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Differential Equations Via Hadamard Approach: Some Existence/Uniqueness Results

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Abstract

In this work, we are concerned with a problem of fractional differential equations involving Hadamard operators. New existence and uniqueness result is discussed. Another existence result using Schaeffer fixed point theorem is also established.

Keywords: *Banach contraction principle, Fixed point theorem, Hadamard fractional operators, Existence and uniqueness.*
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1 Introduction

In recent years, fractional differential equations theory has been acquired much attention due to its applications in a physics, mechanics, chemistry, biology, economics, signal and image processing, etc.. For some practical developments of this fractional theory, we refer the reader to [2,5,6,8,9,14]. Other recent papers on fractional differential equations can be found in [7,10,11,15] and the references therein. It is to note that the most of the above mentioned works are based on Riemann Liouville or Caputo fractional derivatives.
In 1892, Hadamard [12] introduced another class of fractional operators, which differs from the above mentioned ones (Riemann-Liouville, Caputo) because Hadamard operators involve logarithmic functions of arbitrary exponent and named as Hadamard derivative/ Hadamard integral, for more detials, see [1,3,4,13].
Motivated by the Hadamard fractional theory, in this work, taking $1 \leq \alpha \leq$

$2, \beta > 0, 1 < \eta < e, 1 < t < e$, we are concerned with studying the existence and uniqueness as well as the existence of at leat of one solution for the following problem:

$$D^{\alpha}x(t) = f(t, x(t), D^{\alpha-1}x(t)),$$

with the integral conditions:

$$x(1) = 0, AJ^{\beta}x(\eta) + Bx'(e) = c,$$

where the derivative D^{α} and the integral J^{β} are considered in the sense of Hadamard.

2 Preliminaries on Hadamard approach

We introduce some definitions and some auxiliairy results that will be used in the paper. We begin by the following definition:

Definition 2.1 *[13] The Hadamard fractional integral of order $\alpha > 0$ of a function $f \in C([a, b])$,$0 \leq a \leq b \leq \infty$, is defined as*

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{s} \right)^{\alpha - 1} \frac{f(s)}{s} ds, \ \alpha > 0, \ t \geq a, \tag{1}$$

where $\Gamma(\alpha) := \int_0^{\infty} e^{-x} x^{\alpha-1} dx$, and $\log(.) = \log_e(.)$.

We recall also:

Definition 2.2 *[13] Let $0 \leq a \leq b \leq \infty$, $\delta = t\dfrac{d}{dt}$ and $AC_{\delta}^{n}[a, b] = \{f : [a, b] \longmapsto \mathbb{R} : \delta^{n-1}[f(t)] \in AC[a, b]\}$. The Hadamard derivative of order $\alpha > 0$ for a function $f \in AC_{\delta}^{n}[a, b]$is defined as*

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n - \alpha)} \left(t\frac{d}{dt} \right)^n \int_a^t \left(\log \frac{t}{s} \right)^{n - \alpha - 1} \frac{f(s)}{s} ds, \tag{2}$$

where $n - 1 < \alpha < n$,$n = [\alpha] + 1$.

We have also:

Lemma 2.3 *[13] Let $\alpha > 0, g \in L^p([a, b])$, $1 \leq p \leq \infty$ Then*

$$D^{\alpha} J^{\alpha} g(t) = g(t), t \in [a, b].$$

Proposition 2.4 *[13] If $\alpha, \beta > 0$, then*

$$D^{\alpha} \left(\log \frac{t}{a} \right)^{\beta - 1} = \frac{\Gamma(\beta)}{\Gamma(\beta - \alpha)} \left(\log \frac{t}{a} \right)^{\beta - \alpha - 1}$$

$$J^{\alpha} \left(\log \frac{t}{a} \right)^{\beta - 1} = \frac{\Gamma(\beta)}{\Gamma(\beta + \alpha)} \left(\log \frac{t}{a} \right)^{\beta + \alpha - 1}.$$

We need also the lemmas:

Lemma 2.5 *[13] For $\alpha > 0$, a solution of the fractional differential equation $D^\alpha x(t) = 0$ is given by*

$$x(t) = \sum_{j=1}^{n} c_j (\log t)^{\alpha-j}, \tag{3}$$

where $c_j \in \mathbb{R}$, $j = 1, ..., n$, and $n - 1 < \alpha < n$.

