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STUDY OF THE PROPERTIES OF GROWTH AND OSCILLATION OF SOLUTIONS FOR CERTAIN TYPES OF COMPLEX DIFFERENTIAL EQUATIONS

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THESIS

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Abstract

The determination of the order of entire and meromorphic functions plays an important role in Nevanlinna's value distribution theory. Several extensions on the original definition of the order have been made to characterize the asymptotic behavior of fast growing functions. In this thesis, we use a generalized concept of order, called the φ-order, to investigate the properties of growth and oscillation of solutions for homogeneous and non-homogeneous higher order complex linear differential equations. We describe in terms of growth of order and exponent of convergence the relationship between the solutions of differential equations and their coefficients in the complex plane as well as in the unit disc. We provide various generalizations and improvements of many previous works on this subject.

Résumé: La détermination de l'ordre des fonctions entières et méromorphes joue un rôle important dans la théorie de la distribution des valeurs de Nevanlinna. Plusieurs extensions de la définition originale de l'ordre ont été faites pour caractériser le comportement asymptotique des fonctions à croissance rapide. Dans cette thèse, nous utilisons un concept d'ordre généralisé, appelé le φ-ordre, pour étudier les propriétés de croissance et de l'oscillation de solutions pour des équations différentielles linéaires complexes d'ordre supérieur homogènes et non homogènes. Nous décrivons en termes de croissance d'ordre et d'exposant de convergence la relation entre les solutions d'équations différentielles et leurs coefficients sur le plan complexe ainsi que sur le disque unitaire. Nous fournissons diverses généralisations et améliorations de nombreux travaux antérieurs sur ce sujet.

ملخَّص : إن تحديد رتبة الدوال التحليليَة أو الميروموروفيَة تلعب دورًا مهمًا في نظرية توزيع القيم لنيفانلينا. العديد من التعديلات على التعريف الأصلي للرتبة تم اقتراحها لوصف السلوك التقاربي للدوال سريعة التزايد. في هذه الرسالة، نستخدم مفهومًا عامًا للرتبة ، يُطلق عليه الرتبة φ ، للتحقيق في خواص تزايد و تذبذب الحلول لمعادلات تفاضلية خطية عقدية متجانسة وغير متجانسة ذات رتبة عليا. نصف بدلالة رتبة التزايد وقوة التقارب العلاقة بين حلول المعادلات التفاضلية ومعاملاتها في المستوي المركب وكذلك في قرص الوحدة. نقدم العديد من التعميمات والتحسينات لكثير من الأعمال السابقة حول هذا الموضوع.

List of publications

(i) Kara M. A., Belaïdi B., Some estimates of the φ-order and the φ-type of entire and meromorphic functions, *Int. J. Open Problems Complex Analysis,* **10** (2019), No. 3, 42–58. URL: http://www.i-csrs.org

(ii) Kara M. A., Belaïdi B., Growth of φ-order solutions of linear differential equations with meromorphic coefficients on the complex plane, *Ural Math. J.,* **6** (2020), No. 1, 95–113. DOI: 10.15826/umj.2020.1.008

(iii) Kara M. A., Belaïdi B., On the φ-order and the φ-type of analytic and meromorphic functions in the unit disc, *Acta Univ. Apulensis,* **63** (2020), 65–80.
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(iv) Kara M. A., Belaïdi B., Fast growing solutions of linear differential equations with analytic coefficients in the unit disc, *Ann. Univ. Craiova, Math. Computer Sci. Series,* **48** (2021), No. 1, 37–52.
URL: http://inf.ucv.ro/ ami/index.php/ami/article/view/1330

List of communications

(i) On the growth and fixed points of solutions of linear differential equations with entire coefficients. **3rd BYMAT Conference, Bringing Young Mathematicians Together BYMAT 2020**, December 1–3, 2020. University of Valencia, Spain.
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(ii) On the growth and oscillation of fixed points of solutions of complex linear differential equations. **Dynamic Control and Optimization DCO 2021, International Conference on occasion of 65th birthday of Andrey V. Sarychev**, February 3–5, 2021. University of Aveiro, Portugal. URL: https://sites.google.com/view/dco2021/

(iii) Fast growing and fixed points of solutions of complex linear differential equations. **African Conference on Dynamical Systems and Ordinary Differential Equations ACDSODE 21**, March 20–23, 2021. University of Béjaia, Algeria.
URL: https://asdcod21.sciencesconf.org/

(iv) On the Logarithmic Derivative and Complex Differential Equations. **International Workshop on Functional Analysis, Control Systems & Decision Support**, May 25–27, 2021. University of Mostaganem, Algeria.
URL: https://www.univ-mosta.dz/fsei/international-workshop-on-functional-analysis-control-systems-decision-support/

Contents

Notations

$o(.), O(.)$	Landau symbols.
LDE	Linear differential equation.

Sets

$\mathbb{N}$	set of (positive) integer numbers $1, 2, \ldots$
$\mathbb{R}$	set of real numbers.
$\mathbb{C}$	set of the whole complex number z.
Δ	unit disc.
$\mathscr{E}$	set of entire functions.
$\mathscr{M}$	set of meromorphic functions on the whole complex plane.
$\mathscr{E}_\Delta$	set of analytic functions in the unit disc.
$\mathscr{M}_\Delta$	set of meromorphic functions in the unit disc.
Φ	set of positive unbounded non decreasing function φ on $(0, +\infty)$ such that $\varphi(e^t)$ is slowly growing as t tends to $+\infty$.

Functions

$[x]$	integer part of a real number x.
$\log^+$	positive logarithmic function.
$\log_p$	iterated logarithmic function.
$\exp_p$	iterated exponential function.
$M(r, f)$	maximum modulus of f on the disc $\lvert z\rvert \leq r$.
$n(r, f), \overline{n}(r, f)$	unintegrated counting functions.
$N(r, f), \overline{N}(r, f)$	integrated Counting functions.
$m(r, f)$	proximity function.
$T(r, f)$	Nevanlinna characteristic function.
$\delta(a, f)$	deficiency of the point $a \in \mathbb{C} \bigcup \{\infty\}$.
$i(f)$	growth index (finiteness degree) of the iterated p-order of $f \in \mathscr{M}$.
$mes(E)$	finite linear measure of a set $E \subset [0, +\infty)$.
$mes_l(E_0)$	finite logarithmic measure of a set $E_0 \subset [1, +\infty)$.
$mes_l(F_0)$	finite logarithmic measure of a set $F_0 \subset [0, 1)$.
$mes_l(E_1)$	infinite logarithmic measure of a set $E_1 \subset [1, +\infty)$.
$mes_l(F_1)$	infinite logarithmic measure of a set $F_1 \subset [0, 1)$.
$\overline{dens}(F)$	upper density of a set $F \subset (0, +\infty)$.
$\overline{dens}_l(G)$	upper logarithmic density of a set $G \subset (1, +\infty)$.

Orders

$\rho(f), \widetilde{\rho}(f)$	(usual) order of $f \in \mathscr{M}, \mathscr{E}$.
$\sigma(f), \widetilde{\sigma}(f)$	(usual) order of $f \in \mathscr{M}_{\Delta}, \mathscr{E}_{\Delta}$.
$\mu(f), \widetilde{\mu}(f)$	(usual) lower order of $f \in \mathscr{M}, f \in \mathscr{E}$.
$\rho_p(f), \widetilde{\rho}_p(f)$	iterated p-order of $f \in \mathscr{M}, \mathscr{E}$.
$\sigma_p(f), \widetilde{\sigma}_p(f)$	iterated p-order of $f \in \mathscr{M}_{\Delta}, \mathscr{E}_{\Delta}$.
$\rho_{\varphi}^{j}(f), \widetilde{\rho}_{\varphi}^{j}(f)$	φ-order of $f \in \mathscr{M}, \mathscr{E}$ for $j = 0, 1$.
$\sigma_{\varphi}^{j}(f), \widetilde{\sigma}_{\varphi}^{j}(f)$	φ-order of $f \in \mathscr{M}_{\Delta}, \mathscr{E}_{\Delta}$ for $j = 0, 1$.
$\mu_{\varphi}^{j}(f), \widetilde{\mu}_{\varphi}^{j}(f)$	lower φ-order of $f \in \mathscr{M}, \mathscr{E}$ for $j = 0, 1$.
$\nu_{\varphi}^{j}(f), \widetilde{\nu}_{\varphi}^{j}(f)$	lower φ-order of $f \in \mathscr{M}_{\Delta}, \mathscr{E}_{\Delta}$ for $j = 0, 1$.

Types

$\tau(f), \widetilde{\tau}(f)$	(usual) type of $f \in \mathscr{M}, \mathscr{E}$.
$\kappa(f), \widetilde{\kappa}(f)$	(usual) type of $f \in \mathscr{M}_{\Delta}, \mathscr{E}_{\Delta}$.
$\tau_p(f), \widetilde{\tau}_p(f)$	iterated p-type of $f \in \mathscr{M}, \mathscr{E}$.
$\kappa_p(f), \widetilde{\kappa}_p(f)$	iterated p-type of $f \in \mathscr{M}_{\Delta}, \mathscr{E}_{\Delta}$.
$\tau_{\varphi}^{j}(f), \widetilde{\tau}_{\varphi}^{j}(f)$	φ-type of $f \in \mathscr{M}, \mathscr{E}$ for $j = 0, 1$.
$\kappa_{\varphi}^{j}(f), \widetilde{\kappa}_{\varphi}^{j}(f)$	φ-type of $f \in \mathscr{M}_{\Delta}, \mathscr{E}_{\Delta}$ for $j = 0, 1$.
$\underline{\tau}_{\varphi}^{j}(f), \underline{\widetilde{\tau}}_{\varphi}^{j}(f)$	lower φ-type of $f \in \mathscr{M}, \mathscr{E}$ for $j = 0, 1$.
$\underline{\kappa}_{\varphi}^{j}(f), \underline{\widetilde{\kappa}}_{\varphi}^{j}(f)$	lower φ-type of $f \in \mathscr{M}_{\Delta}, \mathscr{E}_{\Delta}$ for $j = 0, 1$.

Convergence exponents

$\lambda(f)$	(usual) convergence exponent of the sequence of zeros of $f \in \mathscr{M}$.
$\overline{\lambda}(f)$	(usual) convergence exponent of the sequence of distinct zeros of $f \in \mathscr{M}$.
$\lambda_p(f)$	iterated p-convergence exponent of the sequence of zeros of $f \in \mathscr{M}$.
$\overline{\lambda}_p(f)$	iterated p-convergence exponent of the sequence of distinct zeros of $f \in \mathscr{M}$.
$\lambda_{\varphi}^{j}(f)$	φ-convergence exponent of the sequence of zeros of $f \in \mathscr{M}$ for $j = 0, 1$.
$\overline{\lambda}_{\varphi}^{j}(f)$	φ-convergence exponent of the sequence of distinct zeros of $f \in \mathscr{M}$ for $j = 0, 1$.
$\underline{\overline{\lambda}}_{\varphi}^{j}(f)$	lower φ-convergence exponent of the sequence of distinct zeros of $f \in \mathscr{M}$ ($j = 0, 1$).

Introduction

The value distribution theory is the study of how often a single-valued complex function assumes some different values. The first well known result in the value distribution theory is the fundamental theorem of Algebra which asserts that every non-constant complex polynomial f of degree $n > 0$ has exactly n zeros by counting multiplicities. Moreover, if a is a complex constant, then the polynomial $f - a$ has also n zeros and f takes the value a exactly n times. Therefore, two polynomials having the same zero points are identically the same except for a non-constant factor. An entire function which is not polynomial is called transcendental entire function and can be viewed by Taylor expansion as a polynomial of infinite degree.

In 1879, Picard proved that any transcendental entire function takes all possible complex finite values infinitely many times, with at most one single exception. For instance, $\exp(z)$ and $\exp(-z)$ are two transcendental entire functions having the same values $-1, +1$ and do not take the value 0. To characterize the asymptotic behavior of an entire function, we use the quantity

$$\mathrm{M}(r, f) := \max\left\{|f(z)| : |z| \leq r\right\}.$$

Certainly, for a polynomial f of degree n, there holds

$$\lim_{r \longrightarrow +\infty} \frac{\log \mathrm{M}(r, f)}{\log r} = n,$$

which means that the maximum modulus of a polynomial f grows as r^n, i.e.,

$$\mathrm{M}(r, f) = \mathrm{O}\left(r^n\right) \quad \text{as} \quad r \longrightarrow +\infty.$$

In order to measure how fast the maximum modulus of an entire function f grows, Borel defined the order of f by

$$\widetilde{\rho}(f) := \inf\left\{\alpha : \log \mathrm{M}(r, f) = \mathrm{O}\left(r^{\alpha}\right)\right\}.$$

As examples, the order of a polynomial is zero, $\widetilde{\rho}\left(e^z\right) = 1$ and $\widetilde{\rho}\left(e^{e^z}\right) = +\infty$. In 1897, Borel reformulated Picard's theorem and proved that for an entire function f of order $\widetilde{\rho}(f)$ $(0 < \widetilde{\rho}(f) < +\infty)$ and for any finite complex number a we have

$$\limsup_{r \to +\infty} \frac{\log n(r, a, f)}{\log r} = \widetilde{\rho}(f)$$

holding possibly for at most one exceptional value a. Here, $n(r, a, f)$ denotes the roots of the equation $f(z) = a$ in the disc $|z| \leq r$, each root being counted according to its multiplicity. This result is known as Picard-Borel theorem.

A single-valued complex function is called meromorphic if the only possible singularities in the complex plane are poles. One can show that a meromorphic function is the quotient of two entire functions. Rational functions are the simplest kinds of meromorphic functions. However, a non-rational function is called transcendental meromorphic function. We can obtain an immediate generalization of Picard's theorem by considering a meromorphic function f that omits one finite value $a \in \mathbb{C}$. Then, $\frac{1}{f-a}$ is an entire function, which by Picard's theorem can omit at most one value. Consequently, any meromorphic function can omit at most two finite values. Due to the presence of poles of a meromorphic function f in the disc $|z| \leq r$ for some finite values of r, the maximum modulus of f may attain the value ∞ and the classical methods used to analyze the asymptotic behavior of meromorphic functions break down. For that purpose, through a series of publications in the years 1922–1925, Rolf Nevanlinna[1] introduced the characteristic function

$$\mathrm{T}(r,f) := m(r,f) + \mathrm{N}(r,f),$$

which makes the value distribution theory more meaningfully and provides a big simplicity. For instance, a meromorphic function f on the complex plane is rational if and only if

$$\lim_{r\to+\infty} \frac{\mathrm{T}(r,f)}{\log r} < \infty.$$

Consequently, if $\log r = o(\mathrm{T}(r,g))$ as $r \longrightarrow +\infty$, then g is transcendental meromorphic function. Nevanlinna also defined the order of meromorphic function f on the complex plane by

$$\rho(f) := \limsup_{r\to+\infty} \frac{\log \mathrm{T}(r,f)}{\log r}.$$

From this new definition of the order, it is clear that if the function f has a finite order $\rho(f)$, then

$$\mathrm{T}(r,f) = \mathrm{O}\left(r^{\rho(f)}\right) \quad \text{as} \quad r \longrightarrow +\infty.$$

This indicates that the order of meromorphic function plays in the value distribution theory of meromorphic functions (also known as Nevanlinna theory) the similar role of the degree of polynomial in the fundamental theorem of Algebra.

Nevanlinna theory becomes the main tool for many studies in a wide variety of areas (see [46, 57, 62, 70]). Particularly, it is a powerful tool for investigating some properties[2] of solutions of certain complex differential equations. The reader can refer to [46, 62] for more details and comprehensive treatment on this subject.

Let us consider the following higher order linear differential equations

$$f^{(k)} + \mathrm{A}_{k-1}(z) f^{(k-1)} + \cdots + \mathrm{A}_0(z) f = 0, \tag{1}$$

$$f^{(k)} + \mathrm{A}_{k-1}(z) f^{(k-1)} + \cdots + \mathrm{A}_0(z) f = \mathrm{F}(z), \tag{2}$$

where $k \geq 2$ and the coefficients $\mathrm{A}_0 \not\equiv 0, \mathrm{A}_1, \ldots, \mathrm{A}_{k-1}, \mathrm{F} \not\equiv 0$ are entire functions. In this case, all solutions of (1) and (2) are also entire. Wittich [68] proved that the coefficients of equation (1) are polynomials if and only if all solutions of (1) are entire functions of finite order. So, if some coefficients of (1) are transcendental, then there exists at least one solution with infinite order. In [23], Frei showed that equation (1) possesses at most

[1] Rolf Nevanlinna (1895–1980) - Finnish mathematician.

[2] "I consider that I understand an equation when I can predict the properties of its solutions, without actually solving it."Paul Dirac (1902–1984) - English Physicist.

s linearly independent solutions of finite order where s is the index of the last transcendental function in the sequence of coefficients $A_0, A_1, \ldots, A_{k-1}$ of (1). After that, Chen and Gao [14] discussed the case when the coefficient A_0 dominates the growth order of other coefficients in equation (1) and obtained the following result:

Theorem A [14]. *Let $A_0, A_1, \ldots, A_{k-1}$ be entire functions satisfying one of the conditions*

(i) $\widetilde{\rho}(A_j) < \widetilde{\rho}(A_0) < \infty$ $(j = 1, \ldots, k-1)$ *or*

(ii) $A_1, \ldots, A_{k-1}$ *are polynomials and* A_0 *is transcendental satisfying* $\widetilde{\rho}(A_0) = +\infty$.

Then, all solutions $f \not\equiv 0$ of (1) satisfy $\widetilde{\rho}(f) = +\infty$.

Later, Chen and Yang [16] obtained the relationship between the growth of the dominant coefficient A_0 and the hyper-order of entire solutions f of (1).

Theorem B [16]. *Let $A_0, A_1, \ldots, A_{k-1}$ be entire functions satisfying*

$$\max\left\{\widetilde{\rho}(A_j) : j = 1, \ldots, k-1\right\} < \widetilde{\rho}(A_0) < +\infty.$$

Then, every non-zero solution f of (1) satisfies $\widetilde{\rho}_2(f) = \widetilde{\rho}(A_0)$.

Example. The function $f(z) = e^{e^z}$ solves the second order linear differential equation

$$f'' - f' - e^{2z} f = 0, \tag{3}$$

where $A_0(z) = -e^{2z}$ and $A_1(z) = -1$ are entire functions satisfying

$$\widetilde{\rho}(A_1) = 0 < \widetilde{\rho}(A_0) = 1.$$

We have $\widetilde{\rho}(f) = +\infty$ and $\widetilde{\rho}_2(f) = 1 = \widetilde{\rho}(A_0)$. One can observe that if we replace the coefficient $A_0(z) = -e^{2z}$ by e^{e^z} in (3), then we obtain from Theorem B that every non-zero solutions of (3) is of infinite hyper-order. For that purpose, Kinnunen [44] employed the concept of iterated p-order for investigating the fast growth of solutions of equations (1) and (2) with entire coefficients. The case when the coefficients are meromorphic functions in the complex plane has been studied by Cao et al. [12]. This extended many previous results obtained for the usual order and the hyper-order.

Theorem C [44]. *Let $A_0, \ldots, A_{k-1}$ be entire functions such that $i(A_0) = p$ $(0 < p < \infty)$. If either*

$$\max\left\{i(A_j) : j = 1, \ldots, k-1\right\} < p$$

or

$$\max\left\{\widetilde{\rho}_p(A_j) : j = 1, \ldots, k-1\right\} < \widetilde{\rho}_p(A_0),$$

then all solutions $f \not\equiv 0$ of equation (1) satisfy $i(f) = p+1$ and $\widetilde{\rho}_{p+1}(f) = \widetilde{\rho}_p(A_0)$.

The oscillation theory of complex differential equations was initiated by Bank and Laine [3, 4] by determining the convergence exponent (denoted by $\lambda(f)$) of the sequence of zeros of solution f of the second order differential equations with entire coefficient $A(z) \not\equiv 0$

$$f^{(k)} + A(z) f = 0. \tag{4}$$

We recall here two key results on the oscillation of solutions of equations (2) and (4).

Theorem D [4]. *Let* A *be a transcendental meromorphic function in the complex plane satisfying* $0 < \rho(A) \leq +\infty$ *and* $\lambda(A) < \rho(A)$. *If* $f \not\equiv 0$ *is a meromorphic solution of* (4), *then*

$$\rho(A) \leq \max\left\{\overline{\lambda}(f), \overline{\lambda}\left(\frac{1}{f}\right)\right\}.$$

Theorem E [44]. *Let* $F, A_0, \ldots, A_{k-1}$ *be entire functions satisfying* $q = i(F)$ *and*

$$p = \max\left\{i(A_j) : j = 0, \ldots, k-1\right\}.$$

If $0 < q < p+1 < +\infty$, $i(A_0) = p$ *and either* $i(A_j) < p$ or $\widetilde{\rho}_p(A_j) < \widetilde{\rho}_p(A_0)$ *for all* $j = 1, \ldots, k-1$, *then every solution* $f \not\equiv 0$ *of* (2) *satisfies*

$$\overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \widetilde{\rho}_{p+1}(f) = \widetilde{\rho}_p(A_0)$$

with at most one exceptional solution.

The concept of iterated $[p,q]$-order is firstly introduced by Juneja et al. [37, 38] and adjusted by Liu et al. [53] in order to study the fast growth and the oscillation of solutions of equations (1) and (2). After that, many researchers have investigated equations (1) and (2) where the coefficients are meromorphic functions in the complex plane [27, 52, 35] or analytic functions in the unit disc [9, 28, 48, 34, 65]. Recently, Chyzhykov and Semochko [20] showed that both the iterated p-order and the iterated $[p,q]$-order do not cover an arbitrary growth of complex functions. They considered a more general scale called the φ-order (cf. [61]) to study the fast growth of entire solutions of equation (1). This new generalized concept of order is quickly adopted by some authors (see [1, 8, 5, 59]).

The main object of this thesis is to make use the concept of φ-order and give the corresponding extensions of the above theorems not only in the complex plane but also in the unit disc. However, this thesis contains an introduction and six chapters organized as follows:

Chapter 1 presents the basic notions and the fundamental results on Nevanlinna theory of meromorphic functions which are useful in the next chapters of this thesis. Here, we do not reproduce the proofs, details can be found in the books [6, 24, 29, 36, 46, 70].

Chapter 2 provides some estimates involving the concepts of φ-order and φ-type of the sum, the product and the derivative of entire and meromorphic functions in the complex plane. Many results in [49, 66, 45] will be revisited and extended.

Chapter 3 investigates the fast growth of solutions of equations (1) and (2) where the coefficients are meromorphic on the complex plane with φ-order. The oscillation of solutions is also studied. We extend and improve many previous results in [20, 8, 5, 12, 53, 52].

Chapter 4 considered the lower φ-order and various conditions on the meromorphic coefficients of equations (1) and (2) to improve some results listed in Chapter 3. We obtain many generalized results of the papers of Hu and Zhang [35] and of Tu and Chen [64].

Chapter 5 gives analogous results of Chapter 2 for the φ-order and the φ-type of analytic and meromorphic functions in the unit disc. We also generalize some results listed in [13, 47].

Chapter 6 is devoted to study the fast growing of solutions of equation (1) when the coefficients are analytic functions in the unit disc with finite φ-order. We give the counterpart for some results obtained in the complex plane (see [5, 8, 20]).

Chapter 1

Basic concepts from Nevanlinna theory

Throughout this thesis, an entire function f and a meromorphic function g in the whole complex plane $\mathbb{C}$ will be denoted respectively by $f \in \mathscr{E}$ and $g \in \mathscr{M}$. If $g \in \mathscr{M}$, we say sometimes that g is just a meromorphic function. Furthermore, an analytic function in the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ will be denoted by $f \in \mathscr{E}_\Delta$ and $g \in \mathscr{M}_\Delta$ will mean a meromorphic function g in Δ. The shorthand notation "$r \longrightarrow 1^-$"when occurs will mean "for all $r \in (0;1)$ sufficiently close to 1 ". The Landau symbols o(.) and O(.) are also used such that:

(i) $f = \mathrm{O}(g)$ if and only if there exists $c > 0$ such that $\left|\frac{f(r)}{g(r)}\right| \leq c$ for sufficiently large r.

(ii) $f = o(g)$ if and only if $\frac{f(r)}{g(r)} \longrightarrow 0$ as $r \longrightarrow +\infty$.

1 Nevanlinna functionals

Firstly, we define the positive logarithmic functions as follows

$$\log^+ u := \max\{0, u\} = \begin{cases} \log u & \text{if } u > 1, \\ 0 & \text{if } 0 < u \leq 1. \end{cases}$$

This function satisfies the following properties:

(i) $\log u = \log^+ u - \log^+ \frac{1}{u}$,

(ii) $|\log u| = \log^+ u + \log^+ \frac{1}{u}$,

(iii) $\log^+\left(\sum_{i=1}^{n} u_i\right) \leq \sum_{i=1}^{n} \log^+ u_i + \log n$,

(iv) $\log^+\left(\prod_{i=1}^{n} u_i\right) \leq \prod_{i=1}^{n} \log^+ u_i$.

Definition 1.1 *Let $a \in \mathbb{C}$ and let $f \in \mathscr{M}$ be a non-constant. We define $n(r, f)$, or sometimes $n(r, \infty, f)$, to be the number of poles of f in the disc $|z| \leq r$ such that each pole is counted according to its multiplicity. The number of distinct poles of f in $|z| \leq r$ is denoted by $\overline{n}(r, f) = \overline{n}(r, \infty, f)$, that is, multiple pole should be counted only once.*

We further define $n\left(r, \frac{1}{f-a}\right)$, or sometimes $n(r, a, f)$, to be the number of zeros of $f - a$ (by counting all multiplicities) in the disc $|z| \leq r$ and $\overline{n}\left(r, \frac{1}{f-a}\right) = \overline{n}(r, a, f)$ to be the number of distinct zeros of $f - a$ (not counting multiplicities) in the disc $|z| \leq r$.

The following result is the key of Nevanlinna's first main theorem.

Theorem 1.1 (Jensen's formula) *Let $f \in \mathscr{M}$ such that $f(0) \neq 0, \infty$. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_m$ be the zeros and the poles of f respectively, each multiple zero and pole is counted according to its multiplicity. Then we have*

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(re^{i\theta}\right)\right| d\theta - \sum_{|a_j|<r} \log\frac{r}{|a_j|} + \sum_{|b_j|<r} \log\frac{r}{|b_j|}.$$

Definition 1.2 *Let $a \in \mathbb{C}$ and let $f \in \mathscr{M}$ be non-constant. The (integrated) Counting function is defined as follows*

$$\mathrm{N}(r,a,f) = \mathrm{N}\left(r, \frac{1}{f-a}\right) := \int_0^r \frac{n(t,a,f) - n(0,a,f)}{t}\, dt + n(0,a,f)\log r,$$

$$\mathrm{N}(r,\infty,f) = \mathrm{N}(r,f) := \int_0^r \frac{n(t,f) - n(0,f)}{t}\, dt + n(0,f)\log r.$$

Similarly, by using $\overline{n}(r,f)$ and $\overline{n}\left(r, \frac{1}{f}\right)$, we generate $\overline{\mathrm{N}}(r,f)$ and $\overline{\mathrm{N}}\left(r, \frac{1}{f-a}\right)$ as follows

$$\overline{\mathrm{N}}(r,a,f) = \overline{\mathrm{N}}\left(r, \frac{1}{f-a}\right) := \int_0^r \frac{\overline{n}(t,a,f) - \overline{n}(0,a,f)}{t}\, dt + \overline{n}(0,a,f)\log r,$$

$$\overline{\mathrm{N}}(r,\infty,f) = \overline{\mathrm{N}}(r,f) := \int_0^r \frac{\overline{n}(t,f) - \overline{n}(0,f)}{t}\, dt + \overline{n}(0,f)\log r.$$

Definition 1.3 *Let $a \in \mathbb{C}$ and let $f \in \mathscr{M}$ be non-constant. The proximity function is defined by*

$$m(r,a,f) = m\left(r, \frac{1}{f-a}\right) := \frac{1}{2\pi}\int_0^{2\pi} \log^+ \frac{1}{\left|f(re^{i\theta}) - a\right|} d\theta,$$

$$m(r,\infty,f) = m(r,f) := \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|f(re^{i\theta})\right| d\theta.$$

Definition 1.4 *Let $f \in \mathscr{M}$ be non-constant. The Nevanlinna characteristic function of f is defined by*

$$\mathrm{T}(r,f) := m(r,f) + \mathrm{N}(r,f).$$

Example 1.1 $\mathrm{T}(r,e^z) = \frac{r}{\pi}$ *and* $\mathrm{T}\left(r,e^{e^z}\right) \sim \frac{e^r}{2\pi^3 r}$ *as* $r \longrightarrow +\infty$.

Proposition 1.1 *Let $n \in \mathbb{N}$ and let $f_i \in \mathscr{M}$ be non-constant $(i = 1, \ldots, n)$. Then, the following statements hold*

(i) $m\left(r, \sum_{i=1}^n f_i\right) \leq \sum_{i=1}^n m(r,f_i) + \log n$ *and* $m\left(r, \prod_{i=1}^n f_i\right) \leq \sum_{i=1}^n m(r,f_i)$,

(ii) $\mathrm{N}\left(r, \sum_{i=1}^n f_i\right) \leq \sum_{i=1}^n \mathrm{N}(r,f_i)$ *and* $\mathrm{N}\left(r, \prod_{i=1}^n f_i\right) \leq \sum_{i=1}^n \mathrm{N}(r,f_i)$,

(iii) $\mathrm{T}\left(r, \sum_{i=1}^n f_i\right) \leq \sum_{i=1}^n \mathrm{T}(r,f_i) + \log n$ *and* $\mathrm{T}\left(r, \prod_{i=1}^n f_i\right) \leq \sum_{i=1}^n \mathrm{T}(r,f_i)$.

Theorem 1.2 (Nevanlinna's first main theorem) *Let $a \in \mathbb{C}$ and let $f \in \mathscr{M}$ be non-constant. Then,*

$$\mathrm{T}\left(r, \frac{1}{f-a}\right) := m\left(r, \frac{1}{f-a}\right) + \mathrm{N}\left(r, \frac{1}{f-a}\right) = \mathrm{T}(r,f) + \mathrm{O}(1) \quad as \quad r \to +\infty.$$

Remark 1.1 *This theorem remains valid for $f \in \mathscr{M}_{\triangle}$ and $r \longrightarrow 1^{-}$.*

Definition 1.5 *Let $f \in \mathscr{E}$. We define the maximum modulus function $\mathrm{M}(r,f)$ by*

$$\mathrm{M}(r,f) := \max\{|f(z)| : |z| \le r\}.$$

Clearly, it follows from the maximum principle that this maximum value is taken on the circle $|z| = r$, i.e.,

$$\mathrm{M}(r,f) = \max\{|f(z)| : |z| = r\}.$$

Example 1.2 $\mathrm{M}(r, e^z) = e^r$, $\mathrm{M}(r, \sin z) = \sinh r$, $\mathrm{M}\left(r, e^{e^z}\right) = e^{e^r}$.

The functions $\mathrm{T}(r,f)$ and $\mathrm{M}(r,f)$ are related by the following theorem.

Theorem 1.3 *Let $f \in \mathscr{E}$. For $0 < r < \mathrm{R}$ we have*

$$\mathrm{T}(r,f) \le \log^{+} \mathrm{M}(r,f) \le \frac{\mathrm{R}+r}{\mathrm{R}-r}\mathrm{T}(\mathrm{R},f).$$

Theorem 1.4 *Let $f \in \mathscr{M}$ be non-constant. The function f is rational if and only if*

$$\mathrm{T}(r,f) = \mathrm{O}(\log r), \quad as \quad r \longrightarrow +\infty.$$

Or equivalently, the function $f \in \mathscr{M}$ is transcendental if and only if

$$\log r = o\left(\mathrm{T}(r,f)\right), \quad as \quad r \longrightarrow +\infty.$$

Theorem 1.5 (Möbius transformation) *Let $f \in \mathscr{M}$ and let $g(z) = \frac{af(z)+b}{cf(z)+d}$ such that $f \not\equiv -\frac{d}{c}$ and a, b, c, d are complex constants satisfying $ad - cb \neq 0$. Then*

$$\mathrm{T}(r,g) = \mathrm{T}(r,f) + \mathrm{O}(1).$$

Theorem 1.6 *[54] Let $n \in \mathbb{N}$ and let $f, a_0, \ldots, a_n \in \mathscr{M}$. If $\mathrm{F} = a_0 + a_1 f + \cdots + a_n f^n$ such that $a_n(z) \not\equiv 0$, then*

$$\mathrm{T}(r,\mathrm{F}) = n\mathrm{T}(r,f) + \mathrm{O}\left(\sum_{k=0}^{n} \mathrm{T}(r,a_k)\right).$$

Theorem 1.7 ([69]) *If $f \in \mathscr{M}$, then*

$$\mathrm{T}(r,f) < \mathrm{O}\left(\mathrm{T}(2r,f') + \log r\right) \quad as \quad r \longrightarrow +\infty.$$

Definition 1.6 *Let $f \in \mathscr{M}$. The deficiency of the point $a \in \mathbb{C} \cup \{\infty\}$ is defined by*

$$\delta(a,f) := \liminf_{r\to+\infty} \frac{m\left(r, \frac{1}{f-a}\right)}{\mathrm{T}\left(r,f\right)} = 1 - \limsup_{r\to+\infty} \frac{\mathrm{N}\left(r, \frac{1}{f-a}\right)}{\mathrm{T}\left(r,f\right)} \quad for \quad a \neq \infty,$$

$$\delta(\infty,f) := \liminf_{r\to+\infty} \frac{m(r,f)}{\mathrm{T}(r,f)} = 1 - \limsup_{r\to+\infty} \frac{\mathrm{N}(r,f)}{\mathrm{T}(r,f)} \quad for \quad a = \infty.$$

Remark 1.2 *Note that $\delta(a,f) \in [0,1]$. If $\delta(a,f) > 0$, the value a is called deficient.*

Example 1.3 *(i) For any $a \in \mathbb{C}$, we have $\delta(a,z) = 0$.*

(ii) $\delta(0,e^z) = \delta(\infty,e^z) = 1$, and for any $a \in \mathbb{C} \setminus \{0\}$ we have $\delta(a,e^z) = 0$.

2 Measures and densities of set

Definition 1.7 *The linear measure of a set* $\mathrm{E} \subset [0,+\infty)$ *is defined by*

$$mes(\mathrm{E}) := \int_{\mathrm{E}} dt.$$

The logarithmic measure of a set $\mathrm{E}_0 \subset [1,+\infty)$ *is defined by*

$$mes_l(\mathrm{E}_0) := \int_{\mathrm{E}_0} \frac{dt}{t}.$$

If $mes(\mathrm{E}) < +\infty$ *and* $mes_l(\mathrm{E}_0) < +\infty$, *we say that the sets* E *and* E_0 *are of finite linear measure and of finite logarithmic measure respectively.*

Definition 1.8 *The logarithmic measure of a set* $\mathrm{F}_0 \subset [0,1)$ *is defined by*

$$mes_l(\mathrm{F}_0) := \int_{\mathrm{F}_0} \frac{dr}{1-r}.$$

Remark 1.3 *In what follows, the sets of finite linear measure and of finite or infinite logarithmic measure are not necessarily the same at each occurrence.*

Definition 1.9 *The upper linear density of a set* $\mathrm{F} \subset (0,+\infty)$ *is defined by*

$$\overline{dens}(\mathrm{F}) := \limsup_{r\to+\infty} \frac{mes(\mathrm{F}\cap[0,r])}{r}.$$

The upper logarithmic density of a set $\mathrm{G} \subset (1,+\infty)$ *is defined by*

$$\overline{dens}_l(\mathrm{G}) := \limsup_{r\to+\infty} \frac{mes_l(\mathrm{G}\cap[1,r])}{\log r}.$$

Proposition 1.2 *[29, 10] Far all sets* $\mathrm{E} \subset [0,+\infty), \mathrm{E}_1 \subset [1,+\infty)$ *and* $\mathrm{G} \subset (1,+\infty)$, *there hold*

(i) if E *is countable, then* $mes(\mathrm{E}) = 0$,

(ii) if $mes_l(\mathrm{E}_1) = \infty$, *then* $mes(\mathrm{E}_1) = \infty$,

(iii) if $\overline{dens}(\mathrm{G}) > 0$, *then* $mes(\mathrm{G}) = \infty$,

(iv) if $\overline{dens}_l(\mathrm{G}) > 0$, *then* $mes_l(\mathrm{G}) = \infty$.

3 Growth and oscillation of functions

Definition 1.10 *Let* $f \in \mathscr{M}$. *The order and the lower order of* f *are respectively defined by*

$$\rho(f) := \limsup_{r\to+\infty} \frac{\log \mathrm{T}(r,f)}{\log r} \quad \textit{and} \quad \mu(f) := \liminf_{r\to+\infty} \frac{\log \mathrm{T}(r,f)}{\log r}.$$

If $f \in \mathscr{E}$, *then the order and the lower order of* f *are respectively defined by*

$$\widetilde{\rho}(f) := \limsup_{r\to+\infty} \frac{\log\log \mathrm{M}(r,f)}{\log r} \quad \textit{and} \quad \widetilde{\mu}(f) := \liminf_{r\to+\infty} \frac{\log\log \mathrm{M}(r,f)}{\log r}.$$

Remark 1.4 *(i) According to Theorem 1.3, one can show that if $f \in \mathscr{E}$ then*

$$\rho(f) = \widetilde{\rho}(f) \quad \text{and} \quad \mu(f) = \widetilde{\mu}(f).$$

(ii) If $\rho(f) = \mu(f)$, then the function f is said to be of regular growth.

