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SOLVABILITY FOR NONLINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS WITH TWO ARBITRARY ORDERS

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Abstract. In this paper we study a coupled system of nonlinear differential equations of two arbitrary orders $\alpha, \beta \in \mathbb{R}^+ - \mathbb{N}$. New existence and uniqueness results are established using Banach fixed point theorem. Other existence results are obtained using Schaefer and Krasnoselskii fixed point theorems. Some illustrative examples are also presented.

1. Introduction

The fractional differential equations theory arises in many scientific disciplines, such as physics, chemistry, control theory, signal processing and biophysics. For more details, we refer the reader to [**7, 9, 11, 12**] and the references therein. Recently, there has been a significant progress in the investigation of these equations, (see [**3, 4, 13, 14**]). Moreover, the study of coupled systems of fractional differential equations is also of great importance. Such systems occur in various problems of applied science. For some recent results on the fractional systems, we refer the reader to ([**1, 2, 5, 6**].
In this paper, we discuss the existence and uniqueness of solutions for the following coupled system of fractional differential equations:

$$\begin{cases} D^{\alpha}u(t) = f_1\left(t, v(t), D^{\alpha-1}v(t), D^{\alpha-2}v(t), ..., D^{\alpha-(n-1)}v(t)\right), t \in [0,1], \\ D^{\beta}v(t) = f_2\left(t, u(t), D^{\beta-1}u(t), D^{\beta-2}u(t), ..., D^{\beta-(n-1)}u(t)\right), t \in [0,1], \\ u(0) = u_0^*,\ u'(0) = u''(0) = ... = u^{(n-2)}(0) = 0, \\ u^{(n-1)}(0) = \gamma I^p u(\eta), \eta \in]0,1[, \\ v(0) = v_0^*,\ v'(0) = ... = v^{(n-2)}(0) = 0, \\ v^{(n-1)}(0) = \delta I^q v(\zeta), \zeta \in]0,1[, \end{cases} \quad (1)$$

where D^{α} and D^{β} denote the Caputo fractional derivatives, p and q are non negative reals numbers, $n-1 < \alpha < n$, $n-1 < \beta < n$, with $n \in \mathbb{N}^*, n \neq 1$, $u_0^*, v_0^* \in \mathbb{R}$, f_1 and f_2 are two functions which will be specified later.
The paper is organized as follows: In section 2, we present some definitions, preliminaries and lemmas. Section 3 is devoted to existence of solutions of problem (1). At the last section, some examples are presented to illustrate our results.

2. Preliminaries

The following notations, definitions and preliminary facts will be used throughout this paper.

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Definition 1. *The Riemann-Liouville fractional integral operator of order $\alpha > 0$, for a continuous function f on $[a,b]$ is defined as:*

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, \alpha > 0, a \leq t \leq b \tag{2}$$

where $\Gamma(\alpha) := \int_0^{\infty} e^{-u} u^{\alpha-1} du.$

Definition 2. *The Caputo fractional derivative of order $\alpha > 0$ for $f \in C^n([a,b])$ is defined as:*

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\, d\tau, n-1 < \alpha < n, n \in N^*, t \in [a,b]. \tag{3}$$

For more details about fractional calculus, we refer the reader to [**10**].

The following lemmas give some properties of Riemann-Liouville fractional integral and Caputo fractional derivative [**7, 9**]:

Lemma 3. *Let $r, s > 0, f \in L_1([a,b])$. Then $I^r I^s f(t) = I^{r+s} f(t), D^s I^s f(t) = f(t), t \in [a,b]$.*

Lemma 4. *Let $s > r > 0, f \in L_1([a,b])$. Then $D^r I^s f(t) = I^{s-r} f(t), t \in [a,b]$.*

To study the coupled system (1), we need the following two lemmas [**7**]:

Lemma 5. *For $\alpha > 0$, the general solution of the equation $D^{\alpha} x(t) = 0$ is given by*

$$x(t) = c_0 + c_1 t + c_2 t^2 + ... + c_{n-1} t^{n-1}, \tag{4}$$

where $c_i \in \mathbb{R}, i = 0,1,2,..,n-1, n = [\alpha]+1$.

Lemma 6. *Let $\alpha > 0$. Then*

$$I^{\alpha} D^{\alpha} x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + ... + c_{n-1} t^{n-1}, \tag{5}$$

for some $c_i \in \mathbb{R}, i = 0,1,2,...,n-1, n = [\alpha]+1$.

We need also the following auxiliary result:

Lemma 7. *Let $g \in C([0,1], \mathbb{R})$. The solution of the problem*

$$D^{\alpha} x(t) = g(t),\ n-1 < \alpha < n,\ n \in \mathbb{N} \tag{6}$$

subject to the conditions

$$x(0) = x_0,\ x'(0) = x''(0) = ... = x^{(n-2)}(0) = 0,$$

and

$$x^{(n-1)}(0) = \gamma I^p x(\eta), \eta \in]0,1[, p > 0$$

is given by:

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)\, ds + x_0$$
$$+ \frac{\gamma \Gamma(p+n) t^{n-1}}{\Gamma(n)(\Gamma(p+n) - \gamma \eta^{p+n-1})} \left(\int_0^{\eta} \frac{(\eta-s)^{p+\alpha-1}}{\Gamma(p+\alpha)} g(s)\, ds + x_0 \frac{\eta^p}{\Gamma(p+1)} \right), \tag{7}$$

provided that $\gamma \neq \frac{\Gamma(p+n)}{\eta^{p+n-1}}$.

