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EXISTENCE RESULTS FOR BOUNDARY - VALUE PROBLEMS WITH NONLINEAR FRACTIONAL DIFFERENTIAL INCLUSIONS AND INTEGRAL CONDITIONS

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Abstract . In this article , the authors establish sufficient conditions for the existence of solutions for a class of boundary value problem for fractional differential inclusions involving the Caputo fractional derivative and nonlinear integral conditions . Both cases of convex and nonconvex valued right hand sides are considered . The topological structure of the set of solutions also examined .

1 . Introduction

This article concerns the existence and uniqueness of solutions of the boundary value problem (BVP for short) with fractional order differential inclusions and nonlinear integral conditions of the form

$$c_D\alpha_y(t) \in F(t,y), \quad \text{for a . e } .t \in J = [0,T], \quad 1 < \alpha \le 2, \quad (1.1)$$

$$y(0) - y'(0) = \int_0^T g(s,y)ds, \quad (1.2)$$

$$y(T) + y'(T) = \int_0^T h(s,y)ds, \quad (1.3)$$

where cD^α is the Caputo fractional derivative , $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map , $(\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R})$, and $g, \quad h \quad : \quad J \times \mathbb{R} \quad \to \mathbb{R}$ are given continuous functions . Differential equations of fractional order have recently proved to be valuable tools in the modelling of many phenomena in various fields of science and engineering . There are numerous applications to problems in viscoelasticity , electro chemistry , control , porous media , electromagnetics , etc .
(see [20 , 30 , 3 1 , 34 , 40 , 4 1 , 45]) . There has been a significant development in ordinary and partial differential equations involving both Riemann - Liouville and Caputo fractional derivatives in recent years ; see the monographs of Kilbas *e t al .*
[38] , Miller and Ross [42] , Samko *e t al .* [50] and the papers of Agarwal *e t al .*
[2] , Benchohra *e t al .* [8] , Benchohra and Hamani [9] , Daftardar - Gejji and Jafari
[1 7] , Delbosco and Rodino [1 9] , Diethelm *e t al .* [20 , 2 1 , 22] , El - Sayed [23 , 24 , 25] ,

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Furati and Tatar [28, 29], Kaufmann and Mboumi [36], Kilbas and Marzan [37], Mainardi [40], Momani and Hadid [43], Momani *et al.* [44], Ouahab [46], Podlubny *et al.* [49], Yu and Gao [52] and the references therein. In [7, 12] the authors studied the existence and uniqueness of solutions of classes of initial value problems for functional differential equations with infinite delay and fractional order, and in [6] a class of perturbed functional differential equations involving the Caputo fractional derivative has been considered. Related problems to (1.1)–(1.3) have been considered by means of different methods by Belarbi *et al.* [5] and Benchohra *et al.* in [10, 11] in the case of $\alpha = 2$.

Applied problems require definitions of fractional derivatives allowing the utilization of physically interpretable initial conditions that contain $y(0), y'(0)$, etc., and the same is true for boundary conditions. Caputo's fractional derivative satisfies these demands. For more details on the geometric and physical interpretation for fractional derivatives of both Riemann-Liouville and Caputo types see [33, 48]. The web site http://people.tuke.sk/igor.podlubny/ authored by Igor Podlubny contains more information on fractional calculus and its applications, and hence it is very useful for those interested in this field.

Boundary value problems with integral boundary conditions constitute a very interesting and important class of problems. They include two, three, multipoint, and nonlocal boundary value problems as special cases. Integral boundary conditions appear in population dynamics [13] and cellular systems [1].

This paper is organized as follows. In Section 2, we introduce some preliminary results needed in the following sections. In Section 3, we present an existence result for the problem (1.1)–(1.3) when the right hand side is convex valued by using the nonlinear alternative of Leray-Schauder type. In Section 4, two results are given for nonconvex valued right hand sides. The first one is based upon a fixed point theorem for contraction multivalued maps due to Covitz and Nadler [16], and the second one on the nonlinear alternative of Leray Schauder type [32] for single-valued maps, combined with a selection theorem due to Bressan-Colombo [14] (also see [27]) for lower semicontinuous multivalued maps with decomposable values. The topological structure of the solutions set is considered in Section 5. An example is presented in the last section. These results extend to the multivalued case some results from the above cited literature, and constitute a new contribution to this emerging field of research.

2. Preliminaries

In this section, we introduce notation, definitions, and preliminary facts that will be used in the remainder of this paper. Let $C(J,\mathbb{R})$ be the Banach space of all continuous functions from J to $\mathbb{R}$ with the norm

$$\|y\|_\infty = \sup\{|y(t)| : 0 \le t \le T\},$$

and let $L^1(J,\mathbb{R})$ denote the Banach space of functions $y : J \to \mathbb{R}$ that are Lebesgue integrable with norm

$$\|y\|_{L^1} = \int_0^T |y(t)|\,dt.$$

We let $L^\infty(J,\mathbb{R})$ be the Banach space of bounded measurable functions $y : J \to \mathbb{R}$ equipped with the norm

