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OSCILLATION OF FIXED POINTS OF SOLUTIONS AND THEIR DERIVATIVES OF SOME HIGHER LINEAR DIFFERENTIAL EQUATIONS

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Abstract In this paper, we investigate the relationship between solutions and their derivatives for the differential equation $f^{(k)}+A_{k-1}f^{(k-1)}+...+A_0f=0$ with $k\geq 2$ and entire functions of finite iterated $p-$order, when A_j $(j=0,1,...,k-1)$ are entire functions of finite iterated $p-$order in order to generalize and extend the results given by Wang and Lü, Liu and Zhang and Belaïdi.

Keywords: linear differential equations, entire solutions, iterated $p-$order, iterated exponent of convergence of the sequence of distinct zeros.

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1. INTRODUCTION AND MAIN RESULT

In this paper, we shall assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of meromorphic functions (see [3], [8]). For the definition of the iterated order of a meromorphic function, we use the same definition as in [4], [2, p. 317], [5, p. 129]. For all $r\in\mathbb{R}$, we define $\exp_1 r:=e^r$ and $\exp_{p+1} r:=\exp\left(\exp_p r\right)$, $p\in\mathbb{N}$. We also define for all r sufficiently large $\log_1 r:=\log r$ and $\log_{p+1} r:=\log\left(\log_p r\right)$, $p\in\mathbb{N}$. Moreover, we denote by $\exp_0 r:=r$, $\log_0 r:=r$, $\log_{-1} r:=\exp_1 r$ and $\exp_{-1} r:=\log_1 r$.

Definition 1.1. ([4], [5]) *Let f be a meromorphic function. The iterated $p-$order $\rho_p(f)$ of f is defined by*

$$\rho_p(f)=\overline{\lim_{r\to+\infty}}\frac{\log_p T(r,f)}{\log r}\quad (p\geq 1 \text{ is an integer}),\tag{1.1}$$

where $T(r,\ f)$ is the Nevanlinna characteristic function of f (see [3], [8]). For $p=1$, this notation is called: order, and for $p=2$: hyper-order.

Definition 1.2. ([4], [5]) *The finiteness degree of the order of a meromorphic function f is defined by*

$$i(f) = \begin{cases} 0, & \text{for } f \text{ rational,} \\ \min\left\{j \in \mathbb{N} : \rho_j(f) < +\infty\right\}, & \text{for } f \text{ transcendental for which} \\ & \text{some } j \in \mathbb{N} \text{ with } \rho_j(f) < +\infty \text{ exists,} \\ +\infty, & \text{for } f \text{ with } \rho_j(f) = +\infty \text{ for all } j \in \mathbb{N}. \end{cases} \tag{1.2}$$

Definition 1.3. ([4]) *Let f be a meromorphic function. The iterated exponent of convergence of the sequence of distinct zeros of $f(z)$ is defined by*

$$\overline{\lambda}_p(f) = \overline{\lim_{r\to+\infty}} \frac{\log_p \overline{N}\left(r, \frac{1}{f}\right)}{\log r}; \; p \geq 1 \text{ is an integer,} \tag{1.3}$$

where $\overline{N}\left(r, \frac{1}{f}\right)$ is the counting function of distinct zeros of $f(z)$ in $\{|z| < r\}$. For $p = 1$, this notation is called: exponent of convergence of the sequence of distinct zeros, and for $p = 2$, we get the hyper-exponent of convergence of the sequence of distinct zeros.

Definition 1.4. ([6]) *Let f be a meromorphic function. Then the iterated exponent of convergence of the sequence of distinct fixed points of $f(z)$ is defined by*

$$\overline{\tau}_p(f) = \overline{\lambda}_p(f - z) = \overline{\lim_{r\to+\infty}} \frac{\log_p \overline{N}\left(r, \frac{1}{f-z}\right)}{\log r}; \; p \geq 1 \text{ is an integer.} \tag{1.4}$$

For $p = 1$, this notation is called: exponent of convergence of the sequence of distinct fixed points. However, for $p = 2$, we get the hyper-exponent of convergence of the sequence of distinct fixed points (see [7]). Thus $\overline{\tau}_p(f) = \overline{\lambda}_p(f - z)$ is an indication of oscillation of distinct fixed points of $f(z)$.

