ROMAI J., v.14, no.1(2018), 89–100

ON INEQUALITIES INVOLVING CONVEX FUNCTIONS AND INTEGRAL CONDITIONS

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Abstract In this paper, using the Riemann-Liouville fractional integral operator, we establish new results that generalize some theorems of the work: [A note on some new fractional results involving convex functions. Acta Math. Univ. Comenianae, Vol. LXXXI, 2, 2012]. We also discuss other integral inequalities generalizing some theorems in the paper: [Some new results of two open problems related to integral inequalities, Journal of Mathematical Inequalities, 10(3), 2016].

Keywords: integral inequality, Riemann-Liouville integral, convex function .
2010 MSC: 26D10, 26A33.

1. INTRODUCTION

In [5] W.J. Liu et al. studied some interesting inequalities for a convex function $(x-a)^{\delta}$ for $\delta \geq 1$ and established the following result:

Theorem 1.1. *Let $\beta > 0$ and $f \geq 0$ be a continuous function on $[a,b]$ with*

$$\int_x^b f^{\min\{1,\beta\}}(t)dt \geq \int_x^b (t-a)^{\min\{1,\beta\}}dt, x \in [a,b]. \quad (1)$$

Then,

$$\int_a^b f^{\alpha+\beta}(x)dx \geq \int_a^b (x-a)^{\alpha} f^{\beta}(x)dx \quad (2)$$

is valid for all $\alpha > 0$.

Then, in 2009, W.J. Liu, Q. Ngo and V.N. Huy [6] proved the following important result includes more general convex function :

Theorem 1.2. *Let f, g, h be positive three continuous functions on $[a,b]$, with $f \leq h$ on $[a,b]$ and such that $\frac{f}{h}$ is decreasing and f, g are increasing. If φ is a* *convex function with* $\varphi(0) = 0$*, then*

$$\frac{\int_a^b f(x)dx}{\int_a^b h(x)dx} \geq \frac{\int_a^b \varphi(f(x))g(x)dx}{\int_a^b \varphi(h(x))g(x)dx}.$$

Recently, Z. Dahmani [1] established generalization for the above theorem, he proved that for any three positive continuous functions f, g and h defined on $[a, b]$, with $f \leq h$, f and g are increasing and $\frac{f}{h}$ is decreasing, then for any $x \in]a, b]$, we have:

$$\frac{J_a^\alpha f(x)}{J_a^\alpha h(x)} \geq \frac{J_a^\alpha[\varphi(f)g](x)}{J_a^\alpha[\varphi(h)g](x)},$$

where $\alpha > 0$ and φ is a positive and convex function, with $\varphi(0) = 0$.
Very recently, A. Kashuri and R. Liko [4] proposed another result, as a response to an open problem posed by Liu et al. in [6]. In fact, for three positive continuous functions f, g and h defined on $[a, b]$, such that $f \leq h$ on $[a, b]$, f, g are increasing and $\frac{f}{h}$ is decreasing, if φ is a positive and convex function, with $\varphi(0) = 0$, the authors of [4] proved that the inequality

$$\frac{\int_a^b f(x)dx}{\int_a^b h(x)dx} \geq \frac{(\int_a^b \varphi(f(x))g(x)dx)^\delta}{(\int_a^b \varphi(h(x))g(x)dx)^\lambda}$$

is valid, under some conditions on $\lambda, \delta, \varphi f(a), \varphi f(b), g(a), g(b)$. Other important results introducing a parameter λ and generalizing Theorem 1.1 are also discussed by the authors of [4].
In this paper, we prove new classical and fractional integral inequalities that generalise some integral results of the papers [1, 4].

2. RIEMANN-LIOUVILLE INTEGRATION

We recall the following definition and some properties.