$$\|y\|_{L^\infty} = \inf\{c > 0 : |y(t)| \le c, \quad \text{a.e. } t \in J\}.$$

Also , $AC^1(J,\mathbb{R})$ will denote the space of functions $y : J \to \mathbb{R}$ that are absolutely continuous and whose first derivative , y', is absolutely continuous . Let $(X, \|\cdot\|)$ be a Banach space and let $P_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed }\}, P_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded }\}, P_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact }\}$, and $P_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex }\}$. A multivalued map $G : X \to P(X)$ is *convex* (*c los ed*) valued if $G(x)$ is convex (closed) for all $x \in X$. We say that G is *bounded on bounded s e ts* if $G(B) = \cup_{x\in B} G(x)$ is bounded in X for all $B \in P_b(X)$ (i . e ., $\sup_{x\in B}\{ \sup \{| y | : y \in G(x)\}\} < \infty)$. The mapping G is called *upper s emi - continuous* (*u . s . c .*) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood N_0 of x_0 such that $G(N_0) \subseteq N$. We say that G is *completely continuous* if $G(\mathcal{B})$ is relatively compact for every $\mathcal{B} \in P_b(X)$. If the multivalued map G is completely continuous with nonempty compact values , then G is u . s . c . if and only if G has a closed graph (i . e ., $x_n \to x_*$, $yn \to y*$, $yn \in G(x_n)$ imply $y* \in G(x_*)$). The mapping G has a *fixed point* if there is $x \in X$ such that $x \in G(x)$. The set of fixed points of the multivalued operator G will be denoted by $FixG$. A multivalued map $G : J \to P_{cl}(\mathbb{R})$ is said to be *measurable* if for every $y \in \mathbb{R}$, the
function

$$t \mapsto d(y, G(t)) = \inf\{| y - z |: z \in G(t)\}$$

is measurable . For more details on multivalued maps see the books of Aubin and Cellina [3] , Aubin and Frankowska [4] , Deimling [1 8] , and Hu and Papageorgiou

[35].

Definition 2 . 1 . A multivalued map $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is Carath é odory if (i) $t \mapsto F(t, u)$ is measurable for each $u \in \mathbb{R}$, and

(i i) $u \mapsto F(t, u)$ is upper semicontinuous for almost all $t \in J$.

For each $y \in C(J,\mathbb{R})$, define the set of *s e lections* for F by

$$S_{F,y} = \{v \in L^1(J,\mathbb{R}) : v(t) \in F(t, y(t)) \text{a.e.} t \in J\}.$$

Let (X, d) be a metric space induced from the normed space $(X, |\cdot|)$. Consider

$$H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{\infty\} \text{givenby}$$
$$H_d(A, B) = \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\},$$

where $d(A, b) = \inf_{a\in A} d(a, b)$ and $d(a, B) = \inf_{b\in B} d(a, b)$. Then $(P_{b,cl}(X), H_d)$ is a metric space and $(P_{cl}(X), H_d)$ is a generalized metric space (see [39]) .

Definition 2 . 2 . A multivalued operator $N : X \to P_{cl}(X)$ is called :

(a) $\gamma-$ Lipschitz if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \le \gamma d(x, y), \quad \text{forall} x, y \in X;$$

(b) a contraction if it is $\gamma-$ Lipschitz with $\gamma < 1$.

The following lemma will be used in the sequel . **Lemma 2 . 3** ([1 6]) **.** *Let* (X, d) *be a complete metric space . If* $N : X \to P_{cl}(X)$ *is a contraction , th en* $FixN \neq \emptyset$. **Definition 2 . 4** ([38 , 47]) **.** The fractional (arbitrary) order integral of the function $h \in L^1([a, b],\mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by

$$I_a^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s) ds,$$

where Γ is the gamma function . When $a = 0$, we write $I^\alpha h(t) = h(t) * \varphi\alpha(t)$, where $\varphi\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0, \varphi\alpha(t) = 0$ for $t \leq 0$, and $\varphi\alpha \to \delta(t)$ as $\alpha \to 0$, where δ is the delta function .

Definition 2 . 5 ([38 , 47]) **.** For a function h given on the interval $[a, b]$, the $\alpha-$ th Riemann - Liouville fractional - order derivative of h is defined by

$$(D^\alpha_{a+}h)(t) = \frac{1}{\Gamma(n-\alpha)}(\frac{d}{dt})^n \int_a^t (t-s)^{n-\alpha-1}h(s)ds.$$

Here $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α. **Definition 2 . 6** ([38]) **.** For a function h given on the interval $[a, b]$, the Caputo fractional - order derivative of h is defined by

$$({}^cD^\alpha_{a+}h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1}h^{(n)}(s)ds,$$

$$\text{where} n = [\alpha] + 1.$$

3 . The Convex Case

In this section , we are concerned with the existence of solutions for the problem (1 . 1) – (1 . 3) when the right hand side has convex values . Initially , we assume that F is a compact and convex valued multivalued map .

Definition 3 . 1 . A function $y \in AC^1(J, \mathbb{R})$ is said to be a solution of (1 . 1) – (1 . 3) , if there exists a function $v \in L^1(J, \mathbb{R})$ with $v(t) \in F(t, y(t))$, for a . e $.t \in J$, such that

$$c_D\alpha_y(t) = v(t), \quad \text{a.e.} \quad t \in J, \quad 1 < \alpha \leq 2,$$

and the function y satisfies conditions (1 . 2) and (1 . 3) .