Example 1.4 *For all $n \in \mathbb{N}$, we have $\rho\left(e^{z^n}\right) = \mu\left(e^{z^n}\right) = n$. $\rho\left(e^{e^z}\right) = \mu\left(e^{e^z}\right) = +\infty$.*

Proposition 1.3 *[51, 70] Let $f_1, f_2 \in \mathscr{M}$. The following statements hold*

(i) $\max\{\rho(f_1 + f_2), \rho(f_1 f_2)\} \leq \max\{\rho(f_1), \rho(f_2)\}$,

(ii) if $\rho(f_1) < \rho(f_2)$, then $\rho(f_1 + f_2) = \rho(f_1 f_2) = \rho(f_2)$,

(iii) $\max\{\mu(f_1 + f_2), \mu(f_1 f_2)\} \leq \max\{\rho(f_1), \mu(f_2)\}$,

(iv) if $\rho(f_1) < \mu(f_2)$, then $\mu(f_1 + f_2) = \mu(f_1 f_2) = \mu(f_2)$,

(v) if $\rho(f_1) < \mu(f_2)$, then $T(r, f_1) = o\left(T(r, f_2)\right)$ as $r \longrightarrow +\infty$.

Theorem 1.8 ([69]) *Let $f \in \mathscr{M}$ and f' be its derivative. Then, $\rho(f) = \rho(f')$.*

Definition 1.11 *[63] Let $f \in \mathscr{M}_{\triangle}$. The order of f is defined by*

$$\sigma(f) := \limsup_{r \longrightarrow 1^-} \frac{\log^+ T(r, f)}{-\log(1-r)}.$$

If $f \in \mathscr{E}_{\triangle}$, then the order of f is defined by

$$\widetilde{\sigma}(f) := \limsup_{r \longrightarrow 1^-} \frac{\log^+ \log^+ M(r, f)}{-\log(1-r)}.$$

Remark 1.5 *If $f \in \mathscr{E}_{\triangle}$, it is well known that $\sigma(f) \leq \widetilde{\sigma}(f) \leq \sigma(f) + 1$, (see [63, Theorem V. 13.]) which are the best possible in the sense that there exist $g \in \mathscr{E}_{\triangle}$ and $h \in \mathscr{E}_{\triangle}$ such that*

$$\widetilde{\sigma}(g) = \sigma(g) \quad \text{and} \quad \widetilde{\sigma}(h) = \sigma(h) + 1.$$

Example 1.5 *Let $\alpha > 1$ be a real constant and let $f(z) = \exp\left\{\frac{1}{(1-z)^\alpha}\right\}$. Then,*

$$\sigma(f) = \alpha - 1 \quad \text{and} \quad \widetilde{\sigma}(f) = \alpha.$$

However, for $f(z) = \exp\left\{\frac{1}{1-z}\right\}$, we have $\sigma(f) = \widetilde{\sigma}(f) = 1$.

Definition 1.12 *[46] Let $f \in \mathscr{M}$. The type of f such that $\rho(f) \in (0, +\infty)$ is defined by*

$$\tau(f) := \limsup_{r \to +\infty} \frac{T(r, f)}{r^{\rho(f)}}.$$

If $f \in \mathscr{E}$ such that $\widetilde{\rho}(f) \in (0, +\infty)$, then the type of f is defined by

$$\widetilde{\tau}(f) := \limsup_{r \to +\infty} \frac{\log M(r, f)}{r^{\widetilde{\rho}(f)}}.$$

Remark 1.6 *In general, $\tau(f) \neq \widetilde{\tau}(f)$ for some $f \in \mathscr{E}$. For instance, $\tau(e^z) = \frac{1}{\pi} < \widetilde{\tau}(e^z) = 1$.*

From [24], if $f \in \mathscr{E}$, the following two estimates hold

$$\begin{cases} \widetilde{\tau}(f) \leq \frac{\pi\rho(f)}{\sin(\pi\rho(f))}\tau(f) & \text{if} \quad 0 < \rho(f) \leq \frac{1}{2}, \\ \widetilde{\tau}(f) \leq \pi\rho(f)\tau(f) & \text{if} \quad \frac{1}{2} \leq \rho(f) < +\infty. \end{cases}$$

Definition 1.13 *[46] Let $f \in \mathscr{M}_{\triangle}$. The type of f such that $\sigma(f) \in (0,+\infty)$ is defined by*

$$\kappa(f) := \limsup_{r \longrightarrow 1^-} (1-r)^{\sigma(f)} \mathrm{T}(r,f).$$

If $f \in \mathscr{E}_{\triangle}$ such that $\widetilde{\sigma}(f) \in (0,+\infty)$, then the type of f is defined by

$$\widetilde{\kappa}(f) := \limsup_{r \longrightarrow 1^-} (1-r)^{\widetilde{\sigma}(f)} \log^+ \mathrm{M}(r,f).$$

Definition 1.14 *Let $f \in \mathscr{M}$ and let $\{z_k, k \in \mathbb{N}\}$ be a sequence of zeros (counted multiplicities) of f such that $\lim\limits_{k \to +\infty} |z_k| = +\infty$. The convergence exponent of the sequence of zeros of f is defined by*

$$\lambda(f) := \inf\left\{\lambda > 0 : \sum_{k \geq 1} |z_k|^{-\lambda} < +\infty\right\}$$

Theorem 1.9 *[46] Let $f \in \mathscr{M}$ with infinitely many zeros and let $\lambda(f)$ be the convergence exponent of the sequence of zeros of f. Then we have*

$$\lambda(f) = \limsup_{r \to +\infty} \frac{\log \mathrm{N}\left(r, \frac{1}{f}\right)}{\log r} = \limsup_{r \to +\infty} \frac{\log n\left(r, \frac{1}{f}\right)}{\log r}.$$

Definition 1.15 *Let $f \in \mathscr{M}$. The convergence exponent $\overline{\lambda}(f)$ of the sequence of distinct zeros of f is defined by*

$$\overline{\lambda}(f) := \limsup_{r \to +\infty} \frac{\log \overline{\mathrm{N}}\left(r, \frac{1}{f}\right)}{\log r}.$$

Example 1.6 *Let $a \in \mathbb{C} \setminus \{0\}$. For $f(z) = e^z - a$, we have $\lambda(f) = 1 = \overline{\lambda}(f)$.*

The following result plays an important role in Nevanlinna theory. It is very useful for many applications in the uniqueness theory of meromorphic functions and in the theory of complex differential equations.

Lemma 1.1 (lemma of logarithmic derivative) *Let $k \in \mathbb{N}$ and let $f \in \mathscr{M}$ be a non-constant meromorphic function. If $\rho(f) < +\infty$, then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = \mathrm{O}(\log r) \quad as \quad r \longrightarrow +\infty.$$

If $\rho(f) = +\infty$, then

$$m\left(r, \frac{f^{(k)}}{f}\right) = \mathrm{O}\left(\log r + \log \mathrm{T}(r,f)\right) \quad as \quad r \longrightarrow +\infty, r \notin \mathrm{E}$$

where $\mathrm{E} \subset [0,+\infty)$ is a set of finite linear measure.

The version of the lemma of logarithmic derivative in the unit disc Δ is stated as follows:

Lemma 1.2 *[31] Let $f \in \mathscr{M}_{\Delta}$ and $k \in \mathbb{N}$. Then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = O\left(\log^{+} T(r,f) + \log\frac{1}{1-r}\right) \quad as \quad r \longrightarrow 1^{-}$$

possibly outside of an exceptional set $F_0 \subset [0,1)$ with finite logarithmic measure. If $\sigma(f) < +\infty$, then

$$m\left(r, \frac{f^{(k)}}{f}\right) = O\left(\log\frac{1}{1-r}\right) \quad as \quad r \longrightarrow 1^{-}.$$

Definition 1.16 *Let $f, g \in \mathscr{M}$. If we have $T(r,g) = o\big(T(r,f)\big)$ as $r \longrightarrow +\infty$ outside of a set $E \subset [0,+\infty)$ of finite linear measure, we say that g is a small function of f and we will simply denote $T(r,g) = S(r,f)$.*

Theorem 1.10 (Valiron-Mokhon'ko) *[46] Let $f \in \mathscr{M}$ and let $R(f) = \frac{P(f)}{Q(f)}$ be an irreducible rational function in f where*

$$P(f) = \sum_{k=0}^{n} a_k f^k \quad and \quad Q(f) = \sum_{j=0}^{m} b_j f^j.$$

If the coefficients $a_k(z)$ and $b_j(z)$ are small functions of f, then we have

$$T\big(r, R(f)\big) = \max\{n,m\}T(r,f) + S(r,f).$$

We now state two important results which allow to avoid the exceptional set when it appears in the demonstrations.

Theorem 1.11 *[2, 26] Let $g : (0,+\infty) \longrightarrow \mathbb{R}$ and $h : (0,+\infty) \longrightarrow \mathbb{R}$ be monotone non-decreasing functions. If either*

(i) $g(r) \leq h(r)$ for all $r \notin E_0 \cup (0,1]$ where $E_0 \subset [1,+\infty)$ is a set of finite logarithmic measure, or

(ii) $g(r) \leq h(r)$ outside of an exceptional set $E \subset [0,+\infty)$ of finite linear measure,

then, for any given constant $\alpha > 1$, there exists $r_0 > 0$ such that $g(r) \leq h(\alpha r)$ for all $r > r_0$.

Theorem 1.12 *[31] Let $g : (0,1) \longrightarrow \mathbb{R}$ and $h : (0,1) \longrightarrow \mathbb{R}$ be monotone non-decreasing functions such that $g(r) \leq h(r)$ possibly outside of an exceptional set $F_0 \subset [0,1)$ with finite logarithmic measure. Then, there exists a constant $d \in (0,1)$ and $s(r) = 1 - d(1-r)$ satisfying $g(r) \leq h(s(r))$ for all $r \in [0,1)$.*

4 The concept of iterated order

For all $r \in \mathbb{R}$, we define the iterated exponential function by

$$\exp_0 r := r \quad \text{and} \quad \exp_p r := \exp\left(\exp_{p-1} r\right), p \in \mathbb{N}.$$

Inductively, for all $r \in (0,+\infty)$, we define the iterated logarithmic function by

$$\log_0 r := r \quad \text{and} \quad \log_p r := \log\left(\log_{p-1} r\right), p \in \mathbb{N}.$$

We also use the conventions $\exp_{-1} r := \log_1 r$ and $\log_{-1} r := \exp_1 r$.

Definition 1.17 *[58] Let $p \in \mathbb{N}$ and let $f \in \mathcal{M}$. The iterated p-order of f is defined by*

$$\rho_p(f) := \limsup_{r\to+\infty} \frac{\log_p \mathrm{T}(r,f)}{\log r}.$$

If $f \in \mathcal{E}$, then the iterated p-order of f is defined as

$$\widetilde{\rho}_p(f) := \limsup_{r\to+\infty} \frac{\log_{p+1} \mathrm{M}(r,f)}{\log r} = \rho_p(f).$$

Remark 1.7 *Note that $\rho_1(f)$ coincides with the usual order $\rho(f)$. For $p=2$, $\rho_2(f)$ is called the hyper-order of f. If we replace the denominator $\log r$ in Definition 1.17 by $\log_q r$, we obtain the iterated $[p,q]$-order [37, 53].*

Example 1.7 *Let $q \in \mathbb{N}$ and $a \in \mathbb{C}\setminus\{0\}$. The iterated p-order of $f(z) = \exp_q(az)$ is*

$$\rho_p(f) = \begin{cases} 0 & \text{if } p < q, \\ 1 & \text{if } p = q, \\ +\infty & \text{if } p > q. \end{cases}$$

Proposition 1.4 *Let $p \in \mathbb{N}$ and let $f \in \mathcal{M}$. If $\rho_p(f) < +\infty$, then $\rho_{p+1}(f) = 0$.*

Definition 1.18 *[46, 44] Let $f \in \mathcal{M}$. The growth index (or finiteness degree) of the iterated p-order of f is defined by*

$$i(f) = \begin{cases} +\infty & \text{if } \rho_j(f) = +\infty \text{ for all } j \in \mathbb{N}, \\ \min\{j \in \mathbb{N} : \rho_j(f) < +\infty\} & \text{if } f \text{ is transcendental and} \\ & \quad \rho_j(f) < +\infty \text{ for some } j \in \mathbb{N}, \\ 0 & \text{if } f \text{ is rational.} \end{cases}$$

Example 1.8 *Let $n \in \mathbb{N}$ and $f(z) = \exp_2(z^n) = e^{e^{z^n}}$. Since $\rho_2(f) = n$, then $i(f) = 2$.*

Definition 1.19 *[13] Let $f \in \mathcal{M}_{\triangle}$ and $p \in \mathbb{N}$. The iterated p-order of f is defined by*

$$\sigma_p(f) := \limsup_{r\longrightarrow 1^-} \frac{\log_p^+ \mathrm{T}(r,f)}{-\log(1-r)}.$$

If $f \in \mathcal{E}_{\triangle}$, then the iterated p-order is defined by

$$\widetilde{\sigma}_p(f) := \limsup_{r\longrightarrow 1^-} \frac{\log_{p+1}^+ \mathrm{M}(r,f)}{-\log(1-r)}.$$

It should be noted that for $p=1$, we have $\sigma_1(f) = \sigma(f)$ and $\widetilde{\sigma}_1(f) = \widetilde{\sigma}(f)$.

Remark 1.8 *By virtue of Theorem 1.3, one can show that, for all $p \geq 2$ and $f \in \mathcal{E}_{\triangle}$, we have $\widetilde{\sigma}_p(f) = \sigma_p(f)$.*

Example 1.9 *Let $\alpha > 1$ be a real constant and $f(z) = \exp_2\left\{\frac{1}{(1-z)^\alpha}\right\}$. Then, $\sigma_2(f) = \widetilde{\sigma}_2(f) = \alpha$.*

Definition 1.20 *[58] Let $f \in \mathscr{M}$. The iterated p-type of f such that $\rho_p(f) \in (0,+\infty)$ is defined by*

$$\tau_p(f) := \limsup_{r\to+\infty} \frac{\log_{p-1} T(r,f)}{r^{\rho_p(f)}}.$$

If $f \in \mathscr{E}$ is transcendental entire function, then the iterated p-type of f such that $\widetilde{\rho}_p(f) \in (0,+\infty)$ is defined by

$$\widetilde{\tau}_p(f) := \limsup_{r\to+\infty} \frac{\log_p M(r,f)}{r^{\widetilde{\rho}_p(f)}}.$$

Remark 1.9 *Let $p \in \mathbb{N}$. If $\widetilde{\tau}_p(f) < +\infty$, then*

$$M(r,f) \sim \exp_p\left\{\widetilde{\tau}_p(f) r^{\widetilde{\rho}_p(f)}\right\} \quad as \quad r \longrightarrow +\infty$$

which describes how fast the entire function f is growing.

Definition 1.21 *[46, 33] Let $f \in \mathscr{M}_{\triangle}$ and $p \in \mathbb{N}$. The iterated p-type of f such that $\sigma_p(f) \in (0,+\infty)$, is defined by*

$$\kappa_p(f) := \limsup_{r\longrightarrow 1^-} (1-r)^{\sigma_p(f)} \log^+_{p-1} T(r,f).$$

If $f \in \mathscr{E}_{\triangle}$ such that $\widetilde{\sigma}_p(f) \in (0,+\infty)$, then the iterated p-type of f is defined by

$$\widetilde{\kappa}_p(f) := \limsup_{r\longrightarrow 1^-} (1-r)^{\widetilde{\sigma}_p(f)} \log^+_{p} M(r,f).$$

Definition 1.22 *[44, 64] Let $f \in \mathscr{M}$ and $p \in \mathbb{N}$. The iterated p-convergence exponent of the sequence of zeros of f is defined by*

$$\lambda_p(f) := \limsup_{r\to+\infty} \frac{\log_p N\left(r, \frac{1}{f}\right)}{\log r}.$$

Similarly, the iterated p-convergence exponent of the sequence of distinct zeros of f is defined by

$$\overline{\lambda}_p(f) := \limsup_{r\to+\infty} \frac{\log_p \overline{N}\left(r, \frac{1}{f}\right)}{\log r}.$$

Example 1.10 *For $f(z) = \exp_2(z) - \exp(z)$, we have $\lambda(f) = +\infty$ and $\lambda_2(f) = 1 = \overline{\lambda}_2(f)$.*

5 The concept of φ-order

Recently, Chyzhykov and Semochko [20] showed that both definitions of iterated p-order and $[p,q]$-order have the disadvantage that they can not cover an arbitrary growth (see [20, Example 1.4]), i.e. there exist some functions with infinite iterated order. They employed a more general scale to measure the fast growth of complex functions called the φ-order (cf. [61]). After that, Belaïdi introduced the definition of the φ-type in [5].

Definition 1.23 *[20] Let $f \in \mathscr{M}$ and let φ be an increasing unbounded function on $(0,+\infty) \longrightarrow (0,+\infty)$. The φ-orders of f are defined by*

$$\rho^0_\varphi(f) := \limsup_{r\to+\infty} \frac{\varphi\left(e^{T(r,f)}\right)}{\log r}, \qquad \rho^1_\varphi(f) := \limsup_{r\to+\infty} \frac{\varphi\left(T(r,f)\right)}{\log r}.$$

If $f \in \mathscr{E}$, then the φ-orders of f are defined by

$$\widetilde{\rho}^{0}_{\varphi}(f) := \limsup_{r\to+\infty} \frac{\varphi\left(\mathrm{M}(r,f)\right)}{\log r}, \qquad \widetilde{\rho}^{1}_{\varphi}(f) := \limsup_{r\to+\infty} \frac{\varphi\left(\log \mathrm{M}(r,f)\right)}{\log r}.$$

Similarly, if replacing the lim sup by the lim inf, we can define the concept of lower φ-order as follows:

Definition 1.24 *[8] Let $f \in \mathscr{M}$ and let φ be an increasing unbounded function on $(0,+\infty)$. The lower φ-orders of f are defined by*

$$\mu^{0}_{\varphi}(f) := \liminf_{r\to+\infty} \frac{\varphi\left(e^{\mathrm{T}(r,f)}\right)}{\log r}, \quad \mu^{1}_{\varphi}(f) := \liminf_{r\to+\infty} \frac{\varphi\left(\mathrm{T}(r,f)\right)}{\log r}.$$

If $f \in \mathscr{E}$, then the lower φ-orders are defined by

$$\widetilde{\mu}^{0}_{\varphi}(f) := \liminf_{r\to+\infty} \frac{\varphi\left(\mathrm{M}(r,f)\right)}{\log r}, \quad \widetilde{\mu}^{1}_{\varphi}(f) := \liminf_{r\to+\infty} \frac{\varphi\left(\log \mathrm{M}(r,f)\right)}{\log r}.$$

Definition 1.25 *[59] Let $f \in \mathscr{M}_{\triangle}$ and let φ be an increasing unbounded function on $(0,+\infty)$. The φ-orders of f are defined by*

$$\sigma^{0}_{\varphi}(f) := \limsup_{r\longrightarrow 1^{-}} \frac{\varphi\left(e^{\mathrm{T}(r,f)}\right)}{-\log(1-r)}, \qquad \sigma^{1}_{\varphi}(f) := \limsup_{r\longrightarrow 1^{-}} \frac{\varphi\left(\mathrm{T}(r,f)\right)}{-\log(1-r)}.$$

If $f \in \mathscr{E}_{\triangle}$, then the φ-orders are defined by

$$\widetilde{\sigma}^{0}_{\varphi}(f) := \limsup_{r\longrightarrow 1^{-}} \frac{\varphi\left(\mathrm{M}(r,f)\right)}{-\log(1-r)}, \qquad \widetilde{\sigma}^{1}_{\varphi}(f) := \limsup_{r\longrightarrow 1^{-}} \frac{\varphi\left(\log^{+} \mathrm{M}(r,f)\right)}{-\log(1-r)}.$$

Definition 1.26 *[40] Let $f \in \mathscr{M}_{\triangle}$ and let φ be an increasing unbounded function on $(0,+\infty)$. The lower φ-orders of f are defined by*

$$\nu^{0}_{\varphi}(f) := \liminf_{r\longrightarrow 1^{-}} \frac{\varphi\left(e^{\mathrm{T}(r,f)}\right)}{-\log(1-r)}, \qquad \nu^{1}_{\varphi}(f) := \liminf_{r\longrightarrow 1^{-}} \frac{\varphi\left(\mathrm{T}(r,f)\right)}{-\log(1-r)}.$$

If $f \in \mathscr{E}_{\triangle}$, then the lower φ-orders of f are defined by

$$\widetilde{\nu}^{0}_{\varphi}(f) := \liminf_{r\longrightarrow 1^{-}} \frac{\varphi\left(\mathrm{M}(r,f)\right)}{-\log(1-r)}, \qquad \widetilde{\nu}^{1}_{\varphi}(f) := \liminf_{r\longrightarrow 1^{-}} \frac{\varphi\left(\log^{+} \mathrm{M}(r,f)\right)}{-\log(1-r)}.$$

We denote by the symbol Φ, the class of positive unbounded increasing functions on $(0,+\infty)$, such that $\varphi\left(e^{t}\right)$ grows slowly, i.e.,

$$\forall c > 0 : \lim_{t\to+\infty} \frac{\varphi\left(e^{ct}\right)}{\varphi\left(e^{t}\right)} = 1.$$

Bandura et al. [1] proved that for any entire transcendental function $f \in \mathscr{E}$ of infinite order, i.e., $\rho(f) = +\infty$, there exists $\varphi \in \Phi$ satisfying $\rho^{0}_{\varphi}(f) < +\infty$.

Example 1.11 *Let $f \in \mathscr{M}$. One can see that $\varphi(r) = \log_{p} r$, $(p \geq 2)$ belongs to the class Φ and $\varphi(r) = \log r \notin \Phi$. Moreover, for $\varphi(r) = \log_{p} r$, the $\rho^{1}_{\varphi}(f)$ order of the function f coincides with its iterated p-order, i.e., $\rho^{1}_{\varphi}(f) = \rho_{p}(f)$ As a particular case, for $\varphi = \log_{2} \in \Phi$ we have $\rho^{0}_{\log_{2}}(f) = \rho(f)$ and $\rho^{1}_{\log_{2}}(f) = \rho_{2}(f)$ which are respectively the usual order and the hyper-order of f.*

Proposition 1.5 *[20] If $\varphi \in \Phi$, then*

$$\forall m>0,\ \forall k\geq 0:\ \frac{\varphi^{-1}(\log x^m)}{x^k}\longrightarrow +\infty,\ x\longrightarrow +\infty. \tag{1.1}$$

$$\forall c>0:\ \frac{\log\varphi^{-1}((1+c)x)}{\log\varphi^{-1}(x)}\longrightarrow +\infty,\ x\longrightarrow +\infty. \tag{1.2}$$

$$\forall c>0:\varphi(cx)\leq\varphi\left(x^c\right)\leq(1+o(1))\varphi(x),\ x\longrightarrow +\infty. \tag{1.3}$$

Proposition 1.6 *[8, 20] If $\varphi\in\Phi$ and $f\in\mathscr{E}$, then for $j=0,1$, we have*

$$\rho_\varphi^j(f)=\widetilde{\rho}_\varphi^j(f)\quad \textit{and}\quad \mu_\varphi^j(f)=\widetilde{\mu}_\varphi^j(f).$$

Proposition 1.7 *[40, 59] If $\varphi\in\Phi$ and $f\in\mathscr{E}_\Delta$, then*

$$\sigma_\varphi^1(f)=\widetilde{\sigma}_\varphi^1(f)\quad \textit{and}\quad \nu_\varphi^1(f)=\widetilde{\nu}_\varphi^1(f).$$

Remark 1.10 *By letting $\varphi(.)=\log_2^+(.)\in\Phi$, we see that $\sigma_\varphi^0(f)=\sigma_{\log_2^+}^0(f)=\sigma(f)$ and $\widetilde{\sigma}_\varphi^0(f)=\widetilde{\sigma}_{\log_2^+}^0(f)=\widetilde{\sigma}(f)$. From Remark 1.5, there exist some analytic functions f in Δ satisfying $\sigma(f)\neq\widetilde{\sigma}(f)$, i.e., $\sigma_\varphi^0(f)\neq\widetilde{\sigma}_\varphi^0(f)$ even if $\varphi\in\Phi$.*

Definition 1.27 *[5] Let $f\in\mathscr{M}$ and let φ be an increasing unbounded function on $(0,+\infty)$. We define the φ-types of f such that $0<\rho_\varphi^i(f)<+\infty\,(i=0,1)$ by*

$$\tau_\varphi^0(f):=\limsup_{r\to+\infty}\frac{e^{\varphi\left(e^{\mathrm{T}(r,f)}\right)}}{r^{\rho_\varphi^0(f)}},\qquad \tau_\varphi^1(f):=\limsup_{r\to+\infty}\frac{e^{\varphi(\mathrm{T}(r,f))}}{r^{\rho_\varphi^1(f)}}.$$

If $f\in\mathscr{E}$, then the φ-types of f such that $0<\widetilde{\rho}_\varphi^i(f)<+\infty\,(i=0,1)$ are defined as

$$\widetilde{\tau}_\varphi^0(f):=\limsup_{r\to+\infty}\frac{e^{\varphi(\mathrm{M}(r,f))}}{r^{\widetilde{\rho}_\varphi^0(f)}},\qquad \widetilde{\tau}_\varphi^1(f):=\limsup_{r\to+\infty}\frac{e^{\varphi(\log\mathrm{M}(r,f))}}{r^{\widetilde{\rho}_\varphi^1(f)}}.$$

Definition 1.28 *[5] Let φ be an increasing unbounded function on $(0,+\infty)$. The lower φ-types of a meromorphic function $f\in\mathscr{M}$ with $0<\mu_\varphi^j(f)<+\infty\,(j=0,1)$ are defined by*

$$\underline{\tau}_\varphi^0(f):=\liminf_{r\to+\infty}\frac{\exp\left\{\varphi\left(e^{\mathrm{T}(r,f)}\right)\right\}}{r^{\mu_\varphi^0(f)}},\quad \underline{\tau}_\varphi^1(f):=\liminf_{r\to+\infty}\frac{\exp\left\{\varphi\left(\mathrm{T}(r,f)\right)\right\}}{r^{\mu_\varphi^1(f)}}.$$

If $f\in\mathscr{E}$ with $0<\widetilde{\mu}_\varphi^j(f)<+\infty\,(j=0,1)$, then the lower φ-types are defined by

$$\underline{\widetilde{\tau}}_\varphi^0(f):=\liminf_{r\to+\infty}\frac{\exp\left\{\varphi\left(\mathrm{M}(r,f)\right)\right\}}{r^{\widetilde{\mu}_\varphi^0(f)}},\quad \underline{\widetilde{\tau}}_\varphi^1(f):=\liminf_{r\to+\infty}\frac{\exp\left\{\varphi\left(\log\mathrm{M}(r,f)\right)\right\}}{r^{\widetilde{\mu}_\varphi^1(f)}}.$$

Definition 1.29 *[42] Let $f\in\mathscr{M}_\Delta$ and let φ be an increasing unbounded function on $(0,+\infty.)$ We define the φ-types of f such that $0<\sigma_\varphi^i(f)<+\infty\,(i=0,1)$ by*

$$\kappa_\varphi^0(f):=\limsup_{r\longrightarrow 1^-}(1-r)^{\sigma_\varphi^0(f)}\exp\left\{\varphi\left(e^{\mathrm{T}(r,f)}\right)\right\},$$

$$\kappa_\varphi^1(f):=\limsup_{r\longrightarrow 1^-}(1-r)^{\sigma_\varphi^1(f)}\exp\left\{\varphi\left(\mathrm{T}(r,f)\right)\right\}.$$

If $f\in\mathscr{E}_\Delta$, then the φ-types such that $0<\widetilde{\sigma}_\varphi^i(f)<+\infty\,(i=0,1)$ are defined by

$$\widetilde{\kappa}_\varphi^0(f):=\limsup_{r\longrightarrow 1^-}(1-r)^{\widetilde{\sigma}_\varphi^0(f)}\exp\left\{\varphi\left(\mathrm{M}(r,f)\right)\right\},$$

$$\widetilde{\kappa}_\varphi^1(f):=\limsup_{r\longrightarrow 1^-}(1-r)^{\widetilde{\sigma}_\varphi^1(f)}\exp\left\{\varphi\left(\log^+\mathrm{M}(r,f)\right)\right\}.$$

Definition 1.30 *[40] Let $f \in \mathscr{M}_{\angle}$ and let φ be an increasing unbounded function on $(0,+\infty)$. We define the lower φ-types of f such that $0 < \nu^{i}_{\varphi}(f) < +\infty\,(i=0,1)$ by*

$$\underline{\kappa}^{0}_{\varphi}(f) := \liminf_{r\longrightarrow 1^{-}} (1-r)^{\nu^{0}_{\varphi}(f)} \exp\left\{\varphi\left(e^{\mathrm{T}(r,f)}\right)\right\},$$

$$\underline{\kappa}^{1}_{\varphi}(f) := \liminf_{r\longrightarrow 1^{-}} (1-r)^{\nu^{1}_{\varphi}(f)} \exp\left\{\varphi\left(\mathrm{T}(r,f)\right)\right\}.$$

If $f \in \mathscr{E}_{\angle}$, then the lower φ-types of f such that $0 < \widetilde{\nu}^{i}_{\varphi}(f) < +\infty\,(i=0,1)$ are defined by

$$\underline{\widetilde{\kappa}}^{0}_{\varphi}(f) := \liminf_{r\longrightarrow 1^{-}} (1-r)^{\widetilde{\nu}^{0}_{\varphi}(f)} \exp\left\{\varphi\left(\mathrm{M}(r,f)\right)\right\},$$

$$\underline{\widetilde{\kappa}}^{1}_{\varphi}(f) := \liminf_{r\longrightarrow 1^{-}} (1-r)^{\widetilde{\nu}^{1}_{\varphi}(f)} \exp\left\{\varphi\left(\log^{+}\mathrm{M}(r,f)\right)\right\}.$$

Proposition 1.8 *[20] Let $\varphi \in \Phi$ and let $f_1, f_2 \in \mathscr{M}$. Then, for $j=0,1$ we have*

(i) if $a \in \mathbb{C}^{\star}$, then $\rho^{j}_{\varphi}(af) = \rho^{j}_{\varphi}(f)$ and $\tau^{j}_{\varphi}(af) = \tau^{j}_{\varphi}(f)$,

(ii) if $f_1 \not\equiv 0$, then $\rho^{j}_{\varphi}\left(\frac{1}{f}\right) = \rho^{j}_{\varphi}(f)$ and $\tau^{j}_{\varphi}\left(\frac{1}{f}\right) = \tau^{j}_{\varphi}(f)$,

(iii) $\rho^{j}_{\varphi}(f_1+f_2) \le \max\left\{\rho^{j}_{\varphi}(f_1), \rho^{j}_{\varphi}(f_2)\right\}$,

(iv) $\rho^{j}_{\varphi}(f_1 f_2) \le \max\left\{\rho^{j}_{\varphi}(f_1), \rho^{j}_{\varphi}(f_2)\right\}$.

Lemma 1.3 *[39] Let $f \in \mathscr{M}$. If $\rho^{0}_{\varphi}(f) < +\infty$, then $\rho^{1}_{\varphi}(f) = 0$.*

Proof. Suppose that $\rho^{0}_{\varphi}(f) = \rho < +\infty$. For any given $\varepsilon > 0$ and sufficiently large r, we have

$$\mathrm{T}(r,f) \le \log\varphi^{-1}\left((\rho+\varepsilon)\log r\right).$$

By Karamata's theorem (cf. [60]), we have $\varphi\left(e^{t}\right) = t^{o(1)}$ as $t \longrightarrow +\infty$. Thus, in view of (1.1) we get

$$\begin{aligned}
\rho^{1}_{\varphi}(f) := \limsup_{r\to+\infty} \frac{\varphi\left(\mathrm{T}(r,f)\right)}{\log r} &= \limsup_{r\to+\infty} \frac{\varphi\left(e^{\log \mathrm{T}(r,f)}\right)}{\log r} \\
&= \limsup_{r\to+\infty} \frac{\left[\log \mathrm{T}(r,f)\right]^{o(1)}}{\log r} \\
&\le \limsup_{r\to+\infty} \frac{\left[\log\log\varphi^{-1}\left((\rho+\varepsilon)\log r\right)\right]^{o(1)}}{\log r} = 0.
\end{aligned}$$

Hence, $\rho^{1}_{\varphi}(f) = 0$. □

Definition 1.31 *[39] Let $f \in \mathscr{M}$ and φ be an increasing unbounded function on $(0,+\infty)$. The φ-convergence exponents of the sequence of zeros of f are defined by*

$$\lambda^{0}_{\varphi}(f) := \limsup_{r\to+\infty} \frac{\varphi\left(e^{\mathrm{N}\left(r,\frac{1}{f}\right)}\right)}{\log r}, \qquad \lambda^{1}_{\varphi}(f) := \limsup_{r\to+\infty} \frac{\varphi\left(\mathrm{N}\left(r,\frac{1}{f}\right)\right)}{\log r}.$$

Similarly, if we replace $\mathrm{N}\left(r,\frac{1}{f}\right)$ *by* $\overline{\mathrm{N}}\left(r,\frac{1}{f}\right)$*, we obtain the* φ*-convergence exponents of the sequence of distinct zeros of* f*.*

$$\overline{\lambda}^{0}_{\varphi}(f) := \limsup_{r\to+\infty}\frac{\varphi\left(e^{\overline{\mathrm{N}}\left(r,\frac{1}{f}\right)}\right)}{\log r}, \qquad \overline{\lambda}^{1}_{\varphi}(f) := \limsup_{r\to+\infty}\frac{\varphi\left(\overline{\mathrm{N}}\left(r,\frac{1}{f}\right)\right)}{\log r}.$$

Definition 1.32 *Let* φ *be an increasing unbounded function on* $(0,+\infty)$*. We define the lower* φ*-convergence exponents of the sequence of zeros of a meromorphic function* $f \in \mathscr{M}$ *by*

$$\underline{\lambda}^{0}_{\varphi}(f) := \liminf_{r\to+\infty}\frac{\varphi\left(e^{\mathrm{N}\left(r,\frac{1}{f}\right)}\right)}{\log r}, \qquad \underline{\lambda}^{1}_{\varphi}(f) := \liminf_{r\to+\infty}\frac{\varphi\left(\mathrm{N}\left(r,\frac{1}{f}\right)\right)}{\log r}.$$

Similarly, the notations $\underline{\overline{\lambda}}^{0}_{\varphi}(f)$ *and* $\underline{\overline{\lambda}}^{1}_{\varphi}(f)$ *can be used to denote the lower* φ*-convergence exponents of the sequence of distinct zeros of* $f \in \mathscr{M}$*.*

Chapter 2

On the φ-order and the φ-type of entire and meromorphic functions

1 Introduction and statement of the main results

Firstly, we recall two classical results investigated the order and the type of the sum and product of entire functions and meromorphic functions in the complex plane.

Theorem 2.1 *[51] Let $f_1, f_2 \in \mathscr{E}$. Then we have*

$$\begin{aligned} \rho(f_1+f_2) &\leq \max\{\rho(f_1),\rho(f_2)\}, \\ \rho(f_1 f_2) &\leq \max\{\rho(f_1),\rho(f_2)\}, \end{aligned}$$

and

$$\begin{aligned} \widetilde{\tau}(f_1+f_2) &\leq \max\{\widetilde{\tau}(f_1),\widetilde{\tau}(f_2)\}, \\ \widetilde{\tau}(f_1 f_2) &\leq \widetilde{\tau}(f_1)+\widetilde{\tau}(f_2). \end{aligned}$$

Theorem 2.2 *[24] Let $f_1, f_2 \in \mathscr{M}$. If $\rho(f_1) < \rho(f_2)$, then*

$$\rho(f_1+f_2) = \rho(f_1 f_2) = \rho(f_2).$$

In [49], Latreuch and Belaïdi established new estimates for the order and type of meromorphic functions and obtained the following results which improved the above two theorems.

Theorem 2.3 *[49] Let $f_1, f_2 \in \mathscr{M}$. Then we have*

(i) if $0 < \rho(f_1) < \rho(f_2) < +\infty$, then $\tau(f_1+f_2) = \tau(f_1 f_2) = \tau(f_2)$,

(ii) if $0 < \rho(f_1) = \rho(f_2) = \rho(f_1+f_2) = \rho(f_1 f_2) < +\infty$, then

$$\begin{aligned} |\tau(f_1)-\tau(f_2)| \leq \tau(f_1+f_2) &\leq \tau(f_1)+\tau(f_2), \\ |\tau(f_1)-\tau(f_2)| \leq \tau(f_1 f_2) &\leq \tau(f_1)+\tau(f_2). \end{aligned}$$

Theorem 2.4 *[49] Let $f_1, f_2 \in \mathscr{M}$. If $0 < \rho(f_1) = \rho(f_2) < +\infty$ and $\tau(f_1) \neq \tau(f_2)$, then*

$$\rho(f_1+f_2) = \rho(f_1 f_2) = \rho(f_1) = \rho(f_2).$$

In [66], Tu et al. generalized theorems 2.3–2.4 from usual concepts of order and type to iterated p-order and iterated p-type.

Theorem 2.5 *[66] Let $f_1, f_2 \in \mathcal{M}$ satisfying $0 < \rho_p(f_1) = \rho_p(f_2) < +\infty$ and $\tau_p(f_1) < \tau_p(f_2)$ for any $p \in \mathbb{N}$. Then we have*

(i) $\rho_p(f_1 + f_2) = \rho_p(f_1 f_2) = \rho_p(f_1) = \rho_p(f_2)$,

(ii) *if $p > 1$, then* $\tau_p(f_1 + f_2) = \tau_p(f_1 f_2) = \tau_p(f_2)$,

(iii) *if $p = 1$, then $\alpha \leq \tau_p(f_1 + f_2) \leq \beta$ and $\alpha \leq \tau_p(f_1 f_2) \leq \beta$ where $\alpha = \tau_p(f_2) - \tau_p(f_1)$ and $\beta = \tau_p(f_1) + \tau_p(f_2)$.*

In this chapter, we give the counterparts of the above theorems for the φ-order and the φ-type concepts. From Theorem 1.8, we know that a meromorphic function $f \in \mathcal{M}$ and its derivative f' have the same order. It is well known also that $\rho_p(f') = \rho_p(f)$ for any $p \in \mathbb{N}$ (see [44, 45]). So, we will show that this assertion remains valid for the φ-order and we derive an estimate for the φ-type of the derivative f'.

