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EXISTENCE OF POSITIVE SOLUTIONS FOR NONLINEAR FRACTIONAL PROBLEM ON THE HALF LINE

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Abstract. This paper deals with the differential equations of fractional order on the half-line. By Leggett-Williams theorem, we present recent results for the existence of positive solutions for a Caputo fractional problem. An illustrative example is also presented.

1. Introduction

In the last few years, fractional differential equations theory have received increasing attention. This theory has been developed very quickly and attracted a considerable interest from researches (see [**1, 2, 3, 6, 7**]).

The motivation for those works stems from both the development of the theory and the applications of such constructions in various sciences such as physics, mechanics, chemistry, engineering, and so on. For an extensive collection of such results, we refer the readers to the monographs by Kilbas and al [**10**], Miller and Ross [16] and Podlubný [**18**].

As one of the focal topics in the research, some kinds of fractional differential equation with specific configurations have been presented. More specifically, In [**4**], the authors investigated the existence and multiplicity of positive solutions of the nonlinear fractional differential equation boundary value problem

$$\begin{cases} D^{\alpha}x(t) + a(t)f(x(t)) = 0 \qquad 0 < t < 1,\ 1 < \alpha \le 2 \\ x(0) = 0,\ x'(1) = 0 \end{cases}$$

By using Krasnosel'skii' s fixed point theorem and Leggett-Williams theorem [**17, 11**], some sufficient conditions for the existence of positive solutions to the above FBVP are obtained. Moreover, the study of positive solution has been studied in [**5, 8, 9, 13, 19**].

To the best of our knowledge, there are few papers devoted to the study of fractional differential equations with a Laplacian operator on the half-line [**12, 14, 15, 20, 21**], where boundary value problems on the half-line have been applied (unsteady flow of gas through a semi-infinite porous medium, the theory of drain flows, etc).

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In this article, we are concerned with the following fractional differential equation:

$$(1)\quad \begin{cases} \Big(p(t)\Phi(D^\alpha x(t))\Big)' + f(t,x(t)) = 0, & t\in[0,\infty),\ 0<\alpha\le 1\\ x(0) = \int\limits_0^\infty g(s)x(s)\mathrm{d}s + \theta, & \\ \lim\limits_{t\to+\infty} p(t)\Phi(D^\alpha x(t)) = \rho, & \end{cases}$$

where $\rho \ge 0$, $\theta \ge 0$, and D^α denotes Caputo fractional derivative of order α, $g\colon [0,\infty) \to [0,\infty)$ is continuous with $\int_0^\infty g(s)\mathrm{d}s < 1$, $p\colon [0,\infty) \to (0,\infty)$ is continuous, and $\Phi(x) = |x|^{q-2}x$ with $q > 1$, and, the inverse function of Φ is $\Phi^{-1}(x) = |x|^{q'-2}x$, where $\frac{1}{q} + \frac{1}{q'} = 1$.

This paper is organized as follows: In section 2, we prepare some material need to prove our main results. In section 3, we obtain existence results of the positive solutions for (1) using Leggett-Williams theorem. In section 4, we give an example to illustrate our results.

2. Preliminaries

In this section, we give some definitions, lemmas and properties which will be used in the next sections.

Definition 1. The Riemann-Liouville fractional integral operator of order $\alpha>0$, for a continuous function f on $[0,\infty)$ is defined as:

$$(2)\qquad J^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)\mathrm{d}\tau \qquad \alpha>0,\ t>0.$$

Definition 2. The Caputo derivative of order α of $f \in C^n([0,\infty[)$ is defined as:

$$(3)\quad D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\mathrm{d}\tau \qquad n-1<\alpha,\ n\in N^*.$$

Definition 3. A function $f\colon [0,\infty)\times\mathbb{R} \to \mathbb{R}$ is called a Carathéodory function if the following conditions are satisfied

(i) For each $u\in\mathbb{R}$, $t \to f(t,u)$ is measurable on $(0,\infty)$,
(ii) For each $t\in[0,\infty)$, $u \to f(t,u)$ is continuous on $\mathbb{R}$,
(iii) For each $r>0$, there exists B_r, $B_r(t)>0$, $t\in[0,\infty)$, $\int_0^\infty B_r(s)\mathrm{d}s < \infty$, such that $|u|\le r$ implies $|f(t,u)| \le B_r(t), t\in[0,\infty)$.

Definition 4. Let X be a real Banach space. The nonempty convex closed subset P of X is called a cone in X if

(i) $ax\in P$ and $x+y\in P$ for all x, $y\in P$ and $a\ge 0$,
(ii) if $x\in P$ and $-x\in P$, then $x=0$.

