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Some Inequalities of Qi Type Using Fractional Integration

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Abstract: In the present paper, we use the Riemann-Liouville fractional integral to establish some integral results for certain classes of functions defined on some intervals of the real line. By introducing parameters α, β and δ, we give some sufficient conditions to generate some fractional inequalities of Qi type, and we give new generalizations for some results of [12,16].

Keywords: Fractional integration, Qi integral inequality

1 Introduction

Inequalities have proved to be one of the most powerful and far-reaching tools for the development of many branches of mathematics. In the last few decades, much significant development in the classical and new inequalities, particularly in analysis has been witnessed. As an example, let us cite the field of integration which is dominated by inequalities involving functions and their integrals [2,4,5,6,9,15]. One of the famous integral inequalities is Feng Qi inequality [14,15]. In [15], Qi proved that

$$\int_a^b [f(\tau)]^{n+2} d\tau \geq \left(\int_a^b f(\tau) d\tau \right)^{n+1} . \tag{1}$$

In [13], the authors established the following inequality:

$$\int_a^b [f(\tau)]^{\beta} d\tau \geq \left(\int_a^b f(\tau) d\tau \right)^{\beta-1} , \tag{2}$$

where $f \in C^1([a,b]), f(a) \geq 0$ and $f'(\tau) > (\beta-2)(\tau-a)^{\beta-3}, \tau \in [a,b]$.

Many researchers have given considerable attention to (2) and a number of extensions, generalizations and variants have appeared in the literature, see [3,10,11,12,14].

In the case of fractional integral, in [7], the authors established some new fractional inequalities based on the paper [1].

The main purpose of this paper is to establish some fractional results of the inequality (2) using Riemann-Liouville fractional integral. Our results have some relationships with some inequalities obtained in [12,16].

2 Basic Definitions

In the following, we will give the necessary notations and basic definitions. For more details, one can consult [6,12].

Definition 1 *A real valued function $f(t), t > 0$ is said to be in the space $C_\mu, \mu \in \mathbb{R}$ if there exists a real number $p > \mu$ such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C(]0, \infty[)$.*

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Definition 2 *A function $f(t), t > 0$ is said to be in the space $C_{\mu}^{n}, n \in \mathbb{N}$, if $f^{(n)} \in C_{\mu}$.*

Definition 3 *The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, for a function $f \in C_{\mu}, (\mu \geq -1)$ is defined as*

$$\begin{gathered} J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau; \quad \alpha > 0,\ t > 0, \\ J^0 f(t) = f(t), \end{gathered} \tag{3}$$

where $\Gamma(\alpha) := \int_0^{\infty} e^{-u} u^{\alpha-1} du$.

For the convenience of establishing the results, we give the semigroup property:

$$J^{\alpha} J^{\beta} f(t) = J^{\alpha+\beta} f(t);\ \alpha \geq 0,\ \beta \geq 0, \tag{4}$$

which implies the commutative property

$$J^{\alpha} J^{\beta} f(t) = J^{\beta} J^{\alpha} f(t). \tag{5}$$

3 Main Results

Theorem 1 *Suppose that $f \in C^1([0,\infty[)$ satisfies $f(0) \geq 0$ and $f^{'}(x) \geq (\beta-2)(\frac{x^{\delta}}{\Gamma(\delta+1)})^{\beta-3} \left(\frac{(x-\tau)^{\delta-1}}{\Gamma(\delta)}\right)^{\alpha-1}$ for $x \in [0,t]; t > 0, \beta \geq 3$. Then for all $\alpha \geq 1$, the inequality*

$$J^{\alpha}\left(f^{\beta}(t)\right) \geq \Gamma^{\beta-2}(\alpha)\left(J^{\alpha} f(t)\right)^{\beta-1} \tag{6}$$

is valid.