Definition 1.5. *The growth index of the iterated convergence exponent of the sequence of zero points of a meromorphic function f with iterated order is defined by*

$$i_\lambda(f) = \begin{cases} 0 & \text{if } n\left(r, \frac{1}{f}\right) = O(\log r) \\ \min\{n \in \mathbb{N} : \lambda_n(f) < \infty\} & \text{if } \lambda_n(f) < \infty \text{ for some } n \in \mathbb{N}. \\ \infty & \text{if } \lambda_n(f) < \infty \text{ for all } n \in \mathbb{N} \end{cases}$$

Similarly, we can define the growth index $i_{\overline{\lambda}}(f)$ of $\overline{\lambda}_p(f)$ and $i_\tau(f), i_{\overline{\tau}}(f)$ of $\tau_p(f), \overline{\tau}_p(f)$.

For $k \geq 2$, we consider the linear differential equation

$$f^{(k)} + A(z) f = 0, \tag{1.5}$$

where $A(z)$ is a transcendental meromorphic function of finite iterated order $\rho_p(A) = \rho > 0$. Many important results have been obtained on the fixed points of general transcendental meromorphic functions for almost four decades (see [11, 13]). However, there are a few studies on the fixed points of solutions of differential equations. In

[15] , Wang and Lü have investigated the fixed points and hyper-order of solutions of second order linear differential equations with meromorphic coefficients and their derivatives. They have obtained the following result:

Theorem A ([15]) *Suppose that* $A(z)$ *is a transcendental meromorphic function satisfying* $\delta(\infty, A) = \underline{\lim}_{r\to+\infty} \frac{m(r,A)}{T(r,A)} = \delta > 0$, $\rho(A) = \rho < +\infty$. *Then every meromorphic solution* $f(z) \not\equiv 0$ *of the equation*

$$f'' + A(z) f = 0 \tag{1.6}$$

is such that f, f' *and* f'' *have infinitely many fixed points and*

$$\overline{\tau}(f) = \overline{\tau}\left(f'\right) = \overline{\tau}\left(f''\right) = \rho(f) = +\infty, \tag{1.7}$$

$$\overline{\tau}_2(f) = \overline{\tau}_2\left(f'\right) = \overline{\tau}_2\left(f''\right) = \rho_2(f) = \rho. \tag{1.8}$$

Theorem A has been generalized to higher order differential equations by Liu and Zhang as follows (see [13]):

Theorem B ([13]) *Suppose that* $k \geq 2$ *and* $A(z)$ *is a transcendental meromorphic function satisfying* $\delta(\infty, A) = \underline{\lim}_{r\to+\infty} \frac{m(r,A)}{T(r,A)} = \delta > 0$, $\rho(A) = \rho < +\infty$. *Then every meromorphic solution* $f(z) \neq 0$ *of* (1.4) , *has the property:* f *and* $f', f'', ..., f^{(k)}$ *all have infinitely many fixed points and*

$$\overline{\tau}(f) = \overline{\tau}\left(f'\right) = \overline{\tau}\left(f''\right) = ... = \overline{\tau}\left(f^{(k)}\right) = \rho(f) = +\infty, \tag{1.9}$$

$$\overline{\tau}_2(f) = \overline{\tau}_2\left(f'\right) = \overline{\tau}_2\left(f''\right) = ... = \overline{\tau}_2\left(f^{(k)}\right) = \rho_2(f) = \rho. \tag{1.10}$$

Theorem A and B have been generalized by B. Belaidi for iterated p-order (see [2]):