Proof. By Lemmas 5,6, the equation (6) implies that:

$$x(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}g(\tau)\,d\tau-c_0-c_1t-c_2t^2-...-c_{n-1}t^{n-1}. \quad (8)$$

For any $k=1,2,...,n-1$, we can write

$$x^{(k)}(0)=-k!c_k,\ \text{for } k=1,2,...,n-1.$$

Using the fact that $x(0)=x_0$, and $x'(0)=...=x^{(n-2)}(0)=0$, we obtain $c_0=-x_0$, and $c_1=c_2=...=c_{n-2}=0$. Thanks to Lemma 3, we get

$$I^p x(t)=\frac{1}{\Gamma(p+\alpha)}\int_0^t (t-s)^{p+\alpha-1}g(s)\,ds+\frac{x_0t^p}{\Gamma(p+1)}-c_{n-1}\frac{\Gamma(n)\,t^{p+n-1}}{\Gamma(p+n)}.$$

Using the conditions of Lemma 7, we have

$$c_{n-1}=-\frac{\gamma\Gamma(p+n)}{\Gamma(n)\left(\Gamma(p+n)-\gamma\eta^{p+n-1}\right)}\left(I^{p+\alpha}g(\eta)+x_0\frac{\eta^p}{\Gamma(p+1)}\right).$$

Substituting $c_0,c_1,c_2,...,c_{n-1}$ in (8), we obtain the desired quantity (7). □

Let us now introduce the spaces:

$$X:=\left\{u\in C([0,1],\mathbb{R});D^{\alpha-1}u,D^{\alpha-2}u,...,D^{\alpha-(n-1)}u\in C([0,1],\mathbb{R})\right\},$$

$$Y:=\left\{v\in C([0,1],\mathbb{R});D^{\beta-1}v,D^{\beta-2}v,...,D^{\beta-(n-1)}v\in C([0,1],\mathbb{R})\right\}.$$

For $n-1<\alpha<n$, we define on X the norm

$$\|u\|_1:=\max\left(\|u\|,\left\|D^{\alpha-1}u\right\|,\left\|D^{\alpha-2}u\right\|,...,\left\|D^{\alpha-(n-1)}u\right\|\right);$$
$$||u||=\sup_{t\in[0,1]}|u(t)|,||D^{\alpha-k}u||=\sup_{t\in[0,1]}\left|D^{\alpha-k}u(t)\right|;k=1,2,...,n-1,\ \text{for all } n\geq 2.$$

We also define on Y the norm

$$\|v\|_{1*}:=\max\left(\|v\|,\left\|D^{\beta-1}v\right\|,\left\|D^{\beta-2}v\right\|,...,\left\|D^{\beta-(n-1)}v\right\|\right),$$
$$||v||=\sup_{t\in[0,1]}|v(t)|,||D^{\beta-h}v||=\sup_{t\in[0,1]}\left|D^{\beta-h}v(t)\right|;h=1,2,...,n-1,\ \text{for all } n\geq 2,$$

where $n-1<\beta<n$.

For the space $X\times Y$, we define the norm

$$\|(u,v)\|_2:=\max\left(\|u\|_1,\|v\|_{1*}\right).$$

It is clear that $\left(X\times Y,\|.\|_2\right)$ is a Banach space.

3. Main Results

We introduce the following quantities:

$$\begin{aligned}
M_1 &: = \frac{1}{\Gamma(\alpha+1)} + \frac{|\gamma|\,\Gamma(p+n)\,\eta^{p+\alpha}}{\Gamma(n)\,|\Gamma(p+n)-\gamma\eta^{p+n-1}|\,\Gamma(p+\alpha+1)},\\
M_2 &: = \frac{1}{\Gamma(\beta+1)} + \frac{|\delta|\,\Gamma(q+n)\,\zeta^{q+\beta}}{\Gamma(n)\,\left|\Gamma(q+n)-\delta\zeta^{q+n-1}\right|\Gamma(q+\beta+1)},\\
M'_k &: = \frac{1}{\Gamma(k+1)} + \frac{|\gamma|\,\Gamma(p+n)\,\eta^{p+\alpha}}{|\Gamma(p+n)-\gamma\eta^{p+n-1}|\,\Gamma(p+\alpha+1)\,\Gamma(n+k-\alpha)},\\
M'_h &: = \frac{1}{\Gamma(h+1)} + \frac{|\delta|\,\Gamma(q+n)\,\zeta^{q+\beta}}{\left|\Gamma(q+n)-\delta\zeta^{q+n-1}\right|\Gamma(q+\beta+1)\Gamma(n+h-\beta)},
\end{aligned}$$

and

$$\begin{aligned}
\omega_1 &= \frac{|\gamma|\,\Gamma(p+n)}{\Gamma(n)\,|\Gamma(p+n)-\gamma\eta^{p+n-1}|},\\
\omega_2 &= \frac{|\delta|\,\Gamma(q+n)}{\Gamma(n)\,|\Gamma(q+n)-\delta\eta^{q+\zeta-1}|},\\
\omega'_k &= \frac{|\gamma|\,\Gamma(p+n)}{|\Gamma(p+n)-\gamma\eta^{p+n-1}|\,\Gamma(n+k-\alpha)},\\
\overline{\omega}'_h &= \frac{|\delta|\,\Gamma(q+n)}{\left|\Gamma(q+n)-\delta\zeta^{q+n-1}\right|\Gamma(n+h-\beta)}.
\end{aligned}$$

Also, we consider the following hypotheses:

