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An Application of Banach Contraction Principle for a Class of Fractional HDEs

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Abstract

This paper deals with a class of fractional hybrid differential equations. We prove an integral representation for the studied class. Then, using the Banach contraction principle, we establish some conditions that guarantee for us the existence of a unique solution.

Keywords: *Hybrid differential equation, Caputo derivative, Fixed point, Existence and uniqueness of solution.*

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1 Introduction

The theory of Fractional differential equations is a very important tool for modeling phenomena in applied sciences and engineering. It has applications in physics, biology, chemistry, engineering, and more others applied domains, we refer the reader to [9], [3], [13],[14] .
On the other hand, the hybrid differential equations is very interesting domain for mathematics and physics, see for instance [1], [2], [5], [7], [8].
In this paper, we are concerned with the following problem:

$$\begin{cases} \mathbf{D}^{\alpha_1}\left(\dfrac{x_1(t)}{f_1(t,x_1(t),x_2(t),...,x_n(t))}\right) &= h_1(t,x_1(t),x_2(t),...,x_n(t)) \\ & +I^{\delta_1}k_1(t,x_1(t),x_2(t),...,x_n(t)), t\in J \\ \mathbf{D}^{\alpha_2}\left(\dfrac{x_(t)}{f_2(t,x_1(t),x_2(t),...,x_n(t))}\right) &= h_2(t,x_1(t),x_2(t),...,x_n(t)) \\ & +I^{\delta_2}k_2(t,x_1(t),x_2(t),...,x_n(t)), t\in J \\ & \cdots \\ \mathbf{D}^{\alpha_n}\left(\dfrac{x_n(t)}{f_n(t,x_1(t),x_2(t),...,x_n(t))}\right) &= h_n(t,x_1(t),x_2(t),...,x_n(t)) \\ & +I^{\delta_n}k_n(t,x_1(t),x_2(t),...,x_n(t)), t\in J \\ x_i(0) &= \theta_i\displaystyle\int_0^{\beta_i}\varphi_i(s)x_i(s)ds, \\ & 0<\beta_i<1, i=1,2,...,n. \end{cases} \tag{1}$$

where, for $i=1,..,n$, the symbols D^{α_i} denote the Caputo fractional derivative with $0<\alpha_i<1$, the symbols I^{δ_i} denote the Riemann-Liouville fractional integral of order δ_i with $0<\delta_i<1$. $J=[0,1]$ represent a time interval, θ_i are real numbers, φ_i are continuous functions on $[0,\beta_i]$, $f_i\in C((J\times\mathbb{R}^n,\mathbb{R}-\{0\})$ and $h_i,k_i\in C((J\times\mathbb{R}^n,\mathbb{R})$.

The paper is organized as follow: Section 2 is devoted to the preliminaries and most important notions used throughout the development of the main results. In section 3, we prove the main result on the uniqueness of one solution for the hybrid problem. In the last section, two open questions are posed.

2 Preliminaries

We introduce some useful definitions and lemmas [6, 10, 11, 12].

Definition 2.1 *Let $\alpha>0$ and $h:[a,b]\longmapsto\mathbb{R}$ be a continuous function. The Riemann-Liouville integral of order α is defined by:*

$$I_a^{\alpha}h(t)=\frac{1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}h(s)ds, \alpha>0, a\leq t\leq b.$$

Definition 2.2 *Let $\alpha > 0, n-1 < \alpha \leq n$ and $h \in C^n([0,T],\mathbb{R})$. The Caputo fractional derivative is defined by:*

$$\begin{aligned} D^{\alpha}h(t) &= I^{n-\alpha}\frac{d^n}{dt^n}(h(t)) \\ &= \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}h^{(n)}(s)ds. \end{aligned}$$

Lemma 2.3 *For $n-1 < \alpha \leq n$ and $h \in C^n([0,T],\mathbb{R})$, the equation $D^{\alpha}h = 0$ has a general solution given by:*

$$h(t) = c_0 + c_1 t + c_2 t^2 + ... + c_{n-1}t^{n-1},$$

where $c_i \in \mathbb{R}, i = 0, 1, 2, .., n-1$.

Lemma 2.4 *Under the assumptions of the above lemma, we have*

$$I^{\alpha}D^{\alpha}h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + ... + c_{n-1}t^{n-1}.$$

3 Main Results

First, we mention that we have noted $x = (x_1, x_2, ..., x_n)$and $x(t) = (x_1(t), x_2(t), ..., x_n(t))$ for the clarity of calculations and reading.

