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Control and Calculation of the $\mathscr{H}_2$-norm for Singular and/or Fractional Models

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THESIS

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Contents

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Poster Presentation

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List of Figures

Notation

$\forall$	:	For all.
$\in$	:	Belongs to.
$\times$	:	Inner product.
$\star$	:	Convolution product.
$p \implies q$	:	Statements p implies statement q.
$\iff$	:	Statements p equivalent statement q.
$\wedge$	:	Conjunction (Boolean operator and).
$\vee$	:	Disjunction (Boolean operator inclusiv or).
δ	:	Dirac Delta function.
$\begin{pmatrix} a \\ b \end{pmatrix}$	:	Binomial coefficients.
$\\|G\\|_2$	:	$\mathscr{H}_2$-norm of the transfer function G.
$\mathbb{N}^*$	:	Set of non-zero natural numbers $\{1,2,3,\cdots\}$.
$\mathbb{N}$	:	Set of natural numbers $\{0,1,2,3,\cdots\}$.
$\mathbb{Z}$	:	Set of integer numbers $\{\cdots,-3,-2,-1,0,+1,+2,+3,\cdots\}$.
$\mathbb{R}$	:	Field of real numbers.
$\mathbb{C}$	:	Field of complex numbers.
$\mathbf{L}_1$	:	Space of functions which is integrable in the sense of Lebesgue.
$n!$	:	Factorial of the natural number n.
$[.]$	:	Integer part of a real number.
$\arg(.)$	:	Argument of complex number.
Re	:	Real part of a complex number.
$\mathbb{R}^{n\times m}$	:	Space of $n \times m$ real matrices.
$\mathbb{R}^n$	:	Space of n-dimensional real vectors.
$\mathbb{C}^n$	:	Space of n-dimensional complex vectors.
$\mathbb{C}^{n\times m}$	:	Space $n \times m$ complex matrices.
Id	:	Identity matrix.
tr(A)	:	Trace of square matrix.
det(.)	:	Determinant of a matrix.
A^{-1}	:	Inverse of matrix A.
A^*	:	Conjugate transpose of matrix A.

$\mathscr{I}^{\alpha}$	:	Riemann-Liouville fractional integral.
$\mathbf{D}_{a,t}^{\alpha}$	:	Fractional order derivative.
${}_{\mathrm{GL}}\mathbf{D}_{0,t}^{\alpha}$	:	Grünwald-Letnikov fractional derivative.
${}_{\mathrm{RL}}\mathbf{D}_{0,t}^{\alpha}$	:	Riemann-Liouville fractional derivative.
$\mathbf{D}^{\alpha}$	:	Caputo fractional derivative.
t	:	Real variable.
s	:	Complex variable.
$\mathscr{T}[.]$	:	Integral transform of an argument.
$\mathscr{B}[.]$	:	Bilateral Laplace transform of an argument.
$\mathscr{L}[.]$	:	Laplace transform of an argument.
$\mathscr{M}[.]$	:	Mellin transform of an argument.
$\Gamma(x)$	:	Gamma function.
$\mathrm{B}(\alpha,\beta)$	:	Beta function.
$\mathrm{E}_{\alpha,\beta}(z)$	:	Two-parameters Mittag-Leffler function.
BIBO	:	Bounded-input, bounded-output.
FDs	:	Fractional derivatives.
LTI	:	Linear time-invariant.
MIMO	:	Multiple-input, multiple-output.
SISO	:	Single-input, single-output.

Introduction

In the engineering and scientific fields, dynamical systems are systems whose state evolves according to fixed rule with respect to time. In fact, they are the basic framework used for modelling, controlling and analysing a large variety of systems and phenomena [16, 30, 41]. An important evolution in the modelling and theory of dynamical systems occurred at the end 1950s and early 1960s, resulting in what is known as the modern theory of dynamical systems and control theory [6, 38]. Besides, a dynamical system can be described by a differential equations [4, 69] or fractional differential equation where the mathematical modelling and simulation of the targeted system and processes are based on the description of their properties in terms of fractional derivation [2, 20, 27, 53, 63].

More thoroughly, in recent years, several processes in physics and engineering systems could be modelled by fractional derivatives (FDs) [1]. Subsequently, various studies have mentioned that almost every field of science and engineering subsumes the application of FDs [30, 47]. For instance, the behaviours of viscoelastic materials [5, 62], electrochemical processes [26], and some other applications of FDs can be found in [1, 24, 30, 48, 65]. For more details and clarifications, the following circuit presents a biomechanics example

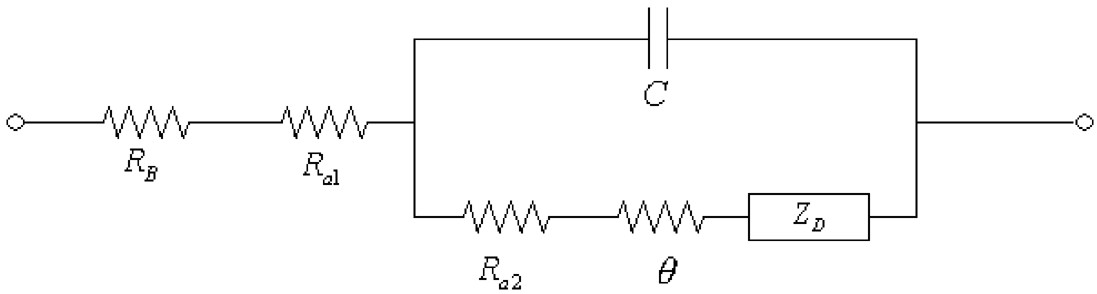

Figure 1: Tissue-electrode circuit model [43]

where R_B is the bulk tissue resistance, R_{a_1}, R_{a_2} are electrode access resistances, θ is the charge transfer resistance, C is the dipole layer capacitance, and Z_D is the fractional Warburg impedance.

The tissue-electrode circuit model given by figure 1 can be described by the following fractional equation [43]

$$C_\alpha \mathbf{D}^{\alpha}_{a,t} v(t) = RC_\alpha \mathbf{D}^{\alpha}_{a,t} i(t) + i(t) + \frac{C_\alpha}{C_\beta} \mathbf{D}^{\alpha-\beta}_{a,t} i(t), \quad 0 < \alpha < 1, \tag{1}$$

where R is resistor, C_α, C_β are two constant phase element impedances, with $\alpha > \beta$, v, i are voltage and current respectively, and set all initial conditions to zero. It must be emphasized that, in control theory, the expression (1) is known as the dynamical equation of the state-space representation. However, there are several other types of dynamical systems representation, among them the transfer function representation appears.

In the engineering fields, a transfer function of an electronic or control system component is a mathematical function, obtained by simple algebraic manipulation of a differential equation that illustrates the system [16, 52]. It represents the ratio between the output of the dynamical system and its input, namely in the Laplace domain with zero initial conditions [50, 59].

The transfer function is very useful in the analysis and the design of linear dynamical systems since it allows the computation of the output of the dynamical system for any input signal. For instance, the fractional-order transfer function associated with the equation (1) is given, using Laplace transform, by

$$\begin{aligned} G(s) &= \frac{V(s)}{I(s)}, \\ &= R + \frac{1}{s^\alpha C_\alpha} + \frac{1}{s^\beta C_\beta}, \end{aligned}$$

for some $s \in \mathbb{C}$. The function G expresses the fractional impedance of the tissue-electrode circuit.

Control theory is a larger branch in mathematics and engineering fields, among its tools, we can find the impulse response energy, known as $\mathcal{H}_2$-norm, which plays an important role and makes a substantial contribution in the field of the dynamical systems theory. The $\mathcal{H}_2$-norm often appears in control theory and can be used to measure the precision of a rational approximation of a transfer function and vice versa [3, 45, 64, 65]. It is defined for a stable, causal, and proper transfer function G, and can be computed in the frequency domain using Plancherel's theorem [45] as

$$\|G\|^2_{\mathcal{H}_2} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} G(j\omega) G^*(j\omega) d\omega,$$

where G^* is the conjugate transpose of G. Moreover, one of the most important problems in modelling and controlling dynamical systems is to calculate the impulse response energy for a transfer function.

Furthermore, several methods have been proposed to compute the $\mathcal{H}_2$-norm for the fractional-order transfer function; most of them use an analytical or algebraic formulation [3, 45, 65]. However, the main goal of the present thesis is to provide an alternative method for computing the $\mathscr{H}_2$-norm for different forms of the fractional-order transfer function associated with the fractional-order dynamical linear time-invariant (LTI) system. In other words, our concern in this thesis is to establish a new method of measuring the $\mathscr{H}_2$-norm for a fractional-order transfer function of the first kind with real parameters, and then, to extend it for the first and second kind fractional-order transfer functions with complex parameters. Indeed, the process is based on the extraction of the state-space model from the fractional-order transfer function, and then, the use of a transformation matrices of the parahermitian matrix [22] for some given conditions, that let the parahermitian matrix, which plays a fundamental role in systems and control theory, invariant, together with a particular integral transforms [19, 68].

This thesis proceeds as follows:

Chapter 1 equips the reader with background information sufficient to understand the technical developments of the succeeding chapters. This chapter includes historical background and complete description of the theory of fractional derivation. Followed by, Laplace and Mellin transforms which are the two important concepts of integral transforms. Finally, some notions of Schur complement are presented.

Chapter 2, we presents some basic tools in the field of control theory. We start by introducing some concepts on the dynamical systems, which are the state-space and the transfer function representations associated with the fractional-order LTI systems. At the end of this chapter, we give some definitions and properties of the $\mathscr{H}_2$-norm since it is the main key of this thesis.

Chapter 3 is devoted to the presentation of the new approach to calculate the $\mathscr{H}_2$-norm for a fractional-order transfer function of the first kind with real parameters. In addition, this chapter extends the proposed method for computing the $\mathscr{H}_2$-norm for a fractional-order transfer function of the first kind with complex parameters. Both proposed methods are based on Caputo fractional derivative, state-space realization, matrices transformation, the parahermitian matrices, and a particular integral transforms. In addition, numerical examples are provided to compare and illustrate the benefits of the proposed approaches.