Lemma 2.6 *[13] Let $\alpha > 0$. We have*

$$J^\alpha D^\alpha x(t) = x(t) + \sum_{j=1}^{n} c_j \log (t)^{\alpha-j}, \tag{4}$$

where $c_j \in \mathbb{R}$, $j = 1, ..., n$, and $n - 1 < \alpha < n$.

In the literature, we can read the following Schaefer fixed point theorem.

Lemma 2.7 *Let E be a Banach space and assume that $T : E \to E$ is a completely continuous operator. If the set $V := \{x \in E : x = \mu T x,\ 0 < \mu < 1\}$ is bounded, then T has a fixed point in E.*

Now, we are ready to prove our first auxiliary "main result":

Lemma 2.8 *Let $f \in C([1, e], \mathbb{R})$. The problem*

$$\begin{cases} D^\alpha x(t) = f(t, x(t), D^{\alpha-1}x(t)), & 1 < t < e, 1 < \alpha \leq 2 \\ x(1) = 0, & AJ^\beta x(\eta) + Bx'(e) = c, \quad \beta > 0, 1 < \eta < e \end{cases} \tag{5}$$

has a unique solution given by:

$$\begin{aligned} x(t) \quad = \quad & J^\alpha f(t, x(t), D^{\alpha-1}x(t)) + (\log t)^{\alpha-1} \left[\frac{c - AJ^{\alpha+\beta} f(\eta, x(\eta), D^{\alpha-1}x(\eta))}{\Omega} \right. \\ & \left. \frac{-BJ^\alpha f(e, x(e), D^{\alpha-1}x(e))}{\Omega} \right], \end{aligned}$$

where
$\Omega = \dfrac{B(\alpha - 1)}{e} + \dfrac{A\Gamma(\alpha)}{\Gamma(\alpha + \beta)} (\log \eta)^{\alpha+\beta-1}$.

Proof: Thanks to Lemma 2.6, we have

$$x(t) = J^\alpha f\left(t, x(t), D^{\alpha-1}x(t)\right) + c_1 (\log)^{\alpha-1} + c_2 (\log)^{\alpha-2} \tag{6}$$

The first boundary condition gives $c_2 = 0$. So we obtain

$$x(t) = J^{\alpha} f(t, x(t), D^{\alpha-1} x\,(t)) + c_1\,(\log)^{\alpha-1}.$$

Thus,

$$J^{\beta} x(\eta) = J^{\alpha+\beta} f(\eta, x(\eta), D^{\alpha-1} x\,(\eta)) + \frac{c_1}{\Gamma(\beta)} \int_1^{\eta} (\log \eta)^{\beta-1} (\log s)^{\alpha-1} \frac{ds}{s}.$$

Using Proposition 2.4, we can write

$$J^{\beta} x(\eta) = J^{\alpha+\beta} f(\eta, x(\eta), D^{\alpha-1} x(\eta)) + c_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)} (\log \eta)^{\beta+\alpha-1} \tag{7}$$

and

$$x'(e) = J^{\alpha-1} f(e, x(e), D^{\alpha-1} x(e)) + \frac{c_1}{\Gamma(\alpha) e}. \tag{8}$$

Using the second boundary condition, we get:

$$c = A J^{\alpha+\beta} f(\eta, x(\eta), D^{\alpha-1} x(\eta)) \quad + \quad A c_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)} (\log \eta)^{\beta+\alpha-1} + B J^{\alpha-1} f(e, x(e), D^{\alpha-1} x(e)) + c_1 \frac{B(\alpha-1)}{e},$$

that is

$$c_1 = \frac{c - A J^{\alpha+\beta} f(\eta, x(\eta), D^{\alpha-1} x(\eta)) - B J^{\alpha} f(e, x(e), D^{\alpha-1} x(e))}{A \frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)} (\log)^{\alpha+\beta-1} + \frac{B(\alpha-1)}{e}}. \tag{9}$$

Finally, substituting the values of c_1 and c_2 in (6), we obtain (5).
This completes the proof.