The following lemma is an auxiliary result that will be used throughout this paper. We prove:

Lemma 3.1 *Let $i = 1, 2, ..., n$ and $0 < \alpha_i, \delta_i < 1$. If $f_i \in C((J \times \mathbb{R}^n, \mathbb{R} - \{0\})$ and $h_i, k_i \in C((J \times \mathbb{R}^n, \mathbb{R})$, then, the solution of the equation*

$$D^{\alpha_i}\left(\frac{x_i(t)}{f_i(t,x(t))}\right) = h_i(t,x(t)) + I^{\delta_i}k_i(t,x(t)) \tag{2}$$

under the condition:

$$x_i(0) = \theta_i \int_0^{\beta_i} \varphi_i(s)x_i(s)ds, 0 < \beta_i < 1, i = 1, 2, ..., n \tag{3}$$

is given by:

$$
\begin{aligned}
x_i(t) &= f_i(t,x(t))\Bigg(\frac{1}{\Gamma(\alpha_i)}\int_0^t (t-\tau)^{\alpha_i-1}h_i(\tau,x(\tau))d\tau \\
&+\frac{1}{\Gamma(\alpha_i+\delta_i)}\int_0^t (t-\tau)^{\alpha_i+\delta_i-1}k_i(\tau,x(\tau))d\tau \\
&+\frac{\theta_i}{f_i(0,x(0))-\theta_i\int_0^{\beta_i} f_i(s,x(s))\varphi_i(s)ds}\int_0^{\beta_i} f_i(s,x(s))\varphi_i(s) \qquad (4)\\
&\times\Bigg[\frac{1}{\Gamma(\alpha_i)}\int_0^t (t-\tau)^{\alpha_i-1}h_i(\tau,x(\tau))d\tau \\
&+\frac{1}{\Gamma(\alpha_i+\delta_i)}\int_0^t (t-\tau)^{\alpha_i+\delta_i-1}k_i(\tau,x(\tau))d\tau\Bigg]ds\Bigg)
\end{aligned}
$$

with: $f_i(0,x(0)) \neq \theta_i \int_0^{\beta_i} f_i(s,x(s))\varphi_i(s)ds.$

Proof

For $i = 1, ..., n$, we consider:

$$D^{\alpha_i}\left(\frac{x_i(t)}{f_i(t,x(t)}\right) = h_i(t,x(t)) + I^{\delta_i}k_i(t,x(t)), t \in J \qquad (5)$$

By using lemmas 2.3 and 2.4, the general solution of (5) is given by:

$$\frac{x_i(t)}{f_i(t,x(t))} = I^{\alpha_i}h_i(t,x(t)) + I^{\alpha_i+\delta_i}k_i(t,x(t)) - c_0 \qquad (6)$$

where $c_0 \in \mathbb{R}$ is an arbitrary constant.

From (6), we get:

$$x_i(t) = f_i(t,x(t))[I^{\alpha_i}h_i(t,x(t)) + I^{\alpha_i+\delta_i}k_i(t,x(t)) - c_0] \qquad (7)$$

On the other hand, we multiply both sides of (7) by $\theta_i\varphi_i(s)$, we get:

$$
\begin{aligned}
\theta_i\varphi_i(s)x_i(s) &= \theta_i\varphi_i(s)f_i(s,x(s)) \\
&\times[I^{\alpha_i}h_i(s,x(s) + I^{\alpha_i+\delta_i}k_i(s,x(s))] - c_0\theta_i f_i(s,x(s))\varphi_i(s) \qquad (8)
\end{aligned}
$$

Then thanks to (8), we can write:

$$\theta_i \int_0^{\beta_i} \varphi_i(s)x_i(s)ds = \theta_i \int_0^{\beta_i} \varphi_i(s)f_i(s,x(s))[I^{\alpha_i}h_i(s,x(s)+I^{\alpha_i+\delta_i}k_i(s,x(s))]ds$$
$$-c_0\int_0^{\beta_i}\theta_i f_i(s,x(s))\varphi_i(s)ds \tag{9}$$

Using (3) and (7), we get:

$$c_0\left(f_i(0,x(0))-\int_0^{\beta_i}\theta_i f_i(s,x(s))\varphi_i(s)ds\right) = \theta_i\int_0^{\beta_i}\varphi_i(s)f_i(s,x(s))$$
$$\times[I^{\alpha_i}h_i(s,x(s)+I^{\alpha_i+\delta_i}k_i(s,x(s))]ds \tag{10}$$

which becomes

$$c_0 = \frac{\theta_i}{\left(f_i(0,x(0))-\int_0^{\beta_i}\theta_i f_i(s,x(s))\varphi_i(s)ds\right)}$$
$$\int_0^{\beta_i}\varphi_i(s)f_i(s,x(s))[I^{\alpha_i}h_i(s,x(s)+I^{\alpha_i+\delta_i}k_i(s,x(s))]ds \tag{11}$$

Replacing c_0 by its value in (7), we obtain (4).