Chapter 4 covers the extension of the method for computing the impulse response energy, proposed in chapter 3, to the second kind fractional-order transfer function with complex parameters. The main advantage of this new process is the use of the state-space model associated with the fractional-order transfer function of the second kind, and therefore, a suitable parahermitian matrix. A numerical example is carried out to compare and to illustrate the effectiveness of the proposed algorithm.

Finally, we will draw a conclusion which highlights the outcomes, perspectives, and some important books and papers used to achieve this thesis.

Chapter 1

Basic notations

1 Introduction

The purpose of this chapter is to provide the reader with sufficient background information to understand the technical development of the following chapters. The first section gathers the tools of fractional calculus needed in this thesis. Two integral transforms are discussed in the second section. Finally, the third section gives some definitions and properties of Schur complement.

2 Fractional calculus

For more than 300 years old, fractional calculus has drawn the attention of many famous mathematicians like *Riemann*, *Liouville*, and *Caputo* [11, 51, 54, 58]. It is a theory of integrals and derivatives of arbitrary real or complex order. At present, a number of fields such as mechanical, electrical engineering, physics, and many others were using fractional calculus [11, 30]. For example, the half-order fractional integral is the natural mathematical connection between thermal or material gradients and the diffusion of heat or ions [56], the voltage-current relation of a semi-infinite lossy transmission line is another example for the application of the fractional derivative in electrical domain [67]. However, in the last years the fractional calculus was applied in biomedicine and biology [27]. Furthermore, the fractional derivative appears in the theory of control of dynamical systems when the controlled system and the controller are described by a fractional differential equation [11]. For instance, the electrical circuit

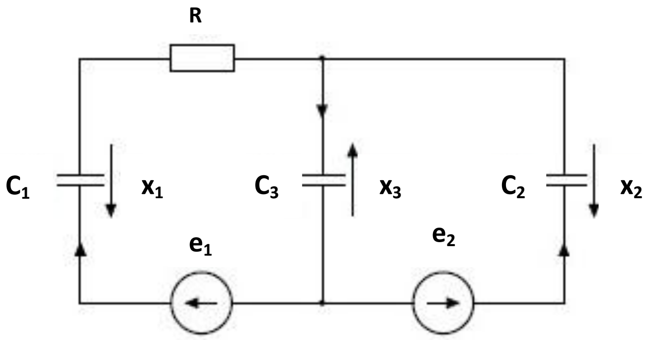

Figure 1.1: Electrical circuit [29]

where R is the resistance, C_1, C_2, C_3 are the capacitances, and e_1, e_2 represent source voltages, can be modelled by the following equations

$$\begin{aligned} e_1(t) &= RC_1\mathbf{D}^{\alpha}_{a,t}x_1(t) + x_1(t) + x_3(t), && (1.1)\\ 0 &= C_1\mathbf{D}^{\alpha}_{a,t}x_1(t) + C_2\mathbf{D}^{\alpha}_{a,t}x_2(t) - C_3\mathbf{D}^{\alpha}_{a,t}x_3(t), && (1.2)\\ e_2(t) &= x_2(t) + x_3(t), && (1.3) \end{aligned}$$

by the use of Kirchoff's laws. The equations (1.1), (1.2), and (1.3) are given by the matrix form

$$E\mathbf{D}^{\alpha}_{a,t}x(t) = Ax(t) + Bu(t),$$

where

$$E = \begin{pmatrix} RC_1 & 0 & 0 \\ C_1 & C_2 & -C_3 \\ 0 & 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix},$$

and

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}, \quad u(t) = \begin{pmatrix} e_1(t) \\ e_2(t) \end{pmatrix}.$$

More than that, in viscoelastic materials, the one degree of freedom model of a passive car suspension

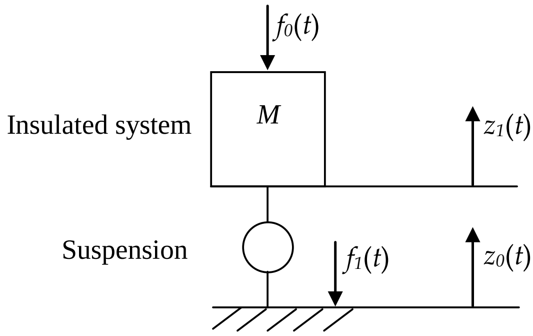

Figure 1.2: One degree of freedom general model of a car suspension [20, 31]

with M is the car quarter mass, z_0 is the profile of the road, f_0 is the efforts applied on the suspension, and z_1, f_1 represent the force generated by the suspension and the vertical movement of the mass respectively, can be represented by

$$\mathbf{D}_{a,t}^{\frac{3}{2}} x(t) = \mathrm{A}x(t) + \mathrm{B}u(t),$$

where

$$\mathrm{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -20^{\frac{3}{2}} \end{pmatrix} \quad \text{and} \quad \mathrm{B} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Note that for both examples, $\mathbf{D}_{a,t}^{\alpha}$ which means the continuous integro-differential operator is defined by [11, 54]

$$\mathbf{D}_{a,t}^{\alpha} = \begin{cases} \dfrac{d^{\alpha}}{dt^{\alpha}} & \alpha > 0, \\ 1 & \alpha = 0, \\ \int_a^t (d\tau)^{\alpha} & \alpha < 0. \end{cases} \tag{1.4}$$

Let us emphasize that, the fractional calculus can be defined as the generalization of integration and differentiation to non-integer order fundamental operator $\mathbf{D}_{a,t}^{\alpha}$, where a and t are the bounds of the operation and $\alpha \in \mathbb{R}$. For this purpose, the most used tools of the fractional calculus will be recalled in this section.

Firstly, we will start by recalling the most important functions used in fractional calculus which are the Euler's Gamma function and the Beta function introduced by the Swiss mathematician *Leonhard Euler* in the 18th century and *Jacques Binet* respectively.

2.1 Some special functions

2.1.1 Euler's Gamma function

Definition 2.1 [30] *A function given by the integral*

$$\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt, \text{ for } x \in \mathbb{C}, \text{with } \mathrm{Re}(x) > 0, \tag{1.5}$$

is called the Euler's Gamma function.

One of the most advantages of the Euler's Gamma function is the generalization of the factorial, written as $n!$, to non-integer values. However, the most important properties of the Euler's Gamma function are described by the following lemmas.

Lemma 2.2 [30] *The Euler's Gamma function satisfies*

$$\Gamma(x+1) = x\Gamma(x), \quad \text{for } x \in \mathbb{C}, \text{with } \mathrm{Re}(x) > 0.$$

Lemma 2.3 [14] *For some* $x \in \mathbb{C}$ *with* $0 < \mathrm{Re}(x) < 1$, *we have,*

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}. \tag{1.6}$$

Example 2.4 *Taking* $x = \frac{1}{2}$ *in* (1.6), *we get*

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

Let us switch now to present some materials of fractional calculus, as shown in the formula (1.4), we can remark that α takes a positive or a negative non-integer values. For that, we will present in this second part of this section, the elementary properties of the fractional calculus when $\alpha < 0$.

2.1.2 Beta function

The Beta function, which is also known as the Euler integral of the first kind is one of the most special types function in fractional calculus [32]. In addition, in many cases, it is more convenient to use the Beta function instead of a certain combination of values of the Euler's Gamma function, it is usually expressed as $\mathrm{B}(x, y)$ and defined by

$$\mathrm{B}(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt, \tag{1.7}$$

where $\mathrm{Re}(x) > 0$ and $\mathrm{Re}(y) > 0$.

The relationship between the Euler's Gamma function and Beta function is given by the following proposition.

Proposition 2.5 [32] *The Beta function can be written in the form of Euler's Gamma function as follows*

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \tag{1.8}$$

where $\mathrm{Re}(x) > 0$, $\mathrm{Re}(y) > 0$, *and* Γ *is Euler's Gamma function defined in* (1.5).

2.2 Fractional integral

More than one definition of fractional integral can be found in the state-of-the-art; like Riemann-Liouville and Wely versions [47]. In the following, we will describe only the one develloped by *Bernhard Riemann* and *Joseph Liouville*, known as Riemann-Liouville fractional integral.

Definition 2.6 [32, 54] *Let* $f \in \mathbf{L}_1[a,b]$. *Then, the Riemann-Liouville fractional integrals* $\mathscr{I}^{\alpha}_{a^+} f$ *and* $\mathscr{I}^{\alpha}_{b^-} f$ *of order* $\alpha \in \mathbb{R}^*_-$ *are defined by*

$$\mathscr{I}^{\alpha}_{a^+} f(t) = \frac{1}{\Gamma(-\alpha)} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha+1}} d\tau,\ t > a, \tag{1.9}$$

and

$$\mathscr{I}^{\alpha}_{b^-} f(t) = \frac{1}{\Gamma(-\alpha)} \int_t^b \frac{f(\tau)}{(\tau-t)^{\alpha+1}} d\tau,\ t < b, \tag{1.10}$$

respectively, with Γ *is Euler's Gamma function defined in* (1.5) . *These integrals are called the left-side and the right-side fractional integrals.*

In the following, we will give some important properties of Riemann-Liouville fractional integral.

2.2.1 Properties of Riemann-Liouville fractional integral

Properties 1.1 [32, 47, 54] *Let* $[a,b] \subset \mathbb{R}$ *be a finite interval,* $f, g \in \mathbf{L}_1[a,b]$, $c_1, c_2 \in \mathbb{R}$, *and* $\alpha, \beta \in \mathbb{R}^*_-$.