For the existence of solutions for the problem (1 . 1) – (1 . 3) , we need the following auxiliary lemmas . **Lemma 3 . 2** ([53]) **.** *Let* $\alpha > 0$; *then the differential equation*

$$c_D\alpha_h(t) = 0$$

has the s o lutions $h(t) = c_0 + c_1t + c_2t^2 + \cdots + c_{n-1}t^{n-1}$, *where* $c_i \in \mathbb{R}$, $i =$

$$0, 1, 2, ..., n-1, and n = [\alpha] + 1.$$

Lemma 3 . 3 ([53]) **.** *Let* $\alpha > 0$; *then*

$$I^{\alpha^c}D^\alpha h(t) = h(t) + c_0 + c_1t + c_2t^2 + \cdots + c_{n-1}t^{n-1}$$

for s ome $c_i \in \mathbb{R}, i = 0, 1, 2, ..., n-1,$ *where* $n = [\alpha] + 1.$

As a consequence of Lemmas 3 . 2 and 3 . 3 , we have the following result which will be useful in the remainder of the paper . **Lemma 3 . 4 .** *Let* $1 < \alpha \leq 2$ *and le t* $\sigma, \rho1, \rho2 : J \to \mathbb{R}$ *be continuous . A function* y *is a s o lution of the fractional integral equation*

$$y(t) = P(t) + \int_0^T G(t,s)\sigma(s)ds, \tag{3.1}$$

where

$$P(t) = \frac{T+1-t}{T+2} \int_0^T \rho 1(s) ds + \frac{t+1}{T+2} \int_0^T \rho 2(s) ds \tag{3.2}$$

and

$$G(t,s) = braceleftmid - braceleftbt - \frac{(t-s)^{\alpha-1}}{(1+t)(^{(T+}_{\Gamma(\alpha)}} \frac{T^{-}_{-} \quad (^{s)^{\alpha-1}}_{(1+}}{2)\Gamma(\alpha)} \frac{t)(T-s)^{\alpha-1}}{T(1+t)(^{(T+2)}_{+_{-}2)\Gamma(\alpha)}} \frac{T^{-}_{-} \quad (T^{s)^{\alpha-2}}_{(1+}}{\Gamma(\alpha-1)} \frac{t)(T-s)^{\alpha-2}}{+,2)\Gamma(\alpha-1)}, \tag{3.3}$$

if and only if y is a s o lution of the fra ctional BVP

$$c_D\alpha_y(t) = \sigma(t), \quad t \in J, \tag{3.4}$$

$$y(0) - y'(0) = \int_0^T \rho 1(s)ds, \tag{3.5}$$

$$y(T) + y'(T) = \int_0^T \rho 2(s)ds. \tag{3.6}$$

Proof . Assume that y satisfies (3 . 4) ; then Lemma 3 . 3 implies

$$y(t) = c_0 + c_1 t + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\sigma(s)ds. \tag{3.7}$$

From (3 . 5) and (3 . 6) , we obtain

$$c_0 - c_1 = \int_0^T \rho 1(s)ds \tag{3.8}$$

and

$$\begin{aligned} c_0 + c_1(T+1) + \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1}\sigma(s)ds \\ + \frac{1}{\Gamma(\alpha-1)} \int_0^T (T-s)^{\alpha-2}\sigma(s)ds \\ = \int_0^T \rho 2(s)ds. \end{aligned} \tag{3.9}$$

Solving (3 . 8) – (3 . 9) , we have

$$\begin{aligned} c_1 = \frac{1}{T+2} \int_0^T \rho 2(s)ds - \frac{1}{T+2} \int_0^T \rho 1(s)ds \\ - \frac{1}{(T+2)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1}\sigma(s)ds \\ - \frac{1}{(T+2)\Gamma(\alpha-1)} \int_0^T (T-s)^{\alpha-2}\sigma(s)ds \end{aligned} \tag{3.10}$$

and

$$
\begin{aligned}
c_0 = & \frac{T+1}{T+2}\int_0^T \rho1(s)ds + \frac{1}{T+2}\int_0^T \rho2(s)ds \\
& -\frac{1}{(T+2)\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\sigma(s)ds \\
& -\frac{1}{(T+2)\Gamma(\alpha-1)}\int_0^T (T-s)^{\alpha-2}\sigma(s)ds.
\end{aligned} \tag{3.11}
$$

From (3 . 7) , (3 . 1 0) , (3 . 1 1) , and the fact that $\int_0^T = \int_0^t + \int_t^T$, we obtain (3 . 1) .

Conversely , if y satisfies equation (3 . 1) , then clearly (3 . 4) – (3 . 6) hold . □

 Remark 3 . 5 .
It is clear that the function $t \mapsto \int_0^T \mid G(t,s) \mid ds$ is continuous on J, and hence is bounded . Thus , we let

$$\tilde{G} := \sup\{\int_0^T \mid G(t,s) \mid ds, t \in J\}.$$

Our first result is based on the nonlinear alternative of Leray - Schauder type for multivalued maps [32] .

Theorem 3 . 6 . *Assume that the following hypotheses hold :*

(H 1) $F : J \times \mathbb{R} \to \mathcal{P}_{cp,c}(\mathbb{R})$ *is a Carath é odory multi - valued map ;*

(H 2) *There exist* $p \in L^\infty(J, \mathbb{R}^+)$ *and a continuous nondecreasing function* ψ :

$$[0, \infty) \to (0, \infty) such that$$

$\| F(t,u) \| \mathcal{P} = \sup \{\mid v \mid : v \in F(t,u)\} \leq p(t)\psi(\mid u \mid)$ *for al l* $t \in J . u \in \mathbb{R}$; (H 3) *There exist* $\phi_g \in L^1(J, \mathbb{R}^+)$ *and a continuous nondecreasing function* ψ^* :

$.The proof will be given in$

several steps .

Step 1 : *$N(y)$ is convex for each $y \in C(J,\mathbb{R})$.* Indeed , if h_1 and h_2 belong to $N(y)$, then there exist $v_1, v_2 \in S_{F,y}$ such that , for all $t \in J$, we have

$$h_i(t) = P_y(t) + \int_0^T G(t,s)v_i(s)ds, \quad i = 1,2.$$

Let $0 \leq d \leq 1$. Then , for each $t \in J$, we have

$$(dh_1 + (1-d)h_2)(t) = P_y(t) + \int_0^T G(t,s)[dv_1(s) + (1-d)v_2(s)]ds.$$

Since $S_{F,y}$ is convex (because F has convex values) , we have $dh_1 + (1-d)h_2 \in N(y)$.