Theorem 2.6 *[43] Let $\varphi \in \Phi$ and let $f_1, f_2 \in \mathcal{M}$. For $j = 0, 1$, if $\rho_\varphi^j(f_1) < \rho_\varphi^j(f_2)$, then*

$$\rho_\varphi^j(f_1 + f_2) = \rho_\varphi^j(f_1 f_2) = \rho_\varphi^j(f_2).$$

Theorem 2.7 *[43] Let $\varphi \in \Phi$ and let $f_1, f_2 \in \mathcal{M}$.*

(i) For $j = 0, 1$, if $0 < \rho_\varphi^j(f_1) < \rho_\varphi^j(f_2) < +\infty$, then

$$\tau_\varphi^j(f_1 + f_2) = \tau_\varphi^j(f_1 f_2) = \tau_\varphi^j(f_2). \tag{2.1}$$

(ii) For $j = 0, 1$, if $0 < \rho_\varphi^j(f_1) = \rho_\varphi^j(f_2) = \rho_\varphi^j(f_1 + f_2) < +\infty$, then

$$\tau_\varphi^j(f_1 + f_2) \leq \max\left\{\tau_\varphi^j(f_1), \tau_\varphi^j(f_2)\right\}.$$

Moreover, if $\tau_\varphi^j(f_1) \neq \tau_\varphi^j(f_2)$, then

$$\tau_\varphi^j(f_1 + f_2) = \max\left\{\tau_\varphi^j(f_1), \tau_\varphi^j(f_2)\right\}. \tag{2.2}$$

(iii) For $j = 0, 1$, if $0 < \rho_\varphi^j(f_1) = \rho_\varphi^j(f_2) = \rho_\varphi^j(f_1 f_2) < +\infty$, then

$$\tau_\varphi^j(f_1 f_2) \leq \max\left\{\tau_\varphi^j(f_1), \tau_\varphi^j(f_2)\right\}.$$

Moreover, if $\tau_\varphi^j(f_1) \neq \tau_\varphi^j(f_2)$, then

$$\tau_\varphi^j(f_1 f_2) = \max\left\{\tau_\varphi^j(f_1), \tau_\varphi^j(f_2)\right\}. \tag{2.3}$$

Corollary 2.1 *[43] Let $\varphi \in \Phi$ and let $f_1, f_2 \in \mathcal{M}$. For $j = 0, 1$, if $0 < \rho_\varphi^j(f_1) = \rho_\varphi^j(f_2) < +\infty$ and $\tau_\varphi^j(f_1) < \tau_\varphi^j(f_2)$, then*

$$\rho_\varphi^j(f_1 + f_2) = \rho_\varphi^j(f_1 f_2) = \rho_\varphi^j(f_1) = \rho_\varphi^j(f_2), \tag{2.4}$$

$$\tau_\varphi^j(f_1 + f_2) = \tau_\varphi^j(f_1 f_2) = \tau_\varphi^j(f_2). \tag{2.5}$$

Theorem 2.8 *[43] Let $\varphi \in \Phi$ and let $f_1, f_2 \in \mathcal{E}$.*

(i) *For* $j=0,1$, *if* $0<\widetilde{\rho}_{\varphi}^{j}(f_1)<\widetilde{\rho}_{\varphi}^{j}(f_2)<+\infty$, *then*

$$\widetilde{\tau}_{\varphi}^{j}(f_1+f_2)=\widetilde{\tau}_{\varphi}^{j}(f_2) \quad \textit{and} \quad \widetilde{\tau}_{\varphi}^{j}(f_1 f_2)\leq \max\left\{\widetilde{\tau}_{\varphi}^{j}(f_1),\widetilde{\tau}_{\varphi}^{j}(f_2)\right\}.$$

(ii) *For* $j=0,1$, *if* $0<\widetilde{\rho}_{\varphi}^{j}(f_1)=\widetilde{\rho}_{\varphi}^{j}(f_2)=\widetilde{\rho}_{\varphi}^{j}(f_1+f_2)<+\infty$, *then*

$$\widetilde{\tau}_{\varphi}^{j}(f_1+f_2)\leq \max\left\{\widetilde{\tau}_{\varphi}^{j}(f_1),\widetilde{\tau}_{\varphi}^{j}(f_2)\right\}.$$

Moreover, if $\widetilde{\tau}_{\varphi}^{j}(f_1)\neq\widetilde{\tau}_{\varphi}^{j}(f_2)$, *then* $\widetilde{\tau}_{\varphi}^{j}(f_1+f_2)=\max\left\{\widetilde{\tau}_{\varphi}^{j}(f_1),\widetilde{\tau}_{\varphi}^{j}(f_2)\right\}$.

(iii) *For* $j=0,1$, *if* $0<\widetilde{\rho}_{\varphi}^{j}(f_1)=\widetilde{\rho}_{\varphi}^{j}(f_2)=\widetilde{\rho}_{\varphi}^{j}(f_1 f_2)<+\infty$, *then*

$$\widetilde{\tau}_{\varphi}^{j}(f_1 f_2)\leq \max\left\{\widetilde{\tau}_{\varphi}^{j}(f_1),\widetilde{\tau}_{\varphi}^{j}(f_2)\right\}.$$

Moreover, if $\widetilde{\tau}_{\varphi}^{j}(f_1)\neq\widetilde{\tau}_{\varphi}^{j}(f_2)$, *then* $\widetilde{\tau}_{\varphi}^{j}(f_1 f_2)=\max\left\{\widetilde{\tau}_{\varphi}^{j}(f_1),\widetilde{\tau}_{\varphi}^{j}(f_2)\right\}$.

Corollary 2.2 *[43] Let* $\varphi\in\Phi$ *and let* $f_1,f_2\in\mathscr{E}$. *For* $j=0,1$, *if* $0<\widetilde{\rho}_{\varphi}^{j}(f_1)=\widetilde{\rho}_{\varphi}^{j}(f_2)<+\infty$ *and* $\widetilde{\tau}_{\varphi}^{j}(f_1)<\widetilde{\tau}_{\varphi}^{j}(f_2)$, *then*

$$\widetilde{\rho}_{\varphi}^{j}(f_1+f_2)=\widetilde{\rho}_{\varphi}^{j}(f_1)=\widetilde{\rho}_{\varphi}^{j}(f_2) \quad \textit{and} \quad \widetilde{\tau}_{\varphi}^{j}(f_1+f_2)=\widetilde{\tau}_{\varphi}^{j}(f_2).$$

Theorem 2.9 *[43] Let* $\varphi\in\Phi$ *and* $f\in\mathscr{M}$. *Then, for any constant* $\alpha>1$ *we have*

$$\rho_{\varphi}^{1}(f')=\rho_{\varphi}^{1}(f)=\rho \quad \textit{and} \quad \frac{\tau_{\varphi}^{1}(f)}{2^{\rho}}\leq\tau_{\varphi}^{1}(f')\leq\alpha^{\rho}\tau_{\varphi}^{1}(f).$$

2 Proofs of main results

2.1 Proof of Theorem 2.6

Since $\rho_{\varphi}^{j}(f_1)<\rho_{\varphi}^{j}(f_2)$, then by Proposition 1.8, it follows that $\rho_{\varphi}^{j}(f_1+f_2)\leq\rho_{\varphi}^{j}(f_2)$ and therefore,

$$\rho_{\varphi}^{j}(f_2)=\rho_{\varphi}^{j}(f_1+f_2-f_1)\leq\max\left\{\rho_{\varphi}^{j}(f_1+f_2),\rho_{\varphi}^{j}(f_1)\right\}.$$

Suppose that $\rho_{\varphi}^{j}(f_1)>\rho_{\varphi}^{j}(f_1+f_2)$. Then

$$\rho_{\varphi}^{j}(f_2)\leq\max\left\{\rho_{\varphi}^{j}(f_1+f_2),\rho_{\varphi}^{j}(f_1)\right\}=\rho_{\varphi}^{j}(f_1)$$

which contradicts the assumption $\rho_{\varphi}^{j}(f_1)<\rho_{\varphi}^{j}(f_2)$. Hence $\rho_{\varphi}^{j}(f_2)\leq\rho_{\varphi}^{j}(f_1+f_2)$ and therefore $\rho_{\varphi}^{j}(f_1+f_2)=\rho_{\varphi}^{j}(f_2)$. Now, we prove that $\rho_{\varphi}^{j}(f_1 f_2)=\rho_{\varphi}^{j}(f_2)$. Indeed, again by Proposition 1.8, it follows that $\rho_{\varphi}^{j}(f_1 f_2)\leq\rho_{\varphi}^{j}(f_2)$ and therefore

$$\rho_{\varphi}^{j}(f_2)=\rho_{\varphi}^{j}\left(f_1 f_2\frac{1}{f_1}\right)\leq\max\left\{\rho_{\varphi}^{j}(f_1 f_2),\rho_{\varphi}^{j}\left(\frac{1}{f_1}\right)\right\}=\max\left\{\rho_{\varphi}^{j}(f_1 f_2),\rho_{\varphi}^{j}(f_1)\right\}.$$

Suppose that $\rho_{\varphi}^{j}(f_1)>\rho_{\varphi}^{j}(f_1 f_2)$. Then

$$\rho_{\varphi}^{j}(f_2)\leq\max\left\{\rho_{\varphi}^{j}(f_1 f_2),\rho_{\varphi}^{j}(f_1)\right\}=\rho_{\varphi}^{j}(f_1).$$

This contradiction leads to $\rho_{\varphi}^{j}(f_2)\leq\rho_{\varphi}^{j}(f_1 f_2)$ and therefore $\rho_{\varphi}^{j}(f_1 f_2)=\rho_{\varphi}^{j}(f_2)$.

2.2 Proof of Theorem 2.7

We will prove the theorem for $j = 1$, the proofs for $j = 0$ are analogous.

(i) The definition of the τ_φ^1-type implies that for any given $\varepsilon > 0$, there exists a sequence $\{r_n, n \geq 1\}$ tending to infinity such that

$$T(r_n, f_2) \geq \varphi^{-1}\left(\log\left(\tau_\varphi^1(f_2) - \varepsilon\right) r_n^{\rho_\varphi^1(f_2)}\right)$$

and

$$T(r, f_1) \leq \varphi^{-1}\left(\log\left(\tau_\varphi^1(f_1) + \varepsilon\right) r^{\rho_\varphi^1(f_1)}\right) \quad \text{as} \quad r \longrightarrow +\infty. \tag{2.6}$$

Thus, by the fact that

$$T(r, f_1 + f_2) \geq T(r, f_2) - T(r, f_1) - \log 2, \tag{2.7}$$

we get from Proposition 1.5 that

$$\begin{aligned} T(r_n, f_1 + f_2) &\geq \varphi^{-1}\left(\log\left(\tau_\varphi^1(f_2) - \varepsilon\right) r_n^{\rho_\varphi^1(f_2)}\right) - \varphi^{-1}\left(\log\left(\tau_\varphi^1(f_1) + \varepsilon\right) r_n^{\rho_\varphi^1(f_1)}\right) - \log 2 \\ &\geq \varphi^{-1}\left(\log\left(\tau_\varphi^1(f_2) - 2\varepsilon\right) r_n^{\rho_\varphi^1(f_2)}\right). \end{aligned} \tag{2.8}$$

It follows from theorem 2.6 that $\rho_\varphi^1(f_1 + f_2) = \rho_\varphi^1(f_2)$. Then, by the monotonicity of φ and (2.8) we obtain

$$\frac{e^{\varphi(T(r_n, f_1+f_2))}}{r_n^{\rho_\varphi^1(f_1+f_2)}} \geq \tau_\varphi^1(f_2) - 2\varepsilon.$$

Since $\varepsilon > 0$ can be arbitrarily chosen, we obtain that

$$\tau_\varphi^1(f_1 + f_2) \geq \tau_\varphi^1(f_2). \tag{2.9}$$

Since $\rho_\varphi^1(f_1 + f_2) = \rho_\varphi^1(f_2) > \rho_\varphi^1(f_1) = \rho_\varphi^1(-f_1)$, we obtain by applying (2.9) that

$$\tau_\varphi^1(f_2) = \tau_\varphi^1(f_1 + f_2 - f_1) \geq \tau_\varphi^1(f_1 + f_2). \tag{2.10}$$

From (2.9) and (2.10), we deduce that $\tau_\varphi^1(f_1 + f_2) = \tau_\varphi^1(f_2)$.

Now we prove that $\tau_\varphi^1(f_1 f_2) = \tau_\varphi^1(f_2)$. By using the fact that

$$T(r, f_1 f_2) \geq T(r, f_1) - T(r, f_2) + O(1) \tag{2.11}$$

and a similar discussion as in the above proof, one can easily show that

$$\tau_\varphi^1(f_1 f_2) \geq \tau_\varphi^1(f_2). \tag{2.12}$$

Since $\rho_\varphi^1(f_1 f_2) = \rho_\varphi^1(f_2) > \rho_\varphi^1(f_1) = \rho_\varphi^1\left(\frac{1}{f_1}\right)$, it follows from (2.12) that

$$\tau_\varphi^1(f_2) = \tau_\varphi^1\left(f_1 f_2 \frac{1}{f_1}\right) \geq \tau_\varphi^1(f_1 f_2)$$

and therefore $\tau_\varphi^1(f_1 f_2) = \tau_\varphi^1(f_2)$.

(ii) By (2.6), Proposition 1.5 and the assumption $0 < \rho_\varphi^1(f_1) = \rho_\varphi^1(f_2) = \rho_\varphi^1(f_1+f_2) < +\infty$, we get

$$\begin{aligned} T(r, f_1+f_2) &\leq T(r,f_1)+T(r,f_2)+O(1) \\ &\leq \varphi^{-1}\left(\log\left(\tau_\varphi^1(f_1)+\varepsilon\right) r^{\rho_\varphi^1(f_1+f_2)}\right) + \varphi^{-1}\left(\log\left(\tau_\varphi^1(f_2)+\varepsilon\right) r^{\rho_\varphi^1(f_1+f_2)}\right) + O(1) \\ &\leq \varphi^{-1}\left(\log\left(\max\left\{\tau_\varphi^1(f_1), \tau_\varphi^1(f_2)\right\}+3\varepsilon\right) r^{\rho_\varphi^1(f_1+f_2)}\right). \end{aligned}$$

By the monotonicity of φ, we get

$$\frac{e^{\varphi(T(r,f_1+f_2))}}{r^{\rho_\varphi^1(f_1+f_2)}} \leq \max\left\{\tau_\varphi^1(f_1), \tau_\varphi^1(f_2)\right\}+3\varepsilon.$$

Hence

$$\tau_\varphi^1(f_1+f_2) \leq \max\left\{\tau_\varphi^1(f_1), \tau_\varphi^1(f_2)\right\}. \tag{2.13}$$

Without loss of generality, we may suppose $\tau_\varphi^1(f_1) < \tau_\varphi^1(f_2)$. Then, by (2.13) and since $\rho_\varphi^1(f_1+f_2) = \rho_\varphi^1(f_1) = \rho_\varphi^1(-f_1)$, it follows that

$$\tau_\varphi^1(f_2) = \tau_\varphi^1(f_1+f_2-f_1) \leq \max\left\{\tau_\varphi^1(f_1+f_2), \tau_\varphi^1(f_1)\right\} = \tau_\varphi^1(f_1+f_2). \tag{2.14}$$

We deduce from (2.13) and (2.14) that $\tau_\varphi^1(f_1+f_2) = \max\left\{\tau_\varphi^1(f_1), \tau_\varphi^1(f_2)\right\}$.

(iii) By a similar discussion as in the above proof and the fact that

$$T(r, f_1 f_2) \leq T(r,f_1)+T(r,f_2),$$

we obtain

$$\tau_\varphi^1(f_1 f_2) \leq \max\left\{\tau_\varphi^1(f_1), \tau_\varphi^1(f_2)\right\}. \tag{2.15}$$

On the other hand, if we suppose that $\tau_\varphi^1(f_1) < \tau_\varphi^1(f_2)$, then by (2.15) and the fact that $\rho_\varphi^1(f_1 f_2) = \rho_\varphi^1(f_1) = \rho_\varphi^1\left(\frac{1}{f_1}\right)$, we obtain

$$\tau_\varphi^1(f_2) = \tau_\varphi^1\left(f_1 f_2 \frac{1}{f_1}\right) \leq \max\left\{\tau_\varphi^1(f_1 f_2), \tau_\varphi^1(f_1)\right\} = \tau_\varphi^1(f_1 f_2). \tag{2.16}$$

It follows from (2.15) and (2.16) that $\tau_\varphi^1(f_1 f_2) = \max\left\{\tau_\varphi^1(f_1), \tau_\varphi^1(f_2)\right\}$.

2.3 Proof of Corollary 2.1

It follows from Proposition 1.8 that

$$\rho_\varphi^j(f_1+f_2) \leq \rho_\varphi^j(f_1) = \rho_\varphi^j(f_2) \quad \text{and} \quad \rho_\varphi^j(f_1 f_2) \leq \rho_\varphi^j(f_1) = \rho_\varphi^j(f_2).$$

By using similar reasoning as in the proof of Theorem 2.7, especially from (2.7) and (2.11), we obtain that

$$\rho_\varphi^j(f_1+f_2) \geq \rho_\varphi^j(f_2) = \rho_\varphi^j(f_1) \quad \text{and} \quad \rho_\varphi^j(f_1 f_2) \geq \rho_\varphi^j(f_2) = \rho_\varphi^j(f_1).$$

Therefore, (2.4) holds. On the other hand, we can see that (2.5) follows immediately from (2.4) and Theorem 2.7.

2.4 Proof of Theorem 2.8

We will prove the theorem for $j=0$, the proofs for $j=1$ are analogous.

(i) The definition of the $\widetilde{\tau}^0_\varphi$-type implies that for any given $\varepsilon>0$ there exists a sequence $\{r_n, n\geq 1\}$ tending to $+\infty$ such that

$$\mathrm{M}(r_n,f_2)\geq \varphi^{-1}\left(\log\left(\widetilde{\tau}^0_\varphi(f_2)-\varepsilon\right)r_n^{\widetilde{\rho}^0_\varphi(f_2)}\right)$$

and

$$\mathrm{M}(r,f_1)\leq \varphi^{-1}\left(\log\left(\widetilde{\tau}^0_\varphi(f_1)+\varepsilon\right)r^{\widetilde{\rho}^0_\varphi(f_1)}\right)\quad \text{as}\quad r\longrightarrow+\infty. \tag{2.17}$$

In each circle $|z|=r_n$ we choose a sequence $\{z_n, n\geq 1\}$ satisfying $|f_2(z_n)|=\mathrm{M}(r_n,f_2)$. Then, by Proposition 1.5 we obtain

$$\begin{aligned}
\mathrm{M}(r_n,f_1+f_2) &\geq |f_1(z_n)+f_2(z_n)|\geq |f_2(z_n)|-|f_1(z_n)|\\
&\geq \mathrm{M}(r_n,f_2)-\mathrm{M}(r_n,f_1)\\
&\geq \varphi^{-1}\left(\log\left(\widetilde{\tau}^0_\varphi(f_2)-\varepsilon\right)r_n^{\widetilde{\rho}^0_\varphi(f_2)}\right)-\varphi^{-1}\left(\log\left(\widetilde{\tau}^0_\varphi(f_1)+\varepsilon\right)r_n^{\widetilde{\rho}^0_\varphi(f_1)}\right)\\
&\geq \varphi^{-1}\left(\log\left(\widetilde{\tau}^0_\varphi(f_2)-2\varepsilon\right)r_n^{\widetilde{\rho}^0_\varphi(f_2)}\right).
\end{aligned}\tag{2.18}$$

It follows from Theorem 2.6 and Proposition 1.6 that $\widetilde{\rho}^0_\varphi(f_1+f_2)=\widetilde{\rho}^0_\varphi(f_2)$. By the monotonicity of φ and (2.18), we get

$$\frac{e^{\varphi(\mathrm{M}(r_n,f_1+f_2))}}{r_n^{\widetilde{\rho}^0_\varphi(f_1+f_2)}}\geq \widetilde{\tau}^0_\varphi(f_2)-2\varepsilon.$$

Since $\varepsilon>0$ is an arbitrary number, then

$$\widetilde{\tau}^0_\varphi(f_1+f_2)\geq\widetilde{\tau}^0_\varphi(f_2) \tag{2.19}$$

Observe that $\widetilde{\rho}^0_\varphi(f_1+f_2)=\widetilde{\rho}^0_\varphi(f_2)>\widetilde{\rho}^0_\varphi(f_1)=\widetilde{\rho}^0_\varphi(-f_1)$. Then, by applying (2.19) we obtain

$$\widetilde{\tau}^0_\varphi(f_2)=\widetilde{\tau}^0_\varphi(f_1+f_2-f_1)\leq\widetilde{\tau}^0_\varphi(f_1+f_2)$$

and therefore $\widetilde{\tau}^0_\varphi(f_1+f_2)=\widetilde{\tau}^0_\varphi(f_2)$. Without loss of generality, we may suppose that $\widetilde{\tau}^1_\varphi(f_1)<\widetilde{\tau}^1_\varphi(f_2)$. We have

$$\begin{aligned}
\mathrm{M}(r,f_1f_2) &\leq \mathrm{M}(r,f_1)\,\mathrm{M}(r,f_2)\\
&\leq \varphi^{-1}\left(\log\left(\widetilde{\tau}^0_\varphi(f_1)+\varepsilon\right)r^{\widetilde{\rho}^0_\varphi(f_1)}\right)\varphi^{-1}\left(\log\left(\widetilde{\tau}^0_\varphi(f_2)+\varepsilon\right)r^{\widetilde{\rho}^0_\varphi(f_2)}\right)\\
&\leq \left[\varphi^{-1}\left(\log\left(\widetilde{\tau}^0_\varphi(f_2)+\varepsilon\right)r^{\widetilde{\rho}^0_\varphi(f_2)}\right)\right]^2.
\end{aligned}$$

Then, by the monotonicity of φ and (1.3) we obtain that

$$\varphi\left(\mathrm{M}(r,f_1f_2)\right)\leq(1+o(1))\log\left(\widetilde{\tau}^0_\varphi(f_2)+\varepsilon\right)r^{\widetilde{\rho}^0_\varphi(f_2)}\leq\log\left(\widetilde{\tau}^0_\varphi(f_2)+2\varepsilon\right)r^{\widetilde{\rho}^0_\varphi(f_2)}.$$

By Theorem 2.6 and Proposition 1.6, it follows that $\widetilde{\rho}^0_\varphi(f_2)=\widetilde{\rho}^0_\varphi(f_1f_2)$. Therefore, $\widetilde{\tau}^0_\varphi(f_1f_2)\leq\widetilde{\tau}^0_\varphi(f_2)$.

(ii) Since $0 < \widetilde{\rho}^0_\varphi(f_1) = \widetilde{\rho}^0_\varphi(f_2) = \widetilde{\rho}^0_\varphi(f_1+f_2) < +\infty$, we get from (2.17) and Proposition 1.5 that

$$\begin{aligned} \mathrm{M}(r, f_1+f_2) &\leq \mathrm{M}(r,f_1)+\mathrm{M}(r,f_2) \\ &\leq \varphi^{-1}\left(\log\left(\widetilde{\tau}^0_\varphi(f_1)+\varepsilon\right) r^{\widetilde{\rho}^0_\varphi(f_1+f_2)}\right) + \varphi^{-1}\left(\log\left(\widetilde{\tau}^0_\varphi(f_2)+\varepsilon\right) r^{\widetilde{\rho}^0_\varphi(f_1+f_2)}\right) \\ &\leq \varphi^{-1}\left(\log\left(\max\left\{\widetilde{\tau}^0_\varphi(f_1), \widetilde{\tau}^0_\varphi(f_2)\right\}+2\varepsilon\right) r^{\widetilde{\rho}^0_\varphi(f_1+f_2)}\right). \end{aligned}$$

Hence

$$\frac{e^{\varphi(\mathrm{M}(r,f_1+f_2))}}{r^{\widetilde{\rho}^0_\varphi(f_1+f_2)}} \leq \max\left\{\widetilde{\tau}^0_\varphi(f_1), \widetilde{\tau}^0_\varphi(f_2)\right\}+2\varepsilon.$$

By arbitrariness of $\varepsilon > 0$, we obtain

$$\widetilde{\tau}^0_\varphi(f_1+f_2) \leq \max\left\{\widetilde{\tau}^0_\varphi(f_1), \widetilde{\tau}^0_\varphi(f_2)\right\}. \tag{2.20}$$

Without loss of generality, we may suppose $\widetilde{\tau}^0_\varphi(f_1) < \widetilde{\tau}^0_\varphi(f_2)$. Then, by (2.20) and the fact that $\widetilde{\rho}^0_\varphi(f_1+f_2) = \widetilde{\rho}^0_\varphi(f_1) = \widetilde{\rho}^0_\varphi(-f_1)$, we obtain

$$\widetilde{\tau}^0_\varphi(f_2) = \widetilde{\tau}^0_\varphi(f_1+f_2-f_1) \leq \max\left\{\widetilde{\tau}^0_\varphi(f_1+f_2), \widetilde{\tau}^0_\varphi(f_1)\right\} = \widetilde{\tau}^0_\varphi(f_1+f_2). \tag{2.21}$$

It follows from (2.20) and (2.21) that $\widetilde{\tau}^0_\varphi(f_1+f_2) = \max\left\{\widetilde{\tau}^0_\varphi(f_1), \widetilde{\tau}^0_\varphi(f_2)\right\}$.

(iii) Since $0 < \widetilde{\rho}^0_\varphi(f_1) = \widetilde{\rho}^0_\varphi(f_2) = \widetilde{\rho}^0_\varphi(f_1 f_2) < +\infty$, then from (2.17) we have

$$\begin{aligned} \mathrm{M}(r, f_1 f_2) &\leq \mathrm{M}(r,f_1)\mathrm{M}(r,f_2) \\ &\leq \varphi^{-1}\left(\log\left(\widetilde{\tau}^0_\varphi(f_1)+\varepsilon\right) r^{\widetilde{\rho}^0_\varphi(f_1 f_2)}\right)\varphi^{-1}\left(\log\left(\widetilde{\tau}^0_\varphi(f_2)+\varepsilon\right) r^{\widetilde{\rho}^0_\varphi(f_1 f_2)}\right) \\ &\leq \left[\varphi^{-1}\left(\log\left(\max\left\{\widetilde{\tau}^0_\varphi(f_1), \widetilde{\tau}^0_\varphi(f_2)\right\}+\varepsilon\right) r^{\widetilde{\rho}^0_\varphi(f_1 f_2)}\right)\right]^2. \end{aligned}$$

By the monotonicity of φ and (1.3), we obtain

$$\frac{e^{\varphi(\mathrm{M}(r,f_1 f_2))}}{r^{\widetilde{\rho}^0_\varphi(f_1 f_2)}} \leq \max\left\{\widetilde{\tau}^0_\varphi(f_1), \widetilde{\tau}^0_\varphi(f_2)\right\}+2\varepsilon.$$

By arbitrariness of $\varepsilon > 0$, we deduce that

$$\widetilde{\tau}^0_\varphi(f_1 f_2) \leq \max\left\{\widetilde{\tau}^0_\varphi(f_1), \widetilde{\tau}^0_\varphi(f_2)\right\}. \tag{2.22}$$

On the other hand, if we suppose that $\widetilde{\tau}^0_\varphi(f_1) < \widetilde{\tau}^0_\varphi(f_2)$, from (2.22) and the fact that $\widetilde{\rho}^0_\varphi(f_1 f_2) = \widetilde{\rho}^0_\varphi(f_1) = \widetilde{\rho}^0_\varphi\left(\frac{1}{f_1}\right)$, we obtain

$$\widetilde{\tau}^0_\varphi(f_2) = \widetilde{\tau}^0_\varphi\left(f_1 f_2 \frac{1}{f_1}\right) \leq \max\left\{\widetilde{\tau}^0_\varphi(f_1 f_2), \widetilde{\tau}^0_\varphi(f_1)\right\} = \widetilde{\tau}^0_\varphi(f_1 f_2). \tag{2.23}$$

From (2.22) and (2.23), we deduce that $\widetilde{\tau}^0_\varphi(f_1 f_2) = \max\left\{\widetilde{\tau}^0_\varphi(f_1), \widetilde{\tau}^0_\varphi(f_2)\right\}$.

2.5 Proof of Corollary 2.2

By Proposition 1.8, we have $\widetilde{\rho}^j_\varphi(f_1+f_2) \leq \widetilde{\rho}^j_\varphi(f_1) = \widetilde{\rho}^j_\varphi(f_2)$. We assume that $\widetilde{\rho}^j_\varphi(f_1+f_2) < \widetilde{\rho}^j_\varphi(f_2)$. Then, it follows from Theorem 2.8 that

$$\widetilde{\tau}^j_\varphi(f_1) = \widetilde{\tau}^j_\varphi(f_1+f_2-f_2) = \widetilde{\tau}^j_\varphi(f_2)$$

which is a contradiction. Hence, $\widetilde{\rho}^j_\varphi(f_1+f_2) = \widetilde{\rho}^j_\varphi(f_1) = \widetilde{\rho}^j_\varphi(f_2)$. This and the second part of Theorem 2.8 yield $\widetilde{\tau}^j_\varphi(f_1+f_2) = \widetilde{\tau}^j_\varphi(f_2)$.

2.6 Proof of Theorem 2.9

The inequality $\rho_{\varphi}^{1}(f') \leq \rho_{\varphi}^{1}(f)$ was proved in [20, Proposition 3.4]. We need to prove the converse inequality. Indeed, from the definition of $\rho_{\varphi}^{1}(f') = \rho$, we have for any given $\varepsilon > 0$ and sufficiently large r that

$$T(r, f') \leq \varphi^{-1}\left(\log r^{\rho+\varepsilon}\right).$$

By (1.1) and Theorem 1.7, we obtain for sufficiently large r

$$T(r, f) < O\left(\varphi^{-1}\left(\log(2r)^{\rho+\varepsilon}\right) + \log r\right) = O\left(\varphi^{-1}\left(\log(2r)^{\rho+2\varepsilon}\right)\right).$$

In view of (1.3) and the monotonicity of φ, we get

$$\varphi\left(T(r, f)\right) \leq (1 + o(1))(\rho + 2\varepsilon)\log 2r \leq (\rho + 3\varepsilon)\log 2r.$$

Since $\varepsilon > 0$ is an arbitrary number, we obtain $\rho_{\varphi}^{1}(f) \leq \rho$ and therefore $\rho_{\varphi}^{1}(f') = \rho_{\varphi}^{1}(f) = \rho$. Similarly, it follows from the definition of $\tau_{\varphi}^{1}(f') = \tau$ that for any given $\varepsilon > 0$ and for all r sufficiently large, we have

$$T(r, f') \leq \varphi^{-1}\left(\log\left(\tau + \varepsilon\right) r^{\rho}\right).$$

By (1.1) and Theorem 1.7, we obtain for sufficiently large r

$$T(r, f) < O\left(\varphi^{-1}\left(\log\left(\tau + \varepsilon\right)(2r)^{\rho}\right) + \log r\right) = O\left(\varphi^{-1}\left(\log\left(\tau + 2\varepsilon\right)(2r)^{\rho}\right)\right).$$

In view of (1.3) and the monotonicity of φ, we get

$$\varphi\left(T(r, f)\right) \leq (1 + o(1))\left(\log\left(\tau + 2\varepsilon\right)(2r)^{\rho}\right) \leq \log\left(\tau + 3\varepsilon\right)(2r)^{\rho}.$$

Then

$$\frac{e^{\varphi\left(T(r,f)\right)}}{r^{\rho}} \leq (\tau + 3\varepsilon)\, 2^{\rho}.$$

By arbitrariness of ε, it is easy to get

$$\tau_{\varphi}^{1}(f) \leq 2^{\rho}\tau \tag{2.24}$$

On the other hand, it follows from Lemma 1.1 that

$$T(r, f') \leq 2T(r, f) + O\left(\log T(r, f) + \log r\right), \quad r \notin E$$

where $E \subset [0, +\infty)$ is a set of finite linear measure. Then, by (2.6) and in view of (1.1) we get

$$T(r, f') \leq O\left(\varphi^{-1}\left(\log\left(\tau_{\varphi}^{1}(f) + 2\varepsilon\right) r^{\rho}\right)\right), \quad r \notin E.$$

By Theorem 1.11 and (1.3), for any given $\alpha > 1$ and sufficiently large r we have

$$\varphi\left(T(r, f')\right) \leq (1 + o(1))\log\left(\tau_{\varphi}^{1}(f) + 2\varepsilon\right)(\alpha r)^{\rho} \leq \log\left(\tau_{\varphi}^{1}(f) + 3\varepsilon\right)(\alpha r)^{\rho}.$$

Then

$$\frac{e^{\varphi\left(T(r,f')\right)}}{r^{\rho}} \leq \left(\tau_{\varphi}^{1}(f) + 3\varepsilon\right)\alpha^{\rho}$$

By arbitrariness of ε, we obtain

$$\tau_{\varphi}^{1}(f') \leq \alpha^{\rho}\tau_{\varphi}^{1}(f), \quad \alpha > 1. \tag{2.25}$$

Finally, we deduce from (2.24) and (2.25) that

$$\frac{\tau_{\varphi}^{1}(f)}{2^{\rho}} \leq \tau \leq \alpha^{\rho}\tau_{\varphi}^{1}(f), \quad \alpha > 1,$$

which completes the proof of Theorem 2.9.

Chapter 3

Fast Growth and oscillation of solutions of LDE with meromorphic coefficients, Part 1

In what follows, the set E will denote a set of r of finite linear measure on $[0,+\infty)$ whereas a set of r of finite and of infinite logarithmic measure on $[1,+\infty)$ will be respectively denoted by E_0 and E_1. All these sets are not necessarily the same at each occurrence.

1 Introduction and statement of the main results

The aim of this chapter is to investigate the following linear differential equations $(k \geq 2)$

$$f^{(k)} + \mathrm{A}_{k-1}(z) f^{(k-1)} + \cdots + \mathrm{A}_0(z) f = 0, \tag{3.1}$$

$$f^{(k)} + \mathrm{A}_{k-1}(z) f^{(k-1)} + \cdots + \mathrm{A}_0(z) f = \mathrm{F}(z), \tag{3.2}$$

where the coefficients $\mathrm{A}_0 \not\equiv 0, \ldots, \mathrm{A}_{k-1}$ and $\mathrm{F} \not\equiv 0$ are meromorphic in the whole complex plane. Departing from some existing results using the concepts of iterated p-order and iterated $[p,q]$-order, we give various generalizations and improvements by considering the concept of φ-order. In [7], Belaïdi considered the growth of solutions of equation (3.1) when the coefficients are entire functions of finite iterated p-order and some coefficients have the same order.

Theorem 3.1 *[7] Let $\mathrm{A}_0, \ldots, \mathrm{A}_{k-1} \in \mathscr{E}$, and let $i(\mathrm{A}_0) = p$ $(0 < p < \infty)$. If the coefficients $\mathrm{A}_j(z)(j = 0, \ldots, k-1)$ satisfy*

$$\max\{\widetilde{\rho}_p(\mathrm{A}_j), j = 1,2,\ldots,k-1\} \leq \widetilde{\rho}_p(\mathrm{A}_0) = \rho (0 < \rho < +\infty)$$

and

$$\max\{\widetilde{\tau}_p(\mathrm{A}_j) : \widetilde{\rho}_p(\mathrm{A}_j) = \widetilde{\rho}_p(\mathrm{A}_0) : j = 1,2,\ldots,k-1\} < \widetilde{\tau}_p(\mathrm{A}_0) = \tau\ (0 < \tau < +\infty),$$

then, all solutions $f \not\equiv 0$ of equation (3.1) satisfy $i(f) = p+1$ and $\widetilde{\rho}_{p+1}(f) = \widetilde{\rho}_p(\mathrm{A}_0) = \rho$.

After that, Cao et al. [12] investigated the growth of meromorphic solutions of equations (3.1) and (3.2) when the coefficients are meromorphic functions of finite iterated p-order on the complex plane which improved many previous results in [7, 44, 64].

Theorem 3.2 *[12] Let $\mathrm{A}_0, \mathrm{A}_1, \ldots, \mathrm{A}_{k-1} \in \mathscr{M}$ such that $i(\mathrm{A}_0) = p$ $(0 < p < +\infty)$ and $\lambda_p\left(\frac{1}{\mathrm{A}_0}\right) < \rho_p(\mathrm{A}_0)$. If either*

$$\max\{i(\mathrm{A}_j) : j = 1,\ldots,k-1\} < p$$

or

$$\max\{\rho_p(A_j) : j = 1,\dots,k-1\} = \rho_p(A_0) = \rho(0 < \rho < +\infty),$$
$$\max\{\tau_p(A_j) : \rho_p(A_j) = \rho_p(A_0) > 0, j = 1,\dots,k-1\} < \tau_p(A_0) = \tau(0 < \tau < +\infty),$$

then, every meromorphic solution $f \not\equiv 0$ whose poles are of uniformly bounded multiplicities of (3.1) *satisfies $i(f) = p+1$ and $\rho_{p+1}(f) = \rho_p(A_0)$.*

The first work that considered the concept of φ-order to study the growth of entire solutions of equation (3.1) was made by Chyzhykov and Semochko [20]. They gave the precise estimate of $\widetilde{\rho}^1_\varphi(f)$ when A_0 strictly dominates the growth of other coefficients.

Theorem 3.3 *[20] Let $\varphi \in \Phi$ and let $A_0, A_1, \dots, A_{k-1} \in \mathscr{E}$ satisfying*

$$\max\left\{\widetilde{\rho}^0_\varphi(A_j) : j = 1,\dots,k-1\right\} < \widetilde{\rho}^0_\varphi(A_0).$$

Then, any non-trivial solution f of (3.1) *satisfies $\widetilde{\rho}^1_\varphi(f) = \widetilde{\rho}^0_\varphi(A_0)$.*

When there are more then one dominant coefficient, Belaïdi [5, 8] introduced the concept of φ-type and obtained the following results.