P. 1. *Linearity*

The left-side and the right-side Riemann-Liouville fractional integrals are linear operators

$$\mathscr{I}^{\alpha}_{a^+}\left(c_1 f + c_2 g\right)(t) = c_1 \mathscr{I}^{\alpha}_{a^+} f(t) + c_2 \mathscr{I}^{\alpha}_{a^+} g(t),$$
$$\mathscr{I}^{\alpha}_{b^-}\left(c_1 f + c_2 g\right)(t) = c_1 \mathscr{I}^{\alpha}_{b^-} f(t) + c_2 \mathscr{I}^{\alpha}_{b^-} g(t).$$

P. 2. ***Semi-group***

$$\begin{aligned}\mathscr{I}^{\alpha}_{a^+}\mathscr{I}^{\beta}_{a^+}f(t) &= \mathscr{I}^{\alpha+\beta}_{a^+}f(t),\\ \mathscr{I}^{\alpha}_{b^-}\mathscr{I}^{\beta}_{b^-}f(t) &= \mathscr{I}^{\alpha+\beta}_{b^-}f(t).\end{aligned}$$

P. 3. ***Commutativity***

$$\mathscr{I}^{\alpha}_{a^+}\mathscr{I}^{\beta}_{a^+}f(t) = \mathscr{I}^{\beta}_{a^+}\mathscr{I}^{\alpha}_{a^+}f(t),$$
$$\mathscr{I}^{\alpha}_{b^-}\mathscr{I}^{\beta}_{b^-}f(t) = \mathscr{I}^{\beta}_{b^-}\mathscr{I}^{\alpha}_{b^-}f(t).$$

Example 2.7 [44] *The effect of the operator $\mathscr{I}^{\alpha}$ on the power function t^{γ} is given by*

$$\mathscr{I}^{\alpha}t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+\alpha)}t^{\gamma+\alpha},$$

for $t > 0$, with $\alpha > 0$ and $\gamma > -1$.

As it is well known, in the ordinary case the integral is the inverse of the derivative, so, this property is preserved in fractional calculus. Several formulas of fractional derivatives appear, the most frequently used are the Grünwald-Letnikov, the Riemann-Liouville, and the Caputo fractional derivatives which are given in the following.

2.3 Fractional derivatives

Let us start with the Grünwald-Letnikov fractional derivative since it is the basic extension of the derivative in fractional calculus.

2.3.1 Grünwald-Letnikov fractional derivative

The first definition of the Grünwald-Letnikov fractional derivative is given as follows.

Definition 2.8 [54] *Let $\alpha > 0$, the α-order Grünwald-Letnikov fractional derivative of function f with respect to t is given by*

$${}_{\mathrm{GL}}\mathbf{D}^{\alpha}_{0,t}f(t) = \lim_{h\to 0} h^{-\alpha}\sum_{j=0}^{\infty}(-1)^j\binom{\alpha}{j}f\left(t-jh\right),$$

where $\binom{\alpha}{j}$ is the binomial coefficients and can be calculated using the relation between Euler's Gamma function and factorial as

$$\binom{\alpha}{j} := \frac{\alpha!}{j!\left(\alpha-j\right)!} = \frac{\Gamma(\alpha+1)}{\Gamma(j+1)\Gamma(\alpha-j+1)}.$$

An alternative of this definition is given by the following definition.

Definition 2.9 [49] *Let n be positive integer, with $n-1 \leq \alpha < n, n \in \mathbb{N}^*$, and $f \in \mathscr{C}^n[0,\mathrm{T}]$. The α-order Grünwald-Letnikov fractional derivative is defined by*

$$ {}_{\mathrm{GL}}\mathbf{D}^{\alpha}_{0,t} f(t) = \sum_{k=0}^{n-1} \frac{t^{k-\alpha} f^{(k)}(0)}{\Gamma(k-\alpha+1)} + \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau. $$

From the definition 2.9, we remark that the function f must be in $\mathscr{C}^n[0,\mathrm{T}]$, where $n-1 \leq \alpha < n, n \in \mathbb{N}^*$, in this framework, *Bernhard Riemann* and *Joseph Liouville* develloped a new definition of the fractional derivative in which, they weakened the condition on the function f.

2.3.2 Riemann-Liouville fractional derivative

Definition 2.10 [32, 54, 60] *The α-order Riemann-Liouville fractional derivative of a continuous function f is given as*

$$ {}_{\mathrm{RL}}\mathbf{D}^{\alpha}_{0,t} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau, \tag{1.11} $$

where $n-1 \leq \alpha < n, n \in \mathbb{N}^*$.

Note, that form (1.11) and for $\alpha = 0$, we obtain,

$$ {}_{\mathrm{RL}}\mathbf{D}^{0}_{0,t} f(t) = f(t), $$

and for $\alpha = 1$, we get,

$$ {}_{\mathrm{RL}}\mathbf{D}^{1}_{0,t} f(t) = f'(t). $$

Example 2.11 [30] *Consider the unit-step function*

$$ \mathbb{1}(t) = \begin{cases} 1 & \text{for } t \geq 0, \\ 0 & \text{for } t < 0. \end{cases} $$

Using (1.11), *we obtain*

$$
\begin{aligned}
{}_{\mathrm{RL}}\mathbf{D}^{\alpha}_{0,t}\mathbb{1}(t) &= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-\tau)^{n-\alpha-1}\,d\tau,\\
&= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\left[\frac{-1}{n-\alpha}(t-\tau)^{n-\alpha}\right]_0^t,\\
&= \frac{1}{\Gamma(n-\alpha)}\frac{1}{n-\alpha}\frac{d^n}{dt^n}t^{n-\alpha},\\
&= \frac{1}{\Gamma(n-\alpha)}\frac{1}{n-\alpha}(n-\alpha)(n-\alpha-1)\cdots(1-\alpha)\,t^{-\alpha},\\
&= \frac{t^{-\alpha}}{\Gamma(1-\alpha)}.
\end{aligned}
$$

Therefore, the α*-order Riemann-Liouville fractional derivative of unit-step function is a decreasing, in time, function.*

The Riemann-Liouville fractional derivative played an important role in the development of the theory of fractional derivatives and integrals and has many applications in pure mathematics. However, among its disadvantages, the Riemann-Liouville of a constant is not zero according to the example 2.11. In addition, the Riemann-Liouville fractional derivative needs initial conditions containing the limit values of Riemann-Liouville fractional derivative of initial time [54], whose physical meanings are not clear. In order to overcome this drawback, a new definition was given by *Michele Caputo* in the 20 century.

2.3.3 Caputo fractional derivative

In 1967, *Caputo* reformulated the definition of the Riemann-Liouville fractional derivative [12], by switching the order of the ordinary derivative with the fractional integral operator. In contrast to the Riemann-Liouville fractional derivative, when solving differential equations using Caputo's definition, it is not necessary to define the fractional order initial conditions because they have the same form as for the integer-order differential equations.

For this reasons, the Caputo fractional derivative, which is denoted by $\mathbf{D}^{\alpha}$ and presented in the following, will be used along this thesis.

Definition 2.12 [54] *The function defined by*

$$
\begin{aligned}
\mathbf{D}^{\alpha}f(t) &= \mathscr{I}^{n-\alpha}\mathbf{D}^{n}f(t),\\
&= \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}f^{(n)}(\tau)d\tau,\quad f^{(n)}(t)=\frac{d^n f(t)}{dt^n},
\end{aligned}
\tag{1.12}
$$

is called the Caputo fractional derivative-integral, where $n-1\le\alpha<n$, $n\in\mathbb{N}^*$.

Example 2.13 *Consider the unit-step function*

$$\mathbb{1}(t) = \begin{cases} 1 & \text{for } t \geq 0, \\ 0 & \text{for } t < 0. \end{cases}$$

Using (1.12)*, we obtain*

$$\begin{aligned} \mathbf{D}^{\alpha} f(t) &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{d^n f(\tau)}{d\tau^n} d\tau, \quad n-1 \leq \alpha < n, n \in \mathbb{N}^*, \\ &= 0. \end{aligned}$$

Thus, the Caputo fractional derivative of constant is equal to zero.

2.4 The relation between the fractional derivatives

It is clear that, from the definitions 2.10 and 2.12, the relationship between the Riemann-Liouville and the Caputo fractional derivatives is

$${}_{\mathrm{RL}}\mathbf{D}^{\alpha}_{0,t} f(t) = \mathbf{D}^{\alpha} f(t) + \sum_{k=0}^{n-1} \frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}(t)\bigg|_{t=0}.$$

However, in the case of Riemann-Liouville and Grünwald-Letnikov both definitions 2.10 and 2.9 provide the same result, for any function f satisfies the above conditions.

3 Integral transforms

In the literature, there are many classes of problems arising in the engineering, the optical, and the physical sciences, that are difficult to solve in their original representation, as for instance, electric current in a simple circuit, the Bernoulli-Euler Beam equation, slowing down of Neutrons [15, 61]. Therefore, integral transform is the effective method and has been successfully used for almost two centuries in solving differential-integral equations and many other problems in applied mathematics, since it allows us to turn a complicated problem into a simplified one.

A general expression of integral transform is defined by [15, 19, 34]

$$\begin{aligned} \mathscr{T}\left[f(t)\right](s) &= \mathrm{F}(s), \\ &= \int_a^b f(t)\mathrm{K}(t,s)dt, \end{aligned}$$

for some $s \in \mathbb{C}$. The unique function F is called the integral transform of the initial function f by the kernel K.

Evidently, $\mathscr{T}$ satisfies the linearity property:

$$\begin{aligned}\mathscr{T}\left[cf(t)+dg(t)\right](s) &= \int_a^b \left(cf(t)+dg(t)\right)\mathrm{K}(t,s)\,dt,\\ &= c\mathscr{T}\left[f(t)\right](s)+d\mathscr{T}\left[g(t)\right](s), \quad c,d\in\mathbb{R}.\end{aligned} \tag{1.13}$$

In addition, we define the inverse operator $\mathscr{T}^{-1}$ such that

$$\mathscr{T}^{-1}\left[\mathrm{F}(s)\right](t) = f(t),$$

which is also a linear operator. Correspondingly,

$$\mathscr{T}^{-1}\mathscr{T} = \mathscr{T}\mathscr{T}^{-1} = \mathrm{Id}$$

which is the identity operator.