Step 2 : *N maps bounded s e ts into bounded s e ts in $C(J,\mathbb{R})$.* Let $B_{\eta^*} = \{y \in C(J,\mathbb{R}) : \ \| y \| \infty \leq \eta^*\}$ be a bounded set in $C(J,\mathbb{R})$ and let $y \in B_{\eta^*}$. Then for each $h \in N(y)$ and $t \in J$, from (H 2) – (H 4) , we have

$$\begin{aligned} | h(t) | &\leq \frac{T+1}{T+2}\int_0^T | g(s,y(s)) | ds + \frac{T+1}{T+2}\int_0^T | h(s,y(s)) | ds \\ &\quad + \int_0^T G(t,s) | v(s)) | ds \\ &\leq \frac{T+1}{T+2}\psi^*(\| y \|_\infty)\int_0^T \phi_g(s)ds + \frac{T+1}{T+2}\text{-}\psi_{(\| y \|_\infty)}\int_0^T \phi h(s)ds \\ &\quad + \psi(\| y \|_\infty) \| p \| L\infty\tilde{G}. \end{aligned}$$

Therefore ,

$$\| h \|_\infty \leq \frac{T+1}{T+2}\psi^*(\eta^*)\int_0^T \phi_g(s)ds + \frac{T+1}{T+2}\text{-}\psi^*_{(\eta)}\int_0^T \phi h(s)ds + \psi(\eta^*) \| p \| L\infty\tilde{G} := \ell.$$

Step 3 : *N maps bounded s e ts into equicontinuous s e ts of $C(J,\mathbb{R})$.* Let $t_1, t_2 \in J$ with $t_1 < t_2$, let B_{η^*} be a bounded set in $C(J,\mathbb{R})$ as in Step 2 , and let $y \in B_{\eta^*}$ and

$h \in N(y)$.Then

$$\begin{aligned} | h(t_2) - h(t_1) | &= \frac{t_2-t_1}{T+2}\int_0^T | g(s,y(s)) | ds + \frac{t_2-t_1}{T+2}\int_0^T | h(s,y(s)) | ds \\ &\quad + \int_0^T | G(t_2,s) - G(t_1,s) || v(s) | ds. \\ &\leq \frac{t_2-t_1}{T+2}\psi^*(\eta^*)\int_0^T \phi_g(s)ds + \frac{t_2-t_1}{T+2}\text{-}\psi^*_{(\eta)}\int_0^T \phi h(s)ds \\ &\quad + \psi(\eta^*) \| p \| L\infty \int_0^T | G(t_2,s) - G(t_1,s) | ds. \end{aligned}$$

As $t_1 \to t_2$, the right - hand side of the above inequality tends to zero . As a conse - quence of Steps 1 to 3 together with the Arzel á– Ascoli theorem , we can conclude that $N : C(J,\mathbb{R}) \to \mathcal{P}(C(J,\mathbb{R}))$ is completely continuous .

Step 4 : *N has a clos ed graph .* Let $yn \to y*, h_n \in N(yn)$, and $h_n \to h_*$. We
need to show that $h_* \in N(y*)$. Now , $h_n \in N(yn)$ implies there exists $v_n \in S_{F,yn}$ such that , for each $t \in J$,

$$h_n(t) = P_{yn}(t) + \int_0^T G(t,s)v_n(s)ds.$$

 We must show that there exists $v_* \in S_{F,y*}$ such that for each $t \in J$,

$$h_*(t) = P_{y*}(t) + \int_0^T G(t,s)v_*(s)ds.$$

Since $F(t,\cdot)$ is upper semicontinuous , for every $\varepsilon > 0$, there exist $n_0(\epsilon) \geq 0$ such that for every $n \geq n_0$, we have

$$v_n(t) \in F(t, yn(t)) \subset F(t, y*(t)) + \varepsilon B(0,1) \quad \text{a.e.} t \in J.$$

Since $F(\cdot,\cdot)$ has compact values , there exists a subsequence $v_{n_m}(\cdot)$ such that

$$v_{n_m}(\cdot) \to v_*(\cdot) \quad \text{as} m \to \infty,$$
$$v_*(t) \in F(t, y*(t)) \quad \text{a.e.} t \in J.$$

For every $w \in F(t, y*(t))$, we have

$$| v_{n_m}(t) - v_*(t) | \leq | v_{n_m}(t) - w | + | w - v_*(t) |,$$

and so

$$| v_{n_m}(t) - v_*(t) | \leq d(v_{n_m}(t), F(t, y*(t))).$$

By an analogous relation obtained by interchanging the roles of v_{n_m} and v_*, it follows that

$$| v_{n_m}(t) - v_*(t) | \leq H_d(F(t, yn(t)), F(t, y*(t))) \leq l(t) \parallel yn - y* \parallel_\infty .$$

Therefore ,

$$\begin{aligned} | h_{n_m}(t) - h_*(t) | \leq & \int_0^T | g(s, yn_m(s)) - g(s, y*(s)) | ds \\ & + \int_0^T | h(s, yn_m(s)) - h(s, y*(s)) | ds \\ & + \int_0^T G(t,s) | v_{n_m}(s) - v_*(s) | ds. \end{aligned}$$

Since

$$\begin{aligned} \int_0^T G(t,s) | v_{n_m}(s) - v_*(s) | ds \leq & \int_0^T G(t,s)l(s)ds \parallel yn_m - y* \parallel_\infty \\ & \leq \tilde{G} \parallel l \parallel L\infty \parallel yn_m - y* \parallel \infty, \end{aligned}$$

and g and h are continuous , $\parallel h_{n_m} - h_* \parallel_\infty \to 0$ as $m \to \infty$.