(H1) There exist non negative real numbers $m_i, n_i, i = 0, 1, ..., n-1$, such that for all $t \in [0,1], (u_0, u_1, ..., u_{n-1}), (v_0, v_1, ..., v_{n-1}) \in \mathbb{R}^n$, we have

$$|f_1(t, u_0, u_1, ..., u_{n-1}) - f_1(t, v_0, v_1, ..., v_{n-1})|$$

$$\leq m_0|u_0 - v_0| + m_1|u_1 - v_1| + ... + m_{n-1}|u_{n-1} - v_{n-1}|,$$

$$|f_2(t, u_0, u_1, ..., u_{n-1}) - f_2(t, v_0, v_1, ..., v_{n-1})|$$

$$\leq n_0|u_0 - v_0| + n_1|u_1 - v_1| + ... + n_{n-1}|u_{n-1} - v_{n-1}|,$$

with $\overline{m} := \max\{m_i\}_{i=0}^{n-1}, \overline{n} := \max\{n_i\}_{i=0}^{n-1}$.

(H2) The functions $f_1, f_2 : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ are continuous.

(H3) There exist positive constants L_1 and L_2, such that for all $t \in [0,1], w \in \mathbb{R}^n$, we have

$$|f_i(t, w)| \leq L_i, i = 1, 2.$$

Our first result is given by:

Theorem 8. *Suppose that* $\gamma \neq \frac{\Gamma(p+n)}{\eta^{p+n-1}}, \delta \neq \frac{\Gamma(q+n)}{\zeta^{q+n-1}}$ *and assume that* $(H1)$ *holds. If*

$$\max(\overline{m}, \overline{n}) \max(M_1, M_2, M'_k, M'_h) < \frac{1}{n}, \tag{9}$$

then the fractional system (1) *has a unique solution on* $[0,1]$.

Proof. Consider the operator $T : X \times Y \to X \times Y$ defined by

$$T(u,v)(t) = \left(T_1(v)(t), T_2(u)(t) \right), \tag{10}$$

where

$$\begin{aligned} T_1(v)(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1\left(s, v(s), D^{\alpha-1}v(s), D^{\alpha-2}v(s), ..., D^{\alpha-(n-1)}v(s)\right) ds \\ &+ u_0^* + \frac{\gamma\Gamma(p+n)\, t^{n-1}}{\Gamma(n)\left(\Gamma(p+n) - \gamma\eta^{p+n-1}\right)} \\ &\times \left(\int_0^\eta \frac{(\eta-s)^{p+\alpha-1}}{\Gamma(p+\alpha)} f_1\left(s, v(s), D^{\alpha-1}v(s), D^{\alpha-2}v(s), ..., D^{\alpha-(n-1)}v(s)\right) ds \right. \\ &\left. + u_0^* \frac{\eta^p}{\Gamma(p+1)} \right), \end{aligned}$$

and

$$\begin{aligned} T_2(u)(t) &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2\left(s, u(s), D^{\beta-1}u(s), D^{\beta-2}u(s), ..., D^{\beta-(n-1)}u(s)\right) ds \\ &+ v_0^* + \frac{\delta\Gamma(q+n)\, t^{n-1}}{\Gamma(n)\left(\Gamma(q+n) - \delta\zeta^{q+n-1}\right)} \\ &\times \left(\int_0^\zeta \frac{(\zeta-s)^{q+\beta-1}}{\Gamma(q+\beta)} f_2\left(s, u(s), D^{\beta-1}u(s), D^{\beta-2}u(s), ..., D^{\beta-(n-1)}u(s)\right) ds \right. \\ &\left. + v_0^* \frac{\zeta^q}{\Gamma(q+1)} \right). \end{aligned}$$

We use Lemma 4. For all $k = 1, 2, ..., n-1$, we have

$$\begin{aligned} D^{\alpha-k}T_1(v)(t) &= \int_0^t \frac{(t-s)^{k-1}}{\Gamma(k)} f_1\left(s, v(s), D^{\alpha-1}v(s), D^{\alpha-2}v(s), ..., D^{\alpha-(n-1)}v(s)\right) ds \\ &+ \frac{\gamma\Gamma(p+n)\, t^{n+k-\alpha-1}}{\left(\Gamma(p+n) - \gamma\eta^{p+n-1}\right)\Gamma(n+k-\alpha)} \\ &\times \left(\int_0^\eta \frac{(\eta-s)^{p+\alpha-1}}{\Gamma(p+\alpha)} f_1\left(s, v(s), D^{\alpha-1}v(s), D^{\alpha-2}v(s), ..., D^{\alpha-(n-1)}v(s)\right) ds \right. \\ &\left. + u_0^* \frac{\eta^p}{\Gamma(p+1)} \right), \end{aligned}$$

and for all $h = 1, 2, ..., n-1$, we can write

$$D^{\beta-h}T_2(u)(t)$$
$$= \int_0^t \frac{(t-s)^{h-1}}{\Gamma(h)} f_2\left(s, u(s), D^{\beta-1}u(s), D^{\beta-2}u(s), ..., D^{\beta-(n-1)}u(s)\right) ds$$
$$+ \frac{\delta\Gamma(q+n) t^{n+h-\beta-1}}{\left(\Gamma(q+n) - \delta\zeta^{q+n-1}\right)\Gamma(n+h-\beta)}$$
$$\times \left(\int_0^\zeta \frac{(\zeta-s)^{q+\beta-1}}{\Gamma(q+\beta)} f_2\left(s, u(s), D^{\beta-1}u(s), D^{\beta-2}u(s), ..., D^{\beta-(n-1)}u(s)\right) ds \right.$$
$$\left. + v_0^* \frac{\zeta^q}{\Gamma(q+1)}\right).$$