By choosing different kernels K and different values for a and b several integral transforms emerged, however, in this section, we will present the two most important and helpful integral transforms that are Laplace and Mellin transforms.

3.1 Laplace transform

The Laplace transform, notated by $\mathscr{L}$ and named after its inventor *Pierre-Simon Laplace*, can be interpreted as a transformation of a function f satisfies the Dirichlet conditions from the continuous-time t to a new function F of complex frequency variable s, according to the following definitions.

3.1.1 Definitions

Definition 3.1 [15] *The bilateral Laplace transform of a function f is defined as*

$$\begin{aligned}\mathscr{B}\left[f(t)\right](s) &= \mathrm{F}(s),\\ &= \int_{-\infty}^{\infty} e^{-st}f(t)dt.\end{aligned} \tag{1.14}$$

If f is causal function, the unilateral Laplace transform is given by

$$\begin{aligned}\mathscr{L}\left[f(t)\right](s) &= \mathrm{F}(s),\\ &= \int_{0}^{\infty} e^{-st}f(t)dt,\end{aligned} \tag{1.15}$$

for some $s\in\mathbb{C}$, where e^{-st} is the kernel of the transform.

Example 3.2 *For $f(t) = t^{\alpha}$ where $\alpha\in\mathbb{R}_+^*$, the Laplace transform (formula (1.15)) of the* *function f is*

$$\mathscr{L}[t^{\alpha}](s) = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}. \tag{1.16}$$

Definition 3.3 [15] *A function f has exponential order a if there exist a constant* $\mathrm{M} > 0$ *such that for some* $\mathrm{T} \leq 0$,

$$\left|f(t)\right| \leq \mathrm{M}e^{at},\ t \leq \mathrm{T}.$$

The existence of the Laplace transform (1.15) is highlighted by the following theorem.

Theorem 3.4 [15] *If f is piecewise continuous function on* $[0,\infty[$ *and of exponential order a, then, the Laplace transform $\mathscr{L}[f]$ exists for* $\mathrm{Re}(s) > a$ *and converges absolutely.*

In order to apply the Laplace transform to several problems, we must invoke its inverse transform.

Definition 3.5 [15] *If the function* F *is the Laplace transform of a function f, then, one can recover the function f by*

$$\begin{aligned} \mathscr{L}^{-1}\left[\mathrm{F}(s)\right](t) &= f(t), \\ &= \frac{1}{2j\pi}\int_{c-j\infty}^{c+j\infty} \mathrm{F}(s)e^{st}\,ds,\ c > 0. \end{aligned}$$

Obviously, $\mathscr{L}$ and $\mathscr{L}^{-1}$ are linear integral operators.

3.1.2 Properties

In this part, we will present some basic properties of the Laplace transform. For this purpose, let us consider the continuous functions f and g in $[0,\infty[$, and having the Laplace transform $\mathscr{L}\left[f(t)\right](s) = \mathrm{F}(s)$ and $\mathscr{L}\left[g(t)\right](s) = \mathrm{G}(s)$ respectively [15].

P. 1. The convolution property: it is given by

$$\mathscr{L}\left[(f\star g)(t)\right](s) = \mathscr{L}\left[f(t)\right](s)\mathscr{L}\left[g(t)\right](s), \tag{1.17}$$

where $\star$ denotes the convolution product defined by

$$\left(f\star g\right)(t) = \int_0^t f(t-\tau)\,g(\tau)\,d\tau = \int_0^t g(t-\tau)\,f(\tau)\,d\tau.$$

P. 2. The derivative property: if $f \in \mathscr{C}^n[0,\infty[$, then,

$$\begin{aligned} \mathscr{L}\left[f^{(n)}(t)\right](s) &= s^n\mathrm{F}(s) - \sum_{k=0}^{n-1} s^{n-k-1}f^{(k)}(0), \\ &= s^n\mathrm{F}(s) - \sum_{k=0}^{n-1} s^k f^{(n-k-1)}(0). \end{aligned}$$

P. 3. The integral property: it is given by

$$\mathscr{L}\left[\int_0^t f(\tau)d\tau\right](s)=\frac{\mathrm{F}(s)}{s}.$$

Another useful property which is needed in this thesis is the connection between the Laplace transform and the fractional derivatives.

3.1.3 The Laplace transform and the fractional derivatives

In this part, we will present the Laplace transform of Riemann-Liouville and Caputo fractional derivatives.

Definition 3.6 [56] *The Laplace transform of fractional derivative, denoted by* $\mathscr{L}[\mathbf{D}^{\alpha}_{0,t}f(t)](s)$, *is given by*

$$\mathscr{L}\left[\mathbf{D}^{\alpha}_{0,t}f(t)\right](s)=s^{\alpha}\mathrm{F}(s)-\sum_{k=0}^{n-1}s^{k}\left[\mathbf{D}^{\alpha-k-1}_{0,t}f(t)\right]_{t=0},\ \ n-1\leq\alpha<n,\ n\in\mathbb{N}^{*}. \tag{1.18}$$

From the expression (1.18), we get the following theorems.

Theorem 3.7 [30, 54] *The Laplace transform of the Riemann-Liouville fractional derivative* (1.11) *with* $n-1\leq\alpha<n,\ n\in\mathbb{N}^{*}$ *has the form*

$$\mathscr{L}\left[{}_{\mathrm{RL}}\mathbf{D}^{\alpha}_{0,t}f(t)\right](s)=s^{\alpha}\mathrm{F}(s)-\sum_{k=1}^{n}s^{k-1}f^{(\alpha-k)}(0), \tag{1.19}$$

where $f^{(\alpha-k)}(0)={}_{\mathrm{RL}}\,\mathbf{D}^{\alpha-k}_{0,t}f(t)\Big|_{t=0}$.

Theorem 3.8 [30, 56] *The Laplace transform of the Caputo fractional derivative* (1.12) *with* $n-1\leq\alpha<n,\ n\in\mathbb{N}^{*}$ *has the form*

$$\mathscr{L}\left[\mathbf{D}^{\alpha}_{0,t}f(t)\right](s)=s^{\alpha}\mathrm{F}(s)-\sum_{k=1}^{n}s^{\alpha-k}f^{(k-1)}(0). \tag{1.20}$$

3.1.4 Application of the Laplace transform

Example 3.9 [15, 40] *Let us consider a nearly simple harmonic vibration equation*

$$\mathbf{D}^{\alpha}y(t)+\omega^{2}y(t)=0,\quad 1\leq\alpha<2, \tag{1.21}$$

where $\omega\in\mathbb{R}^{*}_{+}$ *with the initial conditions* $y(0)=c_0$ *and* $y'(0)=c_1$.

The application of the Laplace transform (formulas (1.20) *and* (1.15)*) to the equation* (1.21) *gives*

$$s^{\alpha}\mathrm{Y}(s)-c_0s^{\alpha-1}-c_1s^{\alpha-2}+\omega^{2}\mathrm{Y}(s)=0,$$

which, for $\left|\omega^2 s^{-\alpha}\right| < 1$*, is equivalent to*

$$Y(s) = c_0 \sum_{k=0}^{\infty} \left(-\omega^2\right)^k s^{-\alpha k-1} + c_1 \sum_{k=0}^{\infty} \left(-\omega^2\right)^k s^{-\alpha k-2}.$$

By the inverse Laplace transformation, we obtain the solution of the fractional differential equation (1.21)

$$y(t) = c_0 E_{\alpha,1}\left(-\omega^2 t^\alpha\right) + c_1 t E_{\alpha,2}\left(-\omega^2 t^\alpha\right).$$

where

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma\left(k\alpha+\beta\right)},$$

is known as, the two parameters Mittag-Leffler function [30] *that plays an important role in the solution of the fractional differential equations.*

In contrast to the Laplace transformation that was introduced to solve physical problems, Mellin's transformation arose in pure and applied mathematical context. The following subsection presents the definition and some properties of the Mellin transformation.

3.2 Mellin transform

Thanks to the Finnish mathematician *Robert Hjalmar Mellin*, the Mellin transform was introduced. Its first occurrence was found in a memoir by *Riemann* in which he used it to study the famous Zeta function [18].

The Mellin transform is an integral transform that can be considered as the multiplicative version of the bilateral Laplace transform.

3.2.1 Definitions

Definition 3.10 [23, 56] *The Mellin transform of a function f defined on the interval* $[0,\infty[$ *is*

$$\begin{aligned} \mathcal{M}\left[f(t)\right](s) &= \int_0^\infty t^{s-1} f(t)\,dt, \\ &= F(s), \end{aligned} \tag{1.22}$$

where $s \in \mathbb{C}$*, and* $\mu_1 < \mathrm{Re}(s) < \mu_2$.

Theorem 3.11 [56] *The Mellin transform* F *of a function f exists, if the function f is piecewise continuous in every closed interval* $[a,b] \subset [0,\infty[$ *and*

$$\int_0^1 \left|f(t)\right| x^{\mu_1-1} dt < \infty, \quad \int_1^\infty \left|f(t)\right| x^{\mu_2-1} dt < \infty.$$

Example 3.12 *For $t > 0$, consider the function*

$$f(t) = \frac{1}{1+t}.$$

Then,

$$\mathcal{M}\left[f(t)\right](s) = \int_0^{\infty} \frac{t^{s-1}}{1+t} dt. \tag{1.23}$$

By substituting $t = \dfrac{v}{1-v}$, *the integral* (1.23) *becomes*

$$\mathcal{M}\left[f(v)\right](s) = \int_0^{1} v^{s-1}(1-v)^{-s} dv, \tag{1.24}$$

with the condition $0 < \mathrm{Re}(s) < 1$.