Step 5 : *A priori bounds on s o lutions .* Let y be a possible solution of the problem
(1 . 1) – (1 . 3) . Then , there exists $v \in S_{F,y}$ such that , for each $t \in J$,

$$\begin{aligned} | y(t) | \leq & \frac{T+1}{T+2} \int_0^T \phi_g(s)\psi^*(| y(s) |)ds + \frac{T+1}{T+2} \int_0^T \phi h(s) \overline{\qquad} \psi(| y(s) |)ds \\ & + \int_0^T G(t,s)p(s)\psi(| y(s) |)ds \\ \leq & \frac{T+1}{T+2}\psi^*(\parallel y \parallel_\infty) \int_0^T \phi_g(s)ds + \frac{T+1}{T+2} \text{-}\psi_(\parallel y \parallel \infty) \int_0^T \phi h(s)ds \\ & + \psi(\parallel y \parallel_\infty)^{\tilde{G}} \parallel p \parallel L\infty. \end{aligned}$$

 Therefore ,

$$\frac{\| y \|_\infty}{a\psi^*(\| y \|_\infty) + b\psi(\| y \|_\infty) + c\tilde{G}\psi(\| y \|_\infty)} \leq 1.$$

Hence , by (3 . 1 2) , there exists M such that $\| y \|_\infty \neq M$. Let

$$U = \{y \in C(J,\mathbb{R}) : \| y \|_\infty < M\}.$$

The operator $N : \bar{U} \to \mathcal{P}(C(J,\mathbb{R}))$ is upper semicontinuous and completely con - tinuous . From the choice of U, there is no $y \in \partial U$ such that $y \in \lambda N(y)$ for some $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray - Schauder type ,

we conclude that N has a fixed point y in U which is a solution of the problem (1 . 1) – (1 . 3) . This completes the proof of the theorem . □

4 . The Nonconvex Case

This section is devoted to proving the existence of solutions for (1 . 1) – (1 . 3) with a nonconvex valued right hand side . Our first result is based on the fixed point theorem for contraction multivalued maps given by Covitz and Nadler [1 6] ; the second one makes use of a selection theorem due to Bressan and Colombo (see [1 4 , 27]) for lower semicontinuous operators with decomposable values combined with the nonlinear Leray - Schauder alternative .

Theorem 4 . 1 . *Assume that (H 5) and the following hypotheses hold :*

(H 7) *There exists a constant* $k^* > 0$ *such that* $| g(t,u) - g(t,\bar{u}) | \leq k^* | u - \bar{u} |$ *for all* $t \in J$ *and* $u, \bar{u} \in \mathbb{R}$.

(H 8) *There exists a constant* $k^{**} > 0$ *such that* $| h(t,u) - h(t,\bar{u}) | \leq k^{**} | u - \bar{u} |$ *for all* $t \in J$ *and* $u, \bar{u} \in \mathbb{R}$.

(H 9) $F : J \times \mathbb{R} \to P_{cp}(\mathbb{R})$ *has the property that* $F(\cdot,u) : J \to P_{cp}(\mathbb{R})$ *is measurable , and integrably bounded for each* $u \in \mathbb{R}$.

If

$$[\frac{T(T+1)}{T+2}k^* + \frac{T(T+1)}{T+2}k^{**} + k\tilde{G}] < 1, \tag{4.1}$$

where $k = \| l \|_{L^\infty}$, *then (1 . 1) – (1 . 3) has at least one solution on* J.

Remark 4 . 2 . For each $y \in C(J,\mathbb{R})$, the set $S_{F,y}$ is nonempty since , by (H 9), F has a measurable selection (see [1 5 , Theorem III . 6]) .

Proof of Theorem 4 . 1 . We shall show that N given in (3 . 1 3) satisfies the assump - tions of Lemma 2 . 3 . The proof will be given in two steps .

Step 1 : $N(y) \in P_{cl}(C(J,\mathbb{R}))$ *for all* $y \in C(J,\mathbb{R})$. Let $(h_n)_{n \geq 0} \in N(y)$ be such that $h_n \to \tilde{h} \in C(J,\mathbb{R})$. Then there exists $v_n \in S_{F,y}$ such that , for each $t \in J$,

$$h_n(t) = P_y(t) + \int_0^T G(t,s)v_n(s)ds.$$

From (H 5) and the fact that F has compact values , we may pass to a subsequence if necessary to obtain that v_n converges weakly to v in $L^1_w(J,\mathbb{R})$ (the space en - dowed with the weak topology) . Using a standard argument , we can show that v_n converges strongly to v and hence $v \in S_{F,y}$. Thus , for each $t \in J$,

$$h_n(t) \to \tilde{h}(t) = P_y(t) + \int_0^T G(t,s)v(s)ds,$$

$$\text{so} \tilde{h} \in N(y).$$

Step 2: *There exists* $\gamma < 1$ *such that*

$$H_d(N(y), N(\overline{y})) \leq \gamma \| y - \bar{y} \|_\infty \quad \text{for all} y, \bar{y} \in C(J, \mathbb{R}).$$