Let ψ be a nonnegative functional on a cone P of a real Banach space X. We define the sets

$$P_r = \{y\in P : |y| < r\},$$
$$P(\psi;a,b) = \{y\in P : a\le\psi(y),\ |y|<b\}.$$

We need also the Leggett-Williams fixed point theorem [**8**]

Theorem 5. *Let $a < b < d < c$ be positive numbers, $T\colon \overline{P_c} \to \overline{P_c}$ be a completely continuous operator, and ψ be a nonnegative continuous concave functional on P, such that $\psi(y) \le \|y\|$ for all $y \in \overline{P_c}$. Suppose that*

- $\{y \in P(\psi; b, d) : \psi(y) > b\} \ne \emptyset,\ \psi(Ty) > b$ *for* $y \in P(\psi; b, d)$,
- $\|Ty\| < a$, *for* $y \in P$, *with* $\|y\| \le a$,
- $\psi(Ty) > a$ *for* $y \in P(\psi; b, c)$ *with* $\|Ty\| > d$.

Then T has at least three fixed points y_1, y_2 and y_3 such that $\|y_1\| < a$, $\psi(y_2) > b$ and $\|y_3\| > a$ with $\psi(y_3) < b$.

We cite also the following three lemmas:

Lemma 6. *For $\alpha > 0$, the general solution of the fractional differential equation $D^\alpha x = 0$ is given by*

$$x(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, \tag{4}$$

where $c_i \in \mathbb{R}$, $i = 0, 1, 2, .., n-1$, $n = [\alpha] + 1$.

Lemma 7. *Let $\alpha > 0$. Then we have*

$$J^\alpha D^\alpha x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, \tag{5}$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \dots, n-1$, $n = [\alpha] + 1$.

Lemma 8. *Let $\beta > \alpha > 0$. Then the formula*

$$D^\alpha J^\beta f(t) = J^{\beta-\alpha} f(t), \qquad t \in [a, b] \tag{6}$$

is valid.

3. Main Results

We introduce the following quantities:

$$\lambda := \lambda(t) = \int_0^t \Phi^{-1}\Big(\frac{1}{p(s)}\Big)\mathrm{d}s.$$

$$\lambda^* := \int_0^\infty \Phi^{-1}\Big(\frac{1}{p(s)}\Big)\mathrm{d}s.$$

For $k > 1$ large enough, we take $\lambda(\frac{1}{k}) < 1$.

We take also

$$\mu := \int_0^{\frac{1}{k}} \Phi^{-1}\Big(\frac{1}{p(s)}\Big)\mathrm{d}s \frac{1}{1 + \int_0^\infty \Phi^{-1}(\frac{1}{p(s)})\mathrm{d}s}.$$

It is clear that $0 < \mu < 1$.

Now, we consider the following Banach space

$$X = \{x \in C[0, \infty) : \lim_{t\to\infty} x(t) < \infty\},$$

with the norm

$$\|x\| = \sup_{t\in[0,\infty)} |x(t)| \qquad \text{for } x \in X,$$

and we define the cone P by

$$P := \Big\{x \in X : x(t) \geq 0,\ x(t) \text{ is non-decreasing on } [0,\infty),\\ \min_{t\in[\frac{1}{k},k]} x(t) \geq \mu \sup_{t\in[0,\infty)} x(t)\Big\}.$$

On P, we define the functional

$$\psi(y) := \min_{t\in[\frac{1}{k},k]} y(t).$$

It is easy to see that ψ is a nonnegative continuous concave functional on P satisfies

$$\psi(y) \leq \|y\|, \qquad y \in P.$$

Now, we introduce the following hypotheses:

($\mathbf{H_1}$) The constant $\lambda^* < \infty$, and the function $p\colon [0,\infty) \to (0,\infty)$ is continuous and satisfies

$$\int_0^\infty g(t) \int_0^t \Phi^{-1}\Big(\frac{1}{p(u)}\Big)\mathrm{d}u\mathrm{d}t < \infty.$$

($\mathbf{H_2}$) The function $f\colon [0,\infty)\times[0,\infty) \to [0,\infty)$ is a Carathéodory function with $f(t,0) \neq 0$ on each subinterval of $[0,\infty)$.