Proof. Since $f^{'}(x) \geq 0$ for $x \in [0,\infty[$ then f is an increasing function on $[0,\infty[$. Hence for any $t > 0$, we can write

$$f(\tau) \leq f(x); \quad \tau \in [0,x],\ \ x \leq t. \tag{7}$$

Multiplying both sides of by $(t-\tau)^{\alpha-1}$, we get

$$(t-\tau)^{\alpha-1} f(\tau) \leq (t-\tau)^{\alpha-1} f(x). \tag{8}$$

Integrating both sides of (8) over $[0,x]$, we obtain

$$\int_0^x (t-\tau)^{\alpha-1} f(\tau)\, d\tau \leq \frac{t^{\alpha}}{\alpha} f(x). \tag{9}$$

Now we define

$$F(x) := \int_0^x (t-\tau)^{\alpha-1} f^{\beta}(\tau)\, d\tau - \left(\int_0^x (t-\tau)^{\alpha-1} f(\tau)\, d\tau\right)^{\beta-1}.$$

Clearly $F(0) = 0$ and

$$\begin{aligned} F^{'}(x) &= (t-x)^{\alpha-1} f^{\beta}(x) - (\beta-1)\left(\int_0^x (t-\tau)^{\alpha-1} f(\tau)\, d\tau\right)^{\beta-2} (t-x)^{\alpha-1} f(x) \\ &= (t-x)^{\alpha-1} f(x)\left[f^{\beta-1}(x) - (\beta-1)\left(\int_0^x (t-\tau)^{\alpha-1} f(\tau)\, d\tau\right)^{\beta-2}\right]. \end{aligned}$$

Setting

$$G(x) = f^{\beta-1}(x) - (\beta-1)\left(\int_0^x (t-\tau)^{\alpha-1} f(\tau)\, d\tau\right)^{\beta-2}.$$

Then we have $G(0) = f^{\beta-1}(0) \geq 0$ and

$$\begin{aligned} G^{'}(x) &= (\beta-1) f^{\beta-2}(x) f^{'}(x) - (\beta-1)(\beta-2)\left(\int_0^x (t-\tau)^{\alpha-1} f(\tau)\, d\tau\right)^{\beta-3} (t-x)^{\alpha-1} f(x) \\ &= (\beta-1) f(x)\left[f^{\beta-3}(x) f^{'}(x) - (\beta-2)\left(\int_0^x (t-\tau)^{\alpha-1} f(\tau)\, d\tau\right)^{\beta-3} (t-x)^{\alpha-1}\right]. \end{aligned}$$

From the conditions of Theorem 1 and inequality (9), we have

$$\begin{aligned} f^{\beta-3}(x) f^{'}(x) &\geq (\beta-2)\left(\frac{t^\alpha}{\alpha} f(x)\right)^{\beta-3} (t-x)^{\alpha-1} \\ &\geq (\beta-2)\left(\int_0^x (t-\tau)^{\alpha-1} f(\tau)\, d\tau\right)^{\beta-3} (t-x)^{\alpha-1}. \end{aligned}$$

Thus $G^{'}(x) \geq 0$ and $G(0) \geq 0$, so we get $G(x) \geq 0$.
On the other hand $F(0) = 0$ and

$$F^{'}(x) = (t-x)^{\alpha-1} f(x) G(x) \geq 0 \text{ for all } x \in [0,t].$$

In particular

$$F(t) = \int_0^t (t-\tau)^{\alpha-1} f^\beta(\tau)\, d\tau - \left(\int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau\right)^{\beta-1} \geq 0,$$

and then

$$\frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f^\beta(\tau)\, d\tau \geq \frac{\Gamma^{\beta-2}(\alpha)}{\Gamma^{\beta-1}(\alpha)} \left(\int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau\right)^{\beta-1}.$$

Theorem 1 is thus proved. ■

Remark 2 *In Theorem 1, if we take $\alpha = 1$, we obtain the inequality (2) on $[0,t]$.*

Theorem 3 *Suppose that α and β are two positive real numbers such that $\alpha > \beta \geq 2$, $m = [\beta]$ and let $f(x) \in C^1[0,\infty[$ satisfying $f'(x) \geq f(x) \geq 0$ and $\left[f^{\alpha-\beta}(x)\right]' \geq (\alpha-\beta)\frac{\beta(\beta-1)...(\beta-m+1)}{(\alpha-1)(\alpha-2)...(\alpha-m+1)}(\frac{x^\delta}{\Gamma(\delta+1)})^{\beta-m}\left(\frac{(x-\tau)^{\delta-1}}{\Gamma(\delta)}\right)^{m-1}$. Then for any $t > 1$ and $\delta > 1$, we have*

$$J^\delta f^\alpha(t) \geq \left(J^\delta f(t)\right)^\beta. \tag{1.2}$$

Proof. Using the fact that $f'(x) \geq f(x) \geq 0$, $[0,x] \subset [0,t]$, we get

$$f(x)\int_0^x \frac{(t-\tau)^{\delta-1}}{\Gamma(\delta)} d\tau \geq \int_0^x \frac{(t-\tau)^{\delta-1}}{\Gamma(\delta)} f(\tau)d\tau, \tau \in [0,x],$$

that is

$$f(x)\frac{x^{\delta-1}}{\Gamma(\delta+1)} \geq \int_0^x \frac{(t-\tau)^{\delta-1}}{\Gamma(\delta)} f(\tau)d\tau.$$