Let $y, \bar{y} \in C(J, \mathbb{R})$ and $h_1 \in N(y)$. Then, there exists $v_1(t) \in F(t, y(t))$ such that,

$$\text{for each} t \in J,$$

$$h_1(t) = P_y(t) + \int_0^T G(t,s) v_1(s) ds.$$

From (H5) it follows that

$$H_d(F(t, y(t)), F(t, \overline{y}(t))) \leq l(t) \mid y(t) - \overline{y}(t) \mid.$$

Hence, there exists $w \in F(t, \overline{y}(t))$ such that

$$\mid v_1(t) - w \mid \leq l(t) \mid y(t) - \overline{y}(t) \mid, \quad t \in J.$$

Consider $U : J \to \mathcal{P}(\mathbb{R})$ given by

$$U(t) = \{ w \in \mathbb{R} : \mid v_1(t) - w \mid \leq l(t) \mid y(t) - \overline{y}(t) \mid \}.$$

Since the multivalued operator $V(t) = U(t) \cap F(t, \overline{y}(t))$ is measurable (see Proposition [15, III.4]), there exists a function $v_2(t)$ which is a measurable selection for V. Thus, $v_2(t) \in F(t, \overline{y}(t))$, and for each $t \in J$,

$$\mid v_1(t) - v_2(t) \mid \leq l(t) \mid y(t) - \overline{y}(t) \mid.$$

For each $t \in J$, define

$$h_2(t) = P_{\bar{y}}(t) + \int_0^T G(t,s) v_2(s) ds,$$

where

$$P_{\bar{y}}(t) = \frac{T+1-t}{T+2} \int_0^T g(s, \overline{y}(s)) ds + \frac{t+1}{T+2} \int_0^T h(s, \overline{y}(s)) ds.$$

Then, for $t \in J$,

$$\begin{aligned}
\mid h_1(t) - h_2(t) \mid &\leq \frac{T+1}{T+2} \int_0^T \mid g(s, y(s)) - g(s, \overline{y}(s)) \mid ds \\
&\quad + \frac{T+1}{T+2} \int_0^T \mid h(s, y(s)) - h(s, \overline{y}(s)) \mid ds \\
&\quad + \int_0^T G(s,t) \mid v_1(s) - v_2(s) \mid ds \\
&\leq \frac{T(T+1)}{T+2} k^* \| y - \overline{y} \|_\infty + \frac{T(T+1)}{T+2} k^{**} \| y - \bar{y} \|_\infty + \tilde{G} k \| y - \overline{y} \|_\infty \\
&\leq \Big[\frac{T(T+1)}{T+2} k^* + \frac{T(T+1)}{T+2} k^{**} + k \tilde{G} \Big] \| y - \overline{y} \|_\infty.
\end{aligned}$$

Therefore ,

$$\| h_1 - h_2 \| \infty \leq [\frac{T(T+1)}{T+2}k^* + \frac{T(T+1)}{T+2}k^{**} + k\tilde{G}] \| y - \overline{y} \| \infty.$$

By an analogous relation , obtained by interchanging the roles of y and $\bar{y}$, it follows that

$$H_d(N(y), N(\overline{y})) \leq [\frac{T(T+1)}{T+2}k^* + \frac{T(T+1)}{T+2}k^{**} + k\tilde{G}] \| y - \overline{y} \| \infty.$$

Therefore , by (4.1), N is a contraction , and so by Lemma 2.3, N has a fixed point y that is a solution to (1 . 1) – (1 . 3) . The proof is now complete . □

Next , we present a result for problem (1 . 1) – (1 . 3) in the spirit of the nonlinear alternative of Leray Schauder type [32] for single - valued maps combined with a selection theorem due to Bressan and Colombo [14] for lower semicontinuous multi - valued maps with decomposable values . Details on multivalued maps with decom - posable values and their properties can be found in the recent book by Fryszkowski [27].

Let A be a subset of $[0,T]\times\mathbb{R}$. We say that A is $\mathcal{L}\otimes\mathcal{B}$ *measurable* if A belongs to the $\sigma-$ algebra generated by all sets of the form $\mathcal{J}\times D$ where $\mathcal{J}$ is Lebesgue measurable in $[0,T]$ and D is Borel measurable in $\mathbb{R}$. A subset A of $L^1([0,T],\mathbb{R})$ is *decomposable* if for all $u,v\in A$ and measurable $\mathcal{J}\subset[0,T], u\chi_{\mathcal{J}}+v\chi_[0,T]-\mathcal{J}\in A$, where χ stands for the characteristic function .

Let $G:X\to\mathcal{P}(X)$ be a multivalued operator with nonempty closed values . We say that G is *lower s emi - continuous (l . s . c .)* if the set $\{x\in X: G(x)\cap B\neq\varnothing\}$ is open for any open set B in X.

Definition 4 . 3 . Let Y be a separable metric space and $N:Y\to\mathcal{P}(L^1([0,T],\mathbb{R}))$ be a multivalued operator . We say N has property (BC) if

(1) N is lower semi - continuous (l . s . c .) ;

(2) N has nonempty closed and decomposable values .

Let $F:[0,T]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ be a multivalued map with nonempty compact values . Assign to F the multivalued operator $\mathcal{F}:C([0,T],\mathbb{R})\to\mathcal{P}(L^1([0,T],\mathbb{R}))$ by

$$\mathcal{F}(y)=\{w\in L^1([0,T],\mathbb{R}): w(t)\in F(t,y(t)) \text{ for a . e } .t\in[0,T]\}.$$

The operator $\mathcal{F}$ is called the Niemytzki operator associated to F.