($\mathbf{H_3}$) There exist $A, c > 0$ such that

$$\left(\rho + \int_0^\infty \frac{Ac}{(1+t)^2}\mathrm{d}s\right)\left[\frac{\int_0^\infty g(u)\left[\theta + J^\alpha\Phi^{-1}(\frac{1}{p(t)})\right]\mathrm{d}u}{1-\int_0^\infty g(s)\mathrm{d}s} + J^\alpha\Phi^{-1}\Big(\frac{1}{p(t)}\Big)\right] + \theta \leq c.$$

($\mathbf{H_4}$) There exist $b > 0, B > 0$, such that

$$\frac{\int_0^\infty g(u)\left[\theta + \frac{1}{\Gamma(\alpha)}\int_0^{\frac{1}{k}}(\frac{1}{k}-\tau)\Phi^{-1}(\frac{\rho}{p(\tau)} + \frac{1}{p(\tau)}\int_\tau^\infty \frac{Bb}{(1+s)^2}\mathrm{d}s)\mathrm{d}\tau\right]\mathrm{d}u}{1-\int_0^\infty g(s)\mathrm{d}s} \geq b.$$

We prove the following two lemmas:

Lemma 9. *Let $0 < \alpha \leq 1$ and suppose that* ($\mathbf{H_1}$) *and* ($\mathbf{H_2}$) *hold. A solution of the problem* (1) *is given by*

$$\begin{aligned} x(t) = {} & \frac{1}{1-\int_0^\infty g(s)\mathrm{d}s}\int_0^\infty g(u)\left[\theta + J^\alpha\Phi^{-1}(\frac{\rho+\int_t^\infty f(s,x(s))\mathrm{d}s}{p(t)})\right]\mathrm{d}u \\ & + \theta + J^\alpha\Phi^{-1}\left[\frac{\rho+\int_t^\infty f(s,x(s))\mathrm{d}s}{p(t)}\right] \end{aligned} \tag{7}$$

Proof. Since f is a Carathéodory function, then we can write

$$p(t)\Phi(D^\alpha x(t)) = \rho + \int_t^\infty f(s,x(s))\mathrm{d}s.$$

Therefore,

$$D^\alpha x(t) = \Phi^{-1}\left[\frac{\rho+\int_t^\infty f(s,x(s))\mathrm{d}s}{p(t)}\right].$$

Thanks to Lemma 7, we deduce that

$$x(t) = x(0) + J^\alpha \Phi^{-1}\left[\frac{\rho + \int_t^\infty f(s,x(s))\mathrm{d}s}{p(t)}\right].$$

The boundary conditions in (1) imply that

$$x(t) = \int_0^\infty g(s)x(s)\mathrm{d}s + \theta + J^\alpha \Phi^{-1}\left[\frac{\rho + \int_t^\infty f(s,x(s))\mathrm{d}s}{p(t)}\right]. \tag{8}$$

Since

$$\int_0^\infty g(s)\mathrm{d}s < 1,$$

then it follows that

$$\begin{aligned}\int_0^\infty g(s)x(s)\mathrm{d}s = &\frac{1}{1-\int_0^\infty g(s)\mathrm{d}s} \\ &\cdot \int_0^\infty g(u)\left[\theta + J^\alpha\Phi^{-1}\left(\frac{\rho + \int_t^\infty f(s,x(s))\mathrm{d}s}{p(t)}\right)\right]\mathrm{d}u.\end{aligned} \tag{9}$$

Combining (8) and (9), we get Lemma 9. □

Lemma 10. *Suppose that* $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$ *hold and* x *is a solution of* (1). *Then we have:*

(a) $D^\alpha x(t) \geq 0$; $t \in [0,\infty[$.
(b) *The function* x *is concave with respect to* λ *on* $[0,\infty)$, *and it is positive with respect to* t *on* $[0,\infty)$.

Proof. (a) Since x is a solution of (1), then for all $t \in [0,\infty)$ we have

$$[p(t)\Phi(D^\alpha x(t))]' \leq 0.$$

Thanks to $(\mathbf{H_2})$,we have

$$\rho - p(t)\Phi(D^\alpha x(t)) \leq 0, \qquad t \in [0,\infty)$$

which satisfies

$$p(t)\Phi(D^\alpha x(t)) \geq 0, \qquad \text{since } \rho \geq 0.$$

Thus

$$D^\alpha x(t) \geq 0 \qquad \text{for all } t \in [0,\infty).$$

(b) Since $D^\alpha x \geq 0$, then to prove that $x > 0$, it suffices to show that $x(0) \geq 0$. We have

$$x(0) = \int_0^\infty g(s)x(s)\mathrm{d}s + \theta \geq x(0)\int_0^\infty g(s)\mathrm{d}s.$$

Since $\int_0^\infty g(s)\mathrm{d}s < 1$, it follows then that $x(0) \geq 0$.
Hence

$$x(t) \geq 0 \quad \text{for } t \in [0,\infty).$$

With help of $(\mathbf{H_2})$ it follows that

$$x(t) > 0 \quad \text{for all } t \in (0,\infty).$$

Finally, we shall prove that x is concave with respect to λ on $[0,\infty)$.