We shall show that T is contractive. Let $(u_1, v_1), (u_2, v_2) \in X \times Y$. Then, for each $t \in [0,1]$, the operator T defined by (10) implies that

$$|T_1(v_2)(t) - T_1(v_1)(t)|$$
$$\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds + \frac{|\gamma|\Gamma(p+n) t^{n-1}}{\Gamma(n)|\Gamma(p+n) - \gamma\eta^{p+n-1}|} \int_0^\eta \frac{(\eta-s)^{p+\alpha-1}}{\Gamma(p+\alpha)} ds$$
$$\times \sup_{0\leq s\leq 1} \left| f_1\left(s, v_2(s), ..., D^{\alpha-(n-1)}v_2(s)\right) - f_1\left(s, v_1(s), ..., D^{\alpha-(n-1)}v_1(s)\right)\right|.$$

Using $(H1)$, we can write:

$$|T_1(v_2) - T_1(v_1)|$$
$$\leq \left(\frac{1}{\Gamma(\alpha+1)} + \frac{|\gamma|\Gamma(p+n)\eta^{p+\alpha}}{\Gamma(n)|\Gamma(p+n) - \gamma\eta^{p+n-1}|\Gamma(p+\alpha+1)}\right)$$
$$\times \left(m_0\|v_2 - v_1\| + ... + m_{n-1}\left\|D^{\alpha-(n-1)}(v_2 - v_1)\right\|\right).$$

Consequently,

$$\|T_1(v_2) - T_1(v_1)\| \leq M_1\overline{m}n\|v_2 - v_1\|_{1*}. \tag{11}$$

Similarly, it can be shown that

$$\|T_2(u_2) - T_2(u_1)\| \leq M_2\overline{n}n\|u_2 - u_1\|_1. \tag{12}$$

On the other hand, for all $k = 1, 2, ..., n-1$, we have

$$\left|D^{\alpha-k}T_1(v_2)(t) - D^{\alpha-k}T_1(v_1)(t)\right|$$
$$\leq \int_0^t \frac{(t-s)^k}{\Gamma(k)} - 1ds + \frac{|\gamma|\Gamma(p+n) t^{n+k-\alpha-1}}{|\Gamma(p+n) - \gamma\eta^{p+n-1}|\Gamma(n+k-\alpha)} \int_0^\eta \frac{(\eta-s)^{p+\alpha-1}}{\Gamma(p+\alpha)} ds$$
$$\times \sup_{0\leq s\leq 1} \left| f_1\left(s, v_2(s), ..., D^{\alpha-(n-1)}v_2(s)\right) - f_1\left(s, v_1(s), ..., D^{\alpha-(n-1)}v_1(s)\right)\right|.$$

This implies that

$$\begin{aligned}&\left|D^{\alpha-k}T_1(v_2)-D^{\alpha-k}T_1(v_1)\right|\\ \leq\ &\left(\frac{1}{\Gamma(k+1)}+\frac{|\gamma|\,\Gamma(p+n)\,\eta^{p+\alpha}}{|\Gamma(p+n)-\gamma\eta^{p+n-1}|\,\Gamma(p+\alpha+1)\,\Gamma(n+k-\alpha)}\right)\\ &\times\left(m_0\left\|v_2-v_1\right\|+...+m_{n-1}\left\|D^{\alpha-(n-1)}(v_2-v_1)\right\|\right).\end{aligned}$$

Therefore,

$$\left\|D^{\alpha-k}T_1(v_2)-D^{\alpha-k}T_1(v_1)\right\|\leq M'_k\overline{m}n\left\|v_2-v_1\right\|_{1*}. \tag{13}$$

With the same arguments, we get

$$\left\|D^{\beta-h}T_2(u_2)-D^{\beta-h}T_2(u_1)\right\|\leq M'_h\overline{n}n\left\|u_2-u_1\right\|_1. \tag{14}$$

Thanks to (11) and (13), we obtain

$$\left\|T_1(v_2)-T_1(v_1)\right\|_1\leq\max\left(M_1,M'_k\right)\overline{m}n\left\|v_2-v_1\right\|_{1*}. \tag{15}$$

Using (12) and (14), we get

$$\left\|T_2(u_2)-T_2(u_{1*})\right\|_1\leq\max\left(M_2,M'_h\right)\overline{n}n\left\|u_2-u_1\right\|_1. \tag{16}$$

Thanks to (15) and (16), we deduce that

$$\left\|T(u_2,v_2)-T(u_1,v_1)\right\|_2\leq$$

$$n\max\left(\overline{m},\overline{n}\right)\max\left(M_1,M_2,M'_k,M'_h\right)\left\|(u_2-u_1,v_2-v_1)\right\|_2. \tag{17}$$

Combining (9) and (17), we conclude that T is a contraction mapping. Hence by Banach fixed point theorem, there exists a unique fixed point which is a solution of (1). □

The second result is the following:

Theorem 9. *Suppose that* $\gamma\neq\frac{\Gamma(p+n)}{\eta^{p+n-1}}$, $\delta\neq\frac{\Gamma(q+n)}{\zeta^{q+n-1}}$ *and assume that* $(H2)$ *and* $(H3)$ *are satisfied. Then the fractional system* (1) *has at least one solution on* $[0,1]$.