Definition 2.1. *[3] The Riemann-Liouville fractional integral operator of order* $\delta \geq 0$*, for a continuous function* f *on* $[a, b]$ *is defined as*

$$J_a^\delta f(x) = \frac{1}{\Gamma(\delta)} \int_a^x (x-u)^{\delta-1} f(u)du; \quad \delta > 0, a < x \leq b,$$
$$J_a^0 f(x) = f(x). \tag{3}$$

We give the semigroup property:

$$J_a^\alpha J_a^\delta f(x) = J_a^{\alpha+\delta} f(x), \alpha \geq 0, \delta \geq 0, \tag{4}$$

In the particular case where $f(x) = (x-a)^\beta$ on $[a,b]$, we have

$$J_a^\delta (x-a)^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\delta+\beta+1)}(x-a)^{\delta+\beta}. \tag{5}$$

3. MAIN RESULTS

Theorem 3.1. *Let f, g and h be three positive continuous functions on $[a,b]$ with $f \leq h$. Suppose that f and g are increasing and $\frac{f}{h}$ is a decreasing function, and assume that φ is a positive and convex function, with $\varphi(0) = 0$. Then for any $x \in]a,b]$, we have:*

$$\frac{J_a^\alpha f(x)}{J_a^\alpha h(x)} \geq \left(\frac{J_a^\alpha[\varphi(f)g](x)}{J_a^\alpha[\varphi(h)g](x)}\right)^\lambda, \tag{6}$$

where $\lambda \geq 1, \alpha > 0$.

Proof. Since $f \leq h$, then for all $\lambda \geq 1$, we can write:

$$\frac{J_a^\alpha f(x)}{J_a^\alpha h(x)} \geq \left(\frac{J_a^\alpha f(x)}{J_a^\alpha h(x)}\right)^\lambda, x \in]a,b]. \tag{7}$$

On the other hand, for any $x \in]a,b]$, we have (see [1]):

$$\frac{J_a^\alpha f(x)}{J_a^\alpha h(x)} \geq \frac{J_a^\alpha[\varphi(f)g](x)}{J_a^\alpha[\varphi(h)g](x)}. \tag{8}$$

Therefore, it yields that

$$\left(\frac{J_a^\alpha f(x)}{J_a^\alpha h(x)}\right)^\lambda \geq \left(\frac{J_a^\alpha[\varphi(f)g](x)}{J_a^\alpha[\varphi(h)g](x)}\right)^\lambda, x \in]a,b]. \tag{9}$$

Using (7) and (9) we obtain (6). ∎

Remark 3.1. *Taking $\lambda = 1$ in Theorem 3.1, we obtain Theorem 3.5 proved in [1].*

Another main result is the following theorem, in which we will generalize a theorem in the paper [4]. We prove:

Theorem 3.2. *Let f, g and h be three positive continuous functions on $[a,b]$, such that $f \leq h$ on $[a,b]$, f and g are increasing and $\frac{f}{h}$ is decreasing. Assume*

that φ is a positive and convex function, with $\varphi(0) = 0$.
In the case where $1 \leq \theta < \lambda$, if $\varphi[f(a)]g(a)J_a^{\alpha}(1) \geq 1$, then, we have:

$$\frac{J_a^{\alpha} f(x)}{J_a^{\alpha} h(x)} \geq \frac{(J_a^{\alpha}[\varphi(f)g](x))^{\theta}}{(J_a^{\alpha}[\varphi(h)g](x))^{\lambda}}, x \in]a, b], \alpha > 0. \tag{10}$$

The same inequality is valid in the case: $1 \leq \lambda < \theta$, under the condition: $\varphi[f(b)]g(b)J_a^{\alpha}(1) \leq 1$.

Proof. We prove the theorem in two steps:
Case 1: For $1 \leq \theta < \lambda$, there exists $s > 0$, such that $\lambda = \theta + s$.
So, we have:

$$\frac{(J_a^{\alpha}[\varphi(f)g](x))^{\theta}}{\left(J_a^{\beta}[\varphi(h)g](x)\right)^{\lambda}} = \left(\frac{J_a^{\alpha}[\varphi(f)g](x)}{J_a^{\alpha}[\varphi(h)g](x)}\right)^{\theta} \times \frac{1}{(J_a^{\alpha}[\varphi(h)g](x))^{s}}$$

Thanks to Theorem 3.1, we obtain

$$\frac{(J_a^{\alpha}[\varphi(f)g](x))^{\theta}}{(J_a^{\alpha}[\varphi(h)g](x))^{\lambda}} \leq \frac{J_a^{\alpha} f(x)}{J_a^{\alpha} h(x)} \times \frac{1}{(J_a^{\alpha}[\varphi(h)g](x))^{s}}.$$

Now, we shall prove that $(J_a^{\alpha}[\varphi(h)g](x))^{s} \geq 1$.