Theorem C ([2]) *Let* $k \geqslant 2$ *and* $A(z)$ *be transcendental meromorphic function of finite iterated order* $\rho_p(A) = \rho > 0$ *such that* $\delta(\infty, A) = \underline{\lim}_{r\to+\infty} \frac{m(r,A)}{T(r,A)} = \delta > 0$.
Suppose, moreover, that either:
(i) *all poles of* f *are of uniformly multiplicity or that*
(ii) $\delta(\infty, f) > 0$.
If $\varphi \neq 0$ *is a meromorphic function with finite* iterated $p-$order $\rho_p(\varphi) < +\infty$, *then every meromorphic solution* $f(z) \neq 0$ *of* (1.5), *satisfies*

$$\overline{\lambda}_p(f - \varphi) = \overline{\lambda}_p\left(f' - \varphi\right) = ... = \overline{\lambda}_p\left(f^{(k)} - \varphi\right) = \rho_p(f) = +\infty, \tag{1.11}$$

and

$$\overline{\lambda}_{p+1}(f-\varphi)=\overline{\lambda}_{p+1}\left(f'-\varphi\right)=...=\overline{\lambda}_{p+1}\left(f^{(k)}-\varphi\right)=\rho_{p+1}(f)=\rho. \tag{1.12}$$

For $k\geq 2$, we consider the linear differential equation

$$f^{(k)}+A_{k-1}f^{(k-1)}+...+A_0 f=0,\ k\geq 2, \tag{1.13}$$

where A_j $(j=0,1,...,k-1)$ are entire functions of finite iterated $p-$order.

The main purpose of this paper is to study the relation between solutions and their derivatives of the differential equation (1.13) and entire functions of finite iterated $p-$order where we generalize and extend the results of Wang and Lü, Liu and Zhang and Belaidi. In fact, we prove the following result:

Theorem 1.1. *Let $k\geq 2$ and $(A_j)_{j=0,1,2,...k-1}$ be entire functions of finite iterated p-order such that $i(A_0)=p; 0<p<\infty$. Assume that*

$$\max\left\{i\left(A_j\right),(j=1,...,k-1)\right\}<i(A_0)$$

or

$$\max\left\{\rho_p\left(A_j\right),(j=1,...,k-1)\right\}<\rho_p(A_0)<+\infty.$$

If $\varphi(z)\neq 0$ is an entire function with $i(\varphi)<p+1$ or $\rho_{p+1}(\varphi)<\rho_p(A_0)$, then every solution $f(z)\neq 0$ of (1.13) satisfies

$$i_{\overline{\lambda}}\left(f^{(i)}-\varphi\right)=i_{\lambda}\left(f^{(i)}-\varphi\right)=i(f)=p+1,\ i\in\mathbb{N} \tag{1.14}$$

and

$$\overline{\lambda}_{p+1}\left(f^{(i)}-\varphi\right)=\overline{\lambda}_{p+1}\left(f^{(i)}-\varphi\right)=\rho_{p+1}(f)=\rho_p(A_0),\ i\in\mathbb{N}. \tag{1.15}$$

For $\varphi(z)=z$ in Theorem 1.1, we obtain the following corollaries:

Corollary 1.1. *Let $k\geq 2$ and $(A_j)_{j=0,1,2,...k-1}$ be entire functions of finite iterated p-order such that $i(A_0)=p(0<p<\infty)$. Assume that*

$$\max\left\{i\left(A_j\right),(j=1,...,k-1)\right\}<i(A_0)$$

or

$$\max\left\{\rho_p\left(A_j\right),(j=1,...,k-1)\right\}<\rho_p(A_0)<+\infty.$$

Then every solution $f(z)\neq 0$ of (1.13), is such that all the derivatives $f^{(i)}$ $(i\in\mathbb{N})$ have infinitely many fixed points and we have

$$i_{\overline{\tau}}\left(f^{(i)}\right)=i_{\tau}\left(f^{(i)}\right)=i(f)=p+1,\ i\in\mathbb{N} \tag{1.16}$$

and

$$\overline{\tau}_{p+1}\left(f^{(i)}\right) = \tau_{p+1}\left(f^{(i)}\right) = \rho_{p+1}(f) = \rho_p(A_0) = \rho, \ i \in \mathbb{N}. \tag{1.17}$$

Corollary 1.2. *Suppose that* $k \geq 2$ *and* $A(z)$ *is a transcendental entire function such that* $0 < \rho_p(A) = \rho < +\infty$. *If* $\varphi(z) \not\equiv 0$ *is an entire function with* $i(\varphi) < p+1$ *or* $\rho_{p+1}(\varphi) < \rho$, *then every solution* $f(z) \not\equiv 0$ *of* (1.5) *satisfies* (1.14) *and* (1.15).