Let us now consider the space defined by:

$$X := \left\{ x \mid x \in C^2([1, e], \mathbb{R}), D^{\alpha-1} x \in C([1, e], \mathbb{R}) \right\}$$

equipped with the norm

$$\| x \|_X = \| x \| + \| D^{\alpha-1} x \|.$$

On this space, we introduce the operator $T : X \longrightarrow X$ as follows:

$$\begin{aligned} (Tx)(t) \quad = \quad & \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} f(s, x(s), D^{\alpha-1} x(s)) \frac{ds}{s} + \frac{(\log t)^{\alpha-1}}{\Omega} \\ & \times \left[c - \frac{A}{\Gamma(\alpha+\beta)} \int_1^{\eta} \left(\log \frac{\eta}{s} \right)^{\alpha+\beta-1} f(s, x(s), D^{\alpha-1} x(s)) \frac{ds}{s} \right. \\ & \left. - \frac{B}{\Gamma(\alpha)} \int_1^{e} \left(\log \frac{e}{s} \right)^{\alpha-1} f(s, x(s), D^{\alpha-1} x(s)) \frac{ds}{s} \right], t \in [1, e], \end{aligned}$$

where,
$\Omega = \dfrac{B(\alpha-1)}{e} + A\dfrac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)}(\log\eta)^{\alpha+\beta-1}.$
This operator will be used to prove our main results by application of fixed point theory on Banach spaces.

3 Main results

We begin by introducing the quantities:

$$M_1 := \frac{1}{\Gamma(\alpha+\beta)} + \frac{1}{\mid\Omega\mid}\left\{\frac{\mid A\mid(\log\eta)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{\mid B\mid}{\Gamma(\alpha)}\right\}$$

and

$$M_2 := 1 + \frac{\Gamma(\alpha)}{\mid\Omega\mid}\left\{\frac{\mid A\mid(\log\eta)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{\mid B\mid}{\Gamma(\alpha)}\right\}.$$

Then, we establish the following existence and uniqueness results by application of Banach contraction principle.

3.1 Existence and Uniqueness

We have:

Theorem 3.1 *Assume that* $f : [1,e]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ *is a continuous function that satisfies:*

$$(H_1)\quad |f(t,x_1,y_1) - f(t,x_2,y_2)| \leq k_1|x_1-x_2| + k_2|y_1-y_2|, \tag{10}$$

for each $t\in[1,e]$ *and* $x_1,y_1,x_2,y_2\in\mathbb{R}$.
If we suppose

$$kM < 1, \tag{11}$$

then, the problem (5) has a unique solution on $[1,e]$,
where $k := \max\{k_1,k_2\}, M := M_1+M_2$.

Proof: To prove this theorem, we need to prove that the operator T has a fixed point in the $C([1,e],\mathbb{R})$. So, we shall prove that T is a contraction mapping on $C([1,e],\mathbb{R})$.