Proof. First of all, we show that the operator T is completely continuous. Note that T is continuous on $X\times Y$ in view of the continuity of f_1 and f_2. We proceed on two steps:

Step 1: Let us take $\sigma>0$ and $B_\sigma:=\{(u,v)\in X\times Y;\left\|(u,v)\right\|_2\leq\sigma\}$. For $(u,v)\in B_\sigma$, using $(H3)$, we can write

$$\left\|T_1(v)\right\|\leq L_1M_1+|u_0^*|+\omega_1\frac{|u_0^*|\,\eta^p}{\Gamma(p+1)}<+\infty, \tag{18}$$

and

$$\left\|T_2(u)\right\|\leq L_2M_2+|v_0^*|+\omega_2\frac{|v_0^*|\,\zeta^q}{\Gamma(q+1)}<+\infty. \tag{19}$$

On the other hand, for all $k=1,2,...,n-1$, we have

$$\left\|D^{\alpha-k}T_1(v)\right\| \tag{20}$$
$$\leq \frac{L_1}{\Gamma(k+1)}+\frac{|\gamma|\,\Gamma(p+n)}{|\Gamma(p+n)-\gamma\eta^{p+n-1}|\,\Gamma(n+k-\alpha)}\left(\frac{L_1\eta^{p+\alpha}}{\Gamma(p+\alpha+1)}+\frac{|u_0^*|\,\eta^p}{\Gamma(p+1)}\right)$$
$$\leq L_1M_k'+\omega_k'\frac{|u_0^*|\,\eta^p}{\Gamma(p+1)}<+\infty, \tag{21}$$

and for all $h=1,2,...,n-1$,

$$\left\|D^{\beta-h}T_2(u)\right\|\leq L_2M_h'+\omega_h'\frac{|v_0^*|\,\zeta^q}{\Gamma(q+1)}<+\infty. \tag{22}$$

Thanks to $(18),(19),(20)$ and (21), we obtain

$$\|T_1(v)\|_1 \leq \max\left(L_1M_1+|u_0|+\omega_1\frac{|u_0^*|\,\eta^p}{\Gamma(p+1)},L_1M_k'+\omega_k'\frac{|u_0^*|\,\eta^p}{\Gamma(p+1)}\right)<+\infty,$$
$$\|T_2(u)\|_{1*} \leq \max\left(L_2M_2+|v_0|+\omega_2\frac{|v_0^*|\,\zeta^q}{\Gamma(q+1)},L_2M_h'+\omega_h'\frac{|v_0^*|\,\zeta^q}{\Gamma(q+1)}\right)<+\infty.$$

These two inequalities imply that

$$\|T(u,v)\|_2<+\infty.$$

Step 2: Let $t_1,t_2\in[0,1]\,,t_1<t_2$ and $(u,v)\in B_\sigma$. We have

$$|\;T_1(v)(t_2)-T_1(v)(t_1)\;|$$
$$\leq \frac{L_1\left(t_2^\alpha-t_1^\alpha\right)}{\Gamma(\alpha+1)}+\frac{|\gamma|\,\Gamma(p+n)\left(t_2^{n-1}-t_1^{n-1}\right)}{\Gamma(n)\,|\Gamma(p+n)-\gamma\eta^{p+n-1}|}\left(\frac{L_1\eta^{p+\alpha}}{\Gamma(p+\alpha+1)}+\frac{|u_0^*|\,\eta^p}{\Gamma(p+1)}\right). \tag{23}$$

The inequality (22) implies that

$$|\quad T_1(v)(t_2)-T_1(v)(t_1)\;| \tag{24}$$
$$\leq \frac{L_1\left(t_2^\alpha-t_1^\alpha\right)}{\Gamma(\alpha+1)}+\omega_1\left(\frac{L_1\eta^{p+\alpha}}{\Gamma(p+\alpha+1)}+\frac{|u_0^*|\,\eta^p}{\Gamma(p+1)}\right)\left(t_2^{n-1}-t_1^{n-1}\right).$$

In the same way, we have

$$|\,T_2(u)(t_2)-T_2(u)(t_1)\;| \tag{25}$$
$$\leq \frac{L_2\left(t_2^\beta-t_1^\beta\right)}{\Gamma(\beta+1)}+\omega_2\left(\frac{L_2\zeta^{q+\beta}}{\Gamma(q+\beta+1)}+\frac{|v_0^*|\,\zeta^q}{\Gamma(q+1)}\right)\left(t_2^{n-1}-t_1^{n-1}\right).$$

On the other hand, for all $k=1,2...,n-1$, we can write

$$|\,D^{\alpha-k}\left(T_1(v)(t_2)-T_1(v)(t_1)\right)| \tag{26}$$
$$\leq \frac{L_1\left(t_2^\alpha-t_1^\alpha\right)}{\Gamma(k+1)}+\omega_k'\left(\frac{L_1\eta^{p+\alpha}}{\Gamma(p+\alpha+1)}+\frac{|u_0^*|\,\eta^p}{\Gamma(p+1)}\right)\left(t_2^{n+k-\alpha-1}-t_1^{n+k-\alpha-1}\right),$$

and for all $h=1,2...,n-1$, we have

$$|\,D^{\beta-k}\left(T_2(v)(t_2)-T_2(v)(t_1)\right)| \tag{27}$$
$$\leq \frac{L_2\left(t_2^\beta-t_1^\beta\right)}{\Gamma(k+1)}+\omega_h'\left(\frac{L_2\zeta^{q+\beta}}{\Gamma(q+\beta+1)}+\frac{|v_0|\,\zeta^q}{\Gamma(q+1)}\right)\left(t_2^{n+h-\beta-1}-t_1^{n+h-\beta-1}\right).$$