We have:

$$\begin{aligned} J_a^{\alpha}[\varphi(h)g](x) &= J_a^{\alpha}[\frac{\varphi(h)}{h} hg](x) \\ &\geq J_a^{\alpha}[\frac{\varphi(h)}{h} fg](x). \end{aligned}$$

Since φ is a convex function, then, for all x, y, we can write

$$(y - x)\varphi'(x) \leq \varphi(y) - \varphi(x).$$

Hence for $y = 0$, we obtain $x\varphi'(x) - \varphi(x) \geq 0$. Therefore, we get $\left(\frac{\varphi(x)}{x}\right)' = \frac{x\varphi'(x) - \varphi(x)}{x^2} \geq 0$, which implies that $\frac{\varphi(x)}{x}$ is an increasing function and by the hypothesis of $f \leq h$, we conclude that $\frac{\varphi[f]}{f} \leq \frac{\varphi[h]}{h}$.
Consequently, we obtain

$$J_a^{\alpha}[\frac{\varphi(h)}{h} fg](x) \geq J_a^{\alpha}[\frac{\varphi(f)}{f} fg](x).$$

On the other hand, since f, g and $\frac{\varphi(t)}{t}$ are increasing, then $[\frac{\varphi(f)}{f} fg](x)$ is increasing. So, we have $\forall x \in [a, b]$, $[\frac{\varphi(f)}{f} fg](x) \geq \varphi[f(a)]g(a)$.
Finally,

$$J_a^\alpha[\frac{\varphi(f)}{f} fg](x) \quad \geq \quad \varphi[f(a)]g(a)J_a^\alpha(1) \geq 1.$$

Case 2: For $1 \leq \lambda < \theta$, there exists $s > 0$, such that $\theta = \lambda + s$.
We have

$$\begin{aligned} \frac{(J_a^\alpha[\varphi(f)g](x))^\theta}{(J_a^\alpha[\varphi(h)g](x))^\lambda} &= \left(\frac{J_a^\alpha[\varphi(f)g](x)}{J_a^\alpha[\varphi(h)g](x)}\right)^\lambda \times \left(J_a^\beta[\varphi(f)g](x)\right)^s. \\ &\leq \frac{J_a^\alpha f(x)}{J_a^\alpha h(x)} \times \left(J_a^\beta[\varphi(f)g](x)\right)^s. \end{aligned}$$

Now, we need to prove that $\left(J_a^\beta[\varphi(f)g](x)\right)^s \leq 1$.

Since $\varphi(f)g$ is increasing on $[a, b]$, we have $[\varphi(f)g](x) \leq \varphi(f(b))g(b)$, $\forall x \in [a, b]$, which implies

$$(J_a^\alpha[\varphi(f)g](x))^s \leq (\varphi(f(b))g(b)J_a^\alpha(1))^s \leq 1.$$

The proof of Theorem 3.2 is thus achieved. ∎

Remark 3.2. *In Theorem 3.2, if we take $\alpha = 1$, we obtain Theorem 2.2 of [4].*

Changing the hypotheses of Theorem 2.1 in [4] by considering two integral conditions on f, we obtain the following result:

Theorem 3.3. *Let $f : [a, b] \to \mathbb{R}^+$ be a continuous function, such that:*

$$\int_x^b (u-a)^{\min(1,\beta)} du \leq \int_x^b f^{\min(1,\beta)}(u) du, x \in [a, b], \beta > 0 \tag{11}$$

and

$$\frac{\Gamma(\alpha)}{\Gamma(\alpha - n + 1)}(b-a)^{n+1} \int_a^b f^\beta(t) dt \leq 1, n = [\alpha], \alpha > 0. \tag{12}$$