For $x, y \in X$ and for each $t \in [1, e]$, we have

$$\begin{aligned}|(Tx)(t)-(Ty)(t)| \leq\ & \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{|f(s,x(s),D^{\alpha-1}x(s))-f(s,y(s),D^{\alpha-1}y(s))|}{s}ds\\ &+\frac{(\log t)^{\alpha-1}}{|\Omega|}\left[\frac{|A|}{\Gamma(\alpha+\beta)}\int_1^\eta\left(\log\frac{\eta}{s}\right)^{\alpha+\beta-1}\right.\\ &\times\frac{|f(s,x(s),D^{\alpha-1}x(s))-f(s,y(s),D^{\alpha-1}y(s))|}{s}ds+\frac{|B|}{\Gamma(\alpha-1)}\\ &\left.\times\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-2}\frac{|f(s,x(s),D^{\alpha-1}x(s))-f(s,y(s),D^{\alpha-1}y(s))|}{s}ds\right].\end{aligned}$$

Thanks to (H_1), we obtain

$$\begin{aligned}|\ (Tx)(t)-(Ty)(t)\ | \leq\ & k\left(\|\ x-y\ \|+\|\ D^{\alpha-1}x-D^{\alpha-1}y\ \|\right)\max_{t\in[1,e]}\left\{\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{1}{s}ds\right.\\ &+\frac{(\log t)^{\alpha-1}}{|\Omega|}\left[\frac{|A|}{\Gamma(\alpha+\beta)}\int_1^\eta\left(\log\frac{\eta}{s}\right)^{\alpha+\beta-1}\frac{1}{s}ds\right.\\ &\left.\left.+\frac{|B|}{\Gamma(\alpha-1)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-2}\frac{1}{s}ds\right]\right\}\end{aligned}$$

Consequently, it yields that

$$\|\ (Tx)-(Ty)\ \|\leq kM_1\left(\|\ x-y\ \|+\|\ D^{\alpha-1}x-D^{\alpha-1}y\ \|\right). \qquad (12)$$

On the other hand, we observe that

$$\begin{aligned}\left|(D^{\alpha-1}Tx)(t)-(D^{\alpha-1}Ty)(t)\right| \leq\ & \int_1^t\frac{|f(s,x(s),D^{\alpha-1}x(s))-f(s,y(s),D^{\alpha-1}y(s))|}{s}ds\\ &+\frac{\Gamma(\alpha)}{(\log t)^2|\Omega|}\left[\frac{|A|}{\Gamma(\alpha+\beta)}\int_1^\eta\left(\log\frac{\eta}{s}\right)^{\alpha+\beta-1}\right.\\ &\times\frac{|f(s,x(s),D^{\alpha-1}x(s))-f(s,y(s),D^{\alpha-1}y(s))|}{s}ds\\ &+\frac{|B|}{\Gamma(\alpha-1)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-2}\\ &\left.\times\frac{|f(s,x(s),D^{\alpha-1}x(s))-f(s,y(s),D^{\alpha-1}y(s))|}{s}ds\right].\end{aligned}$$

By (H_1), we have

$$\begin{aligned}\| (D^{\alpha-1}Tx)(t) - (D^{\alpha-1}Ty)(t) \| &\leq k\left(\| x-y \| + \| D^{\alpha-1}x - D^{\alpha-1}y \|\right) \max_{t\in[1,e]}\left\{\int_1^t \frac{1}{s}ds\right.\\ &+\frac{\Gamma(\alpha)}{(\log t)^2|\Omega|}\left[\frac{|A|}{\Gamma(\alpha+\beta)}\int_1^\eta \left(\log\frac{\eta}{s}\right)^{\alpha+\beta-1}\frac{1}{s}ds\right.\\ &\left.\left.+\frac{|B|}{\Gamma(\alpha-1)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha-2}\frac{1}{s}ds\right]\right\}.\end{aligned}$$

Therefore,

$$\| (D^{\alpha-1}Tx) - (D^{\alpha-1}Ty) \| \leq kM_2\left(\| x-y \| + \| D^{\alpha-1}x - D^{\alpha-1}y \|\right). \quad (13)$$

By (12) and (13), we can write

$$\| (Tx) - (Ty) \|_X \leq kM \| x-y \|_X$$

Thanks to (11), we conclude that T is contractive.
As a consequence of Banach fixe point theorem, we deduce that T has a unique point fixe which is a solution of our problem.