Now, we introduce the following Banach spaces:

$$X_i = \{x_i(t), i = 1,...,n : x_i \in C(J,\mathbb{R})\} \tag{12}$$

with the norm:

$$\|x_i\|_{X_i} = sup\{|x_i(t)| : t \in J\} \tag{13}$$

where $i = 1..,n$.

We bring to the attention that for $i = 1,2,...,n$, $\left(X_i, \|.\|_{X_i}\right)$ is a Banach space.

The product space:

$$\left(\prod_{i=1}^n X_i, \|.\|_{\prod_{i=1}^n X_i}\right) \tag{14}$$

is also a Banach space.

Let $\mathcal{Q}$ be an operator defined by:

$$\mathcal{Q}: \quad \prod_{i=1}^n X_i \to \prod_{i=1}^n X_i$$

$$x(t) \longmapsto \mathcal{Q}x(t)$$

such that for $t \in J$,

$$\mathcal{Q}x(t) = \big(\mathcal{Q}_1 x(t), \mathcal{Q}_2 x(t), ..., \mathcal{Q}_n x(t)\big) \tag{15}$$

where:

$$\begin{aligned} \mathcal{Q}_i x(t) &= f_i(t, x(t)) \Bigg(\frac{1}{\Gamma(\alpha_i)} \int_0^t (t-\tau)^{\alpha_i - 1} h_i(\tau, x(\tau)) d\tau \\ &+ \frac{1}{\Gamma(\alpha_i + \delta_i)} \int_0^t (t-\tau)^{\alpha_i + \delta_i - 1} k_i(\tau, x(\tau)) d\tau \\ &+ \frac{\theta_i}{f_i(0, x(0)) - \theta_i \int_0^{\beta_i} f_i(s, x(s)) \varphi_i(s) ds} \int_0^{\beta_i} f_i(s, x(s)) \varphi_i(s) \\ &\times \Bigg[\frac{1}{\Gamma(\alpha_i)} \int_0^t (t-\tau)^{\alpha_i - 1} h_i(\tau, x(\tau)) d\tau \\ &+ \frac{1}{\Gamma(\alpha_i + \delta_i)} \int_0^t (t-\tau)^{\alpha_i + \delta_i - 1} k_i(\tau, x(\tau)) d\tau \Bigg] ds \Bigg) \end{aligned} \tag{16}$$

Theorem 3.2 *We suppose that:*

H1. *There exist constants* ξ_{ij} , ζ_{ij} *for* $i, j = 1, ..., n$ *such that:*

$$|h_i(t, x_1, ..., x_n) - h_i(t, y_1, ..., y_n)| \leq \sum_{j=1}^{n} \xi_{ij} |x_j - y_j| \tag{17}$$

and

$$|k_i(t, x_1, ..., x_n) - k_i(t, y_1, ..., y_n)| \leq \sum_{j=1}^{n} \zeta_{ij} |x_j - y_j| \tag{18}$$

for all $t \in J$, $x, y \in \mathbb{R}^n$.

H2. *There exist nonnegative constants* $F_i, i = 1, ..., n$ *such that for all* $t \in J$ *and* $x \in \mathbb{R}_n$ $|f_i(t, x(t))| \leq F_i$.