Then for any $\lambda \geq 1$, $b - a \geq 1$, we have

$$\int_a^b f^{\alpha+\beta}(u) du \geq \left(\int_a^b (u-a)^\alpha f^\beta(u) du\right)^\lambda.$$

Proof. For $\lambda \geq 1$, we have

$$\left(\int_a^b (u-a)^\alpha f^\beta(u)du\right)^\lambda = \left(\int_a^b (u-a)^\alpha f^\beta(u)du\right)\left(\int_a^b (u-a)^\alpha f^\beta(u)du\right)^{\lambda-1}.$$

By Theorem 2.1 of [5], we can write $\int_a^b (u-a)^\alpha f^\beta(u)du \leq \int_a^b f^{\alpha+\beta}(u)du$.
Therefore,

$$\left(\int_a^b (u-a)^\alpha f^\beta(u)du\right)^\lambda \leq \int_a^b f^{\alpha+\beta}(u)du\left(\int_a^b (u-a)^\alpha f^\beta(u)du\right)^{\lambda-1}.$$

Now we need to prove that $\int_a^b (u-a)^\alpha f^\beta(u)du \leq 1$.
We have

$$\begin{aligned}\int_a^b (u-a)^\alpha f^\beta(u)du &= -(u-a)^\alpha \int_u^b f^\beta(t)du\Big|_{u=a}^{u=b} \\ &\quad +\alpha\int_a^b (u-a)^{\alpha-1}\int_u^b f^\beta(t)dtdu \\ &= \alpha\int_a^b (u-a)^{\alpha-1}\int_u^b f^\beta(t)dtdu.\end{aligned}$$

Hence, we can write

$$\begin{aligned}\int_a^b (u-a)^\alpha f^\beta(u)du &= \alpha\int_a^b (u-a)^{\alpha-1}\int_u^b f^\beta(t)dtdu \\ &= \alpha(\alpha-1)\int_a^b (u-a)^{\alpha-2}\int_u^b\int_{u_1}^b f^\beta(t)dtdu_1du \\ &\quad \ldots\end{aligned}$$

$$= \alpha(\alpha-1)...(\alpha-n+1)\int_a^b (u-a)^{\alpha-n}$$

$$\times \int_u^b \int_{u_1}^b ... \int_{u_{n-1}}^b f^{\beta}(t)dtdu_{n-1}...du_1du.$$

On the other hand, since $(u-a)^{\alpha-n} \leq (b-a)$, it follows that

$$\begin{aligned}
\int_a^b (u-a)^{\alpha} f^{\beta}(u)du &\leq \alpha(\alpha-1)...(\alpha-n+1)\int_a^b (b-a) \\
&\quad \times \int_u^b \int_{u_1}^b ... \int_{u_{n-1}}^b f^{\beta}(t)dtdu_{n-1}...du_1du \\
&= \alpha(\alpha-1)...(\alpha-n+1)\int_a^b \int_a^b \int_a^b ... \int_a^b f^{\beta}(t)dtdu_{n-1}...du_1du_0du \\
&= \alpha(\alpha-1)...(\alpha-n+1)\int_a^b f^{\beta}(t)dt \\
&\quad \times \int_a^b \int_a^b ... \int_a^b du_{n-1}...du_1du_0du \\
&= \alpha(\alpha-1)...(\alpha-n+1)(b-a)^{n+1}\int_a^b f^{\beta}(t)dt \\
&= \frac{\Gamma(\alpha)}{\Gamma(\alpha-n+1)}(b-a)^{n+1}\int_a^b f^{\beta}(t)dt \leq 1.
\end{aligned}$$

∎

We prove also the following theorem:

Theorem 3.4. *Let $f : [a,b] \longrightarrow \mathbb{R}^+$ be a continuous function, such that:*

$$\int_x^b (u-a)^{\min\{1,\beta\}}du \leq \int_x^b f^{\min\{1,\beta\}}(u)du, x \in [a,b], \beta > 0$$

and

$$\frac{\Gamma(\alpha)}{\Gamma(\alpha-n+1)}(b-a)^{n+1}J_a^\delta f^\beta(b)\leq 1; n=[\alpha], \alpha>0.$$