2. AUXILIARY LEMMATA

To prove our main results, we need the following lemmata.

Lemma 2.1. [6] *Suppose that* $A_0, A_1, ..., A_{k-1}, F(\not\equiv 0)$ *are meromorphic functions and let* f *be a meromorphic solution of the equation*

$$f^{(k)} + A_{k-1}f^{(k-1)} + ... + A_1 f' + A_0 f = F, \tag{2.1}$$

such that $i(f) = \rho + 1\,(0 < p < \infty)$. *If either*

$$\max\left\{i(F), i\left(A_j\right)\ j = 0, 1, ..., k-1\right\} < p+1$$

or

$$\max\left\{\rho_{p+1}(F), \rho_{p+1}\left(A_j\right)\ j = 0, 1, ..., k-1\right\} < \rho_{p+1}(f),$$

then we have $i_{\overline{\lambda}}(f) = i_\lambda(f) = i(f) = p+1$ *and* $\overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \rho_{p+1}(f)$.

Lemma 2.2. *(see Remark 1.3 of* [10]*). If* f *is a meromorphic function with* $i(f) = p$, *then* $\rho_p\left(f'\right) = \rho_p(f)$.

Lemma 2.3. ([10]) *Let* $k \geq 2$ *and* $A_j\ (j = 0, 1, ..., k-1)$ *be entire functions of finite iterated p-order such that* $i(A_0) = p, (0 < p < \infty)$. *Assume that*

$$\max\left\{i\left(A_j\right), (j = 1, ..., k-1)\right\} < i(A_0)$$

or

$$\max\left\{\rho_p\left(A_j\right), (j = 1, ..., k-1)\right\} < \rho_p(A_0) < +\infty.$$

Then every solution $f(z) \not\equiv 0$ *of* (1.13) *satisfies* $i(f) = p+1$ *and* $\rho_{p+1}(f) = \rho_p(A_0)$.

Let $A_j\ (j = 0, 1, ..., k-1)$ be entire functions. We define the following sequence of functions:

$$\begin{cases} A_j^0 = A_j, & j = 0, 1, ..., k-1 \\ A_{k-1}^i = A_{k-1}^{i-1} - \dfrac{\left(A_0^{i-1}\right)'}{A_0^{i-1}}, & i \in \mathbb{N} \\ A_j^i = A_j^{i-1} + A_{j+1}^{i-1}\dfrac{\left(\Psi_{j+1}^{i-1}\right)'}{\Psi_{j+1}^{i-1}}, & j = 0, 1, ..., k-2,\ i \in \mathbb{N}, \end{cases} \tag{2.2}$$

where $\Psi_{j+1}^{i-1} = \dfrac{A_{j+1}^{i-1}}{A_0^{i-1}}$.

Remark 2.1. *In the case where one of functions* A_j^i $(j = 0, 1, ..., k-1)$ *is equal to zero then* $A_j^{i+1} = A_{j-1}^i$ $(j = 0, 1, ..., k-1)$.

Lemma 2.4. *Assume that* f *is an entire solution of* (1.13). *Then* $g_i = f^{(i)}$ *is an entire solution of the equation*

$$g_i^{(k)} + A_{k-1}^i g_i^{(k-1)} + ... + A_0^i g_i = 0, \tag{2.3}$$

where A_j^i $(j = 0, 1, ..., k-1)$ *are given by* (2.2).