3.2 Existence

Our second result will use the Scheafer fixed point theorem. We have:

Theorem 3.2 *Assume that*
(H_2) *: The function f is continuous.*
(H_3) *: There exists $L>0$, such that f is bounded by L.*
Then, the problem (5) *has at least one solution defined on* $[1,e]$.

Proof: We will prove the theorem using the following steps:

Step 1: We remark that The continuity of the functions f implies that T is continuous on X.

Step 2: The operator T is completely continuous.
We define the set $B_r := \{x\in X,\ \|x\|_X \leq r\}$, where $r>0$. For $x\in B_r$, we obtain

$$\begin{aligned}|(Tx)(t)| &\leq \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{|f(s,x(s),D^{\alpha-1}x(s))|}{s}ds\\ &+\frac{(\log t)^{\alpha-1}}{|\Omega|}\left[|c| + \frac{|A|}{\Gamma(\alpha+\beta)}\int_1^\eta \left(\log\frac{\eta}{s}\right)^{\alpha+\beta-1}\right.\\ &\times\frac{|f(s,x(s),D^{\alpha-1}x(s))|}{s}ds\\ &\left.+\frac{|B|}{\Gamma(\alpha-1)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha-2}\frac{|f(s,x(s),D^{\alpha-1}x(s))|}{s}ds\right].\end{aligned}$$

The condition (H_3) allows us to say that

$$\begin{aligned}|(Tx)(t)| &\leq \max_{t\in[1,e]}\left\{\frac{L}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{1}{s}ds\right.\\ &+\frac{(\log t)^{\alpha-1}}{|\Omega|}\left[\frac{L|A|}{\Gamma(\alpha+\beta)}\int_1^\eta\left(\log\frac{\eta}{s}\right)^{\alpha+\beta-1}\frac{1}{s}ds\right.\\ &\left.\left.+\frac{L|B|}{\Gamma(\alpha-1)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-2}\frac{1}{s}ds\right]+\frac{|c|\,(\log t)^{\alpha-1}}{|\Omega|}\right\}.\end{aligned}$$

Therefore,

$$\| \ (Tx) \ \|\leq LM_1+\frac{|c|}{|\Omega|}. \tag{14}$$

For $D^{\alpha-1}$, we have

$$\begin{aligned}\left|(D^{\alpha-1}Tx)(t)\right| &\leq \int_1^t\frac{|f(s,x(s),D^{\alpha-1}x(s))}{s}ds\\ &+\frac{\Gamma(\alpha)}{(\log t)^2|\Omega|}\left[|c|+\frac{|A|}{\Gamma(\alpha+\beta)}\int_1^\eta\left(\log\frac{\eta}{s}\right)^{\alpha+\beta-1}\frac{|f(s,x(s),D^{\alpha-1}x(s))|}{s}ds\right.\\ &\left.+\frac{|B|}{\Gamma(\alpha-1)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-2}\frac{|f(s,x(s),D^{\alpha-1}x(s))|}{s}ds\right].\end{aligned}$$

Thanks to (H_3), it yields that

$$\begin{aligned}\| \ D^{\alpha-1}Tx \ \| &\leq \max_{t\in[1,e]}\left\{\int_1^t\frac{L}{s}ds+\frac{\Gamma(\alpha)}{(\log t)^2|\Omega|}\left[\frac{L|A|}{\Gamma(\alpha+\beta)}\right.\right.\\ &\left.\left.\times\int_1^\eta\left(\log\frac{\eta}{s}\right)^{\alpha+\beta-1}\frac{1}{s}ds+\frac{L|B|}{\Gamma(\alpha-1)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-2}\frac{1}{s}ds\right]+\frac{|c|\Gamma(\alpha)}{(\log t)^2|\Omega|}\right\}.\end{aligned}$$

Hence,

$$\| \ (D^{\alpha-1}Tx)(\|\leq LM_2+\frac{|c|\Gamma(\alpha)}{|\Omega|}. \tag{15}$$

Using (12) and (13), we obtain

$$\| \ Tx \ \|_X\leq L\left(M_1+M_2\right)+2\frac{|c|\Gamma(\alpha)}{|\Omega|}. \tag{16}$$

Therefore,

$$\| \ Tx \ \|_X\leq\infty \tag{17}$$

Hence, the operator T maps bounded sets into bounded sets in X.