Then, for all $\lambda\geq 1, \delta\geq 1, b-a\geq 1$, *we have:*

$$J_a^\delta f^{\alpha+\beta}(b)\geq\left(J_a^\delta(x-a)^\alpha f^\beta(x)|_{x=b}\right)^\lambda.$$

Proof. For $\lambda\geq 1$, we have

$$\left(J_a^\delta(x-a)^\alpha f^\beta(x)|_{x=b}\right)^\lambda=\left(J_a^\delta(x-a)^\alpha f^\beta(x)|_{x=b}\right)\left(J_a^\delta(x-a)^\alpha f^\beta(x)|_{x=b}\right)^{\lambda-1}.$$

Now, we begin by proving that $J_a^\delta(x-a)^\alpha f^\beta(x)|_{x=b}\leq J_a^\delta f^{\alpha+\beta}(b)$.
To do this, we need to prove that

$$J_a^\delta(x-a)^\alpha f^\beta(x)|_{x=b}\geq\frac{\Gamma(\alpha+\beta+1)(b-a)^{\alpha+\beta+\delta}}{\Gamma(\alpha+\beta+\delta+1)}. \tag{13}$$

For $\beta\in]0,1]$, we have

$$\begin{aligned}
J_a^\delta(t-a)^\alpha f^\beta(t) \quad | \quad {}_{t=b} &= \frac{1}{\Gamma(\delta)}\int_a^b(b-x)^{\delta-1}(x-a)^\alpha f^\beta(x)dx\\
&= \frac{1}{\Gamma(\delta)}\Big[(b-x)^{\delta-1}(x-a)^\alpha\int_x^b f^\beta(u)du\mid_{x=a}^{x=b}\\
&\quad+\frac{1}{\Gamma(\delta)}\int_a^b g(x)\left(\int_x^b f^\beta(u)du\right)dx\Big]\\
&= \frac{1}{\Gamma(\delta)}\int_a^b g(x)\left(\int_x^b f^\beta(u)du\right)dx,
\end{aligned}$$

where $g(x)=(\delta-1)(b-x)^{\delta-2}(x-a)^\alpha+\alpha(b-x)^{\delta-1}(x-a)^{\alpha-1}$.
Thanks to the imposed condition, we observe that

$$\begin{aligned}
\tfrac{1}{\Gamma(\delta)}\textstyle\int_a^b g(x)\left(\int_x^b f^\beta(u)du\right)dx &\geq \frac{1}{\Gamma(\delta)}\int_a^b g(x)\left(\int_x^b(u-a)^\beta du\right)dx\\
&= \frac{1}{(\beta+1)\Gamma(\delta)}\int_a^b g(x)\left[(b-a)^{\beta+1}-(x-a)^{\beta+1}\right]dx\\
&= \frac{\Gamma(\alpha+\beta+1)(b-a)^{\alpha+\beta+\delta}}{\Gamma(\alpha+\beta+\delta+1)}.
\end{aligned} \tag{14}$$

If we take $\beta = 1$ in (14), then we get

$$J_a^\delta (t-a)^\alpha f(t)|_{t=b} \geq \frac{\Gamma(\alpha+2)}{\Gamma(\alpha+\delta+2)}(b-a)^{\alpha+\delta+1}. \tag{15}$$

Using the following inequality (see Lemma 2.2 of [7]):

$$ps + qr \geq s^p r^q, \forall p, q, s, r > 0, p + q = 1, \tag{16}$$

with $p = \frac{1}{\beta}$, $q = \frac{\beta-1}{\beta}$, $s = f^\beta(x)$ and $r = (x-a)^{\beta-1}$, we obtain

$$\frac{1}{\beta} f^\beta(x) + \frac{\beta-1}{\beta}(x-a)^\beta \geq f(x)(x-a)^{\beta-1}.$$