Thanks to $(\mathbf{H_1})$, we have $\lambda^* < \infty$, and then $\lambda \in C([0,\infty), [0,\lambda^*[)$.
On the other hand, we have

$$\frac{\mathrm{d}x}{\mathrm{d}\lambda} = \frac{\mathrm{d}x}{\mathrm{d}t}\frac{1}{\Phi^{-1}\big(\frac{1}{p(t)}\big)} \geq 0,$$

$$p(t)\Phi\Big(\frac{\mathrm{d}x}{\mathrm{d}t}\Big) = \Phi\Big(\frac{\mathrm{d}x}{\mathrm{d}\lambda}\Big),$$

and

$$\frac{\mathrm{d}^2x}{\mathrm{d}\lambda^2} = \frac{\Big[p(t)\Phi\Big(\frac{\mathrm{d}x}{\mathrm{d}t}\Big)\Big]'}{\Phi'\Big(\frac{\mathrm{d}x}{\mathrm{d}\lambda}\Big)\frac{\mathrm{d}\lambda}{\mathrm{d}t}}.$$

Using the fact that $[p(t)\Phi(D^\alpha x(t))]' \leq 0$, $\alpha = 1$, $\Phi'(x) > 0$, $x > 0$ and $\frac{\mathrm{d}\lambda}{\mathrm{d}t} > 0$, we obtain $\frac{\mathrm{d}^2x}{\mathrm{d}\lambda^2} \leq 0$.

Hence $x(t)$ is concave with respect to λ on $[0,\infty)$. The proof is complete. □

Now, we define the following nonlinear operator $T\colon P \to X$ by:

$$\begin{aligned} Tx(t) = {} & \frac{1}{1-\int_0^\infty g(s)\mathrm{d}s}\int_0^\infty g(u)\left[\theta + J^\alpha\Phi^{-1}\left(\frac{\rho + \int_t^\infty f(s,x(s))\mathrm{d}s}{p(t)}\right)\right]\mathrm{d}u \\ & + \theta + J^\alpha\Phi^{-1}\left[\frac{\rho + \int_t^\infty f(s,x(s))\mathrm{d}s}{p(t)}\right]. \end{aligned} \tag{10}$$

Then, we prove the following result:

Lemma 11. *Suppose that* $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$ *hold. We have*

(i*) *For $x \in P$, Tx satisfies*

$$(p(t)\Phi(D^\alpha Tx(t)))' + f(t,x(t)) = 0, \quad t \in [0,\infty), \quad 0 < \alpha \leq 1,$$
$$Tx(0) = \int_0^\infty g(s)Tx(s)\mathrm{d}s + \theta, \quad \theta \geq 0,$$
$$\lim_{t\to+\infty} p(t)\Phi(D^\alpha Tx(t)) = \rho, \quad \rho \geq 0.$$

(ii*) *$Tx \in P$ for each $x \in P$.*

(iii*) *x is a bounded positive solution of* (1) *if and only if x is a solution the equation $x = Tx$ in P.*

Proof. The proof of (i*) follows from the definition of T and is omitted.

To show (ii*), we note from $(i*)$ that Tx is a solution of (1). Then, Lemma 10 implies that

$$Tx(t) \geq 0, \quad T'x(t) \geq 0 \qquad \text{for all } t \in [0,\infty),$$

and $Tx(t)$ is concave with respect to λ.

To complete the proof of $TP \subseteq P$, it suffices to prove that

$$\min_{t\in[1/k,k]} Tx(t) \geq \mu \sup_{t\in[0,\infty)} Tx(t). \tag{11}$$

Since $x \in X$ and f is a Carathéodory function, then

$$
\begin{aligned}
Tx(t) \leq & \frac{1}{1-\int_0^\infty g(s)\mathrm{d}s}\int_0^\infty g(u)\left[\theta + J^\alpha \Phi^{-1}(\frac{\rho+\int_t^\infty B_r(s)\mathrm{d}s}{p(t)})\right]\mathrm{d}u \\
& + \theta + \frac{1}{\Gamma(\alpha)}\int_0^\infty (t-\tau)^{\alpha-1}\Phi^{-1}\left[\frac{\rho+\int_\tau^\infty B_r(s)\mathrm{d}s}{p(\tau)}\right]\mathrm{d}\tau < \infty.
\end{aligned}
$$

This means that $\sup_{t\in[0,\infty)} Tx(t)$ exists. So, we shall consider two cases:

Case A. Suppose $Tx(t)$ achieves its maximum at $\sigma \in [0,\infty)$

For $t \in [1/k, k]$, we can write

$$
\begin{aligned}
Tx(t) &\geq Tx(1/k) = Tx(t(\lambda(1/k))) \\
&= Tx\left(t\left(\frac{1-\lambda(1/k)+\lambda(\sigma)}{1+\lambda(\sigma)}\frac{\lambda(1/k)}{1-\lambda(1/k)+\lambda(\sigma)} + \frac{\lambda(1/k)}{1+\lambda(\sigma)}\lambda(\sigma)\right)\right)
\end{aligned}
$$