Step 3: Equi-continuity of $T(B_r)$:
For $t_1, t_2 \in [1, e]$; $t_1 < t_2$, and $x \in B_r$, we have:

$$\begin{aligned} |(Tx)(t_2) - (Tx)(t_1)| &= \Bigg| \frac{1}{\Gamma(\alpha)} \int_1^{t_1} \left(\log \frac{t_2}{s}\right)^{\alpha-1} \frac{f(s, x(s), D^{\alpha-1}x(s))}{s} ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{\alpha-1} \frac{f(s, x(s), D^{\alpha-1}x(s))}{s} ds \\ &- \frac{1}{\Gamma(\alpha)} \int_1^{t_1} \left(\log \frac{t_1}{s}\right)^{\alpha-1} \frac{f(s, x(s), D^{\alpha-1}x(s))}{s} ds \\ &+ \frac{\left((\log t_2)^{\alpha-1} - (\log t_1)^{\alpha-1}\right)}{\Omega} \Bigg[c - \frac{A}{\Gamma(\alpha+\beta)} \\ &\times \int_1^{\eta} \left(\log \frac{\eta}{s}\right)^{\alpha+\beta-1} \frac{f(s, x(s), D^{\alpha-1}x(s))}{s} ds \\ &- \frac{B}{\Gamma(\alpha-1)} \int_1^{e} \left(\log \frac{e}{s}\right)^{\alpha-2} \frac{f(s, x(s), D^{\alpha-1}x(s))}{s} ds \Bigg] \Bigg|. \end{aligned}$$

Thanks to (H_3), we can state that

$$\begin{aligned} |(Tx)(t_2) - (Tx)(t_1)| &\leq \frac{L}{\Gamma(\alpha)} \left| \int_1^{t_1} \left[\left(\log \frac{t_2}{s}\right)^{\alpha-1} - \left(\log \frac{t_1}{s}\right)^{\alpha-1} \right] \frac{1}{s} ds \right| \\ &+ \frac{L}{\Gamma(\alpha)} \left| \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{\alpha-1} \frac{1}{s} ds \right| \\ &+ \frac{|\left((\log t_2)^{\alpha-1} - (\log t_1)^{\alpha-1}\right)|}{|\Omega|} \Bigg[|c| + \frac{L|A|}{\Gamma(\alpha+\beta)} \int_1^{\eta} \left(\log \frac{\eta}{s}\right)^{\alpha+\beta-1} \frac{1}{s} ds \\ &+ \frac{L|B|}{\Gamma(\alpha-1)} \int_1^{e} \left(\log \frac{e}{s}\right)^{\alpha-2} \frac{1}{s} ds \Bigg] \end{aligned}$$

Therefore,

$$\begin{aligned} |(Tx)(t_2) - (Tx)(t_1)| \leq \frac{L}{\Gamma(\alpha+1)} \left[|(\log t_1)^{\alpha} - (\log t_2)^{\alpha} + (\log t_2 - \log t_1)^{\alpha}| + |(\log t_1 - \log t_2)^{\alpha}| \right] \\ + \frac{|\left((\log t_2)^{\alpha-1} - (\log t_1)^{\alpha-1}\right)|}{|\Omega|} \left[|c| + \frac{L|A|}{\Gamma(\alpha+\beta)} \int_1^{\eta} \left(\log \frac{\eta}{s}\right)^{\alpha+\beta-1} \frac{1}{s} ds \right. \\ \left. + \frac{L|B|}{\Gamma(\alpha-1)} \int_1^{e} \left(\log \frac{e}{s}\right)^{\alpha-2} \frac{1}{s} ds \right] \end{aligned} \quad (18)$$