Thus, by the formula (1.7), *we get*

$$\mathcal{M}\left[f(v)\right](s) = \mathrm{B}(s, 1-s),$$

and, from the relation (1.8), *we find*

$$\mathcal{M}\left[f(v)\right](s) = \Gamma(s)\Gamma(1-s),$$

then, using the expression (1.6), *we obtain*

$$\mathcal{M}\left[f(v)\right](s) = \frac{\pi}{\sin(\pi s)}, \quad 0 < \mathrm{Re}(s) < 1.$$

Remark 3.13 [23] *The Mellin transform of the function* $g(t) = (1+t)^{-n}$ *with* $n \in \mathbb{N}^*$, *is presented by the following expression*

$$\mathcal{M}\left[g(t)\right](s) = \frac{\Gamma(s)\Gamma(n-s)}{\Gamma(n)}, \quad \textit{for} \quad 0 < \mathrm{Re}(s) < n.$$

From the expression (1.22), the function f can be reinstated, if it satisfies the Dirichlet conditions in every closed interval $[a, b] \subset [0, \infty[$. Hence, it can be found using the inverse Mellin transform which is defined in the following.

Definition 3.14 [56] *The inverse Mellin transform of* F *is*

$$\begin{aligned} \mathcal{M}^{-1}\left[\mathrm{F}(s)\right](t) &= f(t), \\ &= \frac{1}{2j\pi} \int_{\mu-j\infty}^{\mu+j\infty} t^{-s}\mathrm{F}(s) ds, \quad \mu_1 < \mu < \mu_2, \end{aligned}$$

where $\mu = \mathrm{Re}(s)$.

The basic operational characteristics of the Mellin transformation are summarized in the following.

3.2.2 Properties

Let us consider two functions f and g, having the Mellin transforms $\mathscr{M}\left[f(t)\right](s) = \mathrm{F}(s)$ and $\mathscr{M}\left[g(t)\right](s) = \mathrm{G}(s)$ respectively. The basic properties of Mellin transform are listed as follows [15, 33, 56].

P. 1. Scaling property: for $a > 0$, we have

$$\mathscr{M}[f(at)](s) = a^{-s}\mathrm{F}(s).$$

P. 2. Shifting property: for $a \in \mathbb{R}$, we have

$$\mathscr{M}[t^{a} f(t)](s) = \mathrm{F}(s+a).$$

P. 3. Derivation property: if $f \in \mathscr{C}^{n}[0,\infty[$ with $n \in \mathbb{N}^{*}$, then,

$$\begin{aligned}\mathscr{M}\left[f^{(n)}(t)\right](s) &= (-1)^{n}\frac{\Gamma(s)}{\Gamma(s-n)}\mathrm{F}(s-n)\\ &\quad+\sum_{k=0}^{n-1}(-1)^{k}\frac{\Gamma(s)}{\Gamma(s-k)}f^{(n-k-1)}(t)\,t^{s-k-1}\Big|_{t=0}.\end{aligned}$$

P. 4. Integral property: we have

$$\mathscr{M}\left[\int_{0}^{t} f(\tau)d\tau\right](s) = -\frac{1}{s}\mathrm{F}(s+1).$$

More generally,

$$\mathscr{M}\left[\mathrm{I}_{n} f(t)\right](s) = (-1)^{n}\frac{\Gamma(s)}{\Gamma(s+n)}\mathrm{F}(s+n),$$

where $\mathrm{I}_{n} f(t)$ is the n^{th} repeated integral of f.

3.2.3 The Mellin transform and the fractional integral-derivatives

In the following, we will present the Mellin transform of the Riemann-Liouville fractional integral-derivative and Caputo fractional derivative.

Theorem 3.15 [56] *The Mellin transform of Riemann-Liouville fractional integral* (1.9), *for* $\alpha > 0$ *is given by*

$$\mathscr{M}\left[\mathscr{J}^{\alpha} f(t)\right](s) = \frac{\Gamma(1-s-\alpha)}{\Gamma(1-s)}\mathscr{M}\left[f(t)\right](s+\alpha).$$

Theorem 3.16 [56] *The Mellin transform of Riemann-Liouville fractional derivative* (1.11) *is presented by*

$$\mathscr{M}\left[\,_{\mathrm{RL}}\mathbf{D}_{0,t}^{\alpha} f(t)\right](s) = \sum_{k=0}^{n-1} \frac{\Gamma(1-s+k)}{\Gamma(1-s)} \left[\,_{\mathrm{RL}}\mathbf{D}_{0,t}^{\alpha-k-1} f(t)\, t^{s-k-1}\right]_0^{\infty} + \frac{\Gamma(1-s-\alpha)}{\Gamma(1-s)} \mathscr{M}\left[f(t)\right](s-\alpha),$$

where $n-1 \leq \alpha < n$, *with* $n \in \mathbb{N}^*$.

Theorem 3.17 [56] *The Mellin transform of Caputo fractional derivative* (1.12) *is given by*

$$\mathscr{M}\left[\mathbf{D}^{\alpha} f(t)\right](s) = \sum_{k=0}^{n-1} \frac{\Gamma(\alpha-k-s)}{\Gamma(1-s)} \left[f^{(k)}(t)\, t^{s-\alpha+k}\right]_0^{\infty} + \frac{\Gamma(1-s-\alpha)}{\Gamma(1-s)} \mathscr{M}\left[f(t)\right](s-\alpha),$$

where $n-1 \leq \alpha < n$, *with* $n \in \mathbb{N}^*$.

4 Schur complement

This section presents some details on the Schur complement which is a key tool in matrix theory and widely used for solving different problems. For more details on the Schur complement, we refer the reader to [25, 70].

4.1 Definitions

Let us consider the $(p+q)\times(p+q)$ matrix block M

$$\mathrm{M} = \begin{bmatrix} \mathrm{A} & \mathrm{B} \\ \mathrm{C} & \mathrm{D} \end{bmatrix}, \tag{1.25}$$

where A, B, C, and D are the matrices blocks of dimensions $p\times p$, $p\times q$, $q\times p$, and $q\times q$ respectively.

Definition 4.1 [70] *The Schur complement of the nonsingular matrix* A *in* M *is the matrix*

$$(\mathrm{M}\,/\,\mathrm{A}) = \mathrm{D} - \mathrm{C}\mathrm{A}^{-1}\mathrm{B}. \tag{1.26}$$

Remark 4.2 [22] *The expression* (M / A) *is the Schur complement of the matrix* M *with respect to its top left block entry.*

Definition 4.3 [70] *If* D *is invertible, the Schur complement of* D *in* M *is the matrix*

$$(\mathrm{M}\,/\,\mathrm{D}) = \mathrm{A} - \mathrm{B}\mathrm{D}^{-1}\mathrm{C}.$$

The quotient formula for the Schur complement is given by the following theorem.

Theorem 4.4 [70] *Let* M, A, *and* E *be given square nonsingular matrices such that*

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \textit{and} \quad A = \begin{pmatrix} E & F \\ G & H \end{pmatrix}.$$

Then, A/E *is a nonsingular principal submatrix of* M/E *with*

$$M/A = (M/E)/(A/E).$$

4.2 Some applications of the Schur complement

Some applications of the Schur complement are listed in the following.

Proposition 4.5 [70] *The matrix* M *can be factorized as*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{pmatrix} \begin{pmatrix} I & A^{-1}B \\ 0 & I \end{pmatrix}. \tag{1.27}$$

Remark 4.6 [70] *If the matrices* M, A *and* D *are symmetric and* $B = C^T$, *then, the matrix* M *can be written as*

$$M = \begin{pmatrix} A & C^T \\ C & D \end{pmatrix} = \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & D - CA^{-1}C^T \end{pmatrix} \begin{pmatrix} I & A^{-1}C^T \\ 0 & I \end{pmatrix},$$

which shows that M *is similar to a block-diagonal matrix.*

If the matrix $D - CA^{-1}B$ is invertible, the following theorem presents the inverse of a matrix block M in terms of Schur complement.

Theorem 4.7 [70] *Let* M *be partitioned as in* (1.25) *and suppose that both matrices* M *and* A *are nonsingular. Then,* M/A *is nonsingular and*

$$M^{-1} = \begin{bmatrix} A^{-1} + A^{-1}B(M/A)^{-1}CA^{-1} & -A^{-1}B(M/A)^{-1} \\ -(M/A)^{-1}CA^{-1} & (M/A)^{-1} \end{bmatrix},$$

thus, the $(2,2)$ *block of* M^{-1} *is* $(M/A)^{-1}$.

Theorem 4.8 [70] *Let* M *be a square matrix partitioned as in* (1.25), *if the matrix* D *is nonsingular, then,*

$$\det(M/A) = \det M / \det A.$$

The next example shows the use of the Schur complement for solving linear system equations.

Example 4.9 *Let us consider the linear system equations*

$$\left\{\begin{array}{rcl} \mathrm{A}x+\mathrm{B}y & = & a, \\ \mathrm{C}x+\mathrm{D}y & = & b, \end{array}\right. \tag{1.28}$$

where $x, a \in \mathbb{R}^p$, $y, b \in \mathbb{R}^q$, *and* A, B, C *and* D *are the matrices blocks of an appropriate dimensions, with* $\det \mathrm{D} \neq 0$.

The solution of the linear system equations (1.28) *is given by*

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \left(\mathrm{A}-\mathrm{BD}^{-1}\mathrm{C}\right)^{-1}(a-\mathrm{BD}^{-1}b) \\ \mathrm{D}^{-1}\left(b-\mathrm{C}\left(\mathrm{A}-\mathrm{BD}^{-1}\mathrm{C}\right)^{-1}(a-\mathrm{BD}^{-1}b)\right) \end{pmatrix},$$

where the matrix $\mathrm{A}-\mathrm{BD}^{-1}\mathrm{C}$ *is the Schur Complement of* D *in* M *with*

$$\mathrm{M} = \begin{bmatrix} \mathrm{A} & \mathrm{B} \\ \mathrm{C} & \mathrm{D} \end{bmatrix}.$$

5 Conclusion

The most important tools used in this thesis have been presented in this chapter. Firstly, we have recalled some concepts of the fractional calculus since it provide an excellent instrument for the modelling and the description of the properties of various materials and processes.