Thanks to the concavity of Tx with respect to λ, we have:

$$
\begin{aligned}
Tx(t) &\geq \frac{1-\lambda(1/k)+\lambda(\sigma)}{1+\lambda(\sigma)}Tx\left(t\left(\frac{\lambda(1/k)}{1-\lambda(1/k)+\lambda(\sigma)}\right)\right) + \frac{\lambda(1/k)}{1+\lambda(\sigma)}Tx(t(\lambda(\sigma))) \\
&\geq \frac{\lambda(1/k)}{1+\lambda(\sigma)}Tx(t(\lambda(\sigma))) = \int_0^{1/k}\Phi^{-1}\left(\frac{1}{p(s)}\right)\mathrm{d}s\frac{1}{1+\lambda(\sigma)}Tx(t(\lambda(\sigma))) \\
&\geq \int_0^{1/k}\Phi^{-1}\left(\frac{1}{p(s)}\right)\mathrm{d}s\frac{1}{1+\int_0^\infty \Phi^{-1}(\frac{1}{p(s)})\mathrm{d}s}\sup_{t\in[0,\infty)} Tx(t) \\
&= \mu \sup_{t\in[0,\infty)} Tx(t).
\end{aligned}
$$

Case B. Now, suppose $Tx(t)$ achieves its maximum at ∞ : Choose $\sigma' \in [0,\infty)$, then, with the same arguments as before, we get for $t \in [1/k, k]$ that

$$Tx(t) \geq \mu Tx(\sigma').$$

Let $\sigma' \to \infty$, then for $t \in [1/k, k]$, we can write

$$Tx(t) \geq \mu \sup_{t\in[0,\infty)} Tx(t).$$

Thanks to A and B, we deduce that $Tx \in P$.

The proof of (iii*) is based on Lemma 8 and it can be omitted. □

Our main result is given by:

Theorem 12. *Assume that the hypothesis* $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$ *hold, and there exist constants* a, b *and* c *such that* $0 < a < b < c$.

(D$_1$) $f(t,x) < \dfrac{Aa}{(1+t)^2}$ *for all* $t \in (0,\infty)$ *and* $x \in [0,a]$;

(D$_2$) $f(t,x) > \dfrac{Bb}{(1+t)^2}$ *for all* $t \in [1/k, k]$ *and* $x \in [b,c]$;

(D$_3$) $f(t,x) \leq \dfrac{Ac}{(1+t)^2}$ *for all* $t \in (0,\infty)$ *and* $x \in [0,c]$.

Then, the (1) *has at least three bounded positive solutions* x_1, x_2 *and* x_3 *satisfying* $\|x_1\| < a$, $b < \psi(x_2)$, $a < x_3$ *with* $\psi(x_3) < b$.

Proof. We shall prove that the operator T is completely continuous on P.
It is easy to verify that $T\colon P \to P$ is well defined. We prove that T is continuous and maps bounded sets into pre-compact sets: Let $x_n \to x_0$ as $n \to \infty$ in P, then there exists r_0 such that

$$\sup_{n\geq 0} \|x_n\| < r_0.$$

Hence, we have

$$\int_0^\infty |f(s, x_n(s)) - f(s, x_0(s))| \, \mathrm{d}s \leq \int_0^\infty B_{r_0}(s)\mathrm{d}s.$$

By the Lebesgue dominated convergence theorem, we obtain

$$\int_t^\infty f(u, x_n(u))\mathrm{d}u \to \int_t^\infty f(u, x_0(u))\mathrm{d}u \text{ uniformly as } n \to \infty.$$

Let $\varepsilon > 0$. For all n, we have

$$\Phi(\rho) + \int_s^\infty f(u, x_n(u))\mathrm{d}u \leq \Phi(\rho) + \int_0^\infty B_{r_0}(s)\mathrm{d}s \equiv r.$$

On the other hand, we know that Φ^{-1} is uniformly continuous on $[0, r]$. It follows then that there exists $\delta > 0$, such that for $x, y \in [0, r]$, $|x - y| < \delta$, we have

$$\left|\Phi^{-1}(x) - \Phi^{-1}(y)\right| < \varepsilon.$$

So for the above $\delta > 0$, there exists $N > 0$, such that

$$\left|\rho + \int_t^\infty f(u, x_n(u))\mathrm{d}u - (\rho + \int_t^\infty f(u, x_0(u))\mathrm{d}u)\right| < \delta,$$

where $n > N$, $t \in [0, \infty)$.