We have also,

$$\begin{aligned}
\left|(D^{\alpha-1}Tx)(t_2)-(D^{\alpha-1}Tx)(t_1)\right| \leq\ & \left|\int_1^{t_2}\frac{f(s,x(s),D^{\alpha-1}x(s))}{s}ds-\int_1^{t_1}\frac{f(s,x(s),D^{\alpha-1}x(s))}{s}ds\right| \\
& +\frac{\Gamma(\alpha)}{(|(\log t_2)^2-(\log t_1)^2|)\,|\Omega|}\left[|c|+\frac{|A|}{\Gamma(\alpha+\beta)}\right. \\
& \times\int_1^{\eta}\left(\log\frac{\eta}{s}\right)^{\alpha+\beta-1}\frac{|f(s,x(s),D^{\alpha-1}x(s))|}{s}ds \\
& \left.+\frac{|B|}{\Gamma(\alpha-1)}\int_1^{e}\left(\log\frac{e}{s}\right)^{\alpha-2}\frac{|f(s,x(s),D^{\alpha-1}x(s))|}{s}ds\right].
\end{aligned}$$

By (H_3), we obtain

$$\begin{aligned}
\left|(D^{\alpha-1}Tx)(t_2)-(D^{\alpha-1}Tx)(t_1)\right| \leq\ & \left|\int_{t_1}^{t_2}\frac{L}{s}ds\right|+\frac{(|(\log t_2)^{-2}-(\log t_1)^{-2}|)\Gamma(\alpha)}{|\Omega|} \\
& \times\left[|c|+\frac{L|A|}{\Gamma(\alpha+\beta)}\int_1^{\eta}\left(\log\frac{\eta}{s}\right)^{\alpha+\beta-1}\frac{1}{s}ds\right. \\
& \left.+\frac{L|B|}{\Gamma(\alpha-1)}\int_1^{e}\left(\log\frac{e}{s}\right)^{\alpha-2}\frac{1}{s}ds\right].
\end{aligned}$$

Thus,

$$\begin{aligned}
\left|(D^{\alpha-1}Tx)(t_2)-(D^{\alpha-1}Tx)(t_1)\right| \leq\ & |\log t_2-\log t_1|+\frac{(|(\log t_2)^{-2}-(\log t_1)^{-2}|)\Gamma(\alpha)}{|\Omega|} \\
& \left[|c|+\frac{L|A|}{\Gamma(\alpha+\beta)}\int_1^{\eta}\left(\log\frac{\eta}{s}\right)^{\alpha+\beta-1}\frac{1}{s}ds\right. \\
& \left.+\frac{L|B|}{\Gamma(\alpha-1)}\int_1^{e}\left(\log\frac{e}{s}\right)^{\alpha-2}\frac{1}{s}ds\right].
\end{aligned}$$

Consequently, we obtain

$$\sup_{t\in[1,e]}\mid Tx(t_2)-Tx(t_1)\mid+\sup_{t\in[1,e]}\mid (D^{\alpha-1}Tx)(t_2)-(D^{\alpha-1}Tx)(t_1)\mid\longrightarrow 0\ as\ \ t_2\longrightarrow t_1$$

In these inequalities the right hand sides are independent of x and tend to zero as t_1 tends to t_2.
Then, as a consequence of Steps $2,3$ and by Arzela-Ascoli theorem, we conclude that T is completely continuous.