H3. $\sum_{i=1}^{n}\left(\Phi_i \sum_{j=1}^{n}\xi_{ij} + \Psi_i \sum_{j=1}^{n}\zeta_{ij}\right) < 1$, *where:*

$$\Phi_i := \frac{F_i}{\Gamma(\alpha_i+1)} + \frac{F_i^2|\theta_i|\sup_{s\in J}|\varphi_i(s)|\beta_i^{\alpha_i+1}}{\Gamma(\alpha_i+2)|f_i(0,x(0)) - \theta_i\int_0^{\beta_i} f_i(s,x(s))\varphi_i(s)ds|}$$

$$\Psi_i := \frac{F_i}{\Gamma(\alpha_i+\delta_i+1)} + \frac{F_i^2|\theta_i|\sup_{s\in J}|\varphi_i(s)|\beta_i^{\alpha_i+\delta_i+1}}{\Gamma(\alpha_i+\delta_i+2)|f_i(0,x(0)) - \theta_i\int_0^{\beta_i} f_i(s,x(s))\varphi_i(s)ds|}$$

are satisfied.
Then, there exists a unique solution to (1) provided that θ_i *and* $f_i(0,x(0))$ *satisfy the condition of Lemma 3.1.*

Proof

We need to proceed on two steps:
Step 1: Let $\mathfrak{B}_r$ be given by $\mathfrak{B}_r = \{x \in \prod_{i=1}^{n} X_i : \|x\|_{\prod_{i=1}^{n} X_i} < r\}$ where r is defined by:

$$r \geq \frac{\sum_{i=1}^{n}\Phi_i H_i + \Psi_i K_i}{1 - \sum_{i=1}^{n}(\Phi_i\sum_{j=1}^{n}\xi ij + \Psi_i\sum_{j=1}^{n}\zeta_{ij})} \tag{19}$$

Let $H_i := \sup_{t\in J}|h_i(t,0,...0)| < \infty$ and $K_i := \sup_{t\in J} t \in J|k_i(t,0,...0)| < \infty$, for $i = 1,...,n$.

We notice that using (**H**1), for $x \in \mathfrak{B}_r$, we can write:

$$\begin{aligned} |h_i(t,x_1,...,x_n)| &\leq |h_i(t,x_1,...,x_n) - h_i(t,0,...0)| + |h_i(t,0,...0)| \\ &\leq \sum_{j=1}^{n}\xi_{ij}j|x_j| + H_i \\ &\leq \sum_{j=1}^{n}\xi_{ij}r + H_i \end{aligned} \tag{20}$$

and

$$\begin{aligned} |k_i(t,x_1,...,x_n)| &\leq |k_i(t,x_1,...,x_n) - k_i(t,0,...0)| + |k_i(t,0,...0)| \\ &\leq \sum_{j=1}^{n}\zeta_{ij}|x_j| + K_i \\ &\leq \sum_{j=1}^{n}\zeta_{ij}r + K_i \end{aligned} \tag{21}$$

On the other hand, we have:

$$\begin{aligned}|\mathcal{Q}_i x(t)| \leq\ & |f_i(t,x(t))|\Bigg(\frac{1}{\Gamma(\alpha_i)}\int_0^t (t-\tau)^{\alpha_i-1}|h_i(\tau,x(\tau))|d\tau\\
&+\frac{1}{\Gamma(\alpha_i+\delta_i)}\int_0^t (t-\tau)^{\alpha_i+\delta_i-1}|k_i(\tau,x(\tau))|d\tau\\
&+\frac{|\theta_i|}{|f_i(0,x(0))-\theta_i\int_0^{\beta_i} f_i(s,x(s))\varphi_i(s)ds|}\int_0^{\beta_i}|f_i(s,x(s))||\varphi_i(s)|\\
&\times\Bigg[\frac{1}{\Gamma(\alpha_i)}\int_0^t (t-\tau)^{\alpha_i-1}|h_i(\tau,x(\tau))|d\tau\\
&+\frac{1}{\Gamma(\alpha_i+\delta_i)}\int_0^t (t-\tau)^{\alpha_i+\delta_i-1}|k_i(\tau,x(\tau))|d\tau\Bigg]ds\Bigg)\end{aligned} \tag{22}$$

So, using $(\mathbf{H}1), (\mathbf{H}2), (20)$, and (21), we get:

$$\begin{aligned}|\mathcal{Q}_i x(t)| \leq\ & F_i\Bigg(\frac{1}{\Gamma(\alpha_i)}\int_0^t (t-\tau)^{\alpha_i-1}d\tau\left(\sum_{j=1}^n \xi_{ij}r+H_i\right)\\
&+\frac{1}{\Gamma(\alpha_i+\delta_i)}\int_0^t (t-\tau)^{\alpha_i+\delta_i-1}d\tau\left(\sum_{j=1}^n \zeta_{ij}r+K_i\right)\\
&+\frac{F_i|\theta_i|\sup_{s\in J}|\varphi_i(s)|}{|f_i(0,x(0))-\theta_i\int_0^{\beta_i} f_i(s,x(s))\varphi_i(s)ds|}\\
&\times\int_0^{\beta_i}\Bigg[\frac{1}{\Gamma(\alpha_i)}\int_0^t (t-\tau)^{\alpha_i-1}d\tau\left(\sum_{j=1}^n \xi_{ij}r+H_i\right)\\
&+\frac{1}{\Gamma(\alpha_i+\delta_i)}\int_0^t (t-\tau)^{\alpha_i+\delta_i-1}d\tau\left(\sum_{j=1}^n \zeta_{ij}r+K_i\right)\Bigg]ds\Bigg)\end{aligned} \tag{23}$$

which leads to:

$$\begin{aligned}
\|\mathcal{Q}_i x\|_{X_i} &\leq \left(\frac{F_i}{\Gamma(\alpha_i+1)} + \frac{F_i^2|\theta_i|\sup\limits_{s\in J}|\varphi_i(s)|\beta_i^{\alpha_i+1}}{\Gamma(\alpha_i+2)|f_i(0,x(0)) - \theta_i\int_0^{\beta_i} f_i(s,x(s))\varphi_i(s)ds|}\right)\left(\sum_{j=1}^n \xi_{ij}r + H_i\right) \\
&\quad + \left(\frac{F_i}{\Gamma(\alpha_i+\delta_i+1)} + \frac{F_i^2|\theta_i|\sup\limits_{s\in J}|\varphi_i(s)|\beta_i^{\alpha_i+\delta_i+1}}{\Gamma(\alpha_i+\delta_i+2)|f_i(0,x(0)) - \theta_i\int_0^{\beta_i} f_i(s,x(s))\varphi_i(s)ds|}\right) \\
&\quad \times \left(\sum_{j=1}^n \zeta_{ij}r + K_i\right) \\
&= \Phi_i\left(\sum_{j=1}^n \xi_{ij}r + H_i\right) + \Psi_i\left(\sum_{j=1}^n \zeta_{ij}r + K_i\right)
\end{aligned} \tag{24}$$

for $i = 1, ..., n$.

So (24) implies that:

$$\|\mathcal{Q}_i x\|_{X_i} \leq \Phi_i\left(\sum_{j=1}^n \xi_{ij}r + H_i\right) + \Psi_i\left(\sum_{j=1}^n \zeta_{ij}r + K_i\right), i = 1, ..., n. \tag{25}$$

Hence,

$$\|\mathcal{Q}_i x\|_{\prod_{i=1}^n X_i} \leq r. \tag{26}$$

which leads to the conclusion that $\mathcal{Q}_i(\mathfrak{B}_r) \subset \mathfrak{B}_r$.

Step 2: Let $x, y \in X_i$. For each $t \in J$, we have:

$$\begin{aligned}
|\mathcal{Q}_i x(t) - \mathcal{Q}_i y(t)| \leq\ & |f_i(t,x(t))| \Bigg(\frac{1}{\Gamma(\alpha_i)} \int_0^t (t-\tau)^{\alpha_i - 1} |h_i(\tau, x(\tau)) - h_i(\tau, y(\tau))| d\tau \\
& + \frac{1}{\Gamma(\alpha_i+\delta_i)} \int_0^t (t-\tau)^{\alpha_i+\delta_i-1} |k_i(\tau,x(\tau)) - k_i(\tau, y(\tau))| d\tau \\
& + \frac{F_i |\theta_i|}{|f_i(0,x(0)) - \theta_i \int_0^{\beta_i} f_i(s,x(s))\varphi_i(s) ds|} \int_0^{\beta_i} |\varphi_i(s)| \\
& \times \left[\frac{1}{\Gamma(\alpha_i)} \int_0^t (t-\tau)^{\alpha_i-1} |h_i(\tau,x(\tau)) - h_i(\tau, x(\tau))| d\tau \right. \\
& \left. + \frac{1}{\Gamma(\alpha_i+\delta_i)} \int_0^t (t-\tau)^{\alpha_i+\delta_i-1} |k_i(\tau,x(\tau)) - k_i(\tau,y(\tau))| d\tau \right] ds \Bigg)
\end{aligned} \tag{27}$$