Then for $n > N$, we can write

$$\left|\Phi^{-1}(\rho + \int_t^\infty f(u, x_n(u))\mathrm{d}u) - \Phi^{-1}(\rho + \int_t^\infty f(u, x_0(u))\mathrm{d}u)\right| < \varepsilon.$$

Hence, for $t \in [0, \infty)$, and $n > N$, yields

$$\begin{aligned}
|Tx_n &- Tx_0(t)| \\
&= \frac{1}{1 - \int_0^\infty g(s)\mathrm{d}s} \int_0^\infty g(u) \Big[J^\alpha \Big[\Phi^{-1}\Big(\frac{\rho + \int_t^\infty f(s, x_n(s))\mathrm{d}s}{p(t)}\Big) \\
&\quad - \Phi^{-1}\Big(\frac{\rho + \int_t^\infty f(s, x_0(s))\mathrm{d}s}{p(t)}\Big)\Big]\mathrm{d}u\Big] \\
&\quad + J^\alpha \Big[\Phi^{-1}\Big(\frac{\rho + \int_t^\infty f(s, x_n(s))\mathrm{d}s}{p(t)}\Big) - \Phi^{-1}\Big(\frac{\rho + \int_t^\infty f(s, x_0(s))\mathrm{d}s}{p(t)}\Big)\Big] \\
&\leq \frac{\varepsilon}{1 - \int_0^\infty g(s)\mathrm{d}s} \int_0^\infty g(u) \Big(J^\alpha \Phi^{-1}\Big(\frac{1}{p(t)}\Big)\Big)\mathrm{d}u + \varepsilon J^\alpha \Phi^{-1}\Big(\frac{1}{p(t)}\Big) \\
&\leq \varepsilon \Big[\frac{1}{1 - \int_0^\infty g(s)\mathrm{d}s} \int_0^\infty g(u) \Big(J^\alpha \Phi^{-1}\Big(\frac{1}{p(t)}\Big)\Big)\mathrm{d}u + J^\alpha \Phi^{-1}\Big(\frac{1}{p(t)}\Big)\Big].
\end{aligned}$$

It follows then that

$$\|Tx_n - Tx_0\| \to 0 \text{ uniformly as } n \to \infty.$$

So, T is continuous.

Let Ω be an arbitrary bounded subset of P. First, we shall prove that T is bounded: Since Ω is bounded, then there exists $r > 0$, such that $\|x\| \leq r$, for all $x \in \Omega$. Thanks to Definition 3, we obtain

$$\begin{aligned}
0 \leq Tx(t) &= \frac{1}{1-\int_0^\infty g(s)\mathrm{d}s} \int_0^\infty g(u)\Big[\theta + J^\alpha \Phi^{-1}\Big(\frac{\rho + \int_t^\infty f(s,x(s))\mathrm{d}s}{p(t)}\Big)\Big]\mathrm{d}u \\
&\quad + \theta + J^\alpha \Phi^{-1}\Big[\frac{\rho + \int_t^\infty f(s,x(s))\mathrm{d}s}{p(t)}\Big] \\
&\leq \frac{1}{1-\int_0^\infty g(s)\mathrm{d}s} \int_0^\infty g(u)\Big[\theta + J^\alpha \Phi^{-1}\Big(\frac{\rho + \int_s^\infty B_r(s)\mathrm{d}s}{p(t)}\Big)\Big]\mathrm{d}u \\
&\quad + \theta + \frac{1}{\Gamma(\alpha)} \int_0^\infty (t-\tau)^{\alpha-1}\Phi^{-1}\Big[\frac{\rho + \int_\tau^\infty B_r(s)\mathrm{d}s}{p(\tau)}\Big]\mathrm{d}\tau.
\end{aligned}$$

This means that $T\Omega$ is bounded.

Now we shall prove the equicontinuity of $T(B_r)$. Let us take $t_1,\ t_2 \in (0,\infty)$, $t_1 < t_2$ and $x \in \Omega$. We have

$$\begin{aligned}
&|Tx(t_2) - Tx(t_1)| \\
&\leq \frac{1}{1-\int_0^\infty g(s)\mathrm{d}s} \\
&\quad \times \int_0^\infty g(u)\Big[\int_0^{t_2}\Big(\frac{(t_2-\tau)^{\alpha-1} - (t_1-\tau)^{\alpha-1}}{\Gamma(\alpha)}\Big)\Phi^{-1}\Big(\frac{\rho + \int_\tau^\infty B_r(s)\mathrm{d}s}{p(\tau)}\Big)\mathrm{d}\tau\Big]\mathrm{d}u \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^{t_2}((t_2-\tau)^{\alpha-1} - (t_1-\tau)^{\alpha-1})\Phi^{-1}\Big[\frac{\rho + \int_\tau^\infty B_r(s)\mathrm{d}s}{p(\tau)}\Big]\mathrm{d}\tau.
\end{aligned}$$