Now we define:

$$F(x) := \int_0^x \frac{(t-\tau)^{\delta-1}}{\Gamma(\delta)} f^\alpha(\tau)d\tau - \left(\int_0^x \frac{(t-\tau)^{\delta-1}}{\Gamma(\delta)} f(\tau)d\tau\right)^\beta, \ x \in [0,t].$$

We have:

$$F'(x) = f(x)\frac{(t-\tau)^{\delta-1}}{\Gamma(\delta)} G_1(x),$$

where:

$$G_1(x) = f^{\alpha-1}(x) - \beta\left(\int_0^x \frac{(t-\tau)^{\delta-1}}{\Gamma(\delta)} f(\tau)d\tau\right)^{\beta-1}.$$

The derivative of the function G_1 gives

$$\begin{aligned}
G_1'(x) &= (\alpha-1)\, f^{\alpha-2}(x)f'(x) - \beta(\beta-1)\left(\int_0^x \frac{(t-\tau)^{\delta-1}}{\Gamma(\delta)} f(\tau)d\tau\right)^{\beta-2} f(x)\frac{(x-\tau)^{\delta-1}}{\Gamma(\delta)}\\
&\geq (\alpha-1)\, f^{\alpha-1}(x) - \beta(\beta-1)\left(\int_0^x \frac{(t-\tau)^{\delta-1}}{\Gamma(\delta)} f(\tau)d\tau\right)^{\beta-2} f(x)\frac{(x-\tau)^{\delta-1}}{\Gamma(\delta)}\\
&= f(x)\left((\alpha-1)\, f^{\alpha-2}(x) - \beta(\beta-1)\left(\int_0^x \frac{(t-\tau)^{\delta-1}}{\Gamma(\delta)} f(\tau)d\tau\right)^{\beta-2} \frac{(x-\tau)^{\delta-1}}{\Gamma(\delta)}\right)\\
&= f(x)G_2(x),
\end{aligned}$$

where

$$G_2(x) := \left((\alpha-1)\, f^{\alpha-2}(x) - \beta(\beta-1)\left(\int_0^x \frac{(t-\tau)^{\delta-1}}{\Gamma(\delta)} f(\tau)d\tau\right)^{\beta-2} \frac{(x-\tau)^{\delta-1}}{\Gamma(\delta)}\right).$$

It follows that

$$\begin{aligned}
G_2'(x) &= (\alpha-1)(\alpha-2)\, f^{\alpha-3}(x)f'(x) - \beta(\beta-1)\,((\beta-2)\left(\int_0^x \frac{(t-\tau)^{\delta-1}}{\Gamma(\delta)} f(\tau)d\tau\right)^{\beta-3}\\
&\quad f(x)\left(\frac{(x-\tau)^{\delta-1}}{\Gamma(\delta)}\right)^2 + \left(\int_0^x \frac{(t-\tau)^{\delta-1}}{\Gamma(\delta)} f(\tau)d\tau\right)^{\beta-2}\left(\frac{(x-\tau)^{\delta-1}}{\Gamma(\delta)}\right)')\\
&= (\alpha-1)(\alpha-2)\, f^{\alpha-3}(x)f\prime(x) - \beta(\beta-1)\,((\beta-2)\left(\int_0^x \frac{(t-\tau)^{\delta-1}}{\Gamma(\delta)} f(\tau)d\tau\right)^{\beta-3}\\
&\quad \times f(x)\left(\frac{(x-\tau)^{\delta-1}}{\Gamma(\delta)}\right)^2 + \left(\int_0^x \frac{(t-\tau)^{\delta-1}}{\Gamma(\delta)} f(\tau)d\tau\right)^{\beta-2}\left(\frac{(x-\tau)^{\delta-2}}{\Gamma(\delta-1)}\right)\\
&\geq (\alpha-1)(\alpha-2)\, f^{\alpha-2}(x) - \beta(\beta-1)(\beta-2)\left(\int_0^x \frac{(t-\tau)^{\delta-1}}{\Gamma(\delta)} f(\tau)d\tau\right)^{\beta-3}\\
&\quad \times f(x)\left(\frac{(x-\tau)^{\delta-1}}{\Gamma(\delta)}\right)^2\\
&= f(x)\,(\alpha-1)(\alpha-2)\, f^{\alpha-3}(x) - \beta(\beta-1)(\beta-2)\left(\int_0^x \frac{(t-\tau)^{\delta-1}}{\Gamma(\delta)} f(\tau)d\tau\right)^{\beta-3}\\
&\quad \times\left(\frac{(x-\tau)^{\delta-1}}{\Gamma(\delta)}\right)^2\\
&= f(x)G_3(x).
\end{aligned}$$