Definition 4 . 4 . Let $F:[0,T]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ be a multivalued function with non - empty compact values . We say F is of lower semi - continuous type (l . s . c . type) if its associated Niemytzki operator $\mathcal{F}$ is lower semi - continuous and has nonempty closed and decomposable values .

Next , we state a selection theorem due to Bressan and Colombo .

Theorem 4 . 5 ([1 4]) **.** *Let Y be s eparable metric space and $N:Y\to\mathcal{P}(L^1([0,T],\mathbb{R}))$ be a multivalued operator that has property (BC) . Then N has a continuous s e - lection , i . e . , the re exists a continuous (s ingle - valued) function $\tilde{g}:Y\to L^1([0,1],\mathbb{R})$ such that $\tilde{g}(y)\in N(y)$ for every $y\in Y$.*

Let us introduce the hypotheses

(H 10) $F:[0,T]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ is a nonempty compact valued multivalued map such that :

(a) $(t,u)\mapsto F(t,u)$ is $\mathcal{L}\otimes\mathcal{B}$ measurable ;

(b) $y\mapsto F(t,y)$ is lower semi - continuous for a . e $.t\in[0,T]$;

(H 1 1) for each $q>0$, there exists a function $h_q\in L^1([0,T],\mathbb{R}^+)$ such that $\| F(t,y)\|\mathcal{P}\leq h_q(t)$ for a . e $.t\in[0,T]$ and for $y\in\mathbb{R}$ with $|y|\leq q$.

The following lemma is crucial in the proof of our main theorem .

Lemma 4 . 6 ([26]) **.** *Let $F:[0,T]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ be a multivalued map with nonempty compact values . Assume that* (H 1 0) , (H 1 1) *hold . Then F is of lower s emicontinuous* *type* .

We are now ready for our next main result in this section .

Theorem 4 . 7 . *Suppose that conditions* (H 2) – (H 4) , (H 6) , (H 1 0) , (H 1 1) *are satis - fied . Then the problem* (1 . 1) – (1 . 3) *has at least one s o lution .*

Proof . Conditions (H 1 0) and (H 1 1) imply , by Lemma 4 . 6 , that F is of lower semi - continuous type . By Theorem 4 . 5 , there exists a continuous function $f : C([0,T],\mathbb{R}) \to L^1([0,T],\mathbb{R})$ such that $f(y) \in \mathcal{F}(y)$ for all $y \in C([0,T],\mathbb{R})$. Consider the problem :

$$c_D\alpha_y(t) = f(y)(t), \quad \text{for a . e } .t \in J = [0,T], \quad 1 < \alpha \le 2, \quad (4.2)$$

$$y(0) - y'(0) = \int_0^T g(s,y)ds, \quad (4.3)$$

$$y(0) - y'(0) = \int_0^T g(s,y)ds. \quad (4.4)$$

Observe that if $y \in AC^1([0,T],\mathbb{R})$ is a solution of the problem (4 . 2) – (4 . 4) , then y is a solution to the problem (1 . 1) – (1 . 3) .

We reformulate the problem (4 . 2) – (4 . 4) as a fixed point problem for the operator

$$N_1 : C([0,T,\mathbb{R}) \to C([0,T],\mathbb{R}) \text{definedby :}$$
$$N_1(y)(t) = P_y(t) + \int_0^T G(t,s)f(y)(s)ds,$$

where the functions P_y and G are given by (3 . 1 4) and (3 . 3) , respectively . Using (H 2) – (H 4) and (H 6) , we can easily show (using arguments similar to those in the
proof of Theorem 3 . 6) that the operator N_1 satisfies all conditions in the Leray - Schauder alternative . □

5 . Topological Structure of the Solutions Set

In this section , we present a result on the topological structure of the set of solutions of (1 . 1) – (1 . 3) .

Theorem 5 . 1 . *Assume that* (H 1) *and the following hypotheses hold :*

(H 1 2) *There exists* $p \in C(J,\mathbb{R}^+)$ *such that* $\| F(t,u) \| \mathcal{P} \le p(t)(| u | +1)$ *for al l* $t \in J$

$$andu \in \mathbb{R};$$

(H 1 3) *There exists* $p1 \in C(J,\mathbb{R}^+)$ *such that* $| g(t,u) | \le p1(t)(| u | +1)$ *for al l* $t \in J$

$$andu \in \mathbb{R};$$

(H 1 4) *There exists* $p2 \in C(J,\mathbb{R}^+)$ *such that* $| h(t,u) | \le p2(t)(| u | +1)$ *for all* $t \in J$

$$andu \in \mathbb{R}.$$

If

$$\frac{T(T+1)}{T+2}\frac{M+1}{M}[\| p1 \|_\infty + \| p2 \|_\infty + \tilde{G}\frac{T+2}{T(T+1)} \| p \| L\infty] < 1,$$

then the s o lution s e t of (1 . 1) – (1 . 3) *is nonempty and compact in* $C(J, \mathbb{R})$.
Proof . Let
$S = \{y \in C(J, \mathbb{R}) : y$ is solution of (1 . 1) − (1 . 3) } . From Theorem 3.6, $S \neq \varnothing$.
Now , we prove that S is compact . Let $(yn)_n \in \mathbb{N} \in S$;
then there exists $v_n \in S_{F,yn}$ such that , for $t \in J$,

$$yn(t) = P_{yn}(t) + \int_0^T G(t,s)v_n(s)ds,$$

 where

$$P_{yn}(t) = \frac{T+1-t}{T+2}\int_0^T g(s,yn(s))ds + \frac{t+1}{T+2}\int_0^T h(s,yn(s))ds$$

and the function $G(t,s)$ is given by (3 . 3) .