We see that the function $\varphi(x) = x^{\alpha-1} - (\alpha-1)x$ is decreasing on $[0,1]$, and increasing on $(1,\infty)$. Consequently we can write

$$(t_2-\tau)^{\alpha-1} - (t_1-\tau)^{\alpha-1} \leq (\alpha-1)(t_2-t_1), \quad \text{on } [0,1]$$

and

$$(t_1-\tau)^{\alpha-1} - (t_2-\tau)^{\alpha-1} \leq (\alpha-1)(t_1-t_2), \quad \text{on } (1,\infty).$$

We deduce from these two inequalities that if $t_1, t_2 \in [0,1]$, then

$$\begin{aligned}
&|Tx(t_2) - Tx(t_1)| \\
&\leq \frac{1}{1-\int_0^\infty g(s)\mathrm{d}s}\int_0^\infty g(u)\Big[\frac{(\alpha-1)(t_2-t_1)}{\Gamma(\alpha)}\int_0^{t_2}\Phi^{-1}\Big(\frac{\rho + \int_\tau^\infty B_r(s)\mathrm{d}s}{p(\tau)}\Big)\mathrm{d}\tau\Big]\mathrm{d}u \\
&\quad + \frac{(\alpha-1)(t_2-t_1)}{\Gamma(\alpha)}\int_0^{t_2}\Phi^{-1}\Big[\frac{\rho + \int_\tau^\infty B_r(s)\mathrm{d}s}{p(\tau)}\Big]\mathrm{d}\tau
\end{aligned}$$

and

$$
\begin{aligned}
&|Tx(t_2)-Tx(t_1)|\\
&\le \frac{1}{1-\int_0^\infty g(s)\mathrm{d}s}\int_0^\infty g(u)\Big[\frac{(\alpha-1)(t_1-t_2)}{\Gamma(\alpha)}\int_0^{t_2}\Phi^{-1}\Big(\frac{\rho+\int_\tau^\infty B_r(s)\mathrm{d}s}{p(\tau)}\Big)\mathrm{d}\tau\Big]\mathrm{d}u\\
&\quad+\frac{(\alpha-1)(t_1-t_2)}{\Gamma(\alpha)}\int_0^{t_2}\Phi^{-1}\Big[\frac{\rho+\int_\tau^\infty B_r(s)\mathrm{d}s}{p(\tau)}\Big]\mathrm{d}\tau,
\end{aligned}
$$

provided that t_1, $t_2\in(1,\infty)$.

When $t_1\to t_2$, in the above tow inequalities, we claim that $|Tx(t_2)-Tx(t_1)|$ tend to 0. Consequently $T\Omega$ is equicontinuous. According to the Ascoli-Arzela theorem we deduce that T is completely continuous operator.

Note that $\psi(x)\le\|x\|$ for $x\in\overline{P_c}$. We will show that the conditions of Theorem 5 are satisfied: Put $x\in P_c$. Then $\|x\|\le c$, we find

$$
\begin{aligned}
\|Tx(t)\|&=\sup_{t\in[0,\infty)}Tx(t)\\
&=\frac{1}{1-\int_0^\infty g(s)\mathrm{d}s}\int_0^\infty g(u)\Big[\theta+J^\alpha\Phi^{-1}\Big(\frac{\rho+\int_t^\infty f(s,x(s))\mathrm{d}s}{p(t)}\Big)\Big]\mathrm{d}u\\
&\quad+\theta+J^\alpha\Phi^{-1}\Big[\frac{\rho+\int_t^\infty f(s,x(s))\mathrm{d}s}{p(t)}\Big].
\end{aligned}
$$

Thanks to (D_3), we have

$$
\begin{aligned}
\|Tx(t)\|\le{}&\frac{\big(\rho+\int_0^\infty\frac{Ac}{(1+s)^2}\mathrm{d}s\big)}{1-\int_0^\infty g(s)\mathrm{d}s}\int_0^\infty g(u)\Big[\theta+J^\alpha\Phi^{-1}\Big(\frac{1}{p(t)}\Big)\Big]\mathrm{d}u\\
&+\theta+\Big(\rho+\int_0^\infty\frac{Ac}{(1+s)^2}\mathrm{d}s\Big)J^\alpha\Phi^{-1}\Big(\frac{1}{p(t)}\Big).
\end{aligned}
$$

Using ($\mathbf{H_3}$), we get

$$\|Tx(t)\|\le c$$

This implies that $T\colon P_c\to P_c$.

By the same way, if $x\in P_a$, then with help of (D_1), we obtain $\|Tx\|<a$, and therefore (C_2) is satisfied.