As $t_2 \to t_1$, the right-hand sides of the inequalities (23), (24), (25) and (26) tend to zero. Then, as a consequence of steps 1, 2, and by Arzela-Ascoli theorem, we conclude that T is completely continuous.
Next, we show that the set

$$\Omega := \{(u,v) \in X \times Y, (u,v) = \lambda T(u,v), 0 < \lambda < 1\}$$

is bounded.
Let $(u,v) \in \Omega$, then $(u,v) = \lambda T(u,v)$, for some $0 < \lambda < 1$. Hence, for $t \in [0,1]$, we have:

$$u(t) = \lambda T_1(v)(t), v(t) = \lambda T_2(u)(t).$$

Thanks to $(H3)$ and using (18) and (19), we conclude that

$$\begin{aligned} \|u\| &\leq \lambda\left(L_1M_1 + |u_0^*| + \omega_1 \frac{|u_0^*|\,\eta^p}{\Gamma(p+1)}\right), \\ \|v\| &\leq \lambda\left(L_2M_2 + |v_0^*| + \omega_2 \frac{|v_0^*|\,\zeta^q}{\Gamma(q+1)}\right), \end{aligned} \tag{28}$$

and by (20) and (21), we can state that

$$\begin{aligned} \left\|D^{\alpha-k}u\right\| &\leq \lambda\left(L_1M'_k + \omega'_k \frac{|u_0^*|\,\eta^p}{\Gamma(p+1)}\right), \\ \left\|D^{\beta-1}v\right\| &\leq \lambda\left(L_2M'_h + \omega'_h \frac{|v_0^*|\,\zeta^q}{\Gamma(q+1)}\right), \end{aligned} \tag{29}$$

for all $k, h \in \{= 1, 2, ..., n-1\}$. By (27) and (28), we can state that

$$\begin{aligned} \|u\|_1 &\leq \lambda \max\left(L_1M_1 + |u_0^*| + \omega_1 \frac{|u_0^*|\,\eta^p}{\Gamma(p+1)}, L_1M'_k + \omega'_k \frac{|u_0^*|\,\eta^p}{\Gamma(p+1)}\right) < +\infty, \\ \|v\|_{1*} &\leq \lambda \max\left(L_2M_2 + |v_0^*| + \omega_2 \frac{|v_0^*|\,\zeta^q}{\Gamma(q+1)}, L_2M'_h + \omega'_h \frac{|v_0^*|\,\zeta^q}{\Gamma(q+1)}\right) < +\infty. \end{aligned}$$

Hence,

$$\|(u,v)\|_2 < +\infty.$$

This shows that Ω is bounded. As a conclusion of Schaefer fixed point theorem, we deduce that T has at least one fixed point, which is a solution of (1). $\square$

Our third main result is based on Krasnoselskii theorem [**8**]. We prove the following theorem:

Theorem 10. *Let* $\gamma \neq \frac{\Gamma(p+n)}{\eta^{p+n-1}}, \delta \neq \frac{\Gamma(q+n)}{\zeta^{q+n-1}}$. *Suppose that* $(H1)$, $(H2)$ *and* $(H3)$ *are satisfied, and*

$$\max(\overline{m}, \overline{n}) < \frac{1}{n}. \tag{30}$$

Then, the fractional system (1) *has at least one solution on* $[0,1]$.

Proof. Let us fix $\theta \geq \max\begin{pmatrix} L_1M_1 + |u_0^*| + \omega_1 \frac{|u_0^*|\eta^p}{\Gamma(p+1)}, \\ L_2M_2 + |v_0^*| + \omega_2 \frac{|v_0^*|\zeta^q}{\Gamma(q+1)}, \\ L_1M'_k + \omega'_k \frac{|u_0^*|\eta^p}{\Gamma(p+1)}, \\ L_2M'_h + \omega'_h \frac{|v_0^*|\zeta^q}{\Gamma(q+1)} \end{pmatrix}$ and consider $B_\theta :=$ $\{(u,v) \in X \times Y, ||(u,v)||_2 \leq \theta\}$. On B_θ, we define the operators R and S as follows:

$$R(u,v)(t) = (R_1(v)(t), R_2(u)(t)), \quad S(u,v)(t) = (S_1(v)(t), S_2(u)(t)), \tag{31}$$

where,

$$R_1(v)(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1\left(s, v(s), D^{\alpha-1}v(s), D^{\alpha-2}v(s), ..., D^{\alpha-(n-1)}v(s)\right) ds + u_0^*,$$

$$R_2(u)(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2\left(s, u(s), D^{\beta-1}u(s), D^{\beta-2}u(s), ..., D^{\beta-(n-1)}u(s)\right) ds + v_0^*.$$

and

$$S_1(v)(t) = \frac{\gamma\Gamma(p+n)}{\Gamma(n)\left(\Gamma(p+n) - \gamma\eta^{p+n-1}\right)} t^{n-1}$$
$$\times\left(\int_0^\eta \frac{(\eta-s)^{p+\alpha-1}}{\Gamma(p+\alpha)} f_1\left(s, v(s), D^{\alpha-1}v(s), D^{\alpha-2}v(s), ..., D^{\alpha-(n-1)}v(s)\right) ds\right),$$

$$S_2(u)(t) = \frac{\delta\Gamma(q+n)}{\Gamma(n)\left(\Gamma(q+n) - \delta\zeta^{q+n-1}\right)} t^{n-1}$$
$$\times\left(\int_0^\zeta \frac{(\zeta-s)^{q+\beta-1}}{\Gamma(q+\beta)} f_2\left(s, u(s), D^{\beta-1}u(s), D^{\beta-2}u(s), ..., D^{\beta-(n-1)}u(s)\right) ds\right).$$