Consequently,

$$f^\beta(x) + (\beta-1)(x-a)^\beta \geq \beta f(x)(x-a)^{\beta-1}. \tag{17}$$

Multiplying both sides of (17) by $\frac{1}{\Gamma(\delta)}(b-x)^{\delta-1}(x-a)^\alpha$ and integrating the resulting inequality with respect to x over $[a,b]$, yields

$$J_a^\delta (t-a)^\alpha f^\beta(t)|_{t=b} + (\beta-1) J_a^\delta (t-a)^{\alpha+\beta}|_{t=b} \geq \beta J_a^\delta (t-a)^{\alpha+\beta-1} f(t)|_{t=b}.$$

Then, thanks to (15), we obtain

$$J_a^\delta (t-a)^\alpha f^\beta(t)|_{t=b} + (\beta-1) J_a^\delta (t-a)^{\alpha+\beta}|_{t=b} \geq \beta \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+\beta+\delta+1)}(b-a)^{\alpha+\beta+\delta}.$$

Hence,

$$J_a^\delta (t-a)^\alpha f^\beta(t)|_{t=b} \geq \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+\beta+\delta+1)}(b-a)^{\alpha+\beta+\delta}. \tag{18}$$

Now, we prove that $J_a^\delta (x-a)^\alpha f^\beta(x)|_{x=b} \leq J_a^\delta f^{\alpha+\beta}(b)$.
As before, by Lemma 2.2 of [7]), we get

$$\frac{\beta}{\alpha+\beta} f^{\alpha+\beta}(x) + \frac{\alpha}{\alpha+\beta}(x-a)^{\alpha+\beta} \geq (x-a)^\alpha f^\beta(x). \tag{19}$$

Multiplying both sides of (19) by $\frac{1}{\Gamma(\delta)}(b-x)^{\delta-1}$ and integrating the resulting inequality with respect to x over $[a,b]$, we obtain

$$\frac{\beta}{\alpha+\beta} J_a^\delta f^{\alpha+\beta}(x)|_{x=b} + \frac{\alpha}{\alpha+\beta} J_a^\delta (x-a)^{\alpha+\beta}|_{x=b} \geq J_a^\delta (x-a)^\alpha f^\beta(x)|_{x=b}.$$

Therefore,

$$\beta J_a^{\alpha} f^{\alpha+\beta}(x)|_{x=b}+\alpha J_a^{\delta}(x-a)^{\alpha+\beta}|_{x=b} \geq \alpha J_a^{\delta}(x-a)^{\alpha} f^{\beta}(x)|_{x=b}+\beta J_a^{\delta}(x-a)^{\alpha} f^{\beta}(x)|_{x=b}.$$

Using (18), we obtain

$$\begin{aligned}\beta J_a^{\delta} f^{\alpha+\beta}(x)|_{x=b} + \alpha\frac{\Gamma(\alpha+\beta+1)(b-a)^{\alpha+\beta+\delta}}{\Gamma(\alpha+\beta+\delta+1)} \quad &\geq \quad \alpha\frac{\Gamma(\alpha+\beta+1)(b-a)^{\alpha+\beta+\delta}}{\Gamma(\alpha+\beta+\delta+1)} \\ &+\beta J_a^{\delta}(t-a)^{\alpha} f^{\beta}(t)|_{t=b}.\end{aligned}$$

Hence

$$J_a^{\delta} f^{\alpha+\beta}(b) \geq J_a^{\delta}(t-a)^{\alpha} f^{\beta}(t)|_{t=b}. \tag{20}$$

Now, we need to show that

$$\left(J_a^{\delta}(x-a)^{\alpha} f^{\beta}(x)|_{x=b}\right)^{\lambda-1} \leq 1,$$

which is equivalent to

$$J_a^{\delta}(x-a)^{\alpha} f^{\beta}(x)|_{x=b} \leq 1.$$

An integration by parts allows us to obtain:

$$\begin{aligned} J^{\delta}(x-a)^{\alpha} f^{\beta}(x)|_{x=b} \quad &= \quad \frac{1}{\Gamma(\delta)}\int_a^b (b-u)^{\delta-1}(u-a)^{\alpha} f^{\beta}(u)du \\ &= \quad \frac{1}{\Gamma(\delta)}\Big[-(u-a)^{\alpha}\int_u^b (b-t)^{\delta-1} f^{\beta}(t)du|_{u=a}^{u=b} \\ &\quad +\alpha\int_a^b (u-a)^{\alpha-1}\int_u^b (b-t)^{\delta-1} f^{\beta}(t)dtdu\Big] \\ &= \quad \frac{1}{\Gamma(\delta)}\left[\alpha\int_a^b (u-a)^{\alpha-1}\int_u^b (b-t)^{\delta-1} f^{\beta}(t)dtdu\right] \\ &= \quad \frac{1}{\Gamma(\delta)}\left[\alpha(\alpha-1)\int_a^b (u-a)^{\alpha-2}\int_u^b\int_{u_1}^b (b-t)^{\delta-1} f^{\beta}(t)dtdu_1du\right] \\ &\quad \ldots \end{aligned}$$

$$= \frac{1}{\Gamma(\delta)}\Big[\alpha(\alpha-1)...(\alpha-n+1)\int_a^b (u-a)^{\alpha-n} \times \int_u^b\int_{u_1}^b ... \int_{u_{n-1}}^b (b-t)^{\delta-1} f^\beta(t)dtdu_{n-1}...du_1du\Big].$$

On the other hand, using the fact that $(u-a)^{\alpha-n} \leq (b-a)$, we can write

$$\begin{aligned}
J^\delta(x-a)^\alpha f^\beta(x)|_{x=b} &\leq \frac{1}{\Gamma(\delta)}\Big[\alpha(\alpha-1)...(\alpha-n+1)\int_a^b (b-a) \\
&\quad \times \int_u^b\int_{u_1}^b ... \int_{u_{n-1}}^b (b-t)^{\delta-1} f^\beta(t)dtdu_{n-1}...du_1du\Big] \\
&= \frac{1}{\Gamma(\delta)}\Big[\alpha(\alpha-1)...(\alpha-n+1) \\
&\quad \times \int_a^b\int_a^b\int_u^b\int_{u_1}^b ... \int_{u_{n-1}}^b (b-t)^{\delta-1} f^\beta(t)dtdu_{n-1}...du_1du_0du\Big] \\
&\leq \frac{1}{\Gamma(\delta)}\Big[\alpha(\alpha-1)...(\alpha-n+1) \\
&\quad \times \int_a^b\int_a^b\int_a^b ... \int_a^b (b-t)^{\delta-1} f^\beta(t)dtdu_{n-1}...du_1du_0du\Big] \\
&= \frac{1}{\Gamma(\delta)}\Big[\alpha(\alpha-1)...(\alpha-n+1)\int_a^b (b-t)^{\delta-1} f^\beta(t)dt \\
&\quad \times \int_a^b\int_a^b ... \int_a^b du_{n-1}...du_1du_0du\Big] \\
&= \alpha(\alpha-1)...(\alpha-n+1)(b-a)^{n+1} J_a^\delta f^\beta(b). \\
&= \frac{\Gamma(\alpha)}{\Gamma(\alpha-n+1)}(b-a)^{n+1} J_a^\delta f^\beta(b). \\
&\leq 1.
\end{aligned}$$

Theorem 3.4 is thus proved. ∎

Remark 3.3. *If we take $\delta = 1$ in Theorem 3.4, we obtain Theorem 3.3.*

To finish, we present to the reader the following corollary:

Corollary 3.1. *Let f and h be two positive continuous functions on $[a, b]$, with $f \leq h$, f is increasing and $\frac{f}{h}$ is decreasing. Then, for any $x \in]a, b]$, we have:*

$$\frac{J_a^\delta f(x)}{J_a^\delta h(x)} \geq \left(\frac{J_a^\delta (x-a)^\alpha f^\beta(x)}{J_a^\delta (x-a)^\alpha h^\beta(x)} \right)^\lambda,$$

where $\delta, \alpha, \beta > 0$, $\lambda \geq 1$.

Proof. We take $\varphi(x) = x^\beta$ and $g(x) = (x-a)^\alpha$ in Theorem 3.1. ∎

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