Then, we have introduced two most important integral transforms for our work that are the Laplace and Mellin transforms.

Finally, we conclude this chapter by the definition and some properties of the Schur complement.

As mentioned in the introduction, our problem appears in the fields of control theory. For this purpose, the next chapter concerns the basic tools and techniques of the control theory.

Chapter 2

Basic tools in control theory

1 Introduction

The aim of this chapter is to recall some basic tools in control theory. We will start by introducing the dynamical systems which are primarily the study of the time-evolutionary process in various fields of science and engineering and are described by differential equations. Afterwards, we will present the transfer function, which is useful for the study of LTI dynamical systems. Finally, details on the impulse response energy, known also as the $\mathscr{H}_2$-norm, that can be used to measure the precision of a rational approximation of a fractional-order transfer function and inversely [45, 64, 65] will be given. Note that the concept of the $\mathscr{H}_2$-norm will be widely used in the rest of the present thesis.

2 Dynamical systems

In this section, we will introduce the general notions of continuous-time dynamical systems, for both ordinary and fractional derivatives.

In the engineering area, dynamical systems are the basic framework used for modelling, controlling, and analyzing a large variety of systems and phenomena [42]. They are mainly used in the study of the time-evolutionary process, they can be described by a mathematical model. This model is globally determined by adapted systems of differential equations where its general expression is [7, 38, 69]

$$\dot{x}(t) = f(t; x(t)), \tag{2.1}$$

with $x \in \mathbb{R}^m$ is the state vector, the variable $t \in \mathrm{I} \subset \mathbb{R}_+$ represent the time evolution and the vector-valued function $f : \mathrm{I} \times \mathbb{R}^m \to \mathbb{R}^m$ is a sufficiently smooth function.

The expression (2.1) is said to be an autonomous differential equation or time-invariant system, if the right-hand side of equation (2.1) does not depend on time explicitly. It can be formulated as

$$\dot{x}(t) = f(x(t)),$$

otherwise, the equation (2.1) is called non-autonomous.

In a general context, the concept of a dynamical systems includes a law of the evolution of the state in time and a set of its possible states, this set is known as the state-space of the system [35], its coordinates are state variables. In addition, the state variables are used to describe the state of a dynamical system at any arbitrary time t.

In more explicit manner, the state-space representation consists of both its state equations defined by a set of differential equations called the dynamical equations and a set of algebraic equations called the output equations.

Moreover, one of the most advantages of the state-space representation of continuous-time dynamical linear system is to allow a suitable and a compact way to model and analyze systems with single-input, single-output or multiple inputs and outputs [21, 39]. It is, also, a useful technique for the stability analysis and optimal control problem formulation [16, 50].

More details on the state-space representations are presented in the next parts depending on the type of the dynamical equations.

2.1 Ordinary LTI dynamical systems

This part concerns the state-space representation of the ordinary dynamical systems [21, 39].

2.1.1 State-space representation

Let us recall that the state-space representation of the continuous-time dynamical linear system is a mathematical model of a dynamical system with input, output, and state variables. It is described, in the general case, as follows

$$\begin{aligned} \dot{x}(t) &= f\,(t; x(t), u(t)), \\ y(t) &= g\,(t; x(t), u(t)), \end{aligned} \tag{2.2}$$

where $x \in \mathbb{R}^m$ is the state vector, $u \in \mathbb{R}^q$ is the input vector, and $y(t) \in \mathbb{R}^p$ is the output vector. $f : \mathrm{I} \times \mathbb{R}^m \times \mathbb{R}^q \to \mathbb{R}^m$ and $g : \mathrm{I} \times \mathbb{R}^m \times \mathbb{R}^q \to \mathbb{R}^p$ are Lipchizianne functions with respect to x, continuous in u and continues piecewise on $t \in \mathrm{I}$.

The system (2.2) can be written, under some conditions, in matrix notation as

$$\begin{aligned} \mathrm{E}(t)\,\dot{x}(t) &= \mathrm{A}(t)\,x(t) + \mathrm{B}(t)\,u(t), \\ y(t) &= \mathrm{C}(t)\,x(t) + \mathrm{D}(t)\,u(t), \end{aligned} \tag{2.3}$$

where $\mathrm{E}(t) \in \mathbb{R}^{m\times m}$, $\mathrm{A}(t) \in \mathbb{R}^{m\times m}$ is the state matrix, $\mathrm{B}(t) \in \mathbb{R}^{m\times q}$ is the command matrix,

$C(t) \in \mathbb{R}^{p\times m}$ is the observation matrix, and $D(t) \in \mathbb{R}^{p\times q}$ is the direct transmission matrix.

The system (2.3) is, also, known as MIMO systems for multiples-inputs, multiples-outputs. However, if $q = p = 1$, then, the system (2.3) becomes SISO system for single-input, single-output.

Remark 2.1 *If* $E(t) = \mathrm{Id}$, *then, the system* (2.3) *is called standard dynamical system.*

Remark 2.2 *If* E, A, B, C *and* D *are constant matrices, then, the system* (2.3) *is called LTI dynamical system for linear time-invariant dynamical system.*

Example 2.3 [4] *Let us consider a simplified diagram for a balance system consisting of an inverted pendulum on a cart,*

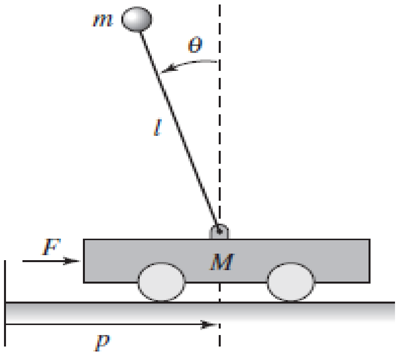

Figure 2.1: Cart-pendulum system [4]

The dynamics of the system can be computed using Newtonian mechanics as

$$\begin{bmatrix} (M+m) & -ml\cos\theta(t) \\ -ml\cos\theta(t) & (J+ml^2) \end{bmatrix} \begin{bmatrix} \ddot{p}(t) \\ \ddot{\theta}(t) \end{bmatrix} + \begin{bmatrix} c\dot{p}(t) + ml\dot{\theta}^2(t)\sin\theta(t) \\ \gamma\dot{\theta}(t) - mgl\sin\theta(t) \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix}, \tag{2.4}$$

where M *is the mass of the base,* m *and* J *are, respectively, the mass and moment of inertia of the system to be balanced,* l *is the distance from the base to the center of mass of the balanced body,* c *and* γ *are coefficients of viscous friction,* g *is the acceleration due to gravity,* p *and* $\dot{p}$ *are, respectively, the position and velocity of the base of the system and* θ, $\dot{\theta}$ *are, respectively, the angle and angular rate of the structure above the base.*

The state-space representation of the dynamical system (2.4) *is given by*

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \dfrac{m^2l^2g}{\mu} & \dfrac{-cJ_t}{\mu} & \dfrac{-\gamma J_t lm}{\mu} \\ 0 & \dfrac{M_t mgl}{\mu} & \dfrac{-clm}{\mu} & \dfrac{-\gamma M_t}{\mu} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ \dfrac{J_t}{\mu} \\ \dfrac{lm}{\mu} \end{bmatrix} u(t) \ , \\ y(t) &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x(t) \ , \end{aligned}$$

with

$$x(t)=\begin{bmatrix} p(t)\\ \theta(t)\\ \dot{p}(t)\\ \dot{\theta}(t)\end{bmatrix},\quad y(t)=\begin{bmatrix} p(t)\\ \theta(t)\end{bmatrix},\quad u(t)=\mathrm{F},$$

and

$$\mathrm{M}_t=\mathrm{M}+m,\quad \mathrm{J}_t=\mathrm{J}+ml^2,\ \textit{and}\quad \mu=\mathrm{M}_t\mathrm{J}_t-m^2l^2.$$

Definition 2.4 [50] *A dynamical system is causal if the output $y(t)$ at a time t_0 depends only on the values of its input $u(t)$ for $t\leq t_0$.*

Definition 2.5 [50] *A dynamical system is strictly proper if the direct transmission matrix* D *is a zero matrix.*

2.1.2 Solvability of LTI dynamical system

Let us consider the LTI dynamical system

$$\mathrm{E}\,\dot{x}(t) = \mathrm{A}x(t)+\mathrm{B}u(t), \tag{2.5}$$

$$y(t) = \mathrm{C}x(t)+\mathrm{D}u(t), \tag{2.6}$$

where $x\in\mathbb{R}^m$ is the state vector, $u\in\mathbb{R}^q$ is the input vector, and $y\in\mathbb{R}^p$ is the output vector. $\mathrm{A}\in\mathbb{R}^{m\times m}$, $\mathrm{B}\in\mathbb{R}^{m\times q}$, $\mathrm{C}\in\mathbb{R}^{p\times m}$, $\mathrm{D}\in\mathbb{R}^{p\times q}$ are, respectively, the state, the command, the observation, the direct transmission matrices, and $\mathrm{E}\in\mathbb{R}^{m\times m}$ with the initial condition $x(0)=x_0$.

Assuming that $\det\mathrm{E}=0$, which means that the system (2.5) is called singular LTI dynamical system, and suppose that the pencil of the pair (E, A) of the system (2.5) is regular, i.e., for some $s\in\mathbb{C}$,

$$\det(s\mathrm{E}-\mathrm{A})\neq 0,$$

then, we can write the resolvent matrix as unique Laurent series [9], as follows

$$(s\mathrm{E}-\mathrm{A})^{-1}=\sum_{i=-\mu}^{\infty}\phi_i s^{-i-1},$$

where μ is nilpotency index of pencil (E, A), and it is given by

$$\mu=\operatorname{rg}\mathrm{E}-\deg[\det(s\mathrm{E}-\mathrm{A})]+1,$$

and ϕ_i is called the fundamental matrix of (2.5). Hence, the solution of the equation (2.5) is given by the following theorem.