Theorem 3.4 *[5] Let $A_0,\dots,A_{k-1}(z)$ be entire functions, and let $\varphi \in \Phi$. If the coefficients $A_j(z)(j = 0,\dots,k-1)$ satisfy*

$$\max\left\{\widetilde{\rho}^0_\varphi\left(A_j\right) : j = 1,\dots,k-1\right\} \le \widetilde{\rho}^0_\varphi\left(A_0\right) = \rho, (0 < \rho < +\infty)$$

and

$$\max\left\{\widetilde{\tau}^0_\varphi\left(A_j\right) : \widetilde{\rho}^0_\varphi\left(A_j\right) = \widetilde{\rho}^0_\varphi\left(A_0\right), j = 1,\dots,k-1\right\} < \widetilde{\tau}^0_\varphi\left(A_0\right) = \tau, (0 < \tau < +\infty),$$

then, all solutions $f \not\equiv 0$ of (3.1) *satisfy $\widetilde{\rho}^1_\varphi\left(f\right) = \widetilde{\rho}^0_\varphi\left(A_0\right)$.*

Theorem 3.5 *[8] Let G be a set of complex numbers z such that*

$$\overline{dens}_l\{|z| : z \in G\} > 0.$$

Let $\varphi \in \Phi$ and let $A_0, A_1, \dots, A_{k-1} \in \mathscr{E}$ satisfying

$$\max\left\{\widetilde{\rho}^0_\varphi(A_j) : j = 0,1,\dots,k-1\right\} \le \alpha, (0 < \alpha < +\infty).$$

Assume that there exists a real number β satisfies $0 < \beta < \alpha$ such that for any given ε $(0 < 2\varepsilon < \alpha - \beta)$, we have

$$T(r, A_j) \le \log\varphi^{-1}(\beta \log r), \quad j = 1,\dots,k-1$$

and

$$T(r, A_0) \ge \log\varphi^{-1}\left((\alpha - \varepsilon)\log r\right)$$

as $|z| \longrightarrow +\infty$ for $z \in G$. Then, every non-zero solution f of equation (3.1) *satisfies $\widetilde{\rho}^1_\varphi(f) = \alpha$.*

Now, we list our main results.

Theorem 3.6 *[39] Let $\varphi \in \Phi$ and let $A_0, A_1, \ldots, A_{k-1} \in \mathscr{M}$. If there exists one coefficient A_s ($s \in \{0,1,\ldots,k-1\}$) such that*

$$\max\left\{\lambda_\varphi^0\left(\frac{1}{A_s}\right), \rho_\varphi^0(A_j) : j = 0,1,\ldots,k-1\,(j \neq s)\right\} < \rho_\varphi^0(A_s) < +\infty,$$

then, every transcendental meromorphic solution f ($f \in \mathscr{M}$) whose poles are of uniformly bounded multiplicities of (3.1) satisfies

$$\rho_\varphi^1(f) \leq \rho_\varphi^0(A_s) \leq \rho_\varphi^0(f).$$

Moreover, if all solutions of (3.1) are meromorphic functions, then there is at least one meromorphic solution, say f_1, satisfies $\rho_\varphi^1(f_1) = \rho_\varphi^0(A_s)$.

Theorem 3.7 *[39] Let $\varphi \in \Phi$ and let $A_0, A_1, \ldots, A_{k-1} \in \mathscr{M}$ satisfying*

$$\max\left\{\lambda_\varphi^0\left(\frac{1}{A_0}\right), \rho_\varphi^0(A_j) : j = 1,\ldots,k-1\right\} < \rho_\varphi^0(A_0) < +\infty.$$

Then, every non-trivial meromorphic solution f ($f \in \mathscr{M}$) whose poles are of uniformly bounded multiplicities of (3.1) satisfies $\rho_\varphi^1(f) = \rho_\varphi^0(A_0)$.

For the case when there exist some other coefficients $A_j(z)$ ($j = 1,\ldots,k-1$) having the same φ-order as the coefficient $A_0(z)$, then we have the following result.

Theorem 3.8 *[39] Let $\varphi \in \Phi$ and let $A_0, A_1, \ldots, A_{k-1} \in \mathscr{M}$ satisfying*

$$\lambda_\varphi^0\left(\frac{1}{A_0}\right) < \rho_\varphi^0(A_0),$$

$$\max\left\{\rho_\varphi^0(A_j) : j = 1,\ldots,k-1\right\} \leq \rho_\varphi^0(A_0) = \rho_0 < +\infty, \tag{3.3}$$

$$\max\left\{\tau_\varphi^0(A_j) : \rho_\varphi^0(A_j) = \rho_\varphi^0(A_0) > 0,\ j = 1,\ldots,k-1\right\} < \tau_\varphi^0(A_0) = \tau_0, (0 < \tau_0 < +\infty). \tag{3.4}$$

Then, every meromorphic solution $f \not\equiv 0$ whose poles are of uniformly bounded multiplicities of (3.1) satisfies $\rho_\varphi^1(f) = \rho_\varphi^0(A_0)$.

Theorem 3.9 *[39] Let $\varphi \in \Phi$ and let $F, A_0, A_1, \ldots, A_{k-1} \in \mathscr{M}$ satisfying $< \rho_\varphi^0(A_0)$ and*

$$\max\left\{\lambda_\varphi^0\left(\frac{1}{A_0}\right), \rho_\varphi^1(F), \rho_\varphi^0(A_j) : j = 1,\ldots,k-1\right\} < \rho_\varphi^0(A_0) < +\infty. \tag{3.5}$$

Then, any non-zero meromorphic solution f ($f \in \mathscr{M}$) whose poles are of uniformly bounded multiplicities of (3.2) satisfies

$$\overline{\lambda}_\varphi^1(f) = \lambda_\varphi^1(f) = \rho_\varphi^1(f) = \rho_\varphi^0(A_0)$$

with at most one exceptional solution f_0 satisfying $\rho_\varphi^1(f_0) < \rho_\varphi^0(A_0)$.

Theorem 3.10 *[39] Let $\varphi \in \Phi$ and $F, A_0, A_1, \ldots, A_{k-1} \in \mathscr{M}$ satisfying*

$$\max\left\{\rho_\varphi^0(A_j) : j = 0,\ldots,k-1\right\} < \rho_\varphi^1(F).$$

If every solution f of (3.2) is meromorphic function whose poles are of uniformly bounded multiplicities, then $\rho_\varphi^1(f) = \rho_\varphi^1(F)$ holds for all solutions of (3.2).

Theorem 3.11 *[39] Let* $G \subset (1,+\infty)$ *be a set of complex numbers z such that*

$$\overline{dens}_l\{|z| : z \in G\} > 0.$$

Let $\varphi \in \Phi$ *and let* $A_0, A_1, \ldots, A_{k-1} \in \mathscr{M}$ *satisfying* $\delta(\infty, A_0) = \delta > 0$ *and*

$$\max\left\{\rho^0_\varphi(A_j) : j = 0,1,\ldots,k-1\right\} \leq \alpha, \ (0 < \alpha < +\infty).$$

Suppose that there exists a real number β *satisfies* $0 < \beta < \alpha$ *such that for any given* ε $(0 < 2\varepsilon < \alpha - \beta)$, *we have*

$$T(r, A_j) \leq \log\varphi^{-1}(\beta \log r), \quad j = 1,\ldots,k-1 \tag{3.6}$$

and

$$T(r, A_0) \geq \log\varphi^{-1}\left((\alpha - \varepsilon)\log r\right) \tag{3.7}$$

as $|z| = r \longrightarrow +\infty$ *for* $z \in G$. *Then, every non-zero meromorphic solution* f $(f \in \mathscr{M})$ *of equation* (3.1) *satisfies* $\rho^1_\varphi(f) = \alpha$.

Theorem 3.12 *[39] Let* $G \subset (1,+\infty)$ *be a set of complex numbers z such that*

$$\overline{dens}_l\{|z| : z \in G\} > 0.$$

Let $\varphi \in \Phi$ *and let* $F, A_0, A_1, \ldots, A_{k-1} \in \mathscr{M}$ *satisfying*

$$\max\left\{\rho^0_\varphi(A_j) : j = 0,1,\ldots,k-1\right\} < \alpha, \ (0 < \alpha < +\infty).$$

Suppose that, there exists a real number β *satisfies* $0 < \beta < \alpha$ *such that for any given* ε $(0 < 2\varepsilon < \alpha - \beta)$, *we have*

$$|A_0(z)| \geq \varphi^{-1}\left((\alpha - \varepsilon)\log r\right) \tag{3.8}$$

and

$$|A_j(z)| \leq \varphi^{-1}(\beta \log r), \quad j = 1,\ldots,k-1 \tag{3.9}$$

as $|z| = r \longrightarrow +\infty$ *for* $z \in G$. *Then, the following statements hold*

(i) If $\rho^1_\varphi(F) \geq \alpha$, *then every meromorphic solution* $f \not\equiv 0$ *whose poles are of uniformly bounded multiplicities of equation* (3.2) *satisfies* $\rho^1_\varphi(f) = \rho^1_\varphi(F)$.

(ii) If $\rho^1_\varphi(F) < \alpha$, *then every meromorphic solution* $f \not\equiv 0$ *whose poles are of uniformly bounded multiplicities of* (3.2) *satisfies*

$$\overline{\lambda}^1_\varphi(f) = \lambda^1_\varphi(f) = \rho^1_\varphi(f) = \alpha$$

with at most one exceptional solution f_0 *satisfying* $\rho^1_\varphi(f_0) < \alpha$.

Remark 3.1 *By setting* $\varphi(r) = \log_{p+1} r$ *in Theorem 3.8, we obtain Theorem 3.2. Namely, Theorem 3.8 and Theorem 3.11 extend respectively Theorem 3.4 and Theorem 3.5 from entire solutions of equation* (3.1) *to meromorphic solutions in the complex plane.*

2 Preliminary lemmas

Lemma 3.1 *[18] Let $f \in \mathscr{M}$ be a solution of equation (3.1) and let $\mathrm{T}(r) = \sum_{j=0}^{k-1} \mathrm{T}(r, \mathrm{A}_j)$ such that not all coefficients A_j are constants. For any given $\gamma > 1$, we have*

$$\log m(r,f) < \mathrm{T}(r)\left[(\log r)\log \mathrm{T}(r)\right]^{\gamma} \quad \text{for} \quad s = 0,$$
$$\log m(r,f) < r^{2s+\gamma-1}\mathrm{T}(r)\left[\log \mathrm{T}(r)\right]^{\gamma} \quad \text{for} \quad s > 0,$$

hold outside of an exceptional set E_s with $\int_{\mathrm{E}_s} t^{s-1}\,dt < +\infty$.

Lemma 3.2 *[22] Let $\mathrm{A}_0, \mathrm{A}_1, \ldots, \mathrm{A}_{k-1} \in \mathscr{M}$ and let $f_1, f_2, \ldots, f_k \in \mathscr{M}$ be linearly independent solutions of equation (3.1). Then, we have*

$$m(r, \mathrm{A}_j) = \mathrm{O}\left(\log \max_{1\le i\le k} \mathrm{T}(r, f_i)\right), \quad j = 0, 1, \ldots, k-1.$$

Lemma 3.3 *[25] Let $\alpha > 1$ be a real constant and let $f \in \mathscr{M}$. Then there exists a set $\mathrm{E}_0 \subset [1, +\infty)$ with finite logarithmic measure such that for all z satisfying $|z| = r \notin [0,1] \bigcup \mathrm{E}_0$, we have*

$$\left|\frac{f^{(j)}(z)}{f^{(i)}(z)}\right| \le \mathrm{B}_\alpha \left[\frac{\mathrm{T}(\alpha r, f)}{r}\left(\log^{\alpha} r\right)\log \mathrm{T}(\alpha r, f)\right]^{j-i}$$

where $\mathrm{B}_\alpha > 0$ is a constant that depends only on α and i, j $(j > i \ge 0)$.

Lemma 3.4 *[39] Let $\varphi \in \Phi$ and $f \in \mathscr{M}$. Then, for all sets $\mathrm{E} \subset [0, +\infty)$ of finite linear measure, there exists a sequence $\{r_n, r_n \notin \mathrm{E}\}$ satisfying*

$$\lim_{r_n\to+\infty} \frac{\varphi\left(\mathrm{T}(r_n, f)\right)}{\log r_n} = \rho^1_\varphi(f) \quad \left(\textit{respectively} \lim_{r_n\to+\infty} \frac{\varphi\left(e^{\mathrm{T}(r_n,f)}\right)}{\log r_n} = \rho^0_\varphi(f)\right).$$

Proof. The definition of $\rho^1_\varphi(f)$ implies that there exists a sequence $\{s_n, n \ge 1\}, s_n \longrightarrow +\infty$ such that

$$\lim_{s_n\to+\infty} \frac{\varphi\left(\mathrm{T}(s_n, f)\right)}{\log s_n} = \rho^1_\varphi(f).$$

Setting $mes(\mathrm{E}) = \delta < +\infty$. Then, for $r_n \in [s_n, s_n + \delta + 1] \setminus \mathrm{E}$, we have

$$\frac{\varphi\left(\mathrm{T}(r_n, f)\right)}{\log r_n} \ge \frac{\varphi\left(\mathrm{T}(s_n, f)\right)}{\log(s_n + \delta + 1)} = \frac{\varphi\left(\mathrm{T}(s_n, f)\right)}{\log s_n + \log\left(1 + \frac{\delta+1}{s_n}\right)}.$$

Thus

$$\lim_{r_n\to+\infty} \frac{\varphi\left(\mathrm{T}(r_n, f)\right)}{\log r_n} \ge \lim_{s_n\to+\infty} \frac{\varphi\left(\mathrm{T}(s_n, f)\right)}{\log s_n + \log\left(1 + \frac{\delta+1}{s_n}\right)} = \rho^1_\varphi(f).$$

On the other hand

$$\lim_{r_n\to+\infty} \frac{\varphi\left(\mathrm{T}(r_n, f)\right)}{\log r_n} \le \limsup_{r\to+\infty} \frac{\varphi\left(\mathrm{T}(r, f)\right)}{\log r} = \rho^1_\varphi(f).$$

Therefore

$$\lim_{r_n\to+\infty} \frac{\varphi\left(\mathrm{T}(r_n, f)\right)}{\log r_n} = \rho^1_\varphi(f).$$

Similar proof for $\rho^0_\varphi(f)$. □

Lemma 3.5 *[39] Let $\varphi \in \Phi$ and let $f \in \mathcal{M}$ satisfying $0 < \rho_\varphi^0(f) < +\infty$ and $0 < \tau_\varphi^0(f) < +\infty$. Then, for any given $\eta < \tau_\varphi^0(f)$, there exists a set $E_1 \subset [0, +\infty)$ of infinite logarithmic measure such that for all $r \in E_1$, we have*

$$\varphi\left(e^{T(r,f)}\right) > \log\left(\eta\, r^{\rho_\varphi^0(f)}\right).$$

Proof. We denote $\rho_\varphi^0(f) = \rho_0$ and $\tau_\varphi^0(f) = \tau_0$. The definition of $\tau_\varphi^0(f)$ implies that there exists a sequence $\{r_m, m \geq 1\}$ tending to $+\infty$ satisfying

$$\left(1 + \frac{1}{m}\right) r_m < r_{m+1} \quad \text{and} \quad \lim_{m \to +\infty} \frac{e^{\varphi\left(e^{T(r_m,f)}\right)}}{r_m^{\rho_0}} = \tau_0.$$

Then, for any given ε $(0 < \varepsilon < \tau_0 - \eta)$, there exists an integer number m_1 such that for all $m \geq m_1$, we have

$$e^{\varphi\left(e^{T(r_m,f)}\right)} > (\tau_0 - \varepsilon) r_m^{\rho_0}. \tag{3.10}$$

Since $\eta < \tau_0 - \varepsilon$, then there exists an integer number m_2 such that for all $m \geq m_2$, we have

$$\left(\frac{m}{m+1}\right)^{\rho_0} > \frac{\eta}{\tau_0 - \varepsilon}. \tag{3.11}$$

Choose $m \geq m_3 = \max\{m_1, m_2\}$, then from (3.10) and (3.11), for any $r \in \left[r_m, \left(1 + \frac{1}{m}\right) r_m\right]$ we get

$$e^{\varphi\left(e^{T(r,f)}\right)} \geq e^{\varphi\left(e^{T(r_m,f)}\right)} > (\tau_0 - \varepsilon) r_m^{\rho_0} \geq (\tau_0 - \varepsilon) \left(\frac{mr}{m+1}\right)^{\rho_0} > \eta\, r^{\rho_0}.$$

Hence

$$\varphi\left(e^{T(r,f)}\right) > \log\left(\eta\, r^{\rho_\varphi^0(f)}\right).$$

Set $E_1 = \bigcup_{m=m_3}^{+\infty} \left[r_m, \left(1 + \frac{1}{m}\right) r_m\right]$, then the logarithmic measure $mes_l(E_1)$ of E_1 satisfies

$$mes_l(E_1) = \sum_{m=m_3}^{+\infty} \int_{r_m}^{\left(1+\frac{1}{m}\right) r_m} \frac{dt}{t} = \sum_{m=m_3}^{+\infty} \log\left(1 + \frac{1}{m}\right) = +\infty.$$

□

Lemma 3.6 *[39] Let $F, A_0, A_1, \ldots, A_{k-1} \in \mathcal{M}$ and let $f \in \mathcal{M}$ be a solution of (3.2). If*

$$\max\left\{\rho_\varphi^1(F), \rho_\varphi^1(A_j) : j = 0, 1, \ldots, k-1\right\} < \rho_\varphi^1(f),$$

then

$$\overline{\lambda}_\varphi^1(f) = \lambda_\varphi^1(f) = \rho_\varphi^1(f).$$

Proof. From equation (3.2), one can see that if z_0 is a zero of f of order $l > k$ and if $A_0, A_1, \ldots, A_{k-1}$ are all analytic at z_0, then z_0 is a zero of F of order at least $l - k$. Consequently, we have

$$n\left(r, \frac{1}{f}\right) \leq k\, \overline{n}\left(r, \frac{1}{f}\right) + n\left(r, \frac{1}{F}\right) + \sum_{j=0}^{k-1} n(r, A_j)$$

and

$$\mathrm{N}\left(r,\frac{1}{f}\right)\le k\overline{\mathrm{N}}\left(r,\frac{1}{f}\right)+\mathrm{N}\left(r,\frac{1}{\mathrm{F}}\right)+\sum_{j=0}^{k-1}\mathrm{N}(r,\mathrm{A}_j). \tag{3.12}$$

We can write equation (3.2) as

$$\frac{1}{f}=\frac{1}{\mathrm{F}}\left(\frac{f^{(k)}}{f}+\mathrm{A}_{k-1}\frac{f^{(k-1)}}{f}+\cdots+\mathrm{A}_1\frac{f'}{f}+\mathrm{A}_0\right). \tag{3.13}$$

By Lemma 1.1 and (3.13), it follows that

$$m\left(r,\frac{1}{f}\right)\le m\left(r,\frac{1}{\mathrm{F}}\right)+\sum_{j=0}^{k-1}m(r,\mathrm{A}_j)+\mathrm{O}\left(\log r+\log\mathrm{T}(r,f)\right) \tag{3.14}$$

holds for all $|z|=r\notin\mathrm{E}$, where E is a set of finite linear measure. By (3.12), (3.14) and Theorem 1.2, we get

$$\begin{aligned}\mathrm{T}(r,f) &= \mathrm{T}\left(r,\frac{1}{f}\right)+\mathrm{O}(1)=m\left(r,\frac{1}{f}\right)+\mathrm{N}\left(r,\frac{1}{f}\right)+\mathrm{O}(1)\\ &\le k\overline{\mathrm{N}}\left(r,\frac{1}{f}\right)+\mathrm{T}(r,\mathrm{F})+\sum_{j=0}^{k-1}\mathrm{T}(r,\mathrm{A}_j)+\mathrm{O}\left(\log r+\log\mathrm{T}(r,f)\right)\end{aligned} \tag{3.15}$$

provided $r\notin\mathrm{E}$ is sufficiently large. Let denote

$$\mu=\max\left\{\rho^1_\varphi(\mathrm{F}),\rho^1_\varphi(\mathrm{A}_j):j=0,1,\ldots,k-1\right\}.$$

According to Lemma 3.4, there exists a sequence $\{r_n,r_n\notin\mathrm{E}\}$ such that

$$\lim_{r_n\to+\infty}\frac{\varphi\left(\mathrm{T}(r_n,f)\right)}{\log r_n}=\rho^1_\varphi(f)=\rho_1.$$

So, if $r_n\notin\mathrm{E}$, then for any given ε $(0<2\varepsilon<\rho_1-\mu)$ there holds

$$\mathrm{T}(r_n,f)\ge\varphi^{-1}\left((\rho_1-\varepsilon)\log r_n\right). \tag{3.16}$$

We have

$$\max\left\{\mathrm{T}(r_n,\mathrm{F}),\mathrm{T}(r_n,\mathrm{A}_j):j=0,1,\ldots,k-1\right\}\le\varphi^{-1}\left((\mu+\varepsilon)\log r_n\right), \tag{3.17}$$

$$\mathrm{O}\left(\log r_n+\log\mathrm{T}(r_n,f)\right)=o\left(\mathrm{T}(r_n,f)\right). \tag{3.18}$$

Since ε $(0<2\varepsilon<\rho_1-\mu)$, then (3.16), (3.17) and Proposition 1.5 yield

$$\begin{aligned}\max\left\{\frac{\mathrm{T}(r_n,\mathrm{F})}{\mathrm{T}(r_n,f)},\frac{\mathrm{T}(r_n,\mathrm{A}_j)}{\mathrm{T}(r_n,f)}:j=0,1,\ldots,k-1\right\} &\le\frac{\exp\left\{\log\varphi^{-1}\left((\mu+\varepsilon)\log r_n\right)\right\}}{\exp\left\{\log\varphi^{-1}\left((\rho_1-\varepsilon)\log r_n\right)\right\}}\\ &=\exp\left\{\log\varphi^{-1}\left((\mu+\varepsilon)\log r_n\right)-\log\varphi^{-1}\left((\rho_1-\varepsilon)\log r_n\right)\right\}\\ =\exp\left\{\left(1-\frac{\log\varphi^{-1}\left((\rho_1-\varepsilon)\log r_n\right)}{\log\varphi^{-1}\left((\mu+\varepsilon)\log r_n\right)}\right)\log\varphi^{-1}\left((\mu+\varepsilon)\log r_n\right)\right\} &\longrightarrow 0\ \text{ as } r_n\longrightarrow+\infty.\end{aligned} \tag{3.19}$$

By substituting (3.18) and (3.19) into (3.15), for sufficiently large $r_n\notin\mathrm{E}$, there holds

$$(1-o(1))\mathrm{T}(r_n,f)\le k\overline{\mathrm{N}}\left(r_n,\frac{1}{f}\right).$$

From this inequality, the monotonicity of φ and (1.3), we obtain $\rho^1_\varphi(f)\le\overline{\lambda}^1_\varphi(f)$. In addition, we have by definition that $\overline{\lambda}^1_\varphi(f)\le\lambda^1_\varphi(f)\le\rho^1_\varphi(f)$. Hence $\overline{\lambda}^1_\varphi(f)=\lambda^1_\varphi(f)=\rho^1_\varphi(f)$. □

3 Proofs of main results

3.1 Proof of Theorem 3.6

(i) We first prove that $\rho^1_\varphi(f) \le \rho^0_\varphi(A_s) \le \rho^0_\varphi(f)$ holds for every transcendental meromorphic function satisfying (3.1). From equation (3.1), we observe that the poles of f can only occur at the poles of $A_0, A_1, \ldots, A_{k-1}$. Since the multiplicities of poles of f are uniformly bounded, then we have

$$\begin{aligned} N(r,f) \le C\overline{N}(r,f) &\le C\sum_{j=0}^{k-1}\overline{N}(r,A_j) \\ &\le C_1 \max\{N(r,A_j) : j = 0,1,\ldots,k-1\} \\ &\le O(T(r,A_s)) \end{aligned}$$

where C and C_1 are positive constants. Hence

$$T(r,f) \le m(r,f) + O(T(r,A_s)).$$

This and Lemma 3.1 give

$$T(r,f) \le m(r,f) + O(T(r,A_s)) \le O\left(e^{T(r,A_s)[(\log r)\log T(r,A_s)]^\gamma}\right),\ \gamma > 1$$

holds outside of an exceptional set E_0 of finite logarithmic measure. By the monotonicity of the function φ and (1.3), we obtain $\rho^1_\varphi(f) \le \rho^0_\varphi(A_s)$.

On the other hand, equation (3.1) can be written as

$$\begin{aligned} -A_s &= \frac{f^{(k)}}{f^{(s)}} + A_{k-1}\frac{f^{(k-1)}}{f^{(s)}} + \cdots + A_{s+1}\frac{f^{(s+1)}}{f^{(s)}} + A_{s-1}\frac{f^{(s-1)}}{f^{(s)}} + \cdots + A_0\frac{f}{f^{(s)}} \\ &= \frac{f}{f^{(s)}}\left(\frac{f^{(k)}}{f} + A_{k-1}\frac{f^{(k-1)}}{f} + \cdots + A_{s+1}\frac{f^{(s+1)}}{f} + A_{s-1}\frac{f^{(s-1)}}{f} + \cdots + A_0\right). \end{aligned}$$

From Lemma 1.1 and the fact that

$$m\left(r,\frac{f}{f^{(s)}}\right) \le T(r,f) + T\left(r,\frac{1}{f^{(s)}}\right) = T(r,f) + T\left(r,f^{(s)}\right) + O(1) = O\left(T(r,f)\right),$$

we get

$$T(r,A_s) \le N(r,A_s) + \sum_{j\ne s} m(r,A_j) + O\left(\log r + \log T(r,f)\right) + O\left(T(r,f)\right) \tag{3.20}$$

holds for all $|z| = r \notin E$ where $E \subset [0,+\infty)$ is a set of finite linear measure. Lemma 3.4 implies that there exists a sequence $\{r_n, n \ge 1\}$, $r_n \longrightarrow +\infty$ such that for $|z_n| = r_n \notin E$

$$\lim_{r_n\to+\infty}\frac{\varphi\left(e^{T(r_n,A_s)}\right)}{\log r_n} = \rho^0_\varphi(A_s) = \rho_0.$$

Clearly

$$T(r_n,A_s) \ge \log\varphi^{-1}\left((\rho_0-\varepsilon)\log r_n\right). \tag{3.21}$$

Under the assumption

$$\eta = \max\left\{\rho^0_\varphi(A_j), \lambda^0_\varphi\left(\frac{1}{A_s}\right) : j \ne s\right\} < \rho^0_\varphi(A_s) = \rho_0 < +\infty,$$

we have

$$\begin{aligned} N(r_n, A_s) &\leq \log\varphi^{-1}\left((\eta+\varepsilon)\log r_n\right), && (3.22)\\ m(r_n, A_j) \leq T(r_n, A_j) &\leq \log\varphi^{-1}\left((\eta+\varepsilon)\log r_n\right),\ j \neq s && (3.23)\end{aligned}$$

provided for any ε $(0 < 2\varepsilon < \rho_0 - \eta)$. Substituting (3.21)–(3.23) into (3.20), we get

$$(1-o(1))\log\varphi^{-1}\left((\rho_0-\varepsilon)\log r_n\right) \leq O\left(\log r_n + \log T(r_n, f)\right) + O\left(T(r_n, f)\right) = O\left(T(r_n, f)\right).$$

Applying (1.3), one can deduce that $\rho_0 = \rho_\varphi^0(A_s) \leq \rho_\varphi^0(f)$.

(ii) Let $\{f_1, f_2, \ldots, f_k\}$ be a meromorphic solution base of equation (3.1). Lemma 3.2 yields

$$e^{m(r,A_s)} \leq O\left(\max_{1\leq i\leq k} T(r, f_i)\right), \quad s \in \{1, 2, \ldots, k-1\}.$$

If $N(r, A_s) \geq m(r, A_s)$, then $T(r, A_s) \leq 2N(r, A_s)$, and therefore $\rho_\varphi^0(A_s) \leq \lambda_\varphi^0\left(\frac{1}{A_s}\right)$. This contradicts the assumption $\lambda_\varphi^0\left(\frac{1}{A_s}\right) < \rho_\varphi^0(A_s)$. Thus, $N(r, A_s) < m(r, A_s)$ and

$$e^{T(r,A_s)} = O\left(e^{m(r,A_s)}\right) \leq O\left(\max_{1\leq i\leq k} T(r, f_i)\right) \quad \text{as} \quad r \longrightarrow +\infty.$$

Hence, there exists at least one solution from $\{f_1, f_2, \ldots, f_k\}$, say f_1, that satisfies

$$e^{T(r,A_s)} \leq O\left(T(r, f_1)\right).$$

By this inequality, (1.3) and the monotonicity of φ, we obtain $\rho_\varphi^0(A_s) \leq \rho_\varphi^1(f_1)$. In the first part, we have proved that $\rho_\varphi^1(f_1) \leq \rho_\varphi^0(A_s)$. Therefore, $\rho_\varphi^1(f_1) = \rho_\varphi^0(A_s)$.

3.2 Proof of Theorem 3.7

Assume that $f \not\equiv 0$ is a meromorphic solution whose poles are of uniformly bounded multiplicities of (3.1). By (3.1), we have

$$A_0 = -\left(\frac{f^{(k)}}{f} + A_{k-1}\frac{f^{(k-1)}}{f} + \cdots + A_1\frac{f'}{f}\right). \tag{3.24}$$

This and Lemma 1.1 yield

$$\begin{aligned} m(r, A_0) &\leq \sum_{j=1}^{k-1} m(r, A_j) + \sum_{j=1}^{k} m\left(r, \frac{f^{(j)}}{f}\right) + O(1)\\ &\leq \sum_{j=1}^{k-1} m(r, A_j) + O\left(\log r + \log T(r, f)\right) && (3.25)\end{aligned}$$

holds possibly outside of an exceptional set $E \subset (0, +\infty)$ of finite linear measure. It follows from (3.25) that

$$\begin{aligned} T(r, A_0) &= m(r, A_0) + N(r, A_0)\\ &\leq N(r, A_0) + \sum_{j=1}^{k-1} m(r, A_j) + O\left(\log r + \log T(r, f)\right) && (3.26)\end{aligned}$$

holds for $r \notin \mathrm{E}$. Lemma 3.4 implies that there exists a sequence $\{r_n, n \geq 1\}$, $r_n \longrightarrow +\infty$ such that for $|z_n| = r_n \notin \mathrm{E}$ we have

$$\lim_{r_n \to +\infty} \frac{\varphi\left(e^{\mathrm{T}(r_n, \mathrm{A}_0)}\right)}{\log r_n} = \rho_\varphi^0(\mathrm{A}_0) = \rho_0$$

Thus

$$\mathrm{T}(r_n, \mathrm{A}_0) \geq \log \varphi^{-1}\left((\rho_0 - \varepsilon)\log r_n\right) \tag{3.27}$$

Since

$$\eta = \max\left\{\rho_\varphi^0(\mathrm{A}_j), \lambda_\varphi^0\left(\frac{1}{\mathrm{A}_0}\right) : j \neq 0\right\} < \rho_\varphi^0(\mathrm{A}_0) = \rho_0 < +\infty,$$

then

$$\mathrm{N}(r_n, \mathrm{A}_0) \leq \log \varphi^{-1}\left((\eta + \varepsilon)\log r_n\right), \tag{3.28}$$

$$m(r_n, \mathrm{A}_j) \leq \mathrm{T}(r_n, \mathrm{A}_j) \leq \log \varphi^{-1}\left((\eta + \varepsilon)\log r_n\right),\ j \neq 0 \tag{3.29}$$

provided for any ε $(0 < 2\varepsilon < \rho_0 - \eta)$. Substituting (3.27)–(3.29) into (3.26), we get

$$(1 - o(1)) \log \varphi^{-1}\left((\rho_0 - \varepsilon)\log r_n\right) \leq \mathrm{O}\left(\log r_n + \log \mathrm{T}(r_n, f)\right).$$

Applying (1.3), one can deduce that $\rho_0 = \rho_\varphi^0(\mathrm{A}_0) \leq \rho_\varphi^1(f)$. On the other hand, from Theorem 3.6, we have $\rho_\varphi^0(\mathrm{A}_0) \geq \rho_\varphi^1(f)$. We deduce finally that any non-zero meromorphic solution f whose poles are of uniformly bounded multiplicities of (3.1) verifies $\rho_\varphi^1(f) = \rho_\varphi^0(\mathrm{A}_0)$.

3.3 Proof of Theorem 3.8

Assume that $f \not\equiv 0$ is a meromorphic solution whose poles are of uniformly bounded multiplicities of (3.1). If

$$\max\left\{\lambda_\varphi^0\left(\frac{1}{\mathrm{A}_0}\right), \rho_\varphi^0(\mathrm{A}_j) : j = 1, \ldots, k-1\right\} < \rho_\varphi^0(\mathrm{A}_0) < +\infty,$$

then Theorem 3.7 yields $\rho_\varphi^1(f) = \rho_\varphi^0(\mathrm{A}_0)$. Assume that $\lambda_\varphi^0\left(\frac{1}{\mathrm{A}_0}\right) < \rho_\varphi^0(\mathrm{A}_0)$ and

$$\max\left\{\rho_\varphi^0(\mathrm{A}_j) : j = 1, \ldots, k-1\right\} = \rho_\varphi^0(\mathrm{A}_0) = \rho_0\ (0 < \rho_0 < +\infty),$$

$$\max\left\{\tau_\varphi^0(\mathrm{A}_j) : \rho_\varphi^0(\mathrm{A}_j) = \rho_\varphi^0(\mathrm{A}_0)\right\} < \tau_\varphi^0(\mathrm{A}_0) = \tau_0\ (0 < \tau_0 < +\infty).$$

Then, there exists a set $\mathrm{J} \subseteq \{1, \ldots, k-1\}$ such that

$$\rho_\varphi^0(\mathrm{A}_j) = \rho_\varphi^0(\mathrm{A}_0) = \rho_0\ (j \in \mathrm{J}) \quad \text{and} \quad \tau_\varphi^0(\mathrm{A}_j) < \tau_\varphi^0(\mathrm{A}_0) = \tau_0\ (j \in \mathrm{J}).$$

Thus, there exist two constants β_1 and β_2 such that

$$\max\left\{\tau_\varphi^0(\mathrm{A}_j) : j \in \mathrm{J}\right\} < \beta_1 < \beta_2 < \tau_\varphi^0(\mathrm{A}_0) = \tau_0.$$

The definition of the type $\tau_\varphi^0(\mathrm{A}_j)$ implies that for sufficiently large r we have

$$e^{m(r, \mathrm{A}_j)} \leq e^{\mathrm{T}(r, \mathrm{A}_j)} < \varphi^{-1}\left(\log\left(\beta_1 r^{\rho_0}\right)\right),\ j \in \mathrm{J} \tag{3.30}$$

and

$$e^{m(r, \mathrm{A}_j)} \leq e^{\mathrm{T}(r, \mathrm{A}_j)} < \varphi^{-1}\left(\log r^{\rho_0^0}\right) < \varphi^{-1}\left(\log\left(\beta_1 r^{\rho_0}\right)\right),\ j \in \{1, \ldots, k-1\} \setminus \mathrm{J}, \tag{3.31}$$

where $0 < \rho_0^0 < \rho_0$. Since $\lambda_0 = \lambda_\varphi^0\left(\frac{1}{A_0}\right) < \rho_\varphi^0(A_0) = \rho_0$, then for any given ε $(0 < 2\varepsilon < \rho_0 - \lambda_0)$ and sufficiently large r, we obtain

$$e^{N(r,A_0)} \leq \varphi^{-1}\left(\log r^{\lambda_0+\varepsilon}\right) < \varphi^{-1}\left(\log r^{\rho_0-\varepsilon}\right) < \varphi^{-1}\left(\log\left(\beta_1 r^{\rho_0}\right)\right). \tag{3.32}$$

Lemma 3.5 implies that there exists a set $E_1 \subset [1,+\infty)$ of infinite logarithmic measure such that for all $r \in E_1$, we have

$$e^{T(r,A_0)} > \varphi^{-1}\left(\log\left(\beta_2 r^{\rho_0}\right)\right). \tag{3.33}$$

By substituting (3.30)–(3.33) into (3.26), for all $r \in E_1 \setminus E$ we obtain

$$(1-o(1))\log\varphi^{-1}\left(\log\left(\beta_2 r^{\rho_0}\right)\right) \leq O\left(\log r + \log T(r,f)\right) \tag{3.34}$$

Since $E_1 \setminus E$ is a set of infinite logarithmic measure, then there exists a sequence of points $|z_n| = r_n \in E_1 \setminus E$ tending to $+\infty$. Hence, by (3.34) we have

$$(1-o(1))\log\varphi^{-1}\left(\log\left(\beta_2 r_n^{\rho_0}\right)\right) \leq O\left(\log r_n + \log T(r_n,f)\right)$$

holds for all z_n such that $|z_n| = r_n \in E_1 \setminus E$ as $|z_n| = r_n \longrightarrow +\infty$. By the monotonicity of φ^{-1} and (1.3), we obtain $\rho_\varphi^0(A_0) \leq \rho_\varphi^1(f)$. By Theorem 3.6, we have $\rho_\varphi^1(f) \leq \rho_\varphi^0(A_0)$. Therefore $\rho_\varphi^1(f) = \rho_\varphi^0(A_0)$ which completes the proof.