Let $(u_1, v_1), (u_2, v_2) \in B_\theta, t \in [0,1]$. Then, thanks to $(H3)$, and using (30), we obtain

$$\begin{aligned} \|R_1(v_1) + S_1(v_2)\| &\leq L_1 M_1 + |u_0^*| + \omega_1 \frac{|u_0^*|\,\eta^p}{\Gamma(p+1)}, \\ \|R_2(u_1) + S_2(u_2)\| &\leq L_2 M_2 + |v_0^*| + \omega_2 \frac{|v_0^*|\,\zeta^q}{\Gamma(q+1)}. \end{aligned} \tag{32}$$

Also, we have

$$\begin{aligned} \left\|D^{\alpha-k}(R_1(v_1) + S_1(v_2))\right\| &\leq L_1 M_k' + \omega_k' \frac{|u_0|\,\eta^p}{\Gamma(p+1)}, \\ \left\|D^{\beta-1}(R_2(u_1) + S_2(u_2))\right\| &\leq L_2 M_h' + \omega_h' \frac{|v_0|\,\zeta^q}{\Gamma(q+1)}. \end{aligned} \tag{33}$$

Using (31) and (32), yield the following two inequalities

$$\|R_1(v_1) + S_1(v_2)\|_1 \leq \max\left(L_1 M_1 + |u_0| + \omega_1 \frac{|u_0|\,\eta^p}{\Gamma(p+1)}, L_1 M_k' + \omega_k' \frac{|u_0|\,\eta^p}{\Gamma(p+1)}\right), \tag{34}$$

and

$$\|R_2(u_1) + S_2(u_2)\|_{1*} \leq \max\left(L_2 M_2 + |v_0| + \omega_2 \frac{|v_0|\,\zeta^q}{\Gamma(q+1)}, L_2 M_h' + \omega_h' \frac{|v_0|\,\zeta^q}{\Gamma(q+1)}\right). \tag{35}$$

By (33) and (34), we obtain

$$\left\|R\left(u_1,v_1\right)+S\left(u_2,v_2\right)\right\|_2\leq \max\begin{pmatrix} L_1M_1+|u_0|+\omega_1\frac{|u_0|\eta^p}{\Gamma(p+1)},\\ L_2M_2+|v_0|+\omega_2\frac{|v_0|\zeta^q}{\Gamma(q+1)},\\ L_1M_k'+\omega_k'\frac{|u_0|\eta^p}{\Gamma(p+1)},\\ L_2M_h'+\omega_h'\frac{|v_0|\zeta^q}{\Gamma(q+1)}\end{pmatrix}\leq\theta. \tag{36}$$

Thanks to (35), we get

$$R\left(u_1,v_1\right)+S\left(u_2,v_2\right)\in B_\theta.$$

Now we prove the contraction of R. Using $(H1)$, we can write

$$\begin{aligned}\left\|R_1\left(v_2\right)-R_1\left(v_1\right)\right\| &\leq \frac{n\overline{m}}{\Gamma\left(\alpha+1\right)}\left\|v_2-v_1\right\|_{1*}, \\ \left\|R_2\left(u_2\right)-R_2\left(u_1\right)\right\| &\leq \frac{n\overline{n}}{\Gamma\left(\beta+1\right)}\left\|u_2-u_1\right\|_1,\end{aligned} \tag{37}$$

and

$$\begin{aligned}\left\|D^{\alpha-k}R_1\left(v_2\right)-D^{\alpha-k}R_1\left(v_1\right)\right\| &\leq \frac{n\overline{m}}{\Gamma\left(k+1\right)}\left\|v_2-v_1\right\|_{1*}, \\ \left\|D^{\beta-h}R_2\left(u_2\right)-D^{\beta-h}R_2\left(u_1\right)\right\| &\leq \frac{n\overline{n}}{\Gamma\left(h+1\right)}\left\|u_2-u_1\right\|_1.\end{aligned} \tag{38}$$

Hence, by (36) and (37), we can write

$$\left\|R\left(u_2,v_2\right)-R\left(u_1,v_1\right)\right\|_2\leq n\max\left(\overline{m},\overline{n}\right)\left\|\left(u_2-u_1,v_2-v_1\right)\right\|_2. \tag{39}$$

Combining (29) and (38), we conclude that R is a contraction mapping. The continuity of f_1 and f_2 given in $(H2)$ implies that the operator S is continuous.

Now, we prove the compactness of the operator S. Let $t_1,t_2\in[0,1]$, $t_1<t_2$ and $(u,v)\in B_\theta$. We have

$$\begin{aligned}&\left\|S_1\left(v\right)\left(t_2\right)-S_1\left(v\right)\left(t_1\right)\right\| \\ \leq\ & L_1\omega_1\frac{\eta^{p+\alpha}}{\Gamma\left(p+\alpha+1\right)}\left(t_2^{n-1}-t_1^{n-1}\right),\end{aligned} \tag{40}$$

$$\begin{aligned}&\left\|S_2\left(u\right)\left(t_2\right)-S_2\left(u\right)\left(t_1\right)\right\| \\ \leq\ & L_2\omega_2\frac{\zeta^{q+\beta}}{\Gamma\left(q+\beta+1\right)}\left(t_2^{n-1}-t_1^{n-1}\right).\end{aligned} \tag{41}$$