Let d be a fixed constant such that $b<d\le c$. Then $\psi(d)\ge d>b$ and $\|d\|=d$, it means $P(\psi,b,d)\neq\emptyset$.

For any $x\in P(\psi,b,d)$, it holds that $\|x\|\le d$ and $\psi(x)=\min_{t\in[\frac{1}{k},k]}x(t)$. Then we have

$$
\begin{aligned}
\psi(Tx)&=\min_{t\in[\frac{1}{k},k]}Tx(t)=Tx\Big(\frac{1}{k}\Big)\\
&=\frac{\int_0^\infty g(u)\left[\theta+\frac{1}{\Gamma(\alpha)}\int_0^{\frac{1}{k}}(\frac{1}{k}-\tau)\Phi^{-1}\big(\frac{\rho+\int_\tau^\infty f(s,x(s))\mathrm{d}s}{p(\tau)}\big)\mathrm{d}\tau\right]\mathrm{d}u}{1-\int_0^\infty g(s)\mathrm{d}s}\\
&\quad+\theta+\frac{1}{\Gamma(\alpha)}\int_0^{\frac{1}{k}}\Big(\frac{1}{k}-\tau\Big)\Phi^{-1}\Big(\frac{\rho+\int_\tau^\infty f(s,x(s))\mathrm{d}s}{p(\tau)}\Big)\mathrm{d}\tau
\end{aligned}
$$

In view of (D_2) and ($\mathbf{H_4}$), we obtain

$$\psi(Tx) \geq \frac{\int_0^\infty g(u)\left[\theta + \frac{1}{\Gamma(\alpha)}\int_0^{\frac{1}{k}}\left(\frac{1}{k}-\tau\right)\Phi^{-1}\left(\frac{\rho+\int_\tau^\infty \frac{Bb}{(1+s)^2}\mathrm{d}s}{p(\tau)}\right)\mathrm{d}\tau\right]\mathrm{d}u}{1-\int_0^\infty g(s)\mathrm{d}s} \geq b.$$

Thus (C_1) is satisfied.

Finally, for any $x \in P(\psi,b,c)$ with $\|Tx\| > d$, then $\|x\| \leq c$ and $\min_{t\in[\frac{1}{k},k]} x(t) \geq b$, by the same method, we can also show that $\psi(Tx) > b$ easily, which means that (C_3) holds.

Therefore, by the conclusion of Theorem 5, the operator T has at least three fixed points. This implies that (1) has at least three solutions. □

4. Example

Let us consider the problem

$$\begin{aligned}
&\left(\tfrac{1}{16}\exp\left(\tfrac{t}{2}\right)\left(D^{\frac{1}{2}}x(t)\right)^4\right)' + \frac{2x+1}{9+t^2} = 0, \quad t\in[0,\infty),\\
&x(0) = \frac{1}{8}\int_0^\infty \exp(-2s)x(s)\mathrm{d}s + 1, \\
&\frac{1}{16}\lim_{t\to+\infty}\exp\left(\frac{t}{2}\right)\left(D^{\frac{1}{2}}x(t)\right)^4 = 1.
\end{aligned} \tag{12}$$

We have

$$p(t) = \frac{1}{16}\exp\left(\frac{t}{2}\right), \qquad \Phi(x) = x^4, \qquad g(t) = \frac{\exp(-2t)}{8},$$
$$f(t,x) = \frac{2x+1}{9+t^2}, \qquad \theta = \rho = 1.$$

It is clear that

$$\lambda = \lambda(t) = 16\left(1-\exp\left(-\frac{t}{8}\right)\right) \quad \text{and} \quad \int_0^\infty g(t)\mathrm{d}t = \frac{1}{8}\int_0^\infty \exp(-2t)\mathrm{d}t = \frac{1}{16} < 1.$$

Through a simple calculation, we get

$$\lambda^* = 16 < \infty, \quad \text{and} \quad \int_0^\infty g(t)\int_0^t \Phi^{-1}\left(\frac{1}{p(u)}\right)\mathrm{d}u\mathrm{d}t = \frac{1}{17} < \infty,$$

The function f is a Carathéodory function and $f(t,0) \neq 0$ on each subinterval of $[0,\infty)$. So ($\mathbf{H_1}$) and ($\mathbf{H_2}$) hold.

Taking $k = 10^{10}$, then $\lambda(\frac{1}{k}) < 1$ and $0 < \mu = \frac{16}{17}(1-\exp(-\frac{1}{8\times 10^{10}})) < 1$.

Next, in order to demonstrate our main result obtained, we choose a, b, c, A, B and C such that ($\mathbf{H_3}$) and ($\mathbf{H_4}$) be satisfied.

By Theorem 5, we conclude that the example (12) has at least three positive solutions.

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