By the same argument as before, we obtain

$$\begin{aligned}
G_{m-1}(x) &= (\alpha-1)(\alpha-2)\ldots(\alpha-m+2)\, f^{\alpha-m+1}(x) - \beta(\beta-1)(\beta-2)\ldots\\
&\quad (\beta-m+2)\left(\int_0^x \frac{(t-\tau)^{\delta-1}}{\Gamma(\delta)} f(\tau)d\tau\right)^{\beta-m+1}\left(\frac{(x-\tau)^{\delta-1}}{\Gamma(\delta)}\right)^{m-2}
\end{aligned}$$

Obviously

$$\begin{aligned} G'_{m-1}(x) &= (\alpha-1)(\alpha-2)\ldots(\alpha-m+1) f^{\alpha-m}(x) f'(x) - \beta(\beta-1)(\beta-2)\ldots(\beta-m+2) \\ &\quad (\beta-m+1)\left(\int_0^x \frac{(t-\tau)^{\delta-1}}{\Gamma(\delta)} f(\tau)d\tau\right)^{\beta-m} f(x)\left(\frac{(x-\tau)^{\delta-1}}{\Gamma(\delta)}\right)^{m-1} \\ &\quad +(\beta-m+1)\left(\int_0^x \frac{(t-\tau)^{\delta-1}}{\Gamma(\delta)} f(\tau)d\tau\right)^{\beta-m+1}\left(\frac{(x-\tau)^{\delta-1}}{\Gamma(\delta)}\right)^{m-3} \\ &\geq (\alpha-1)(\alpha-2)\ldots(\alpha-m+1) f^{\alpha-m}(x) f'(x) - \beta(\beta-1)(\beta-2)\ldots \\ &\quad (\beta-m+2)(\beta-m+1) f^{\beta-m}(x)\left(\frac{x^\delta}{\Gamma(\delta+1)}\right)^{\beta-m} f(x)\left(\frac{(x-\tau)^{\delta-1}}{\Gamma(\delta)}\right)^{m-1} \\ &= f^{\beta-m+1}(x)(\alpha-1)(\alpha-2)\ldots(\alpha-m+1) f^{\alpha-\beta-1}(x) f\prime(x) - \beta(\beta-1)(\beta-2)\ldots \\ &\quad (\beta-m+2)(\beta-m+1)\left(\frac{x^\delta}{\Gamma(\delta+1)}\right)^{\beta-m}\left(\frac{(x-\tau)^{\delta-1}}{\Gamma(\delta)}\right)^{m-1} \\ &= f^{\beta-m+1}(x)\Big[(\alpha-1)(\alpha-2)\ldots(\alpha-m+1)\frac{1}{\alpha-\beta}\left(f^{\alpha-\beta}(x)\right)' - \beta(\beta-1)(\beta-2)\ldots \\ &\quad (\beta-m+2)(\beta-m+1)\left(\frac{x^\delta}{\Gamma(\delta+1)}\right)^{\beta-m}\left(\frac{(x-\tau)^{\delta-1}}{\Gamma(\delta)}\right)^{m-1}\Big]. \end{aligned}$$

We have $G'_{m-1}(x) \geq 0$, it follows that $G_{m-1}(x)$ is increasing on $[0,t]$. Hence $F'(x) \geq 0$. And then $F(x)$ is increasing on $[0,t]$. Finally we can obtain

$$\int_0^x \frac{(t-\tau)^{\delta-1}}{\Gamma(\delta)} f^\alpha(\tau)d\tau - \left(\int_0^x \frac{(t-\tau)^{\delta-1}}{\Gamma(\delta)} f(\tau)d\tau\right)^\beta \geq 0.$$

In particular, for $x=t$, we get (10). ■

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