Theorem 2.6 [9] *The trajectory of the equation* (2.5) *is given by*

$$x(t)=\sum_{i=0}^{\infty}\phi_i\left(\frac{\mathrm{B}}{\Gamma(i+1)}\int_0^t(t-\tau)^i u(\tau)d\tau+\mathrm{E}\frac{t^i}{\Gamma(i+1)}x_0\right)$$
$$+\sum_{i=1}^{\mu}\phi_{-i}\left(\mathrm{B}u^{(i-1)}(t)+\mathrm{E}\delta^{(i-1)}(t)x_0\right),$$

where δ is the Dirac Delta function, μ is the nilpotency index, and ϕ_i are the fundamental matrices.

Example 2.7 [9] *Let us consider the singular dynamical system* (2.5) *with*

$$\mathrm{E}=\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix},\quad \mathrm{A}=\begin{pmatrix}1&0&0\\0&1&0\\0&1&0\end{pmatrix},\quad \mathrm{B}=\begin{pmatrix}1\\0\\1\end{pmatrix},$$

and the initial conditions

$$x_0=\begin{pmatrix}x_{0,1}\\x_{0,2}\\x_{0,3}\end{pmatrix}.$$

For some $s\in\mathbb{C}$, we have

$$\det(s\mathrm{E}-\mathrm{A})=-s.$$

Thus, the system is regular and the nilpotency index is $\mu=2$. Consequently, the fundamental matrices are

$$\phi_0=\begin{pmatrix}0&0&0\\0&0&0\\0&1&0\end{pmatrix},\quad \phi_{-1}=\begin{pmatrix}-1&0&0\\0&0&-1\\0&0&0\end{pmatrix},\ \textit{and}\quad \phi_{-2}=\begin{pmatrix}0&0&-1\\0&0&0\\0&0&0\end{pmatrix}.$$

From theorem 2.6, *we obtain the following result*

$$x(t)=\begin{pmatrix}u(t)-u'(t)-x_{2,0}\delta(t)\\-u(t)\\x_{3,0}\end{pmatrix}.$$

Furthermore, if $\det\mathrm{E}\neq 0$, the system (2.5) turns into

$$\dot{x}(t)=\tilde{\mathrm{A}}x(t)+\tilde{\mathrm{B}}u(t), \tag{2.7}$$

where $\tilde{\mathrm{A}}=\mathrm{E}^{-1}\mathrm{A}$ and $\tilde{\mathrm{B}}=\mathrm{E}^{-1}\mathrm{B}$.

The system (2.7) is called standard LTI dynamical system, its solution is presented by the following theorem.

Theorem 2.8 [9, 28] *The solution of the equation* (2.7) *has the form*

$$x(t) = e^{\tilde{A}t}x_0 + \int_0^t e^{\tilde{A}(t-\tau)}\,\tilde{B}\,u(\tau)\,d\tau, \tag{2.8}$$

where $e^{\tilde{A}t} \in \mathbb{R}^{m\times m}$ *is given by*

$$e^{\tilde{A}t} = \sum_{i=0}^{\infty} \frac{(\tilde{A})^i}{\Gamma(i+1)} t^i.$$

Example 2.9 *Let us consider the following system*

$$E\dot{x}(t) = Ax(t) + Bu(t), \tag{2.9}$$

with the matrices

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

and the initial conditions

$$x_0 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

It is clear that $\det E \neq 0$*. Therefore, the system* (2.9) *becomes*

$$\dot{x}(t) = \tilde{A}x(t) + \tilde{B}u(t),$$

where

$$\tilde{A} = E^{-1}A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \textit{and} \quad \tilde{B} = E^{-1}B = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}.$$

Using (2.8), *it follows that*

$$x(t) = \begin{pmatrix} 1 \\ 1+t \\ -\int_0^t u(\tau)\, d\tau - t - \frac{t^2}{2} - \frac{t^3}{6} \\ 1+t+\frac{t^2}{2} \end{pmatrix}.$$

In the last years, the behaviour of real systems in various fields of science and engineering, including biology, viscoelasticity, electrochemistry and many others, have been modelled by fractional differential equations [20, 29, 31, 48, 65]. The concept of the fractional linear time invariant (LTI) dynamical systems will be detailed in the following.

2.2 Fractional-order LTI dynamical systems

The fractional-order LTI dynamical systems can be described by the fractional-order differential equations as follows [48]

$$\mathrm{H}\left(\mathbf{D}^{\alpha_0\,\alpha_1\,\alpha_2\cdots\alpha_l}\right)\, y(t) \;=\; \mathrm{G}\left(\mathbf{D}^{\beta_0\,\beta_1\,\beta_2\cdots\beta_k}\right)\, u(t), \tag{2.10}$$

where α_i and β_j for $i = 0, \cdots, l$ and $j = 0, \cdots, k$ are, respectively, the fractional order derivatives of the output and input in the sense Caputo fractional derivative. $y \in \mathbb{R}^p$, $u \in \mathbb{R}^q$ are functions of time. H and G are the combination laws of fractional-order derivative operators.

The equation (2.10) can be written as

$$\sum_{i=0}^{l} a_i\, \mathbf{D}^{\alpha_i}\, y(t) = \sum_{j=0}^{k} b_j\, \mathbf{D}^{\beta_j}\, u(t), \tag{2.11}$$

with

$$\mathrm{H}\left(\mathbf{D}^{\alpha_0\,\alpha_1\,\alpha_2\cdots\alpha_l}\right) = \sum_{i=0}^{l} a_i\, \mathbf{D}^{\alpha_i},$$

and

$$\mathrm{G}\left(\mathbf{D}^{\beta_0\,\beta_1\,\beta_2\cdots\beta_k}\right) = \sum_{j=0}^{k} b_j\, \mathbf{D}^{\beta_j}.$$

Definition 2.10 [52] *A fractional-order LTI dynamical system* (2.11) *is said to be commensurate if all the exponents are integer multiples of the same real number* ν. *The derivative order* ν *is called commensurate order of the system, and the equation* (2.11) *becomes*

$$\sum_{i=0}^{l} a_i \mathbf{D}^{i\nu} y(t) = \sum_{j=0}^{k} b_j \mathbf{D}^{j\nu} u(t).$$

2.2.1 State-space models

The state-space representation of fractional-order LTI commensurate dynamical system is described by following equations [29]

$$\begin{aligned} \mathrm{E}\mathbf{D}^{\alpha}x(t) &= \mathrm{A}x(t)+\mathrm{B}u(t), \\ y(t) &= \mathrm{C}x(t)+\mathrm{D}u(t), \end{aligned} \tag{2.12}$$

where $x \in \mathbb{R}^m$ is the state vector, $u \in \mathbb{R}^q$ is the input vector, and $y \in \mathbb{R}^p$ is the output vector, and $\mathrm{A} \in \mathbb{R}^{m\times m}$, $\mathrm{B} \in \mathbb{R}^{m\times q}$, $\mathrm{C} \in \mathbb{R}^{p\times m}$, $\mathrm{D} \in \mathbb{R}^{p\times q}$ are, respectively, the state, the command, the observation, the direct transmission matrices, and $\mathrm{E} \in \mathbb{R}^{m\times m}$; $\mathbf{D}^{\alpha}$ is the α Caputo fractional derivative with $n-1 \le \alpha < n$ and $n \in \mathbb{N}^*$.

Example 2.11 [30] *Let us consider the following fractional electrical circuit*

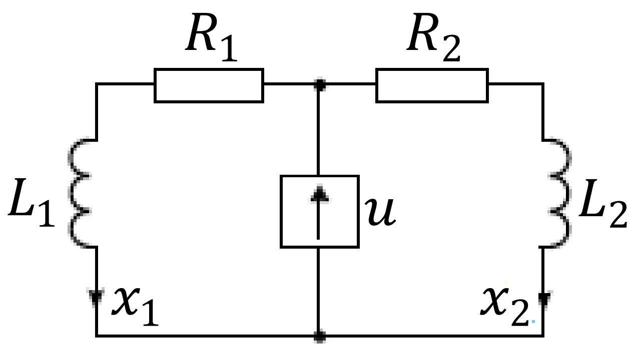

Figure 2.2: Fractional electrical circuit [30]

where $\mathrm{R}_1, \mathrm{R}_2$ *are resistances,* $\mathrm{L}_1, \mathrm{L}_2$ *are inductances, and* u *is source current.*

By the application of Kirchhoff's laws, the fractional electrical circuit can be described by following state equation

$$\mathrm{E}\mathbf{D}^{\alpha}x(t) = \mathrm{A}x(t)+\mathrm{B}u(t), \quad 0<\alpha\le 1,$$

where

$$\mathrm{E} = \begin{pmatrix} \mathrm{L}_1 & -\mathrm{L}_2 \\ 0 & 0 \end{pmatrix}, \quad \mathrm{A} = \begin{pmatrix} -\mathrm{R}_1 & \mathrm{R}_2 \\ -1 & -1 \end{pmatrix}, \quad \textit{and} \quad \mathrm{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

2.2.2 Solvability of fractional-order LTI dynamical systems

Let us consider the fractional-order LTI dynamical system

$$\begin{aligned} \mathrm{E}\mathbf{D}^{\alpha}x(t) &= \mathrm{A}x(t)+\mathrm{B}u(t), \\ y(t) &= \mathrm{C}x(t)+\mathrm{D}u(t), \end{aligned} \tag{2.13}$$

where $x \in \mathbb{R}^m$, $u \in \mathbb{R}^q$, $y \in \mathbb{R}^p$ are state, input and output vectors respectively, and $\mathrm{E}, \mathrm{A} \in \mathbb{R}^{m\times m}$ with $\det \mathrm{E} = 0$, $\mathrm{B} \in \mathbb{R}^{m\times q}$, $\mathrm{C} \in \mathbb{R}^{p\times m}$, and $\mathrm{D} \in \mathbb{R}^{p\times q}$; $\mathbf{D}^\alpha$, for $n-1 \le \alpha < n$ with $n \in \mathbb{N}^*$ is the α Caputo fractional derivative. The initial conditions associated with the system (2.13) are $x(0) = x_0$.