3.4 Proof of Theorem 3.9

Since every solution of equation (3.2) is meromorphic function, then every solution of (3.1) corresponding to equation (3.2) is also meromorphic. Assume that $\{f_1,\ldots,f_k\}$ is a meromorphic solution base of (3.1), then every solution of (3.2) can be written as

$$f = a_1 f_1 + a_2 f_2 + \cdots + a_k f_k, \tag{3.35}$$

where $a_1, a_2, \ldots, a_k$ are meromorphic functions satisfying

$$a_j' = F\,P_j\,W^{-1}(f_1,\ldots,f_k), \quad j = 1,2,\ldots,k \tag{3.36}$$

where P_j are differential polynomials in $\{f_1,\ldots,f_k\}$ and their derivatives, $W^{-1}(f_1,\ldots,f_k)$ is the Wronskian of $\{f_1,\ldots,f_k\}$. Theorem 3.7 gives

$$\rho_\varphi^1(f_j) = \rho_\varphi^0(A_0),\ j = 1,\ldots,k.$$

By Proposition 1.8, Theorem 2.9, (3.35) and (3.36), we obtain

$$\rho_\varphi^1(f) \leq \max\left\{\rho_\varphi^1(F), \rho_\varphi^1(f_j) :\ j = 1,\ldots,k\right\} = \rho_\varphi^0(A_0).$$

In order to show that all solutions f of (3.2) satisfy $\rho_\varphi^1(f) = \rho_\varphi^0(A_0)$ with at most one exceptional solution, say f_1, satisfying $\rho_\varphi^1(f_1) < \rho_\varphi^0(A_0)$, we suppose that there exist two distinct meromorphic solutions f_1 and f_2 of equation (3.2) such that

$$\rho_\varphi^1(f_i) < \rho_\varphi^0(A_0), \quad i = 1,2.$$

Thus, $f = f_1 - f_2$ is also a meromorphic solution of (3.1) that satisfies

$$\rho_\varphi^1(f) = \rho_\varphi^1(f_1 - f_2) \leq \max\left\{\rho_\varphi^1(f_1), \rho_\varphi^1(f_2)\right\} < \rho_\varphi^0(A_0)$$

which contradicts Theorem 3.7. By (3.5) and Lemma 1.3, for every solution f of (3.2) satisfies $\rho_\varphi^1(f) = \rho_\varphi^0(A_0)$ we have

$$\max\left\{\rho_\varphi^1(F), \rho_\varphi^1(A_j) :\ j = 0,1,\ldots,k-1\right\} = \rho_\varphi^1(F) < \rho_\varphi^0(A_0) = \rho_\varphi^1(f).$$

It follows from Lemma 3.6 that $\overline{\lambda}_\varphi^1(f) = \lambda_\varphi^1(f) = \rho_\varphi^1(f)$ which completes the proof of Theorem 3.9.

3.5 Proof of Theorem 3.10

Let f be a meromorphic solution of (1.2) and $\{f_1,\dots,f_k\}$ be a meromorphic solution base of (3.1) corresponding to (3.2). By analogous discussion as in the proof of Theorem 3.9, by Proposition 1.8, Theorem 2.9, (3.35) and (3.36) it follows that

$$\rho_\varphi^1(f) \le \max\left\{\rho_\varphi^1(\mathrm{F}), \rho_\varphi^1(f_j) : j = 1,\dots,k\right\}.$$

From the first part of the proof of Theorem 3.6, one can see that

$$\rho_\varphi^1(f_j) \le \max\left\{\rho_\varphi^0(\mathrm{A}_j) : j = 0,\dots,k-1\right\}, \quad j = 1,\dots,k. \tag{3.37}$$

We obtain from the assumptions of Theorem 3.10 that $\rho_\varphi^1(f_j) \le \rho_\varphi^1(\mathrm{F})$. Thus

$$\rho_\varphi^1(f) \le \rho_\varphi^1(\mathrm{F}).$$

On the other hand, it follows from Proposition 1.8, (3.2) and Theorem 2.9 that

$$\rho_\varphi^1(\mathrm{F}) \le \max\left\{\rho_\varphi^1(f), \rho_\varphi^1(\mathrm{A}_j) : j = 0,\dots,k-1\right\}.$$

Since

$$\rho_\varphi^1(\mathrm{A}_j) \le \rho_\varphi^0(\mathrm{A}_j) < \rho_\varphi^1(\mathrm{F}), \quad j = 0,\dots,k-1$$

then, $\rho_\varphi^1(\mathrm{F}) \le \rho_\varphi^1(f)$. Therefore, $\rho_\varphi^1(f) = \rho_\varphi^1(\mathrm{F})$.

3.6 Proof of Theorem 3.11

Assume that $f \not\equiv 0$ is a meromorphic solution whose poles are of uniformly bounded multiplicities of (3.1). Set $\mathrm{G}_1 = \{|z| = r : z \in \mathrm{G}\}$. Since

$$\overline{dens}_l\{|z| : z \in \mathrm{G}\} > 0,$$

then G_1 is a set of infinite logarithmic measure. Set

$$\delta(\infty, \mathrm{A}_0) := \liminf_{r\to+\infty} \frac{m(r,\mathrm{A}_0)}{\mathrm{T}(r,\mathrm{A}_0)} = \delta > 0. \tag{3.38}$$

Thus

$$m(r,\mathrm{A}_0) > \frac{\delta}{2}\mathrm{T}(r,\mathrm{A}_0) \quad \text{as} \quad r \longrightarrow +\infty. \tag{3.39}$$

By substituting (3.6), (3.7) and (3.39) into (3.25), for any given ε $(0 < 2\varepsilon < \alpha - \beta)$ and sufficiently large r we obtain

$$\begin{aligned}
\frac{\delta}{2}\log\varphi^{-1}\left((\alpha-\varepsilon)\log r\right) &\le \frac{\delta}{2}\mathrm{T}(r,\mathrm{A}_0) \le m(r,\mathrm{A}_0) \\
&\le \sum_{j=1}^{k-1} m(r,\mathrm{A}_j) + \sum_{j=1}^{k} m\left(r,\frac{f^{(j)}}{f}\right) + \mathrm{O}(1) \\
&\le \sum_{j=1}^{k-1} \mathrm{T}(r,\mathrm{A}_j) + \mathrm{O}\left(\log r + \log \mathrm{T}(r,f)\right) \\
&\le (k-1)\log\varphi^{-1}\left(\beta\log r\right) + \mathrm{O}\left(\log r + \log \mathrm{T}(r,f)\right).
\end{aligned}$$

Then

$$(1-o(1))\log\varphi^{-1}\left((\alpha-\varepsilon)\log r\right) \le \mathrm{O}\left(\log r + \log \mathrm{T}(r,f)\right) \tag{3.40}$$

holds for all z such that $|z| = r \in G_1 \setminus E$ as $|z| = r \longrightarrow +\infty$. Since $G_1 \setminus E$ is a set of infinite logarithmic measure, then there exists a sequence of points $|z_n| = r_n \in G_1 \setminus E$ tending to $+\infty$. Hence, by (3.40) we have

$$(1-o(1))\log\varphi^{-1}\left((\alpha-\varepsilon)\log r_n\right) \leq O\left(\log r_n + \log T(r_n, f)\right)$$

holds for all z_n satisfying $|z_n| = r_n \in G_1 \setminus E$ as $|z_n| = r_n \longrightarrow +\infty$. By the monotonicity of φ^{-1} and arbitrariness of ε $(0 < 2\varepsilon < \alpha - \beta)$, one can obtain $\rho_\varphi^1(f) \geq \alpha$.

On the other hand, by a similar proof as in the first part of Theorem 3.6, it follows that $\rho_\varphi^1(f) \leq \alpha$. Therefore $\rho_\varphi^1(f) = \alpha$.

3.7 Proof of Theorem 3.12

(i) If $\rho_\varphi^1(F) \geq \alpha$, then it follows from Theorem 3.10 that $\rho_\varphi^1(f) = \rho_\varphi^1(F)$.

(ii) If $\rho_\varphi^1(F) < \alpha$, we prove that $\rho_1 = \rho_\varphi^1(f) = \alpha$ for every meromorphic solution $f \not\equiv 0$ whose poles are of uniformly bounded multiplicities of (3.1). We show firstly that $\rho_1 = \rho_\varphi^1(f) \geq \alpha$. Without loss of the generality, we suppose the contrary $\rho_1 \leq \beta < \alpha$. Set $G_2 = \{|z| = r : z \in G\}$. Since

$$\overline{dens}_l\{|z| : z \in G\} > 0,$$

then G_2 is a set of infinite logarithmic measure. Lemma 3.3 implies that there exists a constant $B > 0$ and a set $E_0 \subset [1, +\infty)$ of finite logarithmic measure such that for all z satisfying $|z| = r \notin [0,1] \bigcup E_0$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq B\left[T(2r, f)\right]^{k+1}, \quad j = 1, \ldots, k. \tag{3.41}$$

If f is a non-zero meromorphic solution of equation (3.1), then

$$|A_0(z)| \leq \left|\frac{f^{(k)}(z)}{f(z)}\right| + |A_{k-1}(z)|\left|\frac{f^{(k-1)}(z)}{f(z)}\right| + \cdots + |A_1(z)|\left|\frac{f'(z)}{f(z)}\right|. \tag{3.42}$$

By the definition of $\rho_1 = \rho_\varphi^1(f)$ and substituting (3.8), (3.9), (3.41) into (3.42), we obtain

$$\begin{aligned}
\varphi^{-1}\left((\alpha-\varepsilon)\log r\right) &\leq |A_0(z)| \leq k\, B\, \varphi^{-1}(\beta \log r)\left[T(2r, f)\right]^{k+1} \\
&\leq k\, B\, \varphi^{-1}(\beta \log r)\left[\varphi^{-1}\left(\left(\rho_1 + \frac{\varepsilon}{2}\right)\log 2r\right)\right]^{k+1} \\
&\leq \left[\varphi^{-1}\left(\left(\beta + \frac{\varepsilon}{2}\right)\log 2r\right)\right]^{k+2} \\
&\leq \varphi^{-1}\left((\beta+\varepsilon)\log r\right)
\end{aligned} \tag{3.43}$$

holds for all z such that $|z| = r \in G_2 \setminus ([0,1] \bigcup E_0)$ as $|z| = r \longrightarrow +\infty$. Since $G_2 \setminus E_0$ is a set of infinite logarithmic measure, then there exists a sequence of points $|z_n| = r_n \in G_2 \setminus E_0$ tending to $+\infty$. Hence, by (3.43) we have

$$\varphi^{-1}\left((\alpha-\varepsilon)\log r_n\right) \leq \varphi^{-1}\left((\beta+\varepsilon)\log r_n\right)$$

holds for all z_n satisfying $|z_n| = r_n \in G_2 \setminus E_0$ as $|z_n| = r_n \longrightarrow +\infty$. By the monotonicity of φ^{-1} and arbitrariness of ε $(0 < 2\varepsilon < \alpha - \beta)$, one can see that $\alpha \leq \beta$ which contradicts our assumption. Then, $\rho_\varphi^1(f) \geq \alpha$.

On the other hand, by a similar proof in Theorem 3.6 it follows that $\rho_{\varphi}^{1}(f) \leq \alpha$. Therefore $\rho_{\varphi}^{1}(f) = \alpha$.

Now, we prove that all solutions $f \in \mathscr{M}$ of equation (3.2) satisfy $\rho_{\varphi}^{1}(f) = \alpha$ with at most one exceptional solution, say f_0, satisfying $\rho_{\varphi}^{1}(f_0) < \alpha$. Suppose that there exist two distinct meromorphic solutions f_0 and $f_0^{\star}$ of equation (3.2) such that

$$\max\left\{\rho_{\varphi}^{1}(f_0), \rho_{\varphi}^{1}(f_0^{\star})\right\} < \alpha.$$

Then, $f = f_0 - f_0^{\star}$ is also a non-zero meromorphic solution of (3.1) and satisfies

$$\rho_{\varphi}^{1}(f) = \rho_{\varphi}^{1}(f_0 - f_0^{\star}) \leq \max\left\{\rho_{\varphi}^{1}(f_0), \rho_{\varphi}^{1}(f_0^{\star})\right\} < \alpha$$

This is a contradiction. By assumptions of Theorem 3.12, for all solutions $f \in \mathscr{M}$ of equation (3.2) satisfying $\rho_{\varphi}^{1}(f) = \alpha$, we have in view of Lemma 1.3

$$\max\left\{\rho_{\varphi}^{1}(\mathrm{F}), \rho_{\varphi}^{1}(\mathrm{A}_j) : j = 0, 1, \ldots, k-1\right\} = \rho_{\varphi}^{1}(\mathrm{F}) < \alpha = \rho_{\varphi}^{1}(f).$$

It follows from Lemma 3.6 that $\overline{\lambda}_{\varphi}^{1}(f) = \lambda_{\varphi}^{1}(f) = \rho_{\varphi}^{1}(f)$ and therefore

$$\overline{\lambda}_{\varphi}^{1}(f) = \lambda_{\varphi}^{1}(f) = \rho_{\varphi}^{1}(f) = \alpha$$

with at most one exceptional solution f_0 satisfying $\rho_{\varphi}^{1}(f_0) < \alpha$.

Chapter 4

Fast Growth and oscillation of solutions of LDE with meromorphic coefficients, Part 2

1 Introduction and statement of the main results

In Chapter 3, we assumed the multiplicity of poles of the meromorphic solution of equations (3.1) and (3.2) for investigating its properties of growth and oscillation. In this chapter, we continue this investigation by considering other conditions involving the lower φ-order and the lower φ-convergence exponent. Before we state our main results, it is essential to recall some existing results.

Theorem 4.1 *[5] Let* $\varphi \in \Phi$ *and let* $A_0, A_1, \ldots, A_{k-1} \in \mathscr{E}$ *satisfying*

$$\max\left\{\widetilde{\rho}_{\varphi}^{0}\left(A_j\right) : j = 1, \ldots, k-1\right\} \leq \widetilde{\mu}_{\varphi}^{0}\left(A_0\right) \leq \widetilde{\rho}_{\varphi}^{0}\left(A_0\right) < +\infty \left(\widetilde{\mu}_{\varphi}^{0}\left(A_0\right) > 0\right),$$

$$\max\left\{\widetilde{\tau}_{\varphi}^{0}\left(A_j\right) : \widetilde{\rho}_{\varphi}^{0}\left(A_j\right) = \widetilde{\mu}_{\varphi}^{0}\left(A_0\right)\right\} < \widetilde{\underline{\tau}}_{\varphi}^{0}\left(A_0\right) = \underline{\tau}_0 \left(0 < \underline{\tau}_0 < +\infty\right).$$

Then, every non-zero solution f *of* (3.1) *satisfies* $\widetilde{\mu}_{\varphi}^{1}(f) = \widetilde{\mu}_{\varphi}^{0}(A_0) \leq \widetilde{\rho}_{\varphi}^{1}(f) = \widetilde{\rho}_{\varphi}^{0}(A_0)$.

Theorem 4.2 *[8] Let* $\varphi \in \Phi$ *and let* $A_0, A_1, \ldots, A_{k-1} \in \mathscr{E}$ *satisfying*

$$\max\left\{\widetilde{\rho}_{\varphi}^{0}\left(A_j\right) : j = 1, \ldots, k-1\right\} \leq \widetilde{\mu}_{\varphi}^{0}\left(A_0\right) \leq \widetilde{\rho}_{\varphi}^{0}\left(A_0\right) < +\infty$$

$$\limsup_{r \longrightarrow +\infty} \frac{\sum_{j=1}^{k-1} m(r, A_j)}{m(r, A_0)} < 1.$$

Then, every solution $f \not\equiv 0$ *of* (3.1) *satisfies* $\widetilde{\mu}_{\varphi}^{1}(f) = \widetilde{\mu}_{\varphi}^{0}(A_0) \leq \widetilde{\rho}_{\varphi}^{1}(f) = \widetilde{\rho}_{\varphi}^{0}(A_0)$.

The aim of this chapter is to improve the above theorems by considering meromorphic coefficients of (3.1) instead of entire coefficients. We obtain also some results for the φ-convergence exponent and the lower φ-convergence exponent of solutions.

Theorem 4.3 *Let* $\varphi \in \Phi$ *and let* $A_0, A_1, \ldots, A_{k-1} \in \mathscr{M}$ *satisfying*

$$\lambda_{\varphi}^{0}\left(\frac{1}{A_0}\right) < \mu_{\varphi}^{0}(A_0) = \mu_0,$$

$$\max\left\{\rho_{\varphi}^{0}(A_j) : j = 1, \ldots, k-1\right\} \leq \mu_{\varphi}^{0}(A_0) \leq \rho_{\varphi}^{0}(A_0) < +\infty,$$

$$\max\left\{\tau_{\varphi}^{0}(A_j) : \rho_{\varphi}^{0}(A_j) = \mu_{\varphi}^{0}(A_0), j = 1, \ldots, k-1\right\} < \underline{\tau}_{\varphi}^{0}(A_0) = \underline{\tau}_0 \left(0 < \underline{\tau}_0 < +\infty\right).$$

Then, for any non-trivial solution $f \in \mathscr{M}$ *of* (3.1) *satisfying* $\frac{N(r,f)}{\overline{N}(r,f)} < \varphi^{-1}(\alpha \log r)$ *such that* $\alpha \leq \mu_0$, *we have*

$$\overline{\lambda}_{\varphi}^{1}(f-g) = \mu_{\varphi}^{1}(f) = \mu_{\varphi}^{0}(A_0) \leq \rho_{\varphi}^{0}(A_0) = \rho_{\varphi}^{1}(f) = \overline{\lambda}_{\varphi}^{1}(f-g)$$

where $g \in \mathscr{M}(g \not\equiv 0)$ *satisfying* $\rho_{\varphi}^{1}(g) < \mu_{\varphi}^{0}(A_0)$.

Theorem 4.4 *Let* $\varphi \in \Phi$ *and let* $A_0, A_1, \ldots, A_{k-1} \in \mathscr{M}$ *satisfying*

$$\lambda_{\varphi}^{0}\left(\frac{1}{A_0}\right) < \mu_{\varphi}^{0}(A_0) = \mu_0,$$

$$\max\left\{\rho_{\varphi}^{0}(A_j) : j = 1, \ldots, k-1\right\} \leq \mu_{\varphi}^{0}(A_0) \leq \rho_{\varphi}^{0}(A_0) < +\infty,$$

$$\limsup_{r \longrightarrow +\infty} \frac{\sum_{j=1}^{k-1} m(r, A_j)}{m(r, A_0)} < 1.$$

Then, for any non-trivial solution $f \in \mathscr{M}$ *of* (3.1) *satisfying* $\frac{N(r,f)}{\overline{N}(r,f)} < \varphi^{-1}(\alpha \log r)$ *such that* $\alpha \leq \mu_0$, *we have*

$$\overline{\lambda}_{\varphi}^{1}(f-g) = \mu_{\varphi}^{1}(f) = \mu_{\varphi}^{0}(A_0) \leq \rho_{\varphi}^{0}(A_0) = \rho_{\varphi}^{1}(f) = \overline{\lambda}_{\varphi}^{1}(f-g)$$

where $g \in \mathscr{M}(g \not\equiv 0)$ *satisfying* $\rho_{\varphi}^{1}(g) < \mu_{\varphi}^{0}(A_0)$.

Theorem 4.5 *Let* $\varphi \in \Phi$ *and let* $A_0, A_1, \ldots, A_{k-1} \in \mathscr{M}$. *If there exists one coefficient* $A_s(z)$ $(0 \leq s \leq k-1)$ *satisfying*

$$\lambda_{\varphi}^{0}\left(\frac{1}{A_s}\right) < \mu_{\varphi}^{0}(A_s) = \mu_s,$$

$$\max\left\{\rho_{\varphi}^{0}(A_j) : j \neq s\right\} \leq \mu_{\varphi}^{0}(A_s) \leq \rho_{\varphi}^{0}(A_s) < +\infty,$$

$$\max\left\{\tau_{\varphi}^{0}(A_j) : \rho_{\varphi}^{0}(A_j) = \mu_{\varphi}^{0}(A_s) = \mu_s, j \neq s\right\} < \underline{\tau}_{\varphi}^{0}(A_s) = \underline{\tau}_s \left(0 < \underline{\tau}_s < +\infty\right),$$

then, for every non-trivial transcendental meromorphic solution f *of* (3.1) *satisfying* $\frac{N(r,f)}{\overline{N}(r,f)} < \varphi^{-1}(\alpha \log r)$ *such that* $\alpha \leq \mu_s$, *we have*

$$\mu_{\varphi}^{1}(f) \leq \mu_{\varphi}^{0}(A_s) \leq \mu_{\varphi}^{0}(f) \quad \text{and} \quad \rho_{\varphi}^{1}(f) \leq \rho_{\varphi}^{0}(A_s) \leq \rho_{\varphi}^{0}(f).$$

Moreover, any non-transcendental meromorphic solution f *of* (3.1) *is a polynomial of degree* $deg(f) \leq s-1$.

Remark 4.1 *Clearly, Theorems 4.3 – 4.5 improve and generalize Theorems 4.1–4.2 from entire solutions to meromorphic solutions. However, by setting* $\varphi(r) = \log_{p+1}(r), (p \in \mathbb{N})$ *in Theorems 4.3 – 4.5, we obtain the counterpart results in the paper of Hu-Zheng [35] for the case* $p \geq q = 1$.

Remark 4.2 *The condition* $\lambda_{\varphi}^{0}\left(\frac{1}{A_0}\right) < \mu_{\varphi}^{0}(A_0)$ *in Theorems 4.3 and 4.4can be replaced by* $N(r, A_0) = o(m(r, A_0))$ *or* $\delta(\infty, A_0) > 0$.

2 Preliminary lemmas

Lemma 4.1 *[20, 43] Let $\varphi \in \Phi$ and let $f_1, f_2 \in \mathscr{M}$. For $j = 0, 1$, we have*

$$\begin{aligned} \rho_\varphi^j(f_1 + f_2) &\leq \max\left\{\rho_\varphi^j(f_1), \rho_\varphi^j(f_2)\right\}, \\ \rho_\varphi^j(f_1 f_2) &\leq \max\left\{\rho_\varphi^j(f_1), \rho_\varphi^j(f_2)\right\},. \end{aligned}$$

Moreover, if $\rho_\varphi^j(f_1) < \rho_\varphi^j(f_2)$ $(j = 0, 1)$, then $\rho_\varphi^j(f_1 + f_2) = \rho_\varphi^j(f_1 f_2) = \rho_\varphi^j(f_2)$.

Lemma 4.2 *Let $\varphi \in \Phi$ and let $f_1, f_2 \in \mathscr{M}$. For $j = 0, 1$, we have*

$$\begin{aligned} \mu_\varphi^j(f_1 + f_2) &\leq \max\left\{\rho_\varphi^j(f_1), \mu_\varphi^j(f_2)\right\}, \\ \mu_\varphi^j(f_1 f_2) &\leq \max\left\{\rho_\varphi^j(f_1), \mu_\varphi^j(f_2)\right\}. \end{aligned}$$

Moreover, if $\rho_\varphi^j(f_1) < \mu_\varphi^j(f_2)$ $(j = 0, 1)$, then $\mu_\varphi^j(f_1 + f_2) = \mu_\varphi^j(f_1 f_2) = \mu_\varphi^j(f_2)$.

Proof. We prove the lemma only for $j = 0$, the proof for $j = 1$ is similar. Without loss of generality, we assume that $\rho_\varphi^0(f_1) < +\infty$ and $\mu_\varphi^0(f_2) < +\infty$. From the definition of the lower $\mu_\varphi^0(f_2)$, there exists a sequence $r_n \longrightarrow +\infty$ such that

$$\lim_{r_n \to +\infty} \frac{\varphi\left(e^{\mathrm{T}(r_n, f_2)}\right)}{\log r_n} = \mu_\varphi^0(f_2).$$

Then, for any given $\varepsilon > 0$, there exists an integer number m_1 such that

$$\mathrm{T}(r_n, f_2) \leq \log \varphi^{-1}\left(\left(\mu_\varphi^0(f_2) + \varepsilon\right)\log r_n\right)$$

holds for all $n > m_1$. From the definition of the $\rho_\varphi^0(f_1)$, for any given $\varepsilon > 0$, there exists a positive real number r_1 such that

$$\mathrm{T}(r, f_1) \leq \log \varphi^{-1}\left(\left(\rho_\varphi^0(f_1) + \varepsilon\right)\log r\right)$$

holds for all $r > r_1$. Since $r_n \longrightarrow +\infty$, there exists an integer number m_2 such that $r_n > r_1$ and thus

$$\mathrm{T}(r_n, f_1) \leq \log \varphi^{-1}\left(\left(\rho_\varphi^0(f_1) + \varepsilon\right)\log r_n\right)$$

holds for $n > m_2$. In view of Proposition 1.5, for any given $\varepsilon > 0$ and for all $n > \max\{n_1, n_2\}$, we have

$$\begin{aligned} \mathrm{T}(r_n, f_1 + f_2) &\leq \mathrm{T}(r_n, f_1) + \mathrm{T}(r_n, f_2) + \log 2 \\ &\leq \log \varphi^{-1}\left(\left(\rho_\varphi^0(f_1) + \varepsilon\right)\log r_n\right) + \log \varphi^{-1}\left(\left(\mu_\varphi^0(f_2) + \varepsilon\right)\log r_n\right) + \log 2 \\ &\leq \log \varphi^{-1}\left(\left(\max\left\{\rho_\varphi^0(f_1), \mu_\varphi^0(f_2)\right\} + 2\varepsilon\right)\log r_n\right) \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} \mathrm{T}(r_n, f_1 f_2) &\leq \mathrm{T}(r_n, f_1) + \mathrm{T}(r_n, f_2) \\ &\leq \log \varphi^{-1}\left(\left(\max\left\{\rho_\varphi^0(f_1), \mu_\varphi^0(f_2)\right\} + 2\varepsilon\right)\log r_n\right). \end{aligned} \tag{4.2}$$

By arbitrariness of $\varepsilon > 0$ and the monotonicity of φ, we easily obtain from (4.1) and (4.2) that

$$\mu_\varphi^0(f_1+f_2) \le \max\left\{\rho_\varphi^0(f_1), \mu_\varphi^0(f_2)\right\} \tag{4.3}$$

and

$$\mu_\varphi^0(f_1 f_2) \le \max\left\{\rho_\varphi^0(f_1), \mu_\varphi^0(f_2)\right\} \tag{4.4}$$

On the other hand, it is well know that

$$T(r,f_2) = T(r,f_1+f_2-f_1) \le T(r,f_1+f_2)+T(r,f_1)+\log 2 \tag{4.5}$$

and

$$\begin{aligned} T(r,f_2) = T\left(r_n, \frac{f_1 f_2}{f_1}\right) &\le T(r,f_1 f_2)+T\left(r_n,\frac{1}{f_1}\right) \\ &= T(r,f_1 f_2)+T(r,f_1)+O(1). \end{aligned} \tag{4.6}$$

If we suppose that $\mu_\varphi^0(f_2) > \rho_\varphi^0(f_1)$, then by considering (4.5) and (4.6) and a similar discussion as above, we obtain that

$$\mu_\varphi^0(f_2) \le \max\left\{\rho_\varphi^0(f_1), \mu_\varphi^0(f_1+f_2)\right\} = \mu_\varphi^0(f_1+f_2) \tag{4.7}$$

and

$$\mu_\varphi^0(f_2) \le \max\left\{\rho_\varphi^0(f_1), \mu_\varphi^0(f_1 f_2)\right\} = \mu_\varphi^0(f_1 f_2). \tag{4.8}$$

We deduce from (4.3) and (4.7) that $\mu_\varphi^0(f_1+f_2) = \mu_\varphi^0(f_2)$, and by (4.4) and (4.8) we get $\mu_\varphi^0(f_1 f_2) = \mu_\varphi^0(f_2)$. □

Lemma 4.3 *Let $\varphi \in \Phi$ and let $f \in \mathcal{M}$ such that $0 < \rho_\varphi^0(f) < +\infty$. Then, there exists a set $E_1 \subset [1,+\infty)$ of infinite logarithmic measure such that*

$$\tau_\varphi^0(f) = \lim_{\substack{r\to+\infty \\ r\in E_1}} \frac{\exp\left\{\varphi\left(e^{T(r,f)}\right)\right\}}{r^{\rho_\varphi^0(f)}}.$$

Consequently, for any given $\varepsilon > 0$ and sufficiently large $r \in E_1$ we have

$$T(r,f) > \log\varphi^{-1}\left(\log(\tau_\varphi^0(f)-\varepsilon) r^{\rho_\varphi^0(f)}\right)$$

Proof. The definition of $\tau_\varphi^0(f)$ implies that there exists a sequence $\{r_n, n \ge 1\}$ tending to $+\infty$ satisfying $(1+\frac{1}{n})r_n < r_{n+1}$ and

$$\tau_\varphi^0(f) = \lim_{r_n\to+\infty} \frac{\exp\left\{\varphi\left(e^{T(r_n,f)}\right)\right\}}{r_n^{\rho_\varphi^0(f)}}.$$

Then, for any given $\varepsilon > 0$, there exists an integer number n_1 such that for all $n \ge n_1$ and $r \in [r_n, (1+\frac{1}{n})r_n]$ we have

$$\frac{\exp\left\{\varphi\left(e^{T(r_n,f)}\right)\right\}}{r_n^{\rho_\varphi^0(f)}} \left[\frac{\log r_n}{\log(1+\frac{1}{n})r_n}\right]^{\rho_\varphi^0(f)} \le \frac{\exp\left\{\varphi\left(e^{T(r,f)}\right)\right\}}{r^{\rho_\varphi^0(f)}}.$$

Set $\mathrm{E}_1 = \bigcup_{n=n_1}^{+\infty} \left[r_n, (1+\frac{1}{n})r_n\right]$. By the fact that

$$\frac{\log r_n}{\log(1+\frac{1}{n})r_n} \longrightarrow 1 \quad \text{as} \quad r_n \longrightarrow +\infty$$

we get

$$\lim_{\substack{r\to+\infty \\ r\in \mathrm{E}_1}} \frac{\exp\left\{\varphi\left(e^{\mathrm{T}(r,f)}\right)\right\}}{r^{\rho^0_\varphi(f)}} \geq \lim_{r_n\to+\infty} \frac{\exp\left\{\varphi\left(e^{\mathrm{T}(r_n,f)}\right)\right\}}{r_n^{\rho^0_\varphi(f)}} = \tau^0_\varphi(f).$$

The logarithmic measure of E_1 satisfies

$$mes_l(\mathrm{E}_1) = \int_{\mathrm{E}_1} \frac{dr}{r} = \sum_{n=n_1}^{+\infty} \int_{r_n}^{(1+\frac{1}{n})r_n} \frac{dt}{t} = \sum_{n=n_1}^{+\infty} \log\left(1+\frac{1}{n}\right) = +\infty.$$

Obviously

$$\lim_{\substack{r\to+\infty \\ r\in \mathrm{E}_1}} \frac{\exp\left\{\varphi\left(e^{\mathrm{T}(r,f)}\right)\right\}}{r^{\rho^0_\varphi(f)}} \leq \limsup_{\substack{r\to+\infty \\ r\in \mathrm{E}_1}} \frac{\exp\left\{\varphi\left(e^{\mathrm{T}(r,f)}\right)\right\}}{r^{\rho^0_\varphi(f)}} = \tau^0_\varphi(f).$$

Hence

$$\tau^0_\varphi(f) = \lim_{\substack{r\to+\infty \\ r\in \mathrm{E}_1}} \frac{\exp\left\{\varphi\left(e^{\mathrm{T}(r,f)}\right)\right\}}{r^{\rho^0_\varphi(f)}}.$$

Therefore, for any given $\varepsilon > 0$ and sufficiently large $r \in \mathrm{E}_1$ we have

$$\mathrm{T}(r,f) > \log\varphi^{-1}\left(\log\left(\tau^0_\varphi(f)-\varepsilon\right) r^{\rho^0_\varphi(f)}\right).$$

□

By using analogous arguments as in [8, Lemma 2.4] and the above proof, we can easily obtain the following lemma.

Lemma 4.4 *Let $\varphi \in \Phi$ and let $f \in \mathcal{M}$ such that $0 < \mu^0_\varphi(f) < +\infty$. Then, there exists a set $\mathrm{E}_1 \subset [1,+\infty)$ of infinite logarithmic measure such that*

$$\mu^0_\varphi(f) = \lim_{\substack{r\to+\infty \\ r\in \mathrm{E}_1}} \frac{\varphi\left(e^{\mathrm{T}(r,f)}\right)}{\log r}.$$

Consequently, for any given $\varepsilon > 0$ and sufficiently large $r \in \mathrm{E}_1$ we have

$$\mathrm{T}(r,f) < \log\varphi^{-1}\left(\left(\mu^0_\varphi(f)+\varepsilon\right)\log r\right)$$

Lemma 4.5 *Let $\varphi \in \Phi$ and let $\mathrm{F}, \mathrm{A}_0, \mathrm{A}_1, \ldots, \mathrm{A}_{k-1} \in \mathcal{M}$. If $f \in \mathcal{M}$ is a solution of equation (3.2) satisfying*

$$\max\left\{\rho^1_\varphi(\mathrm{F}), \rho^1_\varphi(\mathrm{A}_j) : j = 0,1,\ldots,k-1\right\} < \mu^1_\varphi(f), \tag{4.9}$$

then $\overline{\lambda}^1_\varphi(f) = \underline{\lambda}^1_\varphi(f) = \mu^1_\varphi(f)$.

Proof. We proceed by an analogous manner as in the proof of Lemma 3.6. Set

$$\rho = \max\left\{\rho_\varphi^1(\mathrm{F}), \rho_\varphi^1 \mathrm{A}_j) : j = 0,1,\ldots,k-1\right\} < \mu_\varphi^1 = \mu,$$

then, for any ε $(0 < 2\varepsilon < \mu - \rho)$ and sufficiently large r we get

$$\mathrm{T}(r,f) \geq \varphi^{-1}\left((\mu-\varepsilon)\log r\right) \tag{4.10}$$

and

$$\max\left\{\mathrm{T}(r,\mathrm{F}), \mathrm{T}(r,\mathrm{A}_j) : j = 0,1,\ldots,k-1\right\} \leq \varphi^{-1}\left((\rho+\varepsilon)\log r\right). \tag{4.11}$$

On the other hand, we have

$$\mathrm{O}\left(\log r + \log \mathrm{T}(r,f)\right) = o\left(\mathrm{T}(r,f)\right), \quad \text{as} \quad r \longrightarrow +\infty. \tag{4.12}$$

By (4.10), (4.11) and Proposition 1.5, it follows that

$$\begin{aligned}
\max\left\{\frac{\mathrm{T}(r_n,\mathrm{F})}{\mathrm{T}(r_n,f)}, \frac{\mathrm{T}(r_n,\mathrm{A}_j)}{\mathrm{T}(r_n,f)} : j = 0,1,\ldots,k-1\right\} &\leq \frac{\exp\left\{\log\varphi^{-1}\left((\rho+\varepsilon)\log r\right)\right\}}{\exp\left\{\log\varphi^{-1}\left((\mu-\varepsilon)\log r\right)\right\}} \\
&= \exp\left\{\log\varphi^{-1}\left((\rho+\varepsilon)\log r\right) - \log\varphi^{-1}\left((\mu-\varepsilon)\log r\right)\right\} \\
&= \exp\left\{\left(1 - \frac{\log\varphi^{-1}\left((\mu-\varepsilon)\log r\right)}{\log\varphi^{-1}\left((\rho+\varepsilon)\log r\right)}\right)\log\varphi^{-1}\left((\rho+\varepsilon)\log r\right)\right\} \longrightarrow 0.
\end{aligned} \tag{4.13}$$

provided ε $(0 < 2\varepsilon < \rho_1 - \mu)$ and $r \longrightarrow +\infty$. By substituting (4.12) and (4.13) into (3.15), we obtain

$$(1 - o(1))\mathrm{T}(r,f) \leq k\overline{\mathrm{N}}\left(r, \frac{1}{f}\right), \quad r \notin \mathrm{E}, r \longrightarrow +\infty$$

where $\mathrm{E} \subset [0,+\infty)$ is a set of finite linear measure. By Theorem 1.11, the monotonicity of φ and (1.3), we get $\mu_\varphi^1(f) \leq \underline{\lambda}_\varphi^1(f)$. Since $\mu_\varphi^1(f) \geq \underline{\lambda}_\varphi^1(f) \geq \overline{\lambda}_\varphi^1(f)$, we deduce that $\overline{\lambda}_\varphi^1(f) = \underline{\lambda}_\varphi^1(f) = \mu_\varphi^1(f)$. □

Lemma 4.6 *Let $\varphi \in \Phi$ and let $\mathrm{A}_0,\ldots,\mathrm{A}_{k-1} \in \mathscr{M}$ such that*

$$\max\left\{\rho_\varphi^0\left(\mathrm{A}_j\right) : j \neq s\right\} \leq \mu_\varphi^0\left(\mathrm{A}_s\right) < +\infty.$$

If $f \in \mathscr{M}$ is a non-trivial solution of (3.1) satisfying $\frac{\mathrm{N}(r,f)}{\overline{\mathrm{N}}(r,f)} < \varphi^{-1}(\alpha\log r)$ where $\alpha \leq \mu_\varphi^0(\mathrm{A}_s)$, then $\mu_\varphi^1(f) \leq \mu_\varphi^0(\mathrm{A}_s)$.