Proof. Assume that f is a solution of equation (1.13) and let $g_i = f^{(i)}$. We prove that g_i is an entire solution of the equation (2.11). Our proof is by induction: For $i = 1$, differentiating both sides of (1.13), we obtain

$$f^{(k+1)} + A_{k-1} f^{(k)} + \left(A_{k-1}^{'} + A_{k-2}\right) f^{(k-1)} + ... + \left(A_1^{'} + A_0\right) f^{'} + A_0^{'} f = 0, \tag{2.4}$$

and replacing f by

$$f = -\frac{(f^{(k)} + A_{k-1} f^{(k-1)} + ... + A_1 f^{'})}{A_0},$$

we get

$$f^{(k+1)} + \left(A_{k-1} - \frac{A_0^{'}}{A_0}\right) f^{(k)} + \left(A_{k-1}^{'} + A_{k-2} - A_{k-1}\frac{A_0^{'}}{A_0}\right) f^{(k-1)} ... + \left(A_1^{'} + A_0 - A_1 \frac{A_0^{'}}{A_0}\right) f^{'} = 0.$$

That is

$$g_1^{(k)} + A_{k-1}^1 g_1^{(k-1)} + A_{k-2}^1 g_1^{(k-2)} ... + A_0^1 g_1 = 0.$$

Suppose that the assertion is true for the values which are strictly smaller than a certain i. We suppose g_{i-1} is a solution of the equation

$$g_{i-1}^{(k)} + A_{k-1}^{i-1} g_{i-1}^{(k-1)} + A_{k-2}^{i-1} g_{i-1}^{(k-2)} ... + A_0^{i-1} g_{i-1} = 0. \tag{2.5}$$

Differentiating (2.5), we can write

$$\begin{aligned} g_{i-1}^{(k+1)} + A_{k-1}^{i-1} g_{i-1}^{(k)} + \left(\left(A_{k-1}^{i-1}\right)^{'} + A_{k-2}\right) g_{i-1}^{(k-1)} + ... \\ + \left(\left(A_1^{i-1}\right)^{'} + A_0^{i-1}\right) g_{i-1}^{'} + A_0^{'} g_{i-1} = 0. \end{aligned} \tag{2.6}$$

In (2.6), replacing g_{i-1} by

$$g_{i-1} = -\frac{(g_{i-1}^{(k)} + A_{k-1}^{i-1} g_{i-1}^{(k-1)} + A_{k-2}^{i-1} g_{i-1}^{(k-2)} ... + A\,(g_{i-1})^{'})}{A_0^{i-1}},$$

yields

$$g_{i-1}^{(k+1)}+\left(A_{k-1}^{i-1}-\frac{\left(A_0^{i-1}\right)'}{A_0^{i-1}}\right)g_{i-1}^{(k)}+\left(\left(A_{k-1}^{i-1}\right)'+A_{k-2}-A_{k-1}^{i-1}\frac{\left(A_0^{i-1}\right)'}{A_0^{i-1}}\right)g_{i-1}^{(k-1)}...+$$

$$+\left(\left(A_1^{i-1}\right)'+A_0^{i-1}-A_1^{i-1}\frac{\left(A_0^{i-1}\right)'}{A_0^{i-1}}\right)g_{i-1}'=0. \tag{2.7}$$

That is

$$g_{i-1}^{(k)}+A_{k-1}^{i-1}g_{i-1}^{(k-1)}+A_{k-2}^{i-1}g_{i-1}^{(k-2)}...+A_0^{i-1}g_{i-1}=0.$$

Lemma 2.4 is thus proved. ∎

Lemma 2.5. *Let A_j $(j=0,1,...,k-1)$ be entire functions of finite order. Assume that*

$$\max\left\{i\left(A_j\right),(j=1,...,k-1)\right\}<i\left(A_0\right)$$

or

$$\max\left\{\rho_p\left(A_j\right),(j=1,...,k-1)\right\}<\rho_p\left(A_0\right)<+\infty,$$

and let A_j^i, $(j=0,1,...,k-1)$ be defined as in (2.2). *Then all nontrivial meromorphic solution g of the equation*

$$g^{(k)}+A_{k-1}^i g^{(k-1)}+...+A_0^i g=0,\ k\geq 2 \tag{2.8}$$

satisfy : $i(g)=p+1$ and $\rho_{p+1}(g)=\rho$.