Nevertheless, the solution x is impulse free which is equivalent to the following compatibility conditions

- $\mathrm{E}x(0)$ and $s^{\alpha-k-1}\mathrm{E}x^{(k)}(0)$ exists for all $0 \le k \le n-1$, $n \in \mathbb{N}^*$ and for some $s \in \mathbb{C}$.
- $u(t)$ is provided and $u^{(k)}(0) = 0$ for all $k \in \mathbb{N}$.

Assuming that $\det(s^\alpha \mathrm{E} - \mathrm{A}) \neq 0$, for some $s \in \mathbb{C}$. Hence, the system (2.13) is called singular implicit fractional-order linear system, then, there exists a Laurent series expansion about zero, which is given by [10]

$$\left(s^\alpha \mathrm{E} - \mathrm{A}\right)^{-1} = \sum_{i=-\mu}^{\infty} \phi_i s^{-(i+1)\alpha},$$

where

$$\mu = \mathrm{rg}\,(\mathrm{E}) - \deg\left[\det\left(s^\alpha \mathrm{E} - \mathrm{A}\right)\right] + 1,$$

represents the nilpotency index of the pencil (E, A), and ϕ_i are the fundamental matrices.

Then, the solution of the system (2.13) is given by the following theorem.

Theorem 2.12 [10] *The solution of the implicit fractional dynamical system* (2.13) *with* E *not invertible is given by*

$$\begin{aligned} x(t) = & \sum_{i=0}^{\infty}\sum_{k=0}^{n-1} \frac{t^{i\alpha+k}}{\Gamma(i\alpha+k+1)} \phi_i \mathrm{E}\, x^{(k)}(t)\Big|_{t=0} \\ & + \sum_{i=0}^{\infty} \int_0^t \phi_i \frac{(t-\tau)^{(i+1)\alpha-1}}{\Gamma((i+1)\alpha)} \mathrm{B}u(\tau)\,d\tau \\ & + \sum_{i=1}^{\mu} \phi_{-i} \left(\mathrm{B}u^{(i-1)\alpha}(t) + \sum_{k=0}^{n-1} \delta^{(i\alpha-k-1)} \mathrm{E}\, x^{(k)}(t)\Big|_{t=0} \right), \end{aligned}$$

where δ is the Dirac Delta function, μ is the nilpotent index of the pencil (E, A), ϕ_i *are the fundamental matrices, and Γ is the Euler's Gamma function.*

Example 2.13 [30] *Consider the fractional-order electrical circuit presented in the example* 2.11 *with* $0 < \alpha \le 1$ *and*

$$\mathrm{E} = \begin{pmatrix} \mathrm{L}_1 & -\mathrm{L}_2 \\ 0 & 0 \end{pmatrix}, \quad \mathrm{A} = \begin{pmatrix} -\mathrm{R}_1 & \mathrm{R}_2 \\ -1 & -1 \end{pmatrix}, \quad \mathrm{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and the initial conditions

$$x_0 = \begin{pmatrix} x_{0,1} \\ x_{0,2} \end{pmatrix}.$$

This system is singular, more then that, $\mu = 1$ *and the corresponding fundamental matrices are*

$$\phi_{-1} = \begin{pmatrix} 0 & \dfrac{L_2}{L_1+L_2} \\ \\ 0 & \dfrac{L_1}{L_1+L_2} \end{pmatrix},$$

and

$$\forall\, i \in \mathbb{N},\ \phi_i = \begin{pmatrix} \dfrac{(-1)^i\,(R_1+R_2)^i}{(L_1+L_2)^{i+1}} & \dfrac{(-1)^i\,R_2\,(R_1+R_2)^i}{(L_1+L_2)^{i+1}} + \dfrac{(-1)^{i+1}\,L_2\,(R_1+R_2)^{i+1}}{(L_1+L_2)^{i+2}} \\ \\ \dfrac{(-1)^{i+1}\,(R_1+R_2)^i}{(L_1+L_2)^{i+1}} & \dfrac{(-1)^i\,R_1\,(R_1+R_2)^i}{(L_1+L_2)^{i+1}} + \dfrac{(-1)^{i+1}\,L_1\,(R_1+R_2)^{i+1}}{(L_1+L_2)^{i+2}} \end{pmatrix}.$$

By using theorem 2.12, we find

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix},$$

where

$$\begin{aligned} x_1(t) \quad = \quad & \sum_{i=0}^{\infty} \frac{1}{\Gamma((i+1)\alpha)} \left(\frac{(-1)^i\,R_2\,(R_1+R_2)^i}{(L_1+L_2)^{i+1}} + \frac{(-1)^{i+1}\,L_2\,(R_1+R_2)^{i+1}}{(L_1+L_2)^{i+2}} \right) \\ & \times \int_0^t (t-\tau)^{(i+1)\alpha-1} u(\tau)\,d\tau + \frac{L_2}{L1+L2} u(t) \\ & + \sum_{i=0}^{\infty} \frac{t^{i\alpha}}{\Gamma(i\alpha+1)} \left(\frac{(-1)^i\,L_1\,(R_1+R_2)^i}{(L_1+L_2)^{i+1}} x_{0,1} + \frac{(-1)^{i+1}\,L_2\,(R_1+R_2)^i}{(L_1+L_2)^{i+1}} x_{0,2} \right), \end{aligned}$$

and

$$\begin{aligned} x_2(t) \quad = \quad & \sum_{i=0}^{\infty} \frac{1}{\Gamma((i+1)\alpha)} \left(\frac{(-1)^i\,R_1\,(R_1+R_2)^i}{(L_1+L_2)^{i+1}} + \frac{(-1)^{i+1}\,L_1\,(R_1+R_2)^{i+1}}{(L_1+L_2)^{i+2}} \right) \\ & \times \int_0^t (t-\tau)^{(i+1)\alpha-1} u(\tau)\,d\tau + \frac{L_1}{L1+L2} u(t) \\ & + \sum_{i=0}^{\infty} \frac{t^{i\alpha}}{\Gamma(i\alpha+1)} \left(\frac{(-1)^{i+1}\,L_1\,(R_1+R_2)^i}{(L_1+L_2)^{i+1}} x_{0,1} + \frac{(-1)^{i+2}\,L_2\,(R_1+R_2)^i}{(L_1+L_2)^{i+1}} x_{0,2} \right). \end{aligned}$$

In this case where E can be invertible matrix i. e., $\det \mathrm{E} \neq 0$, the system (2.13) becomes

$$\mathbf{D}^{\alpha} x(t) = \tilde{\mathrm{A}} x(t) + \tilde{\mathrm{B}} u(t), \tag{2.14}$$

where $\tilde{\mathrm{A}} = \mathrm{E}^{-1}\mathrm{A}$ and $\tilde{\mathrm{B}} = \mathrm{E}^{-1}\mathrm{B}$.

Hence, the solution of the equation (2.14) is given by the next theorem.

Theorem 2.14 [31] *The solution of the implicit fractional dynamical system (2.13) is given by*

$$\begin{aligned} x(t) = \sum_{i=0}^{\infty} \frac{\tilde{\mathrm{A}}^{i}\tilde{\mathrm{B}}}{\Gamma((i+1)\alpha)} \int_0^t (t-\tau)^{(i+1)\alpha-1} u(\tau)\, d\tau \\ + \sum_{i=0}^{\infty}\sum_{k=0}^{n-1} \tilde{\mathrm{A}}^{i} \frac{t^{i\alpha+k}}{\Gamma(i\alpha+k+1)} \, x^{(k)}(t)\Big|_{t=0}, \end{aligned}$$

where α is the fractional-order derivatives, and Γ is the Euler's Gamma function.

Example 2.15 *The one degree of freedom general model of a car suspension described in figure 1.2, can be written by the following state equation*

$$\mathbf{D}^{\frac{3}{2}} x(t) = \mathrm{A}x(t) + \mathrm{B}u(t), \tag{2.15}$$

where

$$\mathrm{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -20^{\frac{3}{2}} \end{pmatrix} \quad \textit{and} \quad \mathrm{B} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The solution of (2.15) is given by [31]

$$\begin{aligned} x(t) \quad = \quad & \frac{2\sqrt{\pi}}{\pi}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \int_0^t (t-\tau)^{\frac{1}{2}} u(\tau)\, d\tau + \frac{1}{2}\begin{pmatrix} 0 \\ 1 \\ -20^{\frac{3}{2}} \end{pmatrix} \int_0^t (t-\tau)^2 u(\tau)\, d\tau \\ & + \begin{pmatrix} x_{0,1} \\ x_{0,2} \\ x_{0,3} \end{pmatrix} + \frac{4\sqrt{\pi}}{3\pi}\begin{pmatrix} x_{0,2} \\ x_{0,3} \\ -20^{\frac{3}{2}} x_{0,3} \end{pmatrix} t^{\frac{3}{2}} + \begin{pmatrix} x'_{0,1} \\ x'_{0,2} \\ x'_{0,3} \end{pmatrix} t + \frac{8\sqrt{\pi}}{15\pi}\begin{pmatrix} x'_{0,2} \\ x'_{0,3} \\ -20^{\frac{3}{2}} x'_{0,3} \end{pmatrix} t^{\frac{5}{2}} \\ & + \sum_{i=2}^{\infty} \begin{pmatrix} (-20^{\frac{3}{2}})^{i-2} \\ (-20^{\frac{3}{2}})^{i-1} \\ (-20^{\frac{3}{2}})^{i} \end{pmatrix} \left[\frac{1}{(\frac{3}{2}i+\frac{1}{2})\Gamma(\frac{3}{2}i+\frac{1}{2})} \int_0^t (t-\tau)^{\frac{3}{2}i+\frac{1}{2}} u(\tau)\, d\tau \right. \\ & \left. + \frac{x_{0,3}}{\frac{3}{2}i\Gamma(\frac{3}{2}i)} t^{\frac{3}{2}i} + \frac{x'_{0