Proof. By (3.1), we observe that the poles of f can only occur at the poles of the coefficients $\mathrm{A}_0,\mathrm{A}_1,\ldots,\mathrm{A}_{k-1}$. Since $\frac{\mathrm{N}(r,f)}{\overline{\mathrm{N}}(r,f)} < \varphi^{-1}(\alpha\log r)$, $\left(\alpha \leq \mu_\varphi^0(\mathrm{A}_s)\right)$, then

$$\begin{aligned}
\mathrm{N}(r,f) &\leq \varphi^{-1}(\alpha\log r)\overline{\mathrm{N}}(r,f) \\
&\leq \varphi^{-1}(\alpha\log r)\sum_{j=0}^{k-1}\overline{\mathrm{N}}(r,\mathrm{A}_j) \\
&\leq \varphi^{-1}(\alpha\log r)\sum_{j=0}^{k-1}\mathrm{T}(r,\mathrm{A}_j).
\end{aligned}$$

Hence

$$T(r,f) \le m(r,f) + \varphi^{-1}(\alpha \log r) \sum_{j=0}^{k-1} T(r,A_j). \tag{4.14}$$

Since

$$\max\left\{\rho_\varphi^0(A_j) : j \ne s\right\} \le \mu_\varphi^0(A_s) < +\infty,$$

then for any given $\varepsilon > 0$ we have

$$T(r,A_j) \le \log \varphi^{-1}\left(\left(\mu_\varphi^0(A_s) + \varepsilon\right)\log r\right), \quad j \ne s, r \longrightarrow +\infty. \tag{4.15}$$

By applying Lemma 4.4 to the coefficient A_s and for the above ε, we have

$$T(r,A_s) < \log \varphi^{-1}\left(\left(\mu_\varphi^0(A_s) + \varepsilon\right)\log r\right), \quad r \in E_1, r \longrightarrow +\infty \tag{4.16}$$

where $E_1 \subset [1,+\infty)$ is a set of infinite logarithmic measure. By Lemma 3.1, (4.15) and (4.16), there exists a set $E_0 \subset [1,+\infty)$ of finite logarithmic measure such that for sufficiently large $r \in E_1 \setminus E_0$ we have

$$\begin{aligned} m(r,f) &\le \exp\left\{\left(\sum_{j=0}^{k-1} T(r,A_j)\right)\left[\log r \log \sum_{j=0}^{k-1} T(r,A_j)\right]^\gamma\right\} \\ &\le \varphi^{-1}\left(\left(\mu_\varphi^0(A_s) + 2\varepsilon\right)\log r\right). \end{aligned} \tag{4.17}$$

It follows from (4.14) – (4.17) that

$$\begin{aligned} T(r,f) &\le \varphi^{-1}\left(\left(\mu_\varphi^0(A_s) + 2\varepsilon\right)\log r\right) + k\varphi^{-1}(\alpha \log r)\log \varphi^{-1}\left(\left(\mu_\varphi^0(A_s) + \varepsilon\right)\log r\right) \\ &\le \varphi^{-1}\left(\left(\mu_\varphi^0(A_s) + 3\varepsilon\right)\log r\right), \qquad r \in E_1 \setminus E_0, r \longrightarrow +\infty. \end{aligned} \tag{4.18}$$

Thus, by Theorem 1.11, arbitrariness of ε and the monotonicity of φ, we obtain $\mu_\varphi^1(f) \le \mu_\varphi^0(A_s)$. □

Lemma 4.7 *Let $\varphi \in \Phi$ and let $A_0, A_1, \ldots, A_{k-1} \in \mathcal{M}$ satisfying*

$$\lambda_\varphi^0\left(\frac{1}{A_0}\right) < \mu_\varphi^0(A_0) = \mu_0,$$

$$\max\left\{\rho_\varphi^0(A_j) : j = 1,\ldots,k-1\right\} \le \mu_\varphi^0(A_0) = \mu_0 \ (0 < \mu_0 < +\infty),$$

$$\max\left\{\tau_\varphi^0(A_j) : \rho_\varphi^0(A_j) = \mu_\varphi^0(A_0), j = 1,\ldots,k-1\right\} < \underline{\tau}_\varphi^0(A_0) = \underline{\tau}_0 \ \left(0 < \underline{\tau}_0 < +\infty\right).$$

Then, for any non-trivial solution $f \in \mathcal{M}$ of (3.1) we have $\mu_\varphi^1(f) \ge \mu_\varphi^0(A_0)$.

Proof. Assume that $f \in \mathcal{M}$ is a non-zero solution of equation (3.1). Since $\lambda_\varphi^0\left(\frac{1}{A_0}\right) < \mu_\varphi^0(A_0)$, then

$$N(r,A_0) = o\left(T(r,A_0)\right), \qquad r \longrightarrow +\infty. \tag{4.19}$$

Since $T(r,A_0) = m(r,A_0) + N(r,A_0)$, then by (3.25) and (4.19) we obtain

$$(1 - o(1))T(r,A_0) \le \sum_{j=1}^{k-1} m(r,A_j) + O\left(\log r T(r,f)\right), r \notin E, r \longrightarrow +\infty \tag{4.20}$$

where $E \subset [0,+\infty)$ is a set of finite linear measure. Set

$$a = \max\left\{\rho_{\varphi}^{0}\left(A_j\right) : \rho_{\varphi}^{0}\left(A_j\right) < \mu_{\varphi}^{0}\left(A_0\right) = \mu_0, j = 1,\ldots,k-1\right\}.$$

If $\rho_{\varphi}^{0}\left(A_j\right) < \mu_{\varphi}^{0}\left(A_0\right) = \mu_0$, then for any $\varepsilon(0 < 2\varepsilon < \mu_0 - a)$ and sufficiently large r we have

$$m(r,A_j) \le T(r,A_j) \le \log\varphi^{-1}\left(\log r^{a+\varepsilon}\right) < \log\varphi^{-1}\left(\log r^{\mu_0-\varepsilon}\right),\ j \ne 0. \tag{4.21}$$

Set

$$\tau = \max\left\{\tau_{\varphi}^{0}\left(A_j\right) : \rho_{\varphi}^{0}\left(A_j\right) = \mu_{\varphi}^{0}\left(A_0\right) = \mu_0, j = 1,\ldots,k-1\right\},$$

then $\tau < \underline{\tau}_0 = \underline{\tau}_{\varphi}^{0}\left(A_0\right)$. If

$$\rho_{\varphi}^{0}\left(A_j\right) = \mu_{\varphi}^{0}\left(A_0\right) = \mu_0 \quad \text{and} \quad \tau_{\varphi}^{0}\left(A_j\right) \le \tau < \underline{\tau}_0,$$

then for any $\varepsilon(0 < 2\varepsilon < \underline{\tau}_0 - \tau)$ and sufficiently large r we have

$$m(r,A_j) \le T(r,A_j) \le \log\varphi^{-1}\left(\log(\tau+\varepsilon)r^{\mu_0}\right),\ j \ne 0. \tag{4.22}$$

By the definition of the lower φ-type $\underline{\tau}_{\varphi}^{0}(A_0) = \underline{\tau}_0$, for any given $\varepsilon > 0$ and sufficiently large r we have

$$T(r,A_0) > \log\varphi^{-1}\left(\log(\underline{\tau}_0 - \varepsilon)r^{\mu_0}\right). \tag{4.23}$$

Substituting (4.21) – (4.23) into (4.20), for any $\varepsilon\left(0 < 2\varepsilon < \min\left\{\mu_0 - a, \underline{\tau}_0 - \tau\right\}\right)$ we obtain

$$(1-o(1))\log\varphi^{-1}\left(\log(\underline{\tau}_0 - \varepsilon)r^{\mu_0}\right) \le O\left(\log r + \log T(r,f)\right),\ r \notin E, r \longrightarrow +\infty. \tag{4.24}$$

We deduce from Theorem 1.11, (4.24), the monotonicity of φ^{-1} and (1.3) that

$$\mu_0 = \mu_{\varphi}^{0}(A_0) \le \mu_{\varphi}^{1}(f).$$

□

3 Proofs of main results

3.1 Proof of Theorem 4.3

Firstly, we prove that $\rho_{\varphi}^{1}(f) = \rho_{\varphi}^{0}(A_0)$ and $\mu_{\varphi}^{1}(f) = \mu_{\varphi}^{0}(A_0)$. Since

$$\max\left\{\rho_{\varphi}^{0}\left(A_j\right) : j = 1,\ldots,k-1\right\} \le \mu_{\varphi}^{0}(A_0) \le \rho_{\varphi}^{0}(A_0) < +\infty,$$

then by Lemma 3.1 and (4.14), for any given $\varepsilon > 0$ and sufficiently large $r, r \notin E_0$ we have

$$T(r,f) \le \varphi^{-1}\left(\left(\rho_{\varphi}^{0}(A_0) + 3\varepsilon\right)\log r\right) \tag{4.25}$$

where $E_0 \subset [1,+\infty)$ is a set of finite logarithmic measure. We obtain from Theorem 1.11, the monotonicity of φ and (4.25) that $\rho_{\varphi}^{1}(f) \le \rho_{\varphi}^{0}(A_0)$. Set

$$b = \max\left\{\rho_{\varphi}^{0}\left(A_j\right) : \rho_{\varphi}^{0}\left(A_j\right) < \rho_{\varphi}^{0}\left(A_0\right) = \rho_0, j = 1,\ldots,k-1\right\}.$$

If

$$\rho_{\varphi}^{0}(A_j) < \mu_{\varphi}^{0}\left(A_0\right) \le \rho_{\varphi}^{0}(A_0) = \rho_0 \quad \text{or} \quad \rho_{\varphi}^{0}(A_j) \le \mu_{\varphi}^{0}\left(A_0\right) < \rho_{\varphi}^{0}(A_0) = \rho_0,$$

then for any $\varepsilon(0<2\varepsilon<\rho_0-b)$ and sufficiently large r we have

$$T(r,A_j)\le\log\varphi^{-1}\left(\log r^{\rho+\varepsilon}\right)<\log\varphi^{-1}\left(\log r^{\rho_0-\varepsilon}\right),\ j\neq 0. \tag{4.26}$$

Set

$$\tau=\max\left\{\tau_\varphi^0\left(A_j\right):\rho_\varphi^0\left(A_j\right)=\mu_\varphi^0\left(A_0\right)=\mu_0,j=1,\ldots,k-1\right\},$$

then $\tau<\underline{\tau}_0=\underline{\tau}_\varphi^0\left(A_0\right)$. If

$$\rho_\varphi^0\left(A_j\right)=\mu_\varphi^0\left(A_0\right)=\rho_\varphi^0\left(A_0\right)=\rho_0\quad\text{and}\quad\tau<\underline{\tau}_0\le\tau_0=\tau_\varphi^0\left(A_0\right),$$

then for any $\varepsilon(0<2\varepsilon<\tau_0-\tau)$ we have

$$T(r,A_j)\le\log\varphi^{-1}\left(\log(\tau+\varepsilon)r^{\rho_0}\right),\quad j\neq 0,r\longrightarrow+\infty. \tag{4.27}$$

Applying Lemma 4.3 to the coefficient A_0 gives

$$T(r,A_0)>\log\varphi^{-1}\left(\log(\tau_0-\varepsilon)r^{\rho_0}\right) \tag{4.28}$$

holds for all $r\in E_1, r\longrightarrow+\infty$ where $E_1\subset[1,+\infty)$ is a set of infinite logarithmic measure. Substituting (4.26) – (4.28) into (4.20), it follows for any $\varepsilon\left(0<2\varepsilon<\min\left\{\rho_0-\rho,\tau_0-\tau\right\}\right)$, $r\in E_1\setminus E, r\longrightarrow+\infty$ that

$$(1-o(1))\log\varphi^{-1}\left(\log(\tau_0-\varepsilon)r^{\rho_0}\right)\le O\left(\log r+\log T(r,f)\right) \tag{4.29}$$

where $E\subset[0,+\infty)$ is a set of finite linear measure. By Theorem 1.11, (4.29), the monotonicity of φ^{-1} and (1.3) we obtain $\rho_0=\rho_\varphi^0(A_0)\le\rho_\varphi^1(f)$. Therefore, $\rho_\varphi^1(f)=\rho_\varphi^0(A_0)$. On the other hand, by Lemma 4.6 and Lemma 4.7 we deduce that $\mu_\varphi^1(f)=\mu_\varphi^0(A_0)$.

Secondly, we prove that

$$\underline{\lambda}_\varphi^1(f-g)=\mu_\varphi^1(f)\quad\text{and}\quad\overline{\lambda}_\varphi^1(f-g)=\rho_\varphi^1(f).$$

Let $h=f-g$. Since

$$\rho_\varphi^1(g)<\mu_\varphi^0(A_0)=\mu_\varphi^1(f)\le\rho_\varphi^1(f),$$

it follows from Lemma 4.1 and Lemma 4.2 that

$$\rho_\varphi^1(h)=\rho_\varphi^1(f)=\rho_\varphi^0(A_0)\quad\text{and}\quad\mu_\varphi^1(h)=\mu_\varphi^1(f)=\mu_\varphi^0(A_0).$$

By substituting $f=g+h, f'=g'+h',\ldots,f^{(k)}=g^{(k)}+h^{(k)}$ into (3.1), we obtain

$$h^{(k)}+A_{k-1}(z)h^{(k-1)}+\cdots+A_0(z)h=-\left(g^{(k)}+A_{k-1}(z)g^{(k-1)}+\cdots+A_0(z)g\right) \tag{4.30}$$

If

$$g^{(k)}+A_{k-1}(z)g^{(k-1)}+\cdots+A_0(z)g=G\equiv 0,$$

then by Lemma 4.7 we have $\mu_\varphi^1(g)\ge\mu_\varphi^0(A_0)$ which contradicts the assumption $\rho_\varphi^1(g)<\mu_\varphi^0(A_0)$. Hence $G\not\equiv 0$. By Theorem 2.9 and Lemma 1.3, we get

$$\begin{aligned}\rho_\varphi^1(G)&\le\max\left\{\rho_\varphi^1(g),\rho_\varphi^1(A_j):j=0,1,\ldots,k-1\right\}=\rho_\varphi^1(g)\\&<\mu_\varphi^0(A_0)=\mu_\varphi^1(f)=\mu_\varphi^1(h)\\&\le\rho_\varphi^1(h)=\rho_\varphi^1(f)=\rho_\varphi^0(A_0).\end{aligned}$$

Then, it follows from Lemma 4.5, Lemma 3.6 and (4.30) that

$$\overline{\lambda}_\varphi^1(h)=\lambda_\varphi^1(h)=\rho_\varphi^1(h)=\rho_\varphi^1(f)\quad\text{and}\quad\underline{\lambda}_\varphi^1(h)=\underline{\lambda}_\varphi^1(h)=\mu_\varphi^1(h)=\mu_\varphi^1(f).$$

Therefore

$$\underline{\lambda}_\varphi^1(f-g)=\mu_\varphi^1(f)\quad\text{and}\quad\overline{\lambda}_\varphi^1(f-g)=\rho_\varphi^1(f)$$

which completes the proof of Theorem 4.3.

3.2 Proof of Theorem 4.4

Since

$$\limsup_{r\longrightarrow+\infty}\frac{\sum\limits_{j=1}^{k-1}m(r,\mathrm{A}_j)}{m(r,\mathrm{A}_0)}<1,$$

then there exists $\eta\,(0<\eta<1)$ such that

$$\sum_{j=1}^{k-1}m(r,\mathrm{A}_j)\leq\eta\,m(r,\mathrm{A}_0),\quad r\longrightarrow+\infty \tag{4.31}$$

By (4.31) and (4.20), it follows for all $r\notin \mathrm{E}, r\longrightarrow+\infty$ that

$$(1-o(1))\mathrm{T}(r,\mathrm{A}_0)\leq \mathrm{O}\left(\log r\log\mathrm{T}(r,f)\right) \tag{4.32}$$

where $\mathrm{E}\subset[0,+\infty)$ is a set of finite linear measure. By Theorem 1.11, (4.32), the monotonicity of φ and (1.2) we obtain

$$\rho_{\varphi}^{1}(f)\geq\rho_{\varphi}^{0}(\mathrm{A}_0)\quad\text{and}\quad\mu_{\varphi}^{1}(f)\geq\mu_{\varphi}^{0}(\mathrm{A}_0).$$

From the first part of the proof of Theorem 4.3, we have $\rho_{\varphi}^{1}(f)\leq\rho_{\varphi}^{0}(\mathrm{A}_0)$ and by applying Lemma 4.6 we have $\mu_{\varphi}^{1}(f)\leq\mu_{\varphi}^{0}(\mathrm{A}_0)$. Then, we deduce

$$\mu_{\varphi}^{1}(f)=\mu_{\varphi}^{0}(\mathrm{A}_0)\leq\rho_{\varphi}^{1}(f)=\rho_{\varphi}^{0}(\mathrm{A}_0).$$

The second part of the proof of Theorem 4.3 completes the proof of Theorem 4.4.

3.3 Proof of Theorem 4.5

Firstly, we assume that f is a rational function. If the function f is either polynomial of degree $deg(f)\geq s$ or rational with a pole at z_0 of multiplicity $k\geq 1$, then $f^{(s)}(z)\not\equiv 0$. Since

$$\max\left\{\rho_{\varphi}^{0}(\mathrm{A}_j):j\neq s\right\}\leq\mu_{\varphi}^{0}(\mathrm{A}_s)=\mu_s,$$

then by Lemma 4.7, Lemma 4.1 and (3.1) we get

$$0=\mu_{\varphi}^{0}(0)=\mu_{\varphi}^{0}(f^{(k)}+\mathrm{A}_{k-1}(z)f^{(k-1)}+\cdots+\mathrm{A}_0(z)f)=\mu_{\varphi}^{0}(\mathrm{A}_s)=\mu_s>0$$

and this is a contradiction. Set

$$c=\max\left\{\rho_{\varphi}^{0}\left(\mathrm{A}_j\right):\rho_{\varphi}^{0}\left(\mathrm{A}_j\right)<\mu_{\varphi}^{0}\left(\mathrm{A}_s\right)=\mu_s,j\neq s\right\}.$$

If $\rho_{\varphi}^{0}(\mathrm{A}_j)<\mu_{\varphi}^{0}\left(\mathrm{A}_s\right)=\mu_s$, then for any $\varepsilon(0<2\varepsilon<\mu_s-c)$ we have

$$m(r,\mathrm{A}_j)\leq\mathrm{T}(r,\mathrm{A}_j)\leq\log\varphi^{-1}\left(\log r^{\rho+\varepsilon}\right),\quad j\neq s,r\longrightarrow+\infty. \tag{4.33}$$

Set

$$\tau=\max\left\{\tau_{\varphi}^{0}\left(\mathrm{A}_j\right):\rho=\mu_s,j\neq s\right\},$$

then there exist two constants β_1 and β_2 such that $\tau<\beta_1<\beta_2<\underline{\tau}_s=\underline{\tau}_{\varphi}^{0}\left(\mathrm{A}_s\right)$. If $\rho=\mu_s$, $\tau_{\varphi}^{0}\left(\mathrm{A}_j\right)\leq\tau<\underline{\tau}_s$, then

$$m(r,\mathrm{A}_j)\leq\mathrm{T}(r,\mathrm{A}_j)\leq\log\varphi^{-1}\left(\log\beta_1 r^{\mu_s}\right),\ j\neq s,r\longrightarrow+\infty. \tag{4.34}$$

Since $\lambda_{\varphi}^{0}\left(\frac{1}{A_s}\right) < \mu_{\varphi}^{0}(A_s)$, then

$$N(r,A_s) = o\left(T(r,A_s)\right), \quad r \longrightarrow +\infty. \tag{4.35}$$

By the definition of the lower φ-type $\underline{\tau}_{\varphi}^{0}(A_s) = \underline{\tau}_s$, we have

$$T(r,A_s) > \log\varphi^{-1}\left(\log\beta_2 r^{\mu_s}\right), \quad r \longrightarrow +\infty. \tag{4.36}$$

Since f is rational, then it follows from (3.20) that

$$T(r,A_s) \leq N(r,A_s) + \sum_{j\neq s} m(r,A_j) + O(\log r), \quad r \longrightarrow +\infty. \tag{4.37}$$

By substituting (4.33) – (4.36) into (4.37) and in view of (1.1) we obtain that

$$\begin{aligned}(1-o(1))\log\varphi^{-1}\left(\log\beta_2 r^{\mu_s}\right) &\leq O\left(\log\varphi^{-1}\left(\log\beta_1 r^{\mu_s}\right)\right) + O(\log r)\\ &= O\left(\log\varphi^{-1}\left(\log\beta_1 r^{\mu_s}\right)\right), \quad r \longrightarrow +\infty.\end{aligned}$$

By Theorem 1.11, the monotonicity of φ and (1.3), we obtain $\beta_2 \leq \beta_1$ which is a contradiction. Hence, f must be a polynomial of degree $deg(f) \leq s-1$ when f is a non-transcendental meromorphic solution of (3.1).

Secondly, we assume that f is a transcendental meromorphic solution of (3.1). Then, it follows from (3.20) that

$$T(r,A_s) \leq N(r,A_s) + \sum_{j\neq s} m(r,A_j) + O\left(T(r,f)\right), \quad r \notin E, r \longrightarrow +\infty,. \tag{4.38}$$

where $E \subset [0,+\infty)$ is a set of finite linear measure. By substituting (4.33) – (4.36) into (4.38), we obtain

$$(1-o(1))\log\varphi^{-1}\left(\log(\beta_2-\beta_1) r^{\mu_s}\right) \leq O\left(T(r,f)\right), r \notin E, r \longrightarrow +\infty.$$

By Theorem 1.11, the monotonicity of φ and (1.3), we can deduce that

$$\rho_{\varphi}^{0}(A_s) \leq \rho_{\varphi}^{0}(f) \quad \text{and} \quad \mu_{\varphi}^{0}(A_s) \leq \mu_{\varphi}^{0}(f).$$

Since

$$\max\left\{\rho_{\varphi}^{0}(A_j) : j \neq s\right\} \leq \mu_{\varphi}^{0}(A_s) \leq \rho_{\varphi}^{0}(A_s) < +\infty,$$

then by Lemma 3.1 and (4.14), for any given $\varepsilon > 0$ we have

$$T(r,f) \leq \varphi^{-1}\left(\left(\rho_{\varphi}^{0}(A_s) + 3\varepsilon\right)\log r\right), \quad r \longrightarrow +\infty, r \notin E_0 \tag{4.39}$$

where $E_0 \subset [1,+\infty)$ is a set of finite logarithmic measure. We obtain from Theorem 1.11, the monotonicity of φ and (4.39) that $\rho_{\varphi}^{1}(f) \leq \rho_{\varphi}^{0}(A_s)$. By Lemma 4.6 we have $\mu_{\varphi}^{1}(f) \leq \mu_{\varphi}^{0}(A_s)$. Therefore

$$\rho_{\varphi}^{1}(f) \leq \rho_{\varphi}^{0}(A_s) \leq \rho_{\varphi}^{0}(f) \quad \text{and} \quad \mu_{\varphi}^{1}(f) \leq \mu_{\varphi}^{0}(A_s) \leq \mu_{\varphi}^{0}(f)$$

which completes the proof of Theorem 4.5.

Chapter 5

On the φ-order and the φ-type of analytic and meromorphic functions in the unit disc

1 Introduction and statement of the main results

In [13], Cao and Yi established some results on the sum, the product and the derivative of analytic and meromorphic functions in the unit disc involving the iterated p-order. Later, Latreuch and Belaïdi [47] obtained similar results for the usual type ($p = 1$). We state here some of their results.

Theorem 5.1 *[13] Let $p \in \mathbb{N}$ and let $f_1, f_2 \in \mathcal{M}_{\Delta}$. Then we have*

(i) $\sigma_p(f_1) = \sigma_p\left(\frac{1}{f_1}\right)$, *where* $f_1 \not\equiv 0$, $\sigma_p(af_1) = \sigma_p(f_1)$, *where* $a \in \mathbb{C} \setminus \{0\}$,

(ii) $\sigma_p(f_1') = \sigma_p(f_1)$,

(iii) $\max\{\sigma_p(f_1 + f_2), \sigma_p(f_1 f_2)\} \leq \max\{\sigma_p(f_1), \sigma_p(f_2)\}$,

(iv) if $\sigma_p(f_1) < \sigma_p(f_2)$, *then* $\sigma_p(f_1 + f_2) = \sigma_p(f_1 f_2) = \sigma_p(f_2)$.

Theorem 5.2 *[47] If $f_1, f_2 \in \mathcal{E}_{\Delta}$ satisfying $0 < \widetilde{\sigma}(f_1) = \widetilde{\sigma}(f_2) = \sigma < +\infty$ and $\widetilde{\kappa}(f_1) \neq \widetilde{\kappa}(f_2)$, then*

$$\widetilde{\sigma}(f_1 + f_2) = \sigma \quad \text{and} \quad \widetilde{\kappa}(f_1 + f_2) = \max\left\{\widetilde{\kappa}(f_1), \widetilde{\kappa}(f_2)\right\}.$$

Theorem 5.3 *[47] Let $f_1, f_2 \in \mathcal{M}_{\Delta}$. Then we have*

(i) if $0 < \sigma(f_1) < \sigma(f_2) < +\infty$, *then* $\kappa(f_1 + f_2) = \kappa(f_1 f_2) = \kappa(f_2)$,

(ii) if $0 < \sigma(f_1) = \sigma(f_2) = \sigma(f_1 + f_2) = \sigma(f_1 f_2) < +\infty$, *then*

$$\begin{aligned} |\kappa(f_1) - \kappa(f_2)| \leq \quad & \kappa(f_1 + f_2) \quad \leq \kappa(f_1) + \kappa(f_2), \\ |\kappa(f_1) - \kappa(f_2)| \leq \quad & \kappa(f_1 f_2) \quad \leq \kappa(f_1) + \kappa(f_2). \end{aligned}$$

The main purpose of this chapter is to investigate under suitable conditions the φ-order and the φ-type of $f_1 + f_2$, $f_1 f_2$ and f_1', where f_1, f_2 are analytic or meromorphic functions in the unit disc Δ. We have obtained the following results.

Theorem 5.4 *[42] Let $\varphi \in \Phi$ and let $f_1, f_2 \in \mathcal{M}_{\triangle}$. For $j = 0, 1$, the following statements hold*

(i) $\sigma_{\varphi}^{j}(f_1 + f_2) \leq \max\left\{\sigma_{\varphi}^{j}(f_1), \sigma_{\varphi}^{j}(f_2)\right\}$,

(ii) $\sigma_{\varphi}^{j}(f_1 f_2) \leq \max\left\{\sigma_{\varphi}^{j}(f_1), \sigma_{\varphi}^{j}(f_2)\right\}$,

(iii) for $a \in \mathbb{C} \setminus \{0\}$, we have $\sigma_{\varphi}^{j}(af_1) = \sigma_{\varphi}^{j}(f_1)$ *and* $\kappa_{\varphi}^{j}(af_1) = \kappa_{\varphi}^{j}(f_1)$,

(iv) for $f_1 \not\equiv 0$, we have $\sigma_{\varphi}^{j}\left(\frac{1}{f_1}\right) = \sigma_{\varphi}^{j}(f_1)$ *and* $\kappa_{\varphi}^{j}\left(\frac{1}{f_1}\right) = \kappa_{\varphi}^{j}(f_1)$.

Corollary 5.1 *[42] Let $\varphi \in \Phi$ and let $f_1, f_2 \in \mathcal{M}_{\triangle}$. For $j = 0, 1$, if $\sigma_{\varphi}^{j}(f_1) < \sigma_{\varphi}^{j}(f_2)$, then*

$$\sigma_{\varphi}^{j}(f_1 + f_2) = \sigma_{\varphi}^{j}(f_1 f_2) = \sigma_{\varphi}^{j}(f_2).$$

Theorem 5.5 *[42] Let $\varphi \in \Phi$ and let $f_1, f_2 \in \mathcal{E}_{\triangle}$. For $j = 0, 1$, there hold*

(i) for $a \in \mathbb{C} \setminus \{0\}$, we have $\widetilde{\sigma}_{\varphi}^{j}(af_1) = \widetilde{\sigma}_{\varphi}^{j}(f_1)$ *and* $\widetilde{\kappa}_{\varphi}^{j}(af_1) = \widetilde{\kappa}_{\varphi}^{j}(f_1)$,

(ii) $\max\left\{\widetilde{\sigma}_{\varphi}^{j}(f_1 + f_2), \widetilde{\sigma}_{\varphi}^{j}(f_1 f_2)\right\} \leq \max\left\{\widetilde{\sigma}_{\varphi}^{j}(f_1), \widetilde{\sigma}_{\varphi}^{j}(f_2)\right\}$,

(iii) if $\widetilde{\sigma}_{\varphi}^{j}(f_1) < \widetilde{\sigma}_{\varphi}^{j}(f_2)$, *then* $\widetilde{\sigma}_{\varphi}^{j}(f_1 + f_2) = \widetilde{\sigma}_{\varphi}^{j}(f_2)$.

Theorem 5.6 *[42] Let $\varphi \in \Phi$ and let $f_1, f_2 \in \mathcal{M}_{\triangle}$. For $j = 0, 1$, there hold*

(i) if $0 < \sigma_{\varphi}^{j}(f_1) < \sigma_{\varphi}^{j}(f_2) < +\infty$, *then*

$$\kappa_{\varphi}^{j}(f_1 + f_2) = \kappa_{\varphi}^{j}(f_1 f_2) = \kappa_{\varphi}^{j}(f_2),$$

(ii) if $0 < \sigma_{\varphi}^{j}(f_1) = \sigma_{\varphi}^{j}(f_2) = \sigma_{\varphi}^{j}(f_1 + f_2) < +\infty$, *then*

$$\kappa_{\varphi}^{j}(f_1 + f_2) \leq \max\left\{\kappa_{\varphi}^{j}(f_1), \kappa_{\varphi}^{j}(f_2)\right\}.$$

Moreover, if $\kappa_{\varphi}^{j}(f_1) \neq \kappa_{\varphi}^{j}(f_2)$, *then* $\kappa_{\varphi}^{j}(f_1 + f_2) = \max\left\{\kappa_{\varphi}^{j}(f_1), \kappa_{\varphi}^{j}(f_2)\right\}$.

(iii) if $0 < \sigma_{\varphi}^{j}(f_1) = \sigma_{\varphi}^{j}(f_2) = \sigma_{\varphi}^{j}(f_1 f_2) < +\infty$, *then*

$$\kappa_{\varphi}^{j}(f_1 f_2) \leq \max\left\{\kappa_{\varphi}^{j}(f), \kappa_{\varphi}^{j}(f_2)\right\}.$$

Moreover, if $\kappa_{\varphi}^{j}(f_1) \neq \kappa_{\varphi}^{j}(f_2)$, *then* $\kappa_{\varphi}^{j}(f_1 f_2) = \max\left\{\kappa_{\varphi}^{j}(f_1), \kappa_{\varphi}^{j}(f_2)\right\}$.

Corollary 5.2 *[42] Let $\varphi \in \Phi$ and let $f_1, f_2 \in \mathcal{M}_{\triangle}$. For $j = 0, 1$, if* $0 < \sigma_{\varphi}^{j}(f_1) = \sigma_{\varphi}^{j}(f_2) = \sigma < +\infty$ *and* $\kappa_{\varphi}^{j}(f_1) < \kappa_{\varphi}^{j}(f_2)$, *then*

$$\sigma_{\varphi}^{j}(f_1 + f_2) = \sigma_{\varphi}^{j}(f_1 f_2) = \sigma, \tag{5.1}$$

$$\kappa_{\varphi}^{j}(f_1 + f_2) = \kappa_{\varphi}^{j}(f_1 f_2) = \kappa_{\varphi}^{j}(f_2). \tag{5.2}$$

Theorem 5.7 *[42] Let $\varphi \in \Phi$ and let $f_1, f_2 \in \mathcal{E}_{\triangle}$. For $j = 0, 1$, there hold*

(i) *if* $0<\widetilde{\sigma}_{\varphi}^{j}(f_1)<\widetilde{\sigma}_{\varphi}^{j}(f_2)<+\infty$, *then* $\widetilde{\kappa}_{\varphi}^{j}(f_1+f_2)=\widetilde{\kappa}_{\varphi}^{j}(f_2)$.

(ii) *if* $0<\widetilde{\sigma}_{\varphi}^{j}(f_1)=\widetilde{\sigma}_{\varphi}^{j}(f_2)=\widetilde{\sigma}_{\varphi}^{j}(f_1+f_2)<+\infty$, *then*

$$\widetilde{\kappa}_{\varphi}^{j}(f_1+f_2)\leq\max\left\{\widetilde{\kappa}_{\varphi}^{j}(f_1),\widetilde{\kappa}_{\varphi}^{j}(f_2)\right\}.$$

Moreover, if $\widetilde{\kappa}_{\varphi}^{j}(f_1)\neq\widetilde{\kappa}_{\varphi}^{j}(f_2)$, *then* $\widetilde{\kappa}_{\varphi}^{j}(f_1+f_2)=\max\left\{\widetilde{\kappa}_{\varphi}^{j}(f_1),\widetilde{\kappa}_{\varphi}^{j}(f_2)\right\}$.

(iii) *if* $0<\widetilde{\sigma}_{\varphi}^{j}(f_1)=\widetilde{\sigma}_{\varphi}^{j}(f_2)=\widetilde{\sigma}_{\varphi}^{j}(f_1f_2)<+\infty$, *then*

$$\widetilde{\kappa}_{\varphi}^{j}(f_1f_2)\leq\max\left\{\widetilde{\kappa}_{\varphi}^{j}(f_1),\widetilde{\kappa}_{\varphi}^{j}(f_2)\right\}.$$

Corollary 5.3 *[42] Let* $\varphi\in\Phi$ *and let* $f_1,f_2\in\mathscr{E}_{\triangle}$. *For* $j=0,1$, *if* $0<\widetilde{\sigma}_{\varphi}^{j}(f_1)=\widetilde{\sigma}_{\varphi}^{j}(f_2)<+\infty$ *and* $\widetilde{\kappa}_{\varphi}^{j}(f_1)<\widetilde{\kappa}_{\varphi}^{j}(f_2)$, *then*

$$\widetilde{\sigma}_{\varphi}^{j}(f_1+f_2)=\widetilde{\sigma}_{\varphi}^{j}(f_1)=\widetilde{\sigma}_{\varphi}^{j}(f_2),\tag{5.3}$$

$$\widetilde{\kappa}_{\varphi}^{j}(f_1+f_2)=\widetilde{\kappa}_{\varphi}^{j}(f_2).\tag{5.4}$$

Theorem 5.8 *[42] Let* $\varphi\in\Phi$ *and let* $f\in\mathscr{M}_{\triangle}$. *Then, for* $j=0,1$, *we have* $\sigma_{\varphi}^{j}(f')=\sigma_{\varphi}^{j}(f)$.

Theorem 5.9 *[42] Let* $\varphi\in\Phi$ *and let* $f\in\mathscr{E}_{\triangle}$. *Then, for* $j=0,1$, *we have* $\widetilde{\sigma}_{\varphi}^{j}(f')=\widetilde{\sigma}_{\varphi}^{j}(f)$.

2 Proofs of main results

2.1 Proof of Theorem 5.4

We prove the theorem for $j=1$, the proofs for $j=0$ are similar.

(i) The definition of σ_{φ}^{1}-order implies that for any given $\varepsilon>0$ and for r sufficiently large in $[0,1)$, we have

$$\mathrm{T}(r,f_i)\leq\varphi^{-1}\left(\left(\sigma_{\varphi}^{1}(f_i)+\varepsilon\right)\log\frac{1}{1-r}\right),\ (i=1,2).\tag{5.5}$$

Thus

$$\begin{aligned}\mathrm{T}(r,f_1+f_2) &\leq \mathrm{T}(r,f_1)+\mathrm{T}(r,f_2)+\mathrm{O}(1)\\ &= \mathrm{O}\left(\varphi^{-1}\left(\left(\max\left\{\sigma_{\varphi}^{1}(f_1),\sigma_{\varphi}^{1}(f_2)\right\}+\varepsilon\right)\log\frac{1}{1-r}\right)\right).\end{aligned}$$

By the monotonicity of φ and (1.3), we obtain

$$\frac{\varphi\left(\mathrm{T}(r,f_1+f_2)\right)}{\log\frac{1}{1-r}}\leq\max\left\{\sigma_{\varphi}^{1}(f_1),\sigma_{\varphi}^{1}(f_2)\right\}+\varepsilon.$$

Since $\varepsilon>0$ is arbitrary, then we get $\sigma_{\varphi}^{1}(f_1+f_2)\leq\max\left\{\sigma_{\varphi}^{1}(f_1),\sigma_{\varphi}^{1}(f_2)\right\}$.

(ii)–(iv) follow immediately from the properties

$$\begin{aligned}\mathrm{T}(r,f_1f_2) &\leq \mathrm{T}(r,f_1)+\mathrm{T}(r,f_2),\\ \mathrm{T}(r,af_1) &= \mathrm{T}(r,f_1)+\mathrm{O}(1),\ (a\in\mathbb{C}\setminus\{0\}),\\ \mathrm{T}\left(r,\frac{1}{f_1}\right) &= \mathrm{T}(r,f_1)+\mathrm{O}(1)\end{aligned}$$

and the definitions of φ-order and φ-type.

2.2 Proof of Corollary 5.1

By Theorem 5.4, we have $\sigma^j_\varphi(f_1+f_2)\le\sigma^j_\varphi(f_2)$ and then

$$\sigma^j_\varphi(f_2)=\sigma^j_\varphi(f_1+f_2-f_1)\le\max\left\{\sigma^j_\varphi(f_1+f_2),\sigma^j_\varphi(f_1)\right\}.$$

Suppose that $\sigma^j_\varphi(f_1)>\sigma^j_\varphi(f_1+f_2)$, then

$$\sigma^j_\varphi(f_2)\le\max\left\{\sigma^j_\varphi(f_1+f_2),\sigma^j_\varphi(f_1)\right\}=\sigma^j_\varphi(f_1),$$

Which contradicts the hypothesis $\sigma^j_\varphi(f_1)<\sigma^j_\varphi(f_2)$. Hence, $\sigma^j_\varphi(f_2)\le\sigma^j_\varphi(f_1+f_2)$ and therefore $\sigma^j_\varphi(f_1+f_2)=\sigma^j_\varphi(f_2)$. Similarly, from again Theorem 5.4, we have $\sigma^j_\varphi\left(f_1f_2\right)\le\sigma^j_\varphi\left(f_2\right)$ and by the fact that

$$\sigma^j_\varphi(f_2)=\sigma^j_\varphi\left(f_1f_2\frac{1}{f_1}\right)\le\max\left\{\sigma^j_\varphi(f_1f_2),\sigma^j_\varphi(f_1)\right\},$$

we can get $\sigma^j_\varphi(f_2)\le\sigma^j_\varphi(f_1f_2)$ and therefore $\sigma^j_\varphi(f_1f_2)=\sigma^j_\varphi(f_2)$.

2.3 Proof of Theorem 5.5

(i) It is obvious from the definitions of $\widetilde{\sigma}^j_\varphi(f_1)$ and $\widetilde{\kappa}^j_\varphi(f_1)$, $(j=0,1)$.

(ii) It holds by using the known inequalities

$$|f_1+f_2|\le2\max\left\{|f_1|,|f_2|\right\}\quad\text{and}\quad|f_1f_2|\le\left[\max\left\{|f_1|,|f_2|\right\}\right]^2$$

and by applying (1.3).

(iii) It can be obtained by analogous discussion as in the proof of the first part of Corollary 5.1.

2.4 Proof of Theorem 5.6

We only give the proofs for $j=1$, the proofs for $j=0$ are analogous.