Thanks to (**H**1) and (**H**2), we get:

$$\begin{aligned}
\|\mathcal{Q}_i x - \mathcal{Q}_i y\|_{X_i} \leq\ & F_i \Bigg(\frac{1}{\Gamma(\alpha_i)} \int_0^t (t-\tau)^{\alpha_i-1} d\tau \sum_{j=1}^n \xi_{ij} \|x_j - y_j\| \\
& + \frac{1}{\Gamma(\alpha_i+\delta_i)} \int_0^t (t-\tau)^{\alpha_i+\delta_i-1} d\tau \sum_{j=1}^n \zeta_{ij} \|x_j - y_j\| \\
& + \frac{F_i|\theta_i|}{|f_i(0,x(0)) - \theta_i \int_0^{\beta_i} f_i(s,x(s))\varphi_i(s)ds|} \int_0^{\beta_i} \sup_{s\in J} |\varphi_i(s)| \\
& \times \left[\frac{1}{\Gamma(\alpha_i)} \int_0^t (t-\tau)^{\alpha_i-1} d\tau \sum_{j=1}^n \xi_{ij} \|x_j - y_j\| \right. \\
& \left. + \frac{1}{\Gamma(\alpha_i+\delta_i)} \int_0^t (t-\tau)^{\alpha_i+\delta_i-1} d\tau \sum_{j=1}^n \zeta_{ij} \|x_j - y_j\| \right] ds \Bigg)
\end{aligned} \tag{28}$$

which becomes

$$\begin{aligned}\|\mathcal{Q}_i x-\mathcal{Q}_i y\|_{X_i} &\leq \left(\frac{F_i}{\Gamma(\alpha_i+1)}+\frac{F_i^2|\theta_i|\sup\limits_{s\in J}|\varphi_i(s)|\beta_i^{\alpha_i+1}}{\Gamma(\alpha_i+2)|f_i(0,x(0))-\theta_i\int_0^{\beta_i}f_i(s,x(s))\varphi_i(s)ds|}\right)\\ &\quad\times\left(\sum_{j=1}^n\xi_{ij}\|x_j-y_j\|\right)\\ &\quad+\left(\frac{F_i}{\Gamma(\alpha_i+\delta_i+1)}+\frac{F_i^2|\theta_i|\sup\limits_{s\in J}|\varphi_i(s)|\beta_i^{\alpha_i+\delta_i+1}}{\Gamma(\alpha_i+\delta_i+2)|f_i(0,x(0))-\theta_i\int_0^{\beta_i}f_i(s,x(s))\varphi_i(s)ds|}\right)\\ &\quad\times\left(\sum_{j=1}^n\zeta_{ij}\|x_j-y_j\|\right)\\ &= \Phi_i\left(\sum_{j=1}^n\xi_{ij}\|x_j-y_j\|\right)+\Psi_i\left(\sum_{j=1}^n\zeta_{ij}\|x_j-y_j\|\right)\end{aligned} \tag{29}$$

From (29), we have:

$$\|\mathcal{Q}_i x-\mathcal{Q}_i y\|_{X_i} \leq \left(\Phi_i\sum_{j=1}^n\xi_{ij}+\Psi_i\sum_{j=1}^n\zeta_{ij}\right)\times\left(\sum_{j=1}^n\|x_j-y_j\|\right) \tag{30}$$

for $i=1,...,n$.

Therefore,

$$\|\mathcal{Q}x-\mathcal{Q}y\|_{\sum_{i=1}^n X_i} \leq \sum_{i=1}^n\left(\Phi_i\sum_{j=1}^n\xi_{ij}+\Psi_i\sum_{j=1}^n\zeta_{ij}\right)\times\left(\sum_{j=1}^n\|x_j-y_j\|\right) \tag{31}$$

Since (**H**3) assures that $\sum_{i=1}^n\left(\Phi_i\sum_{j=1}^n\xi_{ij}+\Psi_i\sum_{j=1}^n\zeta_{ij}\right)<1$, then the operator $\mathcal{Q}$ is contractive. Then, according to Banach contraction principle, the system (1) has a unique solution on $[0,1]$.

4 Open Problems

It is to note that, in the future, we will be concerned with the problem (1) for studying the existence of solution via Leray Schauder theorem and/or Kras-

noselskii fixed point lemma.
Open problem A: In this paper, we have presented some conditions to prove the existence and uniqueness of one solution for the problem (1). One first question that needs to be asked is the following:
Is it possible to change the Banach space of the above problem and to present some other conditions assuring the uniqueness of solution?
Open problem B: If we conserve the space and we change its associated norm, what can be the conditions that assure the uniqueness of solution for (1).

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