From (H 1 2) – (H 14) we can prove that there exists a constant $M_1 > 0$ such that

$$\| yn \|_\infty \leq M_1 \quad \text{forall} n \geq 1.$$

As in Step 3 of the proof of Theorem 3 . 6 , we can easily show that the set $\{yn : n \geq 1\}$ is equicontinuous in $C(J,\mathbb{R})$, and so by the Arz $\acute{e}$ la - Ascoli Theorem , we can conclude that there exists a subsequence (denoted again by $\{yn\}$) of $\{yn\}$ converging to y in $C(J,\mathbb{R})$. We shall show that there exist $v(\cdot) \in F(\cdot, y(\cdot))$ such that

$$y(t) = P_y(t) + \int_0^T G(t,s)v(s)ds.$$

Since $F(t,\cdot)$ is upper semicontinuous , for every $\varepsilon > 0$, there exists $n_0(\epsilon) \geq 0$ such that , for every $n \geq n_0$, we have

$$v_n(t) \in F(t, yn(t)) \subset F(t, y(t)) + \varepsilon B(0,1) \quad \text{a.e.} t \in J.$$

Since $F(\cdot,\cdot)$ has compact values , there exists a subsequence $v_{n_m}(\cdot)$ such that

$$v_{n_m}(\cdot) \to v(\cdot) \quad \text{as} m \to \infty,$$
$$v(t) \in F(t, y(t)) \quad \text{a.e.} t \in J.$$

It is clear that the subsequence $v_{n_m}(t)$ is integrally bounded . By the Lebesgue dominated convergence theorem , we have that $v \in L^1(J,\mathbb{R})$, which implies that

$$v \in S_{F,y}.\text{Thus},$$

$$y(t) = P_y(t) + \int_0^T G(t,s)v(s)ds, \quad t \in J.$$

Hence , $S \in \mathcal{P}_{cp}(C(J,\mathbb{R}))$, and this completes the proof of the theorem . □

6 . An Example

As an application of the main results , we consider the fractional differential inclusion

$$c_D \alpha_y(t) \in F(t,y), \quad \text{a.e.} t \in J = [0,1], \quad 1 < \alpha \leq 2, \tag{6.1}$$

$$y(0) - y'(0) = \int_0^1 s^5(1+ \mid y(s) \mid)ds, \tag{6.2}$$

$$y(1) + y'(1) = \int_0^1 s^5(1+ \mid y(s) \mid)ds. \tag{6.3}$$

Set

$$F(t,y) = \{v \in \mathbb{R} : f1(t,y) \leq v \leq f2(t,y)\},$$

where $f1, f2 : J \times \mathbb{R} \to \mathbb{R}$ are measurable in t and Lipschitz continuous in y. We assume that for each $t \in J, f1(t,\cdot)$ is lower semi - continuous (i . e . , the set $\{y \in \mathbb{R} : f1(t,y) > \mu\}$ is open for all $\mu \in \mathbb{R}$), and assume that for each $t \in J, f2(t,\cdot)$ is upper semi - continuous (i . e . , the set $\{y \in \mathbb{R} : f2(t,y) < \mu\}$ is open for each) . Assume that

$$\max (\mid f1(t,y) \mid, \quad \mid f2(t,y) \mid) \leq \tfrac{t}{9}(1+ \mid y \mid) \quad \text{for all } t \in J \text{ and } y \in \mathbb{R}.$$

 From (3.3), G is given by

$$G(t,s) = braceleftmid-braceleftbt\frac{(t-s)^{\alpha-1}}{-(1+t)(^{3\Gamma}_{\Gamma(\alpha)}} \frac{1-^{-s)^{\alpha-1}}_{(1+}}{(\alpha)} \frac{t)(1-s)^{\alpha-1}}{(1+t)(^{3\Gamma(}_{3-\Gamma(\alpha)}} \frac{1-^{-s)^{\alpha-2}}_{(1+}}{\alpha-1)} \frac{t)(1-s)^{\alpha-2},}{3,\Gamma(\alpha-1)} \quad 0_t \leq \leq s^s <^{\leq}$$

We have $T = 1, \phi_g(t) = t^5, \phi h(t) = t^5, a = 1/9, b = 1/9, c = 1/9$, and

$$\psi(y) = 1+y, \quad \psi^*(y) = 1+y, \quad \text{-}\psi_{(y)} = 1+y, \quad \text{forall} y \in [0,\infty).$$

Also ,

$$\int_0^1 G(t,s)ds = \int_0^t G(t,s)ds + \int_t^1 G(t,s)ds$$

$$= \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{(1+t)(1-t)^\alpha}{3\Gamma(\alpha+1)} - \frac{(1+t)}{3\Gamma(\alpha+1)} + \frac{(1+t)(1-t)^{\alpha-1}}{3\Gamma(\alpha)}$$

$$(1+t) \quad (1+t)(1-t)^\alpha \quad (1+t)(1-t)^{\alpha-1}$$

$$3\Gamma(\alpha) \quad 3\Gamma(\alpha+1) \quad 3\Gamma(\alpha) \quad .$$

It is easy to see that

$$\tilde{G} < \frac{3}{\Gamma(\alpha+1)} + \frac{2}{\Gamma(\alpha)} < 5.$$

A simple calculation shows that condition (3 . 1 2) is satisfied for $M > 7/2$. It is clear that F is compact and convex valued , and it is upper semi - continuous (see [1 8]) . Since all the conditions of Theorem 3 . 6 are satisfied , BVP (6 . 1) – (6 . 3) has at least one solution y on J.

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