(i) By the definition of κ^1_φ-type, there exists a sequence $\{r_n,n\ge1\}$ tending to 1^- satisfying $1-(1-\frac{1}{n})(1-r_n<r_{n+1}$ such that for any given $\varepsilon>0$ we have

$$T(r_n,f_2)\ge\varphi^{-1}\left(\log\left(\frac{\kappa^1_\varphi(f_2)-\varepsilon}{(1-r_n)^{\sigma^1_\varphi(f_2)}}\right)\right)\tag{5.6}$$

and for $r\longrightarrow1^-$ there holds

$$T(r,f_i)\le\varphi^{-1}\left(\log\left(\frac{\kappa^1_\varphi(f_i)+\varepsilon}{(1-r)^{\sigma^1_\varphi(f_i)}}\right)\right),\quad i=1,2.\tag{5.7}$$

By Proposition 1.5 and the fact that

$$T(r,f_1+f_2)\ge T(r,f_2)-T(r,f_1)-\log2,\tag{5.8}$$

we get for any given $\varepsilon > 0$ that

$$\begin{aligned} T(r_n, f_1 + f_2) &\geq \varphi^{-1}\left(\log\left(\frac{\kappa^1_\varphi(f_2) - \varepsilon}{(1-r_n)^{\sigma^1_\varphi(f_2)}}\right)\right) - \varphi^{-1}\left(\log\left(\frac{\kappa^1_\varphi(f_1) + \varepsilon}{(1-r_n)^{\sigma^1_\varphi(f_1)}}\right)\right) - \log 2 \\ &\geq \varphi^{-1}\left(\log\left(\frac{\kappa^1_\varphi(f_2) - 2\varepsilon}{(1-r_n)^{\sigma^1_\varphi(f_2)}}\right)\right). \end{aligned} \tag{5.9}$$

Corollary 5.1 yields $\sigma^1_\varphi(f_1 + f_2) = \sigma^1_\varphi(f_2)$. Hence, by (5.9) and the monotonicity of φ, we get

$$(1-r_n)^{\sigma^1_\varphi(f_1+f_2)} \exp\left\{\varphi\left(T(r_n, f_1 + f_2)\right)\right\} \geq \kappa^1_\varphi(f_2) - 2\varepsilon.$$

By arbitrariness of $\varepsilon > 0$, we obtain

$$\kappa^1_\varphi(f_1 + f_2) \geq \kappa^1_\varphi(f_2). \tag{5.10}$$

Since

$$\sigma^1_\varphi(f_1 + f_2) = \sigma^1_\varphi(f_2) > \sigma^1_\varphi(f_1) = \sigma^1_\varphi(-f_1),$$

then from (5.10) it follows that $\kappa^1_\varphi(f_2) = \kappa^1_\varphi(f_1 + f_2 - f) \geq \kappa^1_\varphi(f_1 + f_2)$ and therefore $\kappa^1_\varphi(f_1 + f_2) = \kappa^1_\varphi(f_2)$. On the other hand, by the fact that

$$T(r, f_1 f_2) \geq T(r, f_1) - T(r, f_2) + O(1), \tag{5.11}$$

and a similar discussion as in the above proof, we arrive to

$$\kappa^1_\varphi(f_1 f_2) \geq \kappa^1_\varphi(f_2). \tag{5.12}$$

Since

$$\sigma^1_\varphi(f_1 f_2) = \sigma^1_\varphi(f_2) > \sigma^1_\varphi(f_1) = \sigma^1_\varphi\left(\frac{1}{f_1}\right),$$

then (5.12) gives $\kappa^1_\varphi(f_2) = \kappa^1_\varphi\left(f_1 f_2 \frac{1}{f_1}\right) \geq \kappa^1_\varphi(f_1 f_2)$ and therefore $\kappa^1_\varphi(f_1 f_2) = \kappa^1_\varphi(f_2)$.

(ii) It follows from the assumption $0 < \sigma^1_\varphi(f_1) = \sigma^1_\varphi(f_2) = \sigma^1_\varphi(f_1 + f_2) < +\infty$, (5.7) and Proposition 1.5 that

$$\begin{aligned} T(r, f_1 + f_2) &\leq T(r, f_1) + T(r, f_2) + O(1) \\ &\leq \varphi^{-1}\left(\log\left(\frac{\kappa^1_\varphi(f_1) + \varepsilon}{(1-r)^{\sigma^1_\varphi(f_1)}}\right)\right) + \varphi^{-1}\left(\log\left(\frac{\kappa^1_\varphi(f_2) + \varepsilon}{(1-r)^{\sigma^1_\varphi(f_2)}}\right)\right) + O(1) \\ &\leq \varphi^{-1}\left(\log\left(\frac{\max\left\{\kappa^1_\varphi(f_1), \kappa^1_\varphi(f_2)\right\} + 3\varepsilon}{(1-r)^{\sigma^1_\varphi(f_1+f_2)}}\right)\right). \end{aligned}$$

By the monotonicity of φ and arbitrariness of $\varepsilon > 0$ we obtain

$$\kappa^1_\varphi(f_1 + f_2) \leq \max\left\{\kappa^1_\varphi(f_1), \kappa^1_\varphi(f_2)\right\}. \tag{5.13}$$

Moreover, we may suppose without loss of generality that $\kappa^1_\varphi(f_1) < \kappa^1_\varphi(f_2)$. Then, by (5.13) and since $\sigma^1_\varphi(f_1 + f_2) = \sigma^1_\varphi(f_1) = \sigma^1_\varphi(-f_1)$, we get

$$\kappa^1_\varphi(f_2) = \kappa^1_\varphi(f_1 + f_2 - f_1) \leq \max\left\{\kappa^1_\varphi(f_1 + f_2), \kappa^1_\varphi(f_1)\right\} = \kappa^1_\varphi(f_1 + f_2)$$

and consequently $\kappa^1_\varphi(f_1 + f_2) = \max\left\{\kappa^1_\varphi(f_1), \kappa^1_\varphi(f_2)\right\}$.

(iii) By $T(r, f_1 f_2) \leq T(r, f_1) + T(r, f_2)$ and a similar discussion as in (5.13), we obtain that

$$\kappa_{\varphi}^{1}(f_1 f_2) \leq \max\left\{\kappa_{\varphi}^{1}(f_1), \kappa_{\varphi}^{1}(f_2)\right\}. \tag{5.14}$$

Moreover, if we suppose that $\kappa_{\varphi}^{1}(f_1) < \kappa_{\varphi}^{1}(f_2)$, then by (5.14) and since $\sigma_{\varphi}^{1}(f_1 f_2) = \sigma_{\varphi}^{1}(f_1) = \sigma_{\varphi}^{1}\left(\frac{1}{f_1}\right)$, we arrive to

$$\kappa_{\varphi}^{1}(f_2) = \kappa_{\varphi}^{1}\left(f_1 f_2 \frac{1}{f_1}\right) \leq \max\left\{\kappa_{\varphi}^{1}(f_1 f_2), \kappa_{\varphi}^{1}(f_1)\right\} = \kappa_{\varphi}^{1}(f_1 f_2).$$

Thus, $\kappa_{\varphi}^{1}(f_1 f_2) = \max\left\{\kappa_{\varphi}^{1}(f_1), \kappa_{\varphi}^{1}(f_2)\right\}$.

2.5 Proof of Corollary 5.2

We have from Theorem 5.4 that

$$\sigma_{\varphi}^{j}(f_1 + f_2) \leq \sigma_{\varphi}^{j}(f_1) = \sigma_{\varphi}^{j}(f_2) \quad \text{and} \quad \sigma_{\varphi}^{j}(f_1 f_2) \leq \sigma_{\varphi}^{j}(f_1) = \sigma_{\varphi}^{j}(f_2).$$

Analogous reasoning as in the proof of Theorem 5.6 especially (5.8) and (5.11) leads to

$$\sigma_{\varphi}^{j}(f_1 + f_2) \geq \sigma_{\varphi}^{j}(f_2) = \sigma_{\varphi}^{j}(f_1) \quad \text{and} \quad \sigma_{\varphi}^{j}(f_1 f_2) \geq \sigma_{\varphi}^{j}(f_2) = \sigma_{\varphi}^{j}(f_1).$$

Hence, (5.1) holds. On the other hand, (5.2) is obvious from (5.1) and Theorem 5.6.

2.6 Proof of Theorem 5.7

We will prove the theorem for $j = 0$, the proofs for $j = 1$ are analogous and follow also from Proposition 1.7 and Theorem 5.6.

(i) The definition of the $\widetilde{\kappa}_{\varphi}^{0}$-type implies that for any $\varepsilon > 0$ there exists a sequence $\{r_n : n \geq 1\}$ tending to 1^- satisfying $1 - (1 - \frac{1}{n})(1 - r_n) < r_{n+1}$ such that for any given $\varepsilon > 0$ we have

$$M(r_n, f_2) \geq \varphi^{-1}\left(\log\left(\frac{\widetilde{\kappa}_{\varphi}^{0}(f_2) - \varepsilon}{(1 - r_n)^{\widetilde{\sigma}_{\varphi}^{0}(f_2)}}\right)\right) \tag{5.15}$$

and for all $r \longrightarrow 1^-$

$$M(r, f_i) \leq \varphi^{-1}\left(\log\left(\frac{\widetilde{\kappa}_{\varphi}^{0}(f_i) + \varepsilon}{(1 - r)^{\widetilde{\sigma}_{\varphi}^{0}(f_i)}}\right)\right), (i = 1, 2). \tag{5.16}$$

We consider a sequence $\{z_n : n \geq 1\}$ satisfying $|f_2(z_n)| = M(r_n, f_2)$ in each circle $|z| = r_n$. Then, by (5.15), (5.16) and Proposition 1.5 we obtain

$$\begin{aligned}
M(r_n, f_1 + f_2) &\geq |f_1(z_n) + f_2(z_n)| \geq |f_2(z_n)| - |f_1(z_n)| \\
&\geq M(r_n, f_2) - M(r_n, f_1) \\
&\geq \varphi^{-1}\left(\log\left(\frac{\widetilde{\kappa}_{\varphi}^{0}(f_2) - \varepsilon}{(1 - r_n)^{\widetilde{\sigma}_{\varphi}^{0}(f_2)}}\right)\right) - \varphi^{-1}\left(\log\left(\frac{\widetilde{\kappa}_{\varphi}^{0}(f_1) + \varepsilon}{(1 - r_n)^{\widetilde{\sigma}_{\varphi}^{0}(f_1)}}\right)\right) \\
&\geq \varphi^{-1}\left(\log\left(\frac{\widetilde{\kappa}_{\varphi}^{0}(f_2) - 2\varepsilon}{(1 - r_n)^{\widetilde{\sigma}_{\varphi}^{0}(f_2)}}\right)\right)
\end{aligned} \tag{5.17}$$

provided $\varepsilon > 0$ and $r_n \longrightarrow 1^-$. We have from Theorem 5.5 that $\widetilde{\sigma}^0_\varphi(f_1+f_2) = \widetilde{\sigma}^0_\varphi(f_2)$. Thus, by the monotonicity of φ and (5.17) we obtain

$$(1-r_n)^{\widetilde{\sigma}^0_\varphi(f_1+f_2)} \exp\left\{\varphi\left(\mathrm{M}(r_n, f_1+f_2)\right)\right\} \geq \widetilde{\kappa}^0_\varphi(f_2) - 2\varepsilon.$$

By arbitrariness of $\varepsilon > 0$, we get

$$\widetilde{\kappa}^0_\varphi(f_1+f_2) \geq \widetilde{\kappa}^0_\varphi(f_2). \tag{5.18}$$

On the other hand, by applying (5.18) and since

$$\widetilde{\sigma}^0_\varphi(f_1+f_2) = \widetilde{\sigma}^0_\varphi(f_2) > \widetilde{\sigma}^0_\varphi(f_1) = \widetilde{\sigma}^0_\varphi(-f_1),$$

we obtain

$$\widetilde{\kappa}^0_\varphi(f_2) = \widetilde{\kappa}^0_\varphi(f_1+f_2-f_1) \leq \widetilde{\kappa}^0_\varphi(f_1+f_2)$$

and therefore $\widetilde{\kappa}^0_\varphi(f_1+f_2) = \widetilde{\kappa}^0_\varphi(f_2)$.

(ii) It follows from the assumption $0 < \widetilde{\sigma}^0_\varphi(f_1) = \widetilde{\sigma}^0_\varphi(f_2) = \widetilde{\sigma}^0_\varphi(f_1+f_2) < +\infty$, (5.16) and Proposition 1.5 that

$$\begin{aligned}
\mathrm{M}(r, f_1+f_2) &\leq \mathrm{M}(r, f_1) + \mathrm{M}(r, f_2) \\
&\leq \varphi^{-1}\left(\log\left(\frac{\widetilde{\kappa}^0_\varphi(f_1)+\varepsilon}{(1-r)^{\widetilde{\sigma}^0_\varphi(f_1+f_2)}}\right)\right) + \varphi^{-1}\left(\log\left(\frac{\widetilde{\kappa}^0_\varphi(f_2)+\varepsilon}{(1-r)^{\widetilde{\sigma}^0_\varphi(f_1+f_2)}}\right)\right) \\
&\leq \varphi^{-1}\left(\log\left(\frac{\max\left\{\widetilde{\kappa}^0_\varphi(f_1), \widetilde{\kappa}^0_\varphi(f_2)\right\}+2\varepsilon}{(1-r)^{\widetilde{\sigma}^0_\varphi(f_1+f_2)}}\right)\right).
\end{aligned}$$

The monotonicity of φ and arbitrariness of $\varepsilon > 0$ yield

$$\widetilde{\kappa}^0_\varphi(f_1+f_2) \leq \max\left\{\widetilde{\kappa}^0_\varphi(f_1), \widetilde{\kappa}^0_\varphi(f_2)\right\}. \tag{5.19}$$

Moreover, we may suppose without loss of generality that $\widetilde{\kappa}^0_\varphi(f_1) < \widetilde{\kappa}^0_\varphi(f_2)$. Since

$$\widetilde{\sigma}^0_\varphi(f_1+f_2) = \widetilde{\sigma}^0_\varphi(f_1) = \widetilde{\sigma}^0_\varphi(-f_1),$$

then by applying (5.19) we get

$$\widetilde{\kappa}^0_\varphi(f_2) = \widetilde{\kappa}^0_\varphi(f_1+f_2-f_1) \leq \max\left\{\widetilde{\kappa}^0_\varphi(f_1+f_2), \widetilde{\kappa}^0_\varphi(f_1)\right\} = \widetilde{\kappa}^0_\varphi(f_1+f_2). \tag{5.20}$$

We deduce from (5.19) and (5.20) that $\widetilde{\kappa}^0_\varphi(f_1+f_2) = \max\left\{\widetilde{\kappa}^0_\varphi(f_1), \widetilde{\kappa}^0_\varphi(f_2)\right\}$.

(iii) By the assumption $0 < \widetilde{\sigma}^0_\varphi(f_1) = \widetilde{\sigma}^0_\varphi(f_2) = \widetilde{\sigma}^0_\varphi(f_1 f_2) < +\infty$ and (5.16) it follows that

$$\begin{aligned}
\mathrm{M}(r, f_1 f_2) &\leq \mathrm{M}(r, f_1)\,\mathrm{M}(r, f_2) \\
&\leq \varphi^{-1}\left(\log\left(\frac{\widetilde{\kappa}^0_\varphi(f_1)+\varepsilon}{(1-r)^{\widetilde{\sigma}^0_\varphi(f_1 f_2)}}\right)\right)\varphi^{-1}\left(\log\left(\frac{\widetilde{\kappa}^0_\varphi(f_2)+\varepsilon}{(1-r)^{\widetilde{\sigma}^0_\varphi(f_1 f_2)}}\right)\right) \\
&\leq \left[\varphi^{-1}\left(\log\left(\frac{\max\left\{\widetilde{\kappa}^0_\varphi(f_1), \widetilde{\kappa}^0_\varphi(f_2)\right\}+\varepsilon}{(1-r)^{\widetilde{\sigma}^0_\varphi(f_1 f_2)}}\right)\right)\right]^2.
\end{aligned}$$

By the monotonicity of φ and (1.3), we obtain

$$\begin{aligned}\varphi(\mathrm{M}(r,f_1f_2)) &\leq (1+o(1))\log\left(\frac{\max\left\{\widetilde{\kappa}^0_\varphi(f_1),\widetilde{\kappa}^0_\varphi(f_2)\right\}+\varepsilon}{(1-r)^{\widetilde{\sigma}^0_\varphi(f_1f_2)}}\right)\\ &\leq \log\left(\frac{\max\left\{\widetilde{\kappa}^0_\varphi(f_1),\widetilde{\kappa}^0_\varphi(f_2)\right\}+2\varepsilon}{(1-r)^{\widetilde{\sigma}^0_\varphi(f_1f_2)}}\right).\end{aligned}$$

Since $\varepsilon>0$ is arbitrary, we deduce that $\widetilde{\kappa}^0_\varphi(f_1f_2)\leq\max\left\{\widetilde{\kappa}^0_\varphi(f_1),\widetilde{\kappa}^0_\varphi(f_2)\right\}$.

2.7 Proof of Corollary 5.3

We have from Theorem 5.5 that $\widetilde{\sigma}^j_\varphi(f_1+f_2)\leq\widetilde{\sigma}^j_\varphi(f_1)=\widetilde{\sigma}^j_\varphi(f_2)$. To prove the converse inequality, we assume to the contrary $\widetilde{\sigma}^j_\varphi(f_1+f_2)<\widetilde{\sigma}^j_\varphi(f_2)$. Then, by the first part of Theorem 5.7 we have

$$\widetilde{\kappa}^j_\varphi(f_1)=\widetilde{\kappa}^j_\varphi(f_1+f_2-f_2)=\widetilde{\kappa}^j_\varphi(f_2)$$

which is a contradiction. Hence, $\widetilde{\sigma}^j_\varphi(f_1+f_2)\geq\widetilde{\sigma}^j_\varphi(f_1)$ and therefore (5.3) holds. It is clear that (5.4) follows immediately from assertion (5.3) and the second part of Theorem 5.7.

2.8 Proof of Theorem 5.8

Denote $\sigma^1_\varphi(f)=\sigma$ and $\sigma^1_\varphi(f')=\sigma'$. By (5.5), Lemma 1.2 and (1.1), for all $r\longrightarrow1^-$, $r\notin\mathrm{F}_0\subset[0,1)$ where F_0 is a set of finite logarithmic measure, we have

$$\begin{aligned}\mathrm{T}(r,f') &\leq m\left(r,\frac{f'}{f}\right)+2\mathrm{T}(r,f)\\ &= \mathrm{O}\left(\log\varphi^{-1}\left((\sigma+\varepsilon)\log\frac{1}{1-r}\right)+\log\frac{1}{1-r}\right)+\mathrm{O}\left(\varphi^{-1}\left((\sigma+\varepsilon)\log\frac{1}{1-r}\right)\right)\\ &= \mathrm{O}\left(\varphi^{-1}\left((\sigma+\varepsilon)\log\frac{1}{1-r}\right)\right).\end{aligned}$$

By the monotonicity of φ and (1.3), for all $r\longrightarrow1^-$ and $r\notin\mathrm{F}_0$ we obtain

$$\varphi\left(\mathrm{T}(r,f')\right)\leq(1+o(1))\left((\sigma+\varepsilon)\log\frac{1}{1-r}\right)\leq(\sigma+2\varepsilon)\log\frac{1}{1-r}.$$

By Lemma 1.12, since $\varepsilon>0$ is arbitrary, there holds $\sigma^1_\varphi(f')\leq\sigma^1_\varphi(f)=\sigma$.

Now, we prove $\sigma^1_\varphi(f)\leq\sigma^1_\varphi(f')$. Indeed, by analogy to Theorem 1.7 (see also[21]), we can obtain the following estimate

$$\mathrm{T}(r,f)<\mathrm{O}\left(\mathrm{T}\left(\frac{1+r}{2},f'\right)+\log\frac{1}{1-r}\right)\quad\text{as}\quad r\longrightarrow1^-.$$

From (5.5) and (1.1), it follows that

$$\mathrm{T}(r,f')\leq\varphi^{-1}\left((\sigma'+\varepsilon)\log\frac{1}{1-r}\right)$$

and obviously

$$\begin{aligned}\mathrm{T}(r,f) &\leq \mathrm{O}\left(\varphi^{-1}\left((\sigma'+\varepsilon)\log\frac{2}{1-r}\right)+\log\frac{1}{1-r}\right)\\ &= \mathrm{O}\left(\varphi^{-1}\left((\sigma'+2\varepsilon)\log\frac{1}{1-r}\right)\right).\end{aligned}$$

The monotonicity of φ and (1.3) give

$$\varphi\left(\mathrm{T}(r,f)\right) \leq (1+o(1))\left((\sigma'+2\varepsilon)\log\frac{1}{1-r}\right) \leq (\sigma'+3\varepsilon)\log\frac{1}{1-r}.$$

By arbitrariness of $\varepsilon > 0$, we deduce that $\sigma^1_\varphi(f) \leq \sigma^1_\varphi(f') = \sigma'$ which completes the proof of Theorem 5.8.

2.9 Proof of Theorem 5.9

We prove the theorem for the case $j = 0$. The case for $j = 1$ follows immediately from Proposition 1.7 and Theorem 5.8. For $f \in \mathscr{E}_\Delta$, it is well known that

$$f(z) = f(0) + \int_0^z f'(\zeta)\,d\zeta.$$

Then, for $|z| = r < 1$ we obtain

$$\mathrm{M}(r,f) \leq |f(0)| + r\mathrm{M}(r,f') \leq |f(0)| + \mathrm{M}(r,f').$$

Clearly, it follows from the monotonicity of φ that $\widetilde{\sigma}^0_\varphi(f) \leq \widetilde{\sigma}^0_\varphi(f')$. Now, we prove the reverse inequality. Taking $z_0 = re^{i\theta}$ for any $|z| = r \in [0,1)$ such that $\mathrm{M}(r,f') = |f'(z_0)|$ and $\mathrm{R} = \frac{1+r}{2}$. By Cauchy's integral formula, we have

$$f'(z_0) = \frac{1}{2\pi i}\oint_\Gamma \frac{f(\zeta)}{(\zeta - z_0)^2}\,d\zeta,$$

where $\Gamma = \{\zeta : |\zeta - z_0| = \mathrm{R} - r\}$. Set $\zeta - z_0 = (\mathrm{R}-r)\,e^{i\theta}, (0 \leq \theta \leq 2\pi)$, then $d\zeta = (\mathrm{R}-r)\,ie^{i\theta}d\theta$. Since $\max\{|f(\zeta)| : \zeta \in \Gamma\} \leq \mathrm{M}(\mathrm{R},f)$, then we obtain

$$\begin{aligned}\mathrm{M}(r,f') = |f'(z_0)| &\leq \frac{1}{2\pi}\int_0^{2\pi}\frac{|f(\zeta)|}{|\zeta - z_0|^2}(\mathrm{R}-r)\,d\theta \\ &\leq \frac{\mathrm{M}(\mathrm{R},f)}{\mathrm{R}-r} = \frac{2}{1-r}\mathrm{M}\left(\frac{1+r}{2},f\right). \qquad (5.21)\end{aligned}$$

Now, we consider a set $\mathrm{H} \subset [0,1)$ such that

$$\mathrm{H} = \left\{r \in [0,1) : \log\frac{2}{1-r} < \log\mathrm{M}\left(\frac{1+r}{2},f\right)\right\}.$$

Thus, by the monotonicity of φ, (5.21) and (1.3) we get

$$\begin{aligned}\frac{\varphi\left(\mathrm{M}(r,f')\right)}{\log\frac{1}{1-r}} &\leq \frac{\varphi\left(\exp\left\{\log\frac{2}{1-r} + \log\mathrm{M}\left(\frac{1+r}{2},f\right)\right\}\right)}{\log\frac{1}{1-r}} \\ &\leq \frac{\varphi\left(\exp\left\{2\log\mathrm{M}\left(\frac{1+r}{2},f\right)\right\}\right)}{\log\frac{1}{1-r}} = \frac{\varphi\left(\exp\left\{2\log\mathrm{M}(\mathrm{R},f)\right\}\right)}{\log\frac{2}{1-\mathrm{R}}} \\ &\leq \frac{(1+o(1))\varphi\left(\mathrm{M}(\mathrm{R},f)\right)}{\log\frac{1}{1-\mathrm{R}}}.\end{aligned}$$

Clearly,

$$\widetilde{\sigma}^0_\varphi(f') = \limsup_{r\longrightarrow 1^-}\frac{\varphi\left(\mathrm{M}(r,f')\right)}{-\log(1-r)} \leq \limsup_{\mathrm{R}\longrightarrow 1^-}\frac{(1+o(1))\varphi\left(\mathrm{M}(\mathrm{R},f)\right)}{\log\frac{1}{1-\mathrm{R}}} = \widetilde{\sigma}^0_\varphi(f)$$

holds on H. It remains to estimate $\frac{\varphi(\mathrm{M}(r,f'))}{\log\frac{1}{1-r}}$ on H^c such that

$$\mathrm{H}^c=\left\{r\in[0;1):\log\frac{2}{1-r}\geq\log\mathrm{M}\left(\frac{1+r}{2},f\right)\right\}.$$

In fact, by Karamata's theorem (see [60]) we have $\varphi(e^t)=t^{o(1)}$ as $t\longrightarrow+\infty$. Then, for $r\longrightarrow 1^-$ there holds

$$\frac{\varphi\left(\mathrm{M}(r,f')\right)}{\log\frac{1}{1-r}}\leq\frac{\varphi\left(\exp\left\{\log\frac{2}{1-r}+\log\frac{2}{1-r}\right\}\right)}{\log\frac{1}{1-r}}=\frac{\left[2\log\frac{2}{1-r}\right]^{o(1)}}{\log\frac{1}{1-r}}.$$

It is obvious to see that

$$\widetilde{\sigma}^0_{\varphi}(f'):=\limsup_{r\longrightarrow 1^-}\frac{\varphi\left(\mathrm{M}(r,f')\right)}{-\log(1-r)}=0.$$

We finally deduce that $\widetilde{\sigma}^0_{\varphi}(f')\leq\widetilde{\sigma}^0_{\varphi}(f)$ and therefore $\widetilde{\sigma}^0_{\varphi}(f')=\widetilde{\sigma}^0_{\varphi}(f)$, which completes the proof of Theorem 5.9.

Chapter 6

Fast growing solutions of LDE with analytic coefficients in the unit disc

1 Introduction and statement of the main results

In this chapter, we investigate the fast growth of solutions of the linear differential equation ($k \geq 2$)

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_0(z) f = 0, \tag{6.1}$$

where the coefficients $A_0 \not\equiv 0, \ldots, A_{k-1}$ are analytic functions in the unit disc Δ. In [33], Heittokangas et al. investigated the iterated p-order of solutions of equation (6.1) and obtained the following result.

Theorem 6.1 *[33] Let $p \in \mathbb{N}$. Let $A_0, A_1, \ldots, A_{k-1} \in \mathscr{E}_\Delta$ such that $\widetilde{\sigma}_p(A_j) < \widetilde{\sigma}_p(A_0)$ for $j = 1, \ldots, k-1$, then every solution $f \not\equiv 0$ of (6.1) satisfies $\widetilde{\sigma}_{p+1}(f) = \widetilde{\sigma}_p(A_0)$.*

Observe that $A_0(z)$ is the only one dominant coefficient. In [28], Hamouda gave an improvement of Theorem 6.1 by considering more than one dominant coefficient as follows.

Theorem 6.2 *[28] Let $p \in \mathbb{N}$. Let $A_0, A_1, \ldots, A_{k-1} \in \mathscr{E}_\Delta$ satisfying $\widetilde{\sigma}_p(A_j) \leq \widetilde{\sigma}_p(A_0)$ for $j = 1, \ldots, k-1$, and*

$$\max\left\{\widetilde{\kappa}_p(A_j) : \widetilde{\sigma}_p(A_j) = \widetilde{\sigma}_p(A_0), j = 1, \ldots, k-1\right\} < \widetilde{\kappa}_p(A_0),$$

then any non-zero solution f of (6.1) satisfies $\widetilde{\sigma}_{p+1}(f) = \widetilde{\sigma}_p(A_0)$.

There are many extensions of the above theorems by considering the so called $[p,q]$-order (see some results in [9, 34, 65]). Semochko [59] have used the concept of φ-order to study the fast growing solutions of equation (6.1) which generalizes Theorem 6.1.

Theorem 6.3 *[59] Let $\varphi \in \Phi$ and let $A_0, A_1, \ldots, A_{k-1} \in \mathscr{E}_\Delta$ such that*

$$\max\left\{\widetilde{\sigma}^0_\varphi(A_j), j = 1, \ldots, k-1\right\} < \widetilde{\sigma}^0_\varphi(A_0).$$

Then, all solutions $f \not\equiv 0$ of (6.1) satisfy $\widetilde{\sigma}^1_\varphi(f) = \widetilde{\sigma}^0_\varphi(A_0)$.

The main purpose of this chapter is to generalise Theorems 6.1–6.3 by using the concepts of the (lower) φ-orders and the (lower) φ-types. Our results are also counterparts for theorems listed in [8, 5, 20] where the coefficients $A_j(z)$ in equation (6.1) are entire functions.

Theorem 6.4 *[40] Let $\varphi \in \Phi$ and let $A_0, \ldots, A_{k-1} \in \mathscr{E}_{\triangle}$ such that*

$$\max\left\{\widetilde{\sigma}_{\varphi}^{0}(A_j) : j = 1, \ldots, k-1\right\} < \widetilde{\nu}_{\varphi}^{0}(A_0) \leq \widetilde{\sigma}_{\varphi}^{0}(A_0) < +\infty.$$

Then, all solutions $f \not\equiv 0$ of (6.1) satisfy

$$\widetilde{\nu}_{\varphi}^{0}(A_0) = \widetilde{\nu}_{\varphi}^{1}(f) \leq \widetilde{\sigma}_{\varphi}^{1}(f) = \widetilde{\sigma}_{\varphi}^{0}(A_0).$$

Theorem 6.5 *[40] Let $\varphi \in \Phi$ and let $A_0, \ldots, A_{k-1} \in \mathscr{E}_{\triangle}$ satisfying*

$$\max\left\{\sigma_{\varphi}^{0}(A_j) : j = 1, \ldots, k-1\right\} \leq \nu_{\varphi}^{0}(A_0) < +\infty$$

and

$$\limsup_{r \longrightarrow 1^-} \frac{\sum_{j=1}^{k-1} m(r, A_j)}{m(r, A_0)} < 1.$$

Then, any non-zero solution f of (6.1) satisfies

$$\sigma_{\varphi}^{0}(A_0) \leq \sigma_{\varphi}^{1}(f) = \widetilde{\sigma}_{\varphi}^{1}(f) \leq \widetilde{\sigma}_{\varphi}^{0}(A_0) \quad \text{and} \quad \nu_{\varphi}^{0}(A_0) \leq \nu_{\varphi}^{1}(f) = \widetilde{\nu}_{\varphi}^{1}(f) \leq \widetilde{\nu}_{\varphi}^{0}(A_0).$$

Theorem 6.6 *[40] Let $\varphi \in \Phi$ and let $A_0, \ldots, A_{k-1} \in \mathscr{E}_{\triangle}$ satisfying*

$$\max\left\{\widetilde{\sigma}_{\varphi}^{0}(A_j) : j = 1, \ldots, k-1\right\} \leq \widetilde{\sigma}_{\varphi}^{0}(A_0) = \sigma_0 \ (0 < \sigma_0 < +\infty)$$

and

$$\max\left\{\widetilde{\kappa}_{\varphi}^{0}(A_j) : \widetilde{\sigma}_{\varphi}^{0}(A_j) = \widetilde{\sigma}_{\varphi}^{0}(A_0), j \neq 0\right\} < \widetilde{\kappa}_{\varphi}^{0}(A_0) = \kappa_0 \ (0 < \kappa_0 < +\infty).$$

Then, every non-trivial solution f of (6.1) satisfies $\widetilde{\sigma}_{\varphi}^{1}(f) = \widetilde{\sigma}_{\varphi}^{0}(A_0)$.

Theorem 6.7 *[40] Let $\varphi \in \Phi$ and let $A_0, \ldots, A_{k-1} \in \mathscr{E}_{\triangle}$ satisfying*

$$\max\left\{\widetilde{\sigma}_{\varphi}^{0}(A_j) : j = 1, \ldots, k-1\right\} \leq \widetilde{\nu}_{\varphi}^{0}(A_0) = \nu_0 \ (0 < \nu_0 < +\infty)$$

and

$$\max\left\{\widetilde{\kappa}_{\varphi}^{0}(A_j) : \widetilde{\sigma}_{\varphi}^{0}(A_j) = \widetilde{\nu}_{\varphi}^{0}(A_0), j \neq 0\right\} < \underline{\widetilde{\kappa}}_{\varphi}^{0}(A_0) = \kappa_1 \ (0 < \kappa_1 < +\infty).$$

Then, every non-trivial solution f of (6.1) satisfies $\widetilde{\sigma}_{\varphi}^{1}(f) = \widetilde{\sigma}_{\varphi}^{0}(A_0) \geq \widetilde{\nu}_{\varphi}^{1}(f) = \widetilde{\nu}_{\varphi}^{0}(A_0)$.

Remark 6.1 *Clearly, Theorem 6.6 improves Theorem 6.3 and generalizes Theorem 6.2. Moreover, theorems 6.5 and 6.7 extend respectively theorems 4.2 and 4.1 from entire solutions to analytic solutions in the unit disc.*

2 Preliminary Lemmas

Lemma 6.1 *[19] Let $f \in \mathscr{M}_{\triangle}$ such that $f^{(j)}$ does not vanish identically. Let $\varepsilon > 0$ be a constant, k and j be integers satisfying $0 \leq j < k$, F_0 a set with finite logarithmic measure on $[0,1)$ and $d \in (0,1)$. Then, we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq \left[\left(\frac{1}{1-|z|}\right)^{2+\varepsilon} \max\left\{\log\frac{1}{1-|z|}, T\left(s(|z|), f\right)\right\}\right]^{k-j}, \ |z| \notin F_0,$$

where $s(|z|) = 1 - d(1-|z|)$. Moreover, if $\sigma_1(f) = \sigma < +\infty$, then

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq \left(\frac{1}{1-|z|}\right)^{(k-j)(2+\varepsilon+\sigma)}, \ |z| \notin F_0.$$

Lemma 6.2 *[40] Let $\varphi \in \Phi$ and $k \in \mathbb{N}$. Let $f \in \mathcal{M}_{\angle}$ of order $\sigma_{\varphi}^{1}(f) = \sigma_1$. Then, for any given $\varepsilon > 0$ we have*

$$m\left(r, \frac{f^{(k)}}{f}\right) = O\left(\log^{+} \varphi^{-1}\left((\sigma_1 + \varepsilon)\log\frac{1}{1-r}\right)\right)$$

holds for all $r \to 1^{-}$ outside of a set $F_0 \subset [0,1)$ with finite logarithmic measure.