Also, we have

$$\begin{aligned}&\left\|D^{\alpha-k}S_1\left(v\right)\left(t_2\right)-D^{\alpha-k}S_1\left(v\right)\left(t_1\right)\right\| \\ \leq\ & L_1\omega_k'\frac{\eta^{p+\alpha}}{\Gamma\left(p+\alpha+1\right)}\left(t_2^{n+k-\alpha-1}-t_1^{n+k-\alpha-1}\right),\end{aligned} \tag{42}$$

$$\begin{aligned}&\left\|D^{\beta-h}S_2\left(u\right)\left(t_2\right)-D^{\beta-h}S_2\left(u\right)\left(t_1\right)\right\| \\ \leq\ & L_2\omega_h'\frac{\zeta^{q+\beta}}{\Gamma\left(q+\beta+1\right)}\left(t_2^{n+h-\beta-1}-t_1^{n+h-\beta-1}\right).\end{aligned} \tag{43}$$

The right hand side of (39), (40), (41) and (42) are independent of (u,v) and tend to zero as $t_1 \to t_2$, so S is relatively compact on B_θ. Then by Ascolli-Arzella theorem, the operator S is compact. Finally, by Krasnoselskii theorem, we conclude that there exists a solution to (1). Theorem 10 is thus proved. □

4. Examples

Example 1: Consider the following fractional differential system:

$$\begin{cases} D^{\frac{7}{2}}u(t) = \dfrac{e^{-t^2}|v(t)|}{16+e^t} + \dfrac{\sin\left(D^{\frac{5}{2}}v(t) + D^{\frac{3}{2}}v(t) + D^{\frac{1}{2}}v(t)\right)}{32(\pi t^2+1)}, t\in[0,1], \\ D^{\frac{7}{2}}v(t) = \dfrac{|u(t)| + \left|D^{\frac{5}{2}}u(t)\right| + \left|D^{\frac{3}{2}}u(t)\right| + \left|D^{\frac{1}{2}}u(t)\right|}{e(\pi t+20)\left(e^t + |u(t)| + \left|D^{\frac{5}{2}}u(t)\right| + \left|D^{\frac{3}{2}}u(t)\right| + \left|D^{\frac{1}{2}}u(t)\right|\right)}, t\in[0,1], \\ u(0) = \sqrt{2}, v(0) = \sqrt{3}, \\ u'(0) = u''(0) = 0, u'''(0) = 4I^{\frac{1}{2}}u(\eta), \\ v'(0) = v''(0) = 0, v'''(0) = 4I^{\frac{1}{2}}v(\xi), \end{cases} \tag{44}$$

where, $\alpha = \beta = \frac{7}{2}, p = q = \frac{1}{2}, \gamma = \delta = 4, \eta = \xi = \frac{2}{5}$. For this example, we have

$$M_1 = M_2 = \frac{16}{105\sqrt{\pi}} + \frac{28\sqrt{\pi}}{375\left|105\sqrt{\pi} - 64\left(\frac{2}{5}\right)^{3,5}\right|} = 0.086,$$

$$M_1' := \frac{1}{\Gamma(2)} + \frac{28\sqrt{\pi}}{125\left|105\sqrt{\pi} - 64\left(\frac{2}{5}\right)^{3,5}\right|\Gamma\left(\frac{3}{2}\right)} = 1.0021,$$

$$M_2' := \frac{1}{\Gamma(3)} + \frac{28\sqrt{\pi}}{125\left|105\sqrt{\pi} - 64\left(\frac{2}{5}\right)^{3,5}\right|\Gamma\left(\frac{5}{2}\right)} = 0.5016,$$

$$M_3' := \frac{1}{\Gamma(4)} + \frac{28\sqrt{\pi}}{125\left|105\sqrt{\pi} - 64\left(\frac{2}{5}\right)^{3,5}\right|\Gamma\left(\frac{7}{2}\right)} = 0.166$$

We have also $\gamma \neq \frac{\Gamma(p+n)}{\eta^{p+n-1}}$, $\delta \neq \frac{\Gamma(q+n)}{\zeta^{q+n-1}}$ and

$$\max(\overline{m}, \overline{n}) \max(M_1, M_2, M_k', M_h') = \frac{1}{16} \times 1.0021 < \frac{1}{4},$$

The conditions of the Theorem 8 hold. Therefore, the problem (43) has a unique solution on $[0,1]$.

Example 2: Consider the following problem:

$$\begin{cases} D^{\frac{11}{4}}u(t) = \dfrac{e^{-t}}{16 + \sin(v(t)) + \cos\left(D^{\frac{7}{4}}v(t) + D^{\frac{3}{4}}v(t)\right)}, t\in[0,1], \\ D^{\frac{20}{7}}v(t) = \dfrac{e^{-2t}\sin(u(t))}{16 + \cos\left(D^{\frac{13}{7}}u(t) + D^{\frac{6}{7}}u(t)\right)}, t\in[0,1], \\ u(0) = 0,\ u'(0) = \sqrt{2}I^3 u(\eta), \\ v(0) = 0,\ v'(0) = \sqrt{2}I^2 v(\xi). \end{cases} \tag{45}$$

For this example, we have $\alpha = \frac{11}{4}, \beta = \frac{20}{7}$, $p = 3$, $q = 2$, $\gamma = \delta = \sqrt{2}$, $\eta = \frac{4}{5}$, $\xi = \frac{1}{5}$, and for all $(u, v, z) \in \mathbb{R}^3$

$$f_1(t,u,v,z) = \frac{e^{-t}}{16 + \sin u + \cos(v+z)},$$
$$f_2(t,u,v,z) = \frac{e^{-2t}\sin u}{16 + \cos(v+z)}.$$

It's clear that f_1 and f_2 are continuous and bounded functions. Thus the conditions of Theorem 9 hold, then the problem (44) has at least one solution on $[0, 1]$.

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