Step 4: The set defined by

$$\Delta:=\{(x)\in X;x=\lambda T\left(x\right),0<\lambda<1\} \tag{19}$$

is bounded:
Let $x \in \Delta$, then $x = T(x)$, for some $0 < \lambda < 1$. Thus, for each $t \in [1, e]$, we have:

$$\begin{aligned} x(t) &= \lambda \left[\frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} f(s, x(s), D^{\alpha-1}x(s)) \frac{ds}{s} + \frac{(\log t)^{\alpha-1}}{\Omega} \right. \\ & \left\{ c - \frac{A}{\Gamma(\alpha+\beta)} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{\alpha+\beta-1} f(s, x(s), D^{\alpha-1}x(s)) \frac{ds}{s} \right. \\ & \left. \left. - \frac{B}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} f(s, x(s), D^{\alpha-1}x(s)) \frac{ds}{s} \right\} \right]. \end{aligned}$$

Thanks to (H_3), we can write

$$\begin{aligned} \frac{1}{\lambda} |x(t)| &\leq \max_{t \in [1,e]} \left\{ \frac{L}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{1}{s} ds \right. \\ & + \frac{(\log t)^{\alpha-1}}{|\Omega|} \left[\frac{L|A|}{\Gamma(\alpha+\beta)} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{\alpha+\beta-1} \frac{1}{s} ds \right. \\ & \left. \left. + \frac{L|B|}{\Gamma(\alpha-1)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-2} \frac{1}{s} ds \right] + \frac{|c| (\log t)^{\alpha-1}}{|\Omega|} \right\}. \end{aligned}$$

Therefore,

$$| x(t) | \leq \lambda \left(LM_1 + \frac{|c|}{|\Omega|} \right). \tag{20}$$

On the other hand,

$$\begin{aligned} \frac{1}{\lambda} \left|(D^{\alpha-1}x(t)\right| &\leq \int_1^t \frac{|f(s, x(s), D^{\alpha-1}x(s))}{s} ds \\ & + \frac{\Gamma(\alpha)}{(\log t)^2 |\Omega|} \left[|c| + \frac{|A|}{\Gamma(\alpha+\beta)} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{\alpha+\beta-1} \frac{|f(s, x(s), D^{\alpha-1}x(s))|}{s} ds \right. \\ & \left. + \frac{|B|}{\Gamma(\alpha-1)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-2} \frac{|f(s, x(s), D^{\alpha-1}x(s))|}{s} ds \right]. \end{aligned}$$

The condition (H_3) implies that

$$\begin{aligned} \frac{1}{\lambda} | (D^{\alpha-1}x)(t) | &\leq \max_{t \in [1,e]} \left\{ \int_1^t \frac{L}{s} ds + \frac{\Gamma(\alpha)}{(\log t)^2 |\Omega|} \left[\frac{L|A|}{\Gamma(\alpha+\beta)} \right. \right. \\ & \left. \left. \times \int_1^\eta \left(\log \frac{\eta}{s}\right)^{\alpha+\beta-1} \frac{1}{s} ds + \frac{L|B|}{\Gamma(\alpha-1)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-2} \frac{1}{s} ds \right] + \frac{|c|\Gamma(\alpha)}{(\log t)^2 |\Omega|} \right\}. \end{aligned}$$

Hence, we can write

$$| (D^{\alpha-1}x)(t) | \leq \lambda \left(LM_2 + \frac{|c|\Gamma(\alpha)}{|\Omega|} \right). \tag{21}$$

It follows from (20) and (21) that

$$\| x \|_X \leq \lambda \left(L\left(M_1 + M_2\right) + 2\frac{|c|\Gamma(\alpha)}{|\Omega|} \right). \tag{22}$$

Thus,

$$\| x \|_X \leq \infty \tag{23}$$

Consequently, Δ is bounded.
As a conclusion of Schaefer fixed point theorem, we deduce that T has at least one fixed point, which is a solution of (1).

4 Open Problem

Is it possible to extend the above results in the case of coupled order of Hadamard integration?
The same question can be posed with the $(k, s)-$Riemann-Liouville integrals and with mixed Riemann-Liouville integrals.

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