Proof. We proceed by mathematical induction. Firstly, for $k = 1$, the definition of $\sigma_{\varphi}^{1}(f) = \sigma_1$ implies that for any $\varepsilon > 0$ and for all $r \to 1^{-}$, we have

$$T(r, f) \leq \varphi^{-1}\left((\sigma_1 + \varepsilon)\log\frac{1}{1-r}\right). \tag{6.2}$$

It follows from Lemma 1.2, (1.1) and (6.2) that

$$m\left(r, \frac{f'}{f}\right) = O\left(\log^{+} \varphi^{-1}\left((\sigma_1 + \varepsilon)\log\frac{1}{1-r}\right)\right),\ r \notin F_0, \tag{6.3}$$

where $F_0 \subset [0,1)$ with $\int_{F_0} \frac{dr}{1-r} < +\infty$. Secondly, we assume for any $\varepsilon > 0$ that

$$m\left(r, \frac{f^{(k)}}{f}\right) = O\left(\log^{+} \varphi^{-1}\left((\sigma_1 + \varepsilon)\log\frac{1}{1-r}\right)\right),\ r \notin F_0.$$

Since

$$N\left(r, f^{(k)}\right) \leq (k+1)N(r, f) \quad \text{and} \quad m\left(r, f^{(k)}\right) \leq m\left(r, \frac{f^{(k)}}{f}\right) + m(r, f),$$

then

$$\begin{aligned} T\left(r, f^{(k)}\right) &= m\left(r, f^{(k)}\right) + N\left(r, f^{(k)}\right) \\ &\leq (k+1)T(r, f) + O\left(\log^{+} \varphi^{-1}\left((\sigma_1 + \varepsilon)\log\frac{1}{1-r}\right)\right) \\ &= O\left(\varphi^{-1}\left((\sigma_1 + \varepsilon)\log\frac{1}{1-r}\right)\right),\ r \notin F_0. \end{aligned}$$

In view of (6.3), it follows that

$$m\left(r, \frac{f^{(k+1)}}{f^{(k)}}\right) = O\left(\log^{+} \varphi^{-1}\left((\sigma_1 + \varepsilon)\log\frac{1}{1-r}\right)\right),\ r \notin F_0.$$

Thus, for $r \notin F_0$ we get

$$\begin{aligned} m\left(r, \frac{f^{(k+1)}}{f}\right) &\leq m\left(r, \frac{f^{(k+1)}}{f^{(k)}}\right) + m\left(r, \frac{f^{(k)}}{f}\right) \\ &= O\left(\log^{+} \varphi^{-1}\left((\sigma_1 + \varepsilon)\log\frac{1}{1-r}\right)\right). \end{aligned}$$

□

Lemma 6.3 *[32] Let $A_0, A_1, \ldots, A_{k-1}$ be analytic functions in $\Delta_R = \{z \in \mathbb{C} : |z| < R\}$, where $0 < R \leq +\infty$ and f be a solution of (6.1) in Δ_R, $1 \leq p < +\infty$. Then, for all $0 \leq r < R$ we have*

$$[m_p(r, f)]^p \leq C\left(1 + \sum_{j=0}^{k-1} \int_0^{2\pi} \int_0^r \left|A_j\left(se^{i\theta}\right)\right|^{\frac{1}{k-j}} ds\, d\theta\right),$$

where $C>0$ *is a constant depending on* p *and on the initial values of* $f(z)$ *at the point* z_θ *where* $A_j(z_\theta)\neq 0$ *for some* $j=0,1,\dots,k-1$ *and where*

$$[m_p(r,f)]^p=\frac{1}{2\pi}\int_0^{2\pi}\left[\log^+\left|f\left(re^{i\theta}\right)\right|\right]^p d\theta.$$

Lemma 6.4 *[32] Let* f *be a solution of* (6.1) *in* $\Delta_R=\{z\in\mathbb{C}:|z|<R\}$, *where* $0<R\leq+\infty$. *Let* $n_c\in\{1,2,\dots,k\}$ *be the number of non-zero coefficients* $A_j(j=0,1,\dots,k-1)$, *and let* $\theta\in[0,2\pi]$ *and* $\varepsilon>0$. *If* $z_0=s\,e^{i\theta}\in\Delta_R$ *is such that* $A_j\neq 0$ *for some* $j=0,1,\dots,k-1$, *then for all* $s\leq r<R$, *we have*

$$\left|f\left(r\,e^{i\theta}\right)\right|\leq C\exp\left\{n_c\int_s^r\max_{j=0,1,\dots,k-1}\left|A_j\left(t\,e^{i\theta}\right)\right|^{\frac{1}{k-j}}dt\right\},$$

where C *is a constant satisfying*

$$C\leq(1+\varepsilon)\max_{j=0,1,\dots,k-1}\frac{\left|f^{(j)}(z_0)\right|}{n_c^j\max\limits_{n=0,1,\dots,k-1}|A_n(z_0)|^{\frac{j}{k-n}}}.$$

Lemma 6.5 *[40] Let* $\varphi\in\Phi$ *and let* $f\in\mathcal{M}_\Delta$ *such that* $\nu_\varphi^1(f)=\nu_1<+\infty$. *Then, there exists a set* $F_1\subset[0,1)$ *of infinite logarithmic measure such that for all* $r\in F_1, r\to 1^-$ *and for any* $\varepsilon>0$ *we have*

$$T(r,f)<\varphi^{-1}\left((\nu_1+\varepsilon)\log\frac{1}{1-r}\right).$$

Proof. The definition of $\nu_\varphi^1(f)=\nu_1$ implies that there exists a sequence $\{r_n,n\geq 1\}$ tending to 1^- satisfying for any constant $d\in(0,1):\ 1-d(1-r_n)<r_{n+1}$ and

$$\lim_{r_n\to 1^-}\frac{\varphi\left(T(r_n,f)\right)}{-\log(1-r_n)}=\nu_1.$$

Thus, for any given $\varepsilon>0$ there exists an integer number n_1 such that for all $n\geq n_1$ we have

$$T(r_n,f)<\varphi^{-1}\left((\nu_1+\varepsilon)\log\frac{1}{1-r_n}\right).$$

Set $F_1=\bigcup\limits_{n=n_1}^{+\infty}\left[1-\frac{1-r_n}{d},r_n\right]$. For all $r\in F_1\subset[0,1)$, we obtain

$$\begin{aligned}T(r,f)\leq T(r_n,f) &< \varphi^{-1}\left(\left(\nu_1+\frac{\varepsilon}{2}\right)\log\frac{1}{1-r_n}\right)\\ &\leq \varphi^{-1}\left(\left(\nu_1+\frac{\varepsilon}{2}\right)\log\frac{1}{d(1-r)}\right)\\ &< \varphi^{-1}\left((\nu_1+\varepsilon)\log\frac{1}{1-r}\right),\end{aligned}$$

such that

$$mes_l(F_1)=\int_{F_1}\frac{dr}{1-r}=\sum_{n=n_1}^{+\infty}\int_{1-\frac{1-r_n}{d}}^{r_n}\frac{dt}{1-t}=\sum_{n=n_1}^{+\infty}\log\frac{1}{d}=+\infty.$$

□

By a similar proof, one can easily prove the following lemma.

Lemma 6.6 *Let $\varphi \in \Phi$ and let $f \in \mathscr{E}_{\Delta}$ such that $\widetilde{\nu}^0_{\varphi}(f) = \nu_0 < +\infty$. Then, there exists a set $F_1 \subset [0,1)$ of infinite logarithmic measure such that for all $r \in F_1$, $r \to 1^-$ and for any $\varepsilon > 0$ we have*

$$M(r,f) < \varphi^{-1}\left((\nu_0+\varepsilon)\log\frac{1}{1-r}\right).$$

Lemma 6.7 *[40] Let $\varphi \in \Phi$ and let $A_0,\dots,A_{k-1} \in \mathscr{E}_{\Delta}$. Assume that*

$$\max\left\{\widetilde{\sigma}^0_{\varphi}(A_j) : j \neq s\right\} \leq \widetilde{\nu}^0_{\varphi}(A_s) < +\infty.$$

If $f \not\equiv 0$ is a solution of (6.1), then $\widetilde{\nu}^1_{\varphi}(f) \leq \widetilde{\nu}^0_{\varphi}(A_s)$.

Proof. By Lemma 6.3 we have

$$\begin{aligned} T(r,f) = m(r,f) &\leq C\left(1+\sum_{j=0}^{k-1}\int_0^{2\pi}\int_0^r \left|A_j\left(se^{i\theta}\right)\right|^{\frac{1}{k-j}} ds\,d\theta\right) \\ &\leq 2\pi C\left(1+\sum_{j=0}^{k-1} rM(r,A_j)\right). \end{aligned} \tag{6.4}$$

Set $\alpha = \max\left\{\widetilde{\sigma}^0_{\varphi}(A_j) : j \neq s\right\}$. By the definition of $\widetilde{\sigma}^0_{\varphi}(A_j)$, for any $\varepsilon > 0$ and $r \to 1^-$ we have

$$M(r,A_j) \leq \varphi^{-1}\left(\left(\alpha+\frac{\varepsilon}{2}\right)\log\frac{1}{1-r}\right), \quad j \neq s. \tag{6.5}$$

By Lemma 6.6, there exists a set $F_1 \subset [0,1)$ with $\int_{F_1}\frac{dr}{1-r} = +\infty$ such that

$$M(r,A_s) < \varphi^{-1}\left(\left(\widetilde{\nu}^0_{\varphi}(A_s)+\frac{\varepsilon}{2}\right)\log\frac{1}{1-r}\right) \quad r \in F_1,\ r \to 1^-. \tag{6.6}$$

By (6.4)–(6.6), we have

$$T(r,f) \leq O\left(\varphi^{-1}\left(\left(\widetilde{\nu}^0_{\varphi}(A_s)+\varepsilon\right)\log\frac{1}{1-r}\right)\right), \quad r \in F_1,\ r \to 1^-. \tag{6.7}$$

Then, it follows from (1.3), (6.7), by arbitrariness of $\varepsilon > 0$ and the monotonicity of φ^{-1} that $\widetilde{\nu}^1_{\varphi}(f) \leq \widetilde{\nu}^0_{\varphi}(A_s)$. □

Lemma 6.8 *[40] Let $\varphi \in \Phi$ and let $A_0,\dots,A_{k-1} \in \mathscr{E}_{\Delta}$. Then, every non-zero solution f of (6.1) satisfies*

$$\widetilde{\sigma}^1_{\varphi}(f) \leq \max\left\{\widetilde{\sigma}^0_{\varphi}(A_j) : j = 0,1,\dots,k-1\right\}.$$

Proof. Set

$$\beta = \max\left\{\widetilde{\sigma}^0_{\varphi}(A_j) : j = 0,1,\dots,k-1\right\}.$$

By the definition of $\widetilde{\sigma}^0_{\varphi}(A_j)$ for any $\varepsilon > 0$ and $r \to 1^-$ we have

$$M(r,A_j) \leq \varphi^{-1}\left(\left(\beta+\frac{\varepsilon}{2}\right)\log\frac{1}{1-r}\right), \quad j = 0,1\dots,k-1. \tag{6.8}$$

By (6.4) and (6.8), we have

$$T(r,f) = m(r,f) \leq O\left(\varphi^{-1}\left((\beta+\varepsilon)\log\frac{1}{1-r}\right)\right), \quad r \to 1^-. \tag{6.9}$$

It follows from (1.3), (6.9), by the arbitrariness of $\varepsilon > 0$ and the monotonicity of φ^{-1} that

$$\widetilde{\sigma}_{\varphi}^{1}(f) \leq \beta = \max\left\{\widetilde{\sigma}_{\varphi}^{0}\left(A_j\right) : j = 0, 1, \ldots, k-1\right\}.$$

□

Lemma 6.9 *[40] Let $\varphi \in \Phi$ and let $f \in \mathscr{E}_{\Delta}$ be analytic function in Δ satisfying $0 < \widetilde{\sigma}_{\varphi}^{0}(f) = \sigma_0 < +\infty$ and $0 < \widetilde{\kappa}_{\varphi}^{0}(f) = \kappa_0 < +\infty$. Then, for any given $0 < \beta < \widetilde{\kappa}_{\varphi}^{0}(f)$, there exists a set $F_1 \subset [0,1)$ of infinite logarithmic measure such that for all $r \in F_1$, we have*

$$\varphi\left(M(r,f)\right) > \log\frac{\beta}{(1-r)^{\sigma_0}}.$$

Proof. The definition of $\widetilde{\kappa}_{\varphi}^{0}(f)$ implies that there exists a sequence $\{r_n, n \geq 1\}$ tending to 1^- satisfying $1 - \left(1 - \frac{1}{n}\right)(1 - r_n) < r_{n+1}$ and

$$\lim_{n\to+\infty}(1-r_n)^{\sigma_0}\exp\left\{\varphi\left(M(r_n,f)\right)\right\} = \kappa_0.$$

Thus, for any given $\varepsilon > 0$, there exists an integer number n_1 such that for all $n \geq n_1$ we have

$$\exp\left\{\varphi\left(M(r_n,f)\right)\right\} > \frac{\kappa_0 - \varepsilon}{(1-r_n)^{\sigma_0}}. \tag{6.10}$$

For any given β such that $0 < \beta < \kappa_0 - \varepsilon$, there exists an integer n_2 such that for all $n \geq n_2$, we have

$$\left(1 - \frac{1}{n}\right)^{\sigma_0} > \frac{\beta}{\kappa_0 - \varepsilon}. \tag{6.11}$$

For all $n \geq n_3 = \max\{n_1, n_2\}$ and for any $r \in \left[r_n, 1 - (1 - \frac{1}{n})(1 - r_n)\right]$, by (6.10) and (6.11), it follows that

$$\begin{aligned}\exp\left\{\varphi\left(M(r,f)\right)\right\} &\geq \exp\left\{\varphi\left(M(r_n,f)\right)\right\} > \frac{\kappa_0-\varepsilon}{(1-r_n)^{\sigma_0}} \\ &\geq \frac{\kappa_0-\varepsilon}{(1-r)^{\sigma_0}}\left(1-\frac{1}{n}\right)^{\sigma_0} > \frac{\beta}{(1-r)^{\sigma_0}}.\end{aligned}$$

Hence

$$\varphi\left(M(r,f)\right) > \log\frac{\beta}{(1-r)^{\sigma_0}}.$$

Set $F_1 = \bigcup_{n=n_3}^{+\infty}\left[r_n, 1 - (1 - \frac{1}{n})(1 - r_n)\right]$. Clearly, F_1 satisfies

$$mes_l(F_1) = \int_{F_1}\frac{dr}{1-r} = \sum_{n=n_3}^{+\infty}\int_{r_n}^{1-(1-\frac{1}{n})(1-r_n)}\frac{dr}{1-r} = \sum_{n=n_3}^{+\infty}\log\frac{n}{n-1} = +\infty.$$

□

Lemma 6.10 *[59] Let $\varphi \in \Phi$ and $f \in \mathscr{E}_{\Delta}$ such that $0 < \sigma_{\varphi}^{0}(f) = \sigma_0 < +\infty$. Then, for any given $0 < \beta < \sigma_0$, there exists a set $F_1 \subset [0,1)$ of infinite logarithmic measure such that for all $r \in F_1$, we have*

$$\varphi\left(e^{T(r,f)}\right) > \beta\log\frac{1}{1-r}.$$

3 Proofs of main results

3.1 Proof of Theorem 6.4

Suppose that f is a non-zero solution of (6.1). By Theorem 6.3, we have $\widetilde{\sigma}^1_{\varphi}(f) = \widetilde{\sigma}^0_{\varphi}(A_0)$. So, we only need to prove that $\widetilde{\nu}^1_{\varphi}(f) = \widetilde{\nu}^0_{\varphi}(A_0)$. Firstly, we prove the inequality $\nu_1 = \widetilde{\nu}^1_{\varphi}(f) \geq \widetilde{\nu}^0_{\varphi}(A_0) = \nu_0$. Suppose that to the contrary, $\nu_1 = \widetilde{\nu}^1_{\varphi}(f) < \widetilde{\nu}^0_{\varphi}(A_0) = \nu_0$. Set

$$\beta = \max\left\{\widetilde{\sigma}^0_{\varphi}(A_j) : j = 1,\ldots,k-1\right\}.$$

We can suppose with out reducing the generality that $\nu_1 \leq \beta < \nu_0$. By equation (6.1), we have

$$|A_0(z)| \leq \left|\frac{f^{(k)}(z)}{f(z)}\right| + |A_{k-1}(z)|\left|\frac{f^{(k-1)}(z)}{f(z)}\right| + \cdots + |A_1(z)|\left|\frac{f'(z)}{f(z)}\right|. \tag{6.12}$$

For any given ε $(0 < 3\varepsilon < \nu_0 - \beta)$, we have from the definitions of $\widetilde{\nu}^0_{\varphi}(A_0)$ and $\widetilde{\sigma}^0_{\varphi}(A_j)$ that for $r \to 1^-$

$$|A_0(z)| \geq \varphi^{-1}\left((\nu_0 - \varepsilon)\log\frac{1}{1-r}\right) \tag{6.13}$$

and

$$|A_j(z)| \leq \varphi^{-1}\left((\beta + \varepsilon)\log\frac{1}{1-r}\right), \quad j = 1,\ldots,k-1. \tag{6.14}$$

By Lemma 6.1, for $j = 1,\ldots,k$ and $|z| \notin F_0$, where F_0 is a set of finite logarithmic measure on $[0,1)$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq \left[\left(\frac{1}{1-|z|}\right)^{2+2\varepsilon} T\left(s(|z|), f\right)\right]^j.$$

It follows from Lemma 6.5 and Proposition 1.7 that there exists a set F_1 of infinite logarithmic measure on $[0,1)$ such that for all $|z| \in F_1 \setminus F_0$ we have

$$\begin{aligned}\left|\frac{f^{(j)}(z)}{f(z)}\right| &\leq \left[\left(\frac{1}{1-|z|}\right)^{2+2\varepsilon} T\left(s(|z|), f\right)\right]^j \\ &\leq \left[\left(\frac{1}{1-|z|}\right)^{2+2\varepsilon} \varphi^{-1}\left((\nu_1 + \varepsilon)\log\frac{1}{1-s(|z|)}\right)\right]^j.\end{aligned} \tag{6.15}$$

Since $F_1 \setminus F_0$ is a set of infinite logarithmic measure, there exists a sequence of points $r_n = |z_n| \in F_1 \setminus F_0$ tending to 1. Set $R_n = s(|z_n|) = 1 - d(1-|z_n|)$, $d \in (0,1)$. We have $1 - |z_n| = \frac{1-R_n}{d}$, $d \in (0,1)$. By substituting (6.13)–(6.15) into (6.12) for the above ε $(0 < 3\varepsilon < \nu_0 - \beta)$ and via (1.2), we obtain for $R_n \longrightarrow 1^-$ and $R_n \in F_1 \setminus F_0$ that

$$\begin{aligned}&\varphi^{-1}\left((\nu_0 - \varepsilon)\log\frac{d}{1-R_n}\right) \\ &\leq k\left[\left(\frac{d}{1-R_n}\right)^{2+2\varepsilon} \varphi^{-1}\left((\nu_1 + \varepsilon)\log\frac{1}{1-R_n}\right)\right]^k \varphi^{-1}\left((\beta + \varepsilon)\log\frac{d}{1-R_n}\right) \\ &\leq \left[\varphi^{-1}\left((\beta + \varepsilon)\log\frac{d}{1-R_n}\right)\right]^{k+2} \leq \varphi^{-1}\left((\beta + 2\varepsilon)\log\frac{d}{1-R_n}\right).\end{aligned}$$

By arbitrariness of ε $(0 < 3\varepsilon < \nu_0 - \beta)$ and the monotonicity of φ^{-1} we obtain the contradiction $\nu_0 \leq \beta$. Thus $\nu_1 = \widetilde{\nu}^1_{\varphi}(f) \geq \widetilde{\nu}^0_{\varphi}(A_0) = \nu_0$.

Now, we prove the converse inequality $\widetilde{\nu}^1_{\varphi}(f) \le \widetilde{\nu}^0_{\varphi}(A_0)$. Let $\theta_0 \in [0,2\pi]$ be such that $\left|f\left(re^{i\theta_0}\right)\right| = M(r,f)$. By Lemma 6.4, we have

$$\begin{aligned} M(r,f) &\le C\exp\left\{n_c\int_s^r \max_{j=0,1,\dots,k-1}\left|A_j\left(te^{i\theta_0}\right)\right|^{\frac{1}{k-j}}dt\right\} \\ &\le C\exp\left\{n_c\int_s^r \max_{j=0,1,\dots,k-1}\left|M(r,A_j)\right|^{\frac{1}{k-j}}dt\right\} \\ &\le C\exp\left\{n_c(r-s)\max_{j=0,1,\dots,k-1}M(r,A_j)\right\}. \end{aligned} \tag{6.16}$$

The definition of $\widetilde{\sigma}^0_{\varphi}(A_j)$ implies that for any $\varepsilon > 0$ and $r \to 1^-$, we have

$$M(r,A_j) \le \varphi^{-1}\left(\left(\beta+\frac{\varepsilon}{2}\right)\log\frac{1}{1-r}\right), \quad j = 1,\dots,k-1, \tag{6.17}$$

where $\beta = \max\left\{\widetilde{\sigma}^0_{\varphi}(A_j) : j = 1,\dots,k-1\right\}$. From Lemma 6.6, there exists a set $F_1 \subset [0,1)$ of infinite logarithmic measure such that for $r \in F_1$, $r \to 1^-$ we have

$$M(r,A_0) \le \varphi^{-1}\left(\left(\widetilde{\nu}^0_{\varphi}(A_0)+\frac{\varepsilon}{2}\right)\log\frac{1}{1-r}\right). \tag{6.18}$$

We deduce from (6.16)–(6.18) that for any $\varepsilon > 0$ and all $r \in F_1$, $r \to 1^-$

$$\log M(r,f) \le \varphi^{-1}\left(\left(\widetilde{\nu}^0_{\varphi}(A_0)+\varepsilon\right)\log\frac{1}{1-r}\right).$$

By arbitrariness of $\varepsilon > 0$ and the monotonicity of φ^{-1}, we obtain $\widetilde{\nu}^1_{\varphi}(f) \le \widetilde{\nu}^0_{\varphi}(A_0)$. Then, Theorem 6.4 is proved.

3.2 Proof of Theorem 6.5

Let f be a non-zero solution of equation (6.1). First, we prove $\sigma^0_{\varphi}(A_0) \le \sigma^1_{\varphi}(f)$. Suppose that to the contrary, $\sigma^1_{\varphi}(f) < \sigma^0_{\varphi}(A_0)$. Let β_1 and β_2 be two real constants satisfying $\sigma^1_{\varphi}(f) < \beta_1 < \beta_2 < \sigma^0_{\varphi}(A_0)$. Equation (6.1) can be written as

$$A_0 = -\left(\frac{f^{(k)}}{f} + A_{k-1}\frac{f^{(k-1)}}{f} + \cdots + A_1\frac{f'}{f}\right). \tag{6.19}$$

This with Lemma 1.2 give

$$\begin{aligned} m(r,A_0) &\le \sum_{j=1}^{k-1} m(r,A_j) + \sum_{j=1}^{k} m\left(r,\frac{f^{(j)}}{f}\right) + O(1) \\ &\le \sum_{j=1}^{k-1} m(r,A_j) + O\left(\log^+ T(r,f) + \log\frac{1}{1-r}\right) \end{aligned}$$

holds for all $r \to 1^-$ outside of an exceptional set $F_0 \subset [0,1)$ of finite linear measure. Assume that

$$\limsup_{r\to 1^-}\frac{\sum_{j=1}^{k-1} m(r,A_j)}{m(r,A_0)} = \nu < \lambda < 1.$$

Thus

$$\sum_{j=1}^{k-1} m(r,A_j) < \lambda m(r,A_0), \quad r \to 1^-.$$

Hence, for $r \to 1^-$, $r \notin F_0$ we have

$$(1-\lambda)m(r,A_0) < O\left(\log^+ T(r,f) + \log\frac{1}{1-r}\right). \tag{6.20}$$

Then, by Lemma 6.2 we get that

$$\begin{aligned} T(r,A_0) &= m(r,A_0) \le O\left(\log^+ T(r,f) + \log\frac{1}{1-r}\right) \\ &= O\left(\log^+ \varphi^{-1}\left(\beta_1 \log\frac{1}{1-r}\right)\right) \end{aligned} \tag{6.21}$$

holds for $r \to 1^-$ and $r \notin F_0$. It follows from Lemma 6.10 that there exists a set F_1 of infinite logarithmic measure on $[0,1)$ such that

$$T(r,A_0) > \log^+ \varphi^{-1}\left(\beta_2 \log\frac{1}{1-r}\right), \quad r \to 1^-. \tag{6.22}$$

Since $F_1 \setminus F_0$ is a set of infinite logarithmic measure, there exists a sequence of points $r_n = |z_n| \in F_1 \setminus F_0$ tending to 1. By substituting (6.22) into (6.21), we obtain for all $r_n = |z_n| \in F_1 \setminus F_0$, $r_n \to 1^-$ and any given ε, $0 < \varepsilon < \beta_2 - \beta_1$

$$\begin{aligned} \log^+ \varphi^{-1}\left(\beta_2 \log\frac{1}{1-r_n}\right) &\le O\left(\log^+ \varphi^{-1}\left(\beta_1 \log\frac{1}{1-r_n}\right)\right) \\ &\le \log^+ \varphi^{-1}\left((\beta_1+\varepsilon)\log\frac{1}{1-r_n}\right). \end{aligned} \tag{6.23}$$

By arbitrariness of ε ($0 < \varepsilon < \beta_2 - \beta_1$) and the monotonicity of φ^{-1}, from (6.23) we obtain $\beta_2 \le \beta_1$. This contradiction proves the inequality $\sigma_\varphi^0(A_0) \le \sigma_\varphi^1(f)$. On the other hand, it follows from Lemma 6.8 that $\sigma_\varphi^1(f) \le \widetilde{\sigma}_\varphi^0(A_0)$. Therefore,

$$\sigma_\varphi^0(A_0) \le \sigma_\varphi^1(f) = \widetilde{\sigma}_\varphi^1(f) \le \widetilde{\sigma}_\varphi^0(A_0).$$

Now, we prove $\nu_\varphi^0(A_0) \le \nu_\varphi^1(f)$. Suppose that to the contrary, $\nu_\varphi^1(f) < \nu_\varphi^0(A_0)$. Let α_1 and α_2 be two real constants satisfying $\nu_\varphi^1(f) < \alpha_1 < \alpha_2 < \nu_\varphi^0(A_0)$. It follows from Lemma 6.5 that there exists a set F_1 of infinite logarithmic measure on $[0,1)$ such that for $r \in F_1$, $r \to 1^-$ we have

$$T(r,f) < \varphi^{-1}\left(\alpha_1 \log\frac{1}{1-r}\right) \tag{6.24}$$

and for $r \to 1^-$ we have

$$T(r,A_0) > \log^+ \varphi^{-1}\left(\alpha_2 \log\frac{1}{1-r}\right). \tag{6.25}$$

Since $F_1 \setminus F_0$ is a set of infinite logarithmic measure, there exists a sequence of points $r_n = |z_n| \in F_1 \setminus F_0$ tending to 1. By substituting (6.24) and (6.25) into (6.20), we obtain for all $r_n = |z_n| \in F_1 \setminus F_0$, $r_n \to 1^-$ and any given ε ($0 < \varepsilon < \alpha_2 - \alpha_1$)

$$\begin{aligned} \log^+ \varphi^{-1}\left(\alpha_2 \log\frac{1}{1-r_n}\right) &\le O\left(\log^+ \varphi^{-1}\left(\alpha_1 \log\frac{1}{1-r_n}\right)\right) \\ &\le \log^+ \varphi^{-1}\left((\alpha_1+\varepsilon)\log\frac{1}{1-r_n}\right). \end{aligned} \tag{6.26}$$

By arbitrariness of ε $(0<\varepsilon<\alpha_2-\alpha_1)$ and the monotonicity of φ^{-1}, from (6.26) we obtain $\alpha_2\le\alpha_1$. This contradiction proves the inequality $\nu^0_\varphi(A_0)\le\nu^1_\varphi(f)$. By Lemma 6.7, we obtain $\nu^1_\varphi(f)\le\widetilde{\nu}^0_\varphi(A_0)$ and therefore

$$\nu^0_\varphi(A_0)\le\nu^1_\varphi(f)=\widetilde{\nu}^1_\varphi(f)\le\widetilde{\nu}^0_\varphi(A_0).$$

3.3 Proof of Theorem 6.6

If

$$\max\left\{\widetilde{\sigma}^0_\varphi(A_j):j=1,\dots,k-1\right\}<\widetilde{\sigma}^0_\varphi(A_0)<+\infty,$$

then by Theorem 6.3 we get $\widetilde{\sigma}^1_\varphi(f)=\widetilde{\sigma}^0_\varphi(A_0)$. Suppose now

$$\max\left\{\widetilde{\sigma}^0_\varphi(A_j):j=1,\dots,k-1\right\}=\widetilde{\sigma}^0_\varphi(A_0)=\sigma_0\ (0<\sigma_0<+\infty)$$

and

$$\max\left\{\widetilde{\kappa}^0_\varphi(A_j):\widetilde{\sigma}^0_\varphi(A_j)=\widetilde{\sigma}^0_\varphi(A_0),\ j\ne0\right\}<\widetilde{\kappa}^0_\varphi(A_0)=\kappa_0\ (0<\kappa_0<+\infty).$$

Clearly, there exists a set $J\subset\{1,\dots,k-1\}$ such that $\widetilde{\sigma}^0_\varphi(A_j)=\widetilde{\sigma}^0_\varphi(A_0)=\sigma_0$ and $\widetilde{\kappa}^0_\varphi(A_j)<\widetilde{\sigma}^0_\varphi(A_0)=\kappa_0$ for all $j\in J$ as well as $\widetilde{\sigma}^0_\varphi(A_j)<\widetilde{\sigma}^0_\varphi(A_0)=\sigma_0$ for all $j\in\{1,\dots,k-1\}\setminus J$. First, we prove $\widetilde{\sigma}^1_\varphi(f)\ge\widetilde{\sigma}^0_\varphi(A_0)$. Suppose that to the contrary, $\sigma_1=\widetilde{\sigma}^1_\varphi(f)<\widetilde{\sigma}^0_\varphi(A_0)=\sigma_0$. Let β_1 and β_2 be two real constants satisfying

$$\max\left\{\widetilde{\kappa}^0_\varphi(A_j):j\in J\right\}<\beta_1<\beta_2<\kappa_0.$$

Then, by the definition of $\widetilde{\sigma}^0_\varphi(A_j)$, for all $r\longrightarrow1^-$, we have

$$|A_j(z)|\le\varphi^{-1}\left(\log\frac{\beta_1}{(1-r)^{\sigma_0}}\right),\quad j\in J \tag{6.27}$$

and

$$|A_j(z)|\le\varphi^{-1}\left(\gamma\log\frac{1}{1-r}\right)\le\varphi^{-1}\left(\log\frac{\beta_1}{(1-r)^{\sigma_0}}\right),\quad j\in\{1,\dots,k-1\}\setminus J, \tag{6.28}$$

where $0<\gamma<\sigma_0$. It follow from Lemma 6.9 that there exists a set $F_1\subset[0,1)$ of infinite logarithmic measure such that for all $r\in F_1$, we have

$$|A_0(z)|=M(r,A_0)>\varphi^{-1}\left(\log\frac{\beta_2}{(1-r)^{\sigma_0}}\right). \tag{6.29}$$

By Lemma 6.1, for $j=1,\dots,k$ and $|z|\notin F_0$, where F_0 is a set of finite logarithmic measure on $[0,1)$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right|\le\left[\left(\frac{1}{1-|z|}\right)^{2+2\varepsilon}T(s(|z|),f)\right]^j.$$

It follows from the definition of $\sigma^1_\varphi(f)$ and Proposition 1.7 that for any given ε $(0<\varepsilon<\sigma_0-\sigma_1)$ and for all $|z|\notin F_0$ we have

$$\begin{aligned}\left|\frac{f^{(j)}(z)}{f(z)}\right| &\le \left[\left(\frac{1}{1-|z|}\right)^{2+2\varepsilon}T\left(s(|z|),f\right)\right]^j\\ &\le \left[\left(\frac{1}{1-|z|}\right)^{2+2\varepsilon}\varphi^{-1}\left((\sigma_1+\varepsilon)\log\frac{1}{1-s(|z|)}\right)\right]^j.\end{aligned} \tag{6.30}$$

Since $F_1 \setminus F_0$ is a set of infinite logarithmic measure, there exists a sequence of points $r_n = |z_n| \in F_1 \setminus F_0$ tending to 1. Set $R_n = s(|z_n|) = 1 - d(1-|z_n|)$, $d \in (0,1)$. Substituting (6.27)–(6.30) into (6.12) for any given ε $\left(0 < \varepsilon < \min\left\{\frac{\beta_2-\beta_1}{2}, \sigma_0 - \sigma_1\right\}\right)$, we obtain for $R_n \longrightarrow 1^-$ and $R_n \in F_1 \setminus F_0$ that

$$\begin{aligned}
&\varphi^{-1}\left(\log \frac{\beta_2 d^{\sigma_0}}{(1-R_n)^{\sigma_0}}\right) \\
&\leq k\left[\left(\frac{d}{1-R_n}\right)^{2+2\varepsilon} \varphi^{-1}\left((\sigma_1+\varepsilon)\log\frac{1}{1-R_n}\right)\right]^k \varphi^{-1}\left(\log\frac{\beta_1 d^{\sigma_0}}{(1-R_n)^{\sigma_0}}\right) \\
&\leq \left[\varphi^{-1}\left(\log\frac{(\beta_1+\varepsilon)\, d^{\sigma_0}}{(1-R_n)^{\sigma_0}}\right)\right]^{k+2} \leq \varphi^{-1}\left(\log\frac{(\beta_1+2\varepsilon)\, d^{\sigma_0}}{(1-R_n)^{\sigma_0}}\right).
\end{aligned}$$

The last estimate is verified in view of (1.2). By arbitrariness of ε $\left(0 < \varepsilon < \min\left\{\frac{\beta_2-\beta_1}{2}, \sigma_0-\sigma_1\right\}\right)$ and the monotonicity of φ^{-1} we obtain the contradiction $\beta_2 \leq \beta_1$. Thus $\widetilde{\sigma}^1_\varphi(f) \geq \widetilde{\sigma}^0_\varphi(A_0)$. It follows from Lemma 6.8 that

$$\widetilde{\sigma}^1_\varphi(f) \leq \max\left\{\widetilde{\sigma}^0_\varphi(A_j) : j = 0,1,\ldots,k-1\right\} = \widetilde{\sigma}^0_\varphi(A_0).$$

Therefore, $\widetilde{\sigma}^1_\varphi(f) = \widetilde{\sigma}^0_\varphi(A_0)$ and Theorem 6.6 is proved.

3.4 Proof of Theorem 6.7

Since $\widetilde{\nu}^0_\varphi(A_0) \leq \widetilde{\sigma}^0_\varphi(A_0)$, then by Theorem 6.6 we obtain $\widetilde{\sigma}^1_\varphi(f) = \widetilde{\sigma}^0_\varphi(A_0)$. If

$$\max\left\{\widetilde{\sigma}^0_\varphi(A_j) : j \neq 0\right\} < \widetilde{\nu}^0_\varphi(A_0),$$

then by Theorem 6.4 we have $\widetilde{\nu}^1_\varphi(f) = \widetilde{\nu}^0_\varphi(A_0)$. Assume that $\widetilde{\sigma}^0_\varphi(A_j) = \widetilde{\nu}^0_\varphi(A_0) = \nu_0$ and $\widetilde{\kappa}^0_\varphi(A_j) \leq \kappa < \underline{\widetilde{\kappa}}^0_\varphi(A_0) = \kappa_1$. We first prove the inequality $\nu_1 = \widetilde{\nu}^1_\varphi(f) \geq \widetilde{\nu}^0_\varphi(A_0) = \nu_0$. Suppose that to the contrary, $\nu_1 < \nu_0$. It follows from the definitions of $\widetilde{\kappa}^0_\varphi(A_j)$ and $\underline{\widetilde{\kappa}}^0_\varphi(A_0)$ that for any given $\varepsilon(0 < 3\varepsilon < \kappa_1 - \kappa)$ and for all $r \longrightarrow 1^-$, we have

$$|A_j(z)| \leq \varphi^{-1}\left(\log\frac{\kappa+\varepsilon}{(1-r)^{\nu_0}}\right) \tag{6.31}$$

and

$$|A_0(z)| \geq \varphi^{-1}\left(\log\frac{\kappa_1-\varepsilon}{(1-r)^{\nu_0}}\right). \tag{6.32}$$

By Lemma 6.1, for $j = 1,\ldots,k$ and $|z| \notin F_0$, where F_0 is a set of finite logarithmic measure on $[0,1)$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq \left[\left(\frac{1}{1-|z|}\right)^{2+2\varepsilon} T\left(s(|z|), f\right)\right]^j.$$

It follows from Proposition 1.7 and Lemma 6.5 that there exists a set $F_1 \subset [0,1)$ of infinite logarithmic measure such that for any given $\varepsilon(0 < \varepsilon < \nu_0 - \nu_1)$ and for all $|z| \in F_1 \setminus F_0$ we have

$$\begin{aligned}
\left|\frac{f^{(j)}(z)}{f(z)}\right| &\leq \left[\left(\frac{1}{1-|z|}\right)^{2+2\varepsilon} T\left(s(|z|), f\right)\right]^j \\
&\leq \left[\left(\frac{1}{1-|z|}\right)^{2+2\varepsilon} \varphi^{-1}\left((\nu_1+\varepsilon)\log\frac{1}{1-s(|z|)}\right)\right]^j.
\end{aligned} \tag{6.33}$$

Since $F_1 \setminus F_0$ is a set of infinite logarithmic measure, there exists a sequence of points $r_n = |z_n| \in F_1 \setminus F_0$ tending to 1. Set $R_n = s(|z_n|) = 1 - d(1-|z_n|)$, $d \in (0,1)$. Substituting (6.31)–(6.33) into (6.12) for any given ε $\left(0 < \varepsilon < \min\left\{\nu_0 - \nu_1, \frac{\kappa_1 - \kappa}{3}\right\}\right)$, we obtain for $R_n \longrightarrow 1^-$ and $R_n \in F_1 \setminus F_0$ that

$$\begin{aligned}
&\varphi^{-1}\left(\log\frac{(\kappa_1-\varepsilon)d^{\nu_0}}{(1-R_n)^{\nu_0}}\right)\\
&\le k\left[\left(\frac{d}{1-R_n}\right)^{2+2\varepsilon}\varphi^{-1}\left((\nu_1+\varepsilon)\log\frac{1}{1-R_n}\right)\right]^k\varphi^{-1}\left(\log\frac{(\kappa+\varepsilon)d^{\nu_0}}{(1-R_n)^{\nu_0}}\right)\\
&\le\left[\varphi^{-1}\left(\log\frac{(\kappa+\varepsilon)d^{\nu_0}}{(1-R_n)^{\nu_0}}\right)\right]^{k+2}\le\varphi^{-1}\left(\log\frac{(\kappa+2\varepsilon)d^{\nu_0}}{(1-R_n)^{\nu_0}}\right).
\end{aligned}$$

The last estimate is verified in view of (1.2). By arbitrariness of ε $\left(0 < \varepsilon < \min\left\{\nu_0 - \nu_1, \frac{\kappa_1 - \kappa}{3}\right\}\right)$ and the monotonicity of φ^{-1} we obtain the contradiction $\kappa_1 \le \kappa$. Thus $\widetilde{\nu}^1_\varphi(f) \ge \widetilde{\nu}^0_\varphi(A_0)$. It follows from Lemma 6.7 that $\widetilde{\nu}^1_\varphi(f) \le \widetilde{\nu}^0_\varphi(A_0)$. Therefore, $\widetilde{\nu}^1_\varphi(f) = \widetilde{\nu}^0_\varphi(A_0)$.

Conclusion

In this thesis, we have discussed some properties of the φ-order and the φ-type of analytic and meromorphic functions. We have applied those generalized concepts in the investigation of the properties of growth and oscillation of solutions of higher order linear differential equations of the form ($k \geq 2$)

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_0(z) f = 0,$$
$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_0(z) f = F(z),$$

where the coefficients $F \not\equiv 0, A_0 \not\equiv 0, A_1(z), \ldots, A_{k-1}(z)$ are meromorphic functions on the complex plane. We have related the growth and the exponent of convergence of the solutions to the growth of coefficients. We have obtained similar results when the coefficients are analytic in th unit disc. Nevertheless, this thesis shows that there are still much work to be done on this subject and arises many interesting questions and open problems such as:

Problem 1. Can we extend the results of Chapter 6 by considering meromorphic coefficients with finite φ-order in the unit disc?

Problem 2. What can be said about the growth and the oscillation of solutions of linear differential equations of the form

$$A_k(z) f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_0(z) f = 0$$

where the coefficients $A_0 \not\equiv 0, \ldots, A_k \not\equiv 1$ are entire functions with φ-order? Note that in this case the solutions can be meromorphic even if the coefficients are entire.

Problem 3. How about employing the concept of φ-order for investigating the fast growth of meromorphic solutions of linear difference equations of the form (see [17])

$$A_k(z) f(z+k) + \cdots + A_1(z) f(z+1) + A_0(z) f(z) = 0,$$

where all the coefficients $A_0, \ldots, A_k$ are non-zero entire functions?

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