_1 = 0, \quad b_1 = \frac{A_2(1+2DA_3Z_\pm^2)}{2A_3B_1Z_\pm}, \quad \mu = \frac{A_2Z_\pm}{2}, \\ c = \frac{(-B_0+DA_2B_1Z_\pm^2)-2B_0DA_3Z_\pm^2}{1+2DA_3Z_\pm^2}. \tag{57}$$

(v) The fifth set

$$a_0 = 2a_1 = 2b_1 = -\frac{A_2}{2A_3}, \quad \mu = \frac{A_2R_\pm}{4}, \quad c = \frac{A_2-2B_0A_3R_\pm+A_2B_1R_\pm}{2A_3R_\pm}. \tag{58}$$

(vi) The sixth set

$$a_0 = -2a_1 = -2b_1 = -\frac{A_2}{2A_3}, \quad \mu = \frac{A_2Z_\pm}{4}, \quad c = \frac{A_2-2B_0A_3Z_\pm+A_2B_1Z_\pm}{2A_3Z_\pm}. \tag{59}$$

This in turn gives the two kink solutions

$$u_1(x,t) = -\frac{A_2}{2A_3} + \frac{A_2(1+2DA_3R_\pm^2)}{2A_3B_1R_\pm}\tanh\Big[\frac{A_2R_\pm}{2} \\ (x - \frac{(-B_0+DA_2B_1R_\pm^2)-2B_0DA_3R_\pm^2}{1+2DA_3R_\pm^2}t)\Big], \tag{60}$$

$$u_2(x,t) = -\frac{A_2}{2A_3} + \frac{A_2(1+2DA_3Z_\pm^2)}{2A_3B_1Z_\pm}\tanh\Big[\frac{A_2Z_\pm}{2} \\ (x - \frac{(-B_0+DA_2B_1Z_\pm^2)-2B_0DA_3Z_\pm^2}{1+2DA_3Z_\pm^2}t)\Big], \tag{61}$$

and the traveling wave solutions

$$u_3(x,t) = -\frac{A_2}{2A_3} + \frac{A_2(1+2DA_3R_\pm^2)}{2A_3B_1R_\pm}\coth\Big[\frac{A_2R_\pm}{2} \\ (x - \frac{(-B_0+DA_2B_1R_\pm^2)-2B_0DA_3R_\pm^2}{1+2DA_3R_\pm^2}t)\Big], \tag{62}$$

$$u_4(x,t) = -\frac{A_2}{2A_3} + \frac{A_2(1+2DA_3Z_\pm^2)}{2A_3B_1Z_\pm}\coth\Big[\frac{A_2Z_\pm}{2} \\ (x - \frac{(-B_0+DA_2B_1Z_\pm^2)-2B_0DA_3Z_\pm^2}{1+2DA_3Z_\pm^2}t)\Big], \tag{63}$$

$$u_5(x,t) = -\frac{A_2}{2A_3} - \frac{A_2}{4A_3}\tanh\Big[\frac{A_2R_\pm}{4}(x - \frac{A_2-2B_0A_3R_\pm+A_2B_1R_\pm}{2A_3R_\pm}t)\Big] \\ -\frac{A_2}{4A_3}\coth\Big[\frac{A_2R_\pm}{2}(x - \frac{A_2-2B_0A_3R_\pm+A_2B_1R_\pm}{2A_3R_\pm}t)\Big], \tag{64}$$

$$u_6(x,t) = -\frac{A_2}{2A_3} + \frac{A_2}{4A_3}\tanh\Big[\frac{A_2Z_\pm}{4}(x - \frac{A_2-2B_0A_3Z_\pm+A_2B_1Z_\pm}{2A_3Z_\pm}t)\Big] \\ +\frac{A_2}{4A_3}\coth\Big[\frac{A_2Z_\pm}{2}(x - \frac{A_2-2B_0A_3Z_\pm+A_2B_1Z_\pm}{2A_3Z_\pm}t)\Big]. \tag{65}$$

7 Conclusion

The extended tanh method was successfully used to establish traveling wave solutions. The work confirms the power of this method to handle nonlinear evolution equations. The availability of computer systems like Mathematica and Maple facilitate the tedious algebraic calculations.

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