Proof. Let $\{f_1,f_2,...,f_k\}$ be a fundamental system of solutions of (1.13). We show that $\left\{f_1^{(i)},f_2^{(i)},...,f_k^{(i)}\right\}$ is a fundamental system of solutions of (2.8). By Lemma 2.4, it follows that $f_1^{(i)},f_2^{(i)},...,f_k^{(i)}$ is a solutions (2.8) . Let $\alpha_1,\alpha_2,...,\alpha_k$ be constants such that

$$\alpha_1 f_1^{(i)}+\alpha_2 f_2^{(i)}+...+\alpha_k f_k^{(i)}=0.$$

Then, we have

$$\alpha_1 f_1+\alpha_2 f_2+...+\alpha_k f_k=P(z),$$

where $P(z)$ is a polynomial of degree less than i. Since $\alpha_1 f_1+\alpha_2 f_2+...+\alpha_k f_k$ is a solution of (1.13), then P is a solution of (1.13), and by Lemma 2.3, we conclude that P is an infinite solution of (1.13); this leads to a contradiction. Therefore, P is a trivial solution. We deduce that $\alpha_1 f_1+\alpha_2 f_2+...+\alpha_k f_k=0$. Using the fact that $\{f_1,f_2,...,f_k\}$ is a fundamental system of solutions of (1.13), we get $\alpha_1=\alpha_2=...=\alpha_k=0$. Now, let g be a non trivial solution of (2.8). Then, using the fact that $\left\{f_1^{(i)},f_2^{(i)},...,f_k^{(i)}\right\}$ is a fundamental solution of (2.8), we claim that there exist constants $\alpha_1, \alpha_2, ..., \alpha_k$ not all equal to zero, such that $g = \alpha_1 f_1^{(i)} + \alpha_2 f_2^{(i)} + ... + \alpha_k f_k^{(i)}$. Let $h = \alpha_1 f_1 + \alpha_2 f_2 + ... + \alpha_k f_k$, h be a solution of (1.13) and $h^{(i)} = g$. Hence, by Lemma 2.2, we have $\rho_{p+1}(h) = \rho_{p+1}(g)$, and by Lemma 2.3, we have $i(h) = i(g) = p+1$ and $\rho_{p+1}(h) = \rho_{p+1}(g) = \rho$. ∎

3. PROOF OF THEOREM 1.1

Assume that f is a solution of equation (1.13). By Lemma 2.3, we can write $i(f) = p+1$, $\rho_{p+1}(f) = \rho_p(A_0)$. Taking $g_i = f^{(i)}$, then, using Lemma 2.2, we have $i(g_i) = p+1$, $\rho_{p+1}(g_i) = \rho_p(A_0)$. Now, let $w(z) = g_i(z) - \varphi(z)$, where φ is an entire function with $\rho_{p+1}(\varphi) < \rho_p(A_0)$.
Then $i(w) = i(g_i) = p+1$, and $\rho_{p+1}(w) = \rho_{p+1}(g_i) = \rho_{p+1}(f) = \rho(A_0)$.
In order to prove $i_{\overline{\lambda}}(g_i - \varphi) = i_\lambda(g_i - \varphi) = p+1$ and $\overline{\lambda}_{p+1}(g_i - \varphi) = \lambda_{p+1}(g_i - \varphi) = \rho(A_0)$, we need to prove only $i_{\overline{\lambda}}(w) = i_\lambda(w) = p+1$ and $\overline{\lambda}_{p+1}(w) = \rho(A_0)$. Using the fact that $g_i = w + \varphi$, and by Lemma 2.4 we get

$$w^{(k)} + A_{k-1}^i w^{(k-1)} + ... + A_0^i w = -\left(\varphi^{(k)} + A_{k-1}^i \varphi^{(k-1)} + ... + A_0^i \varphi\right) = F. \tag{3.1}$$

By $\rho_p\left(A_j^i\right) < \infty$, $\rho_{p+1}(\varphi) < \rho_p(A_0)$ and Lemma 2.3, we get $F \not\equiv 0$ and $\rho_{p+1}(F) < \infty$. By Lemma 2.4 $i_{\overline{\lambda}}(w) = i_\lambda(w) = p+1$ and $\overline{\lambda}_{p+1}(w) = \lambda_{p+1}(w) = \rho_{p+1}(w) = \rho(A_0)$. The proof of theorem 1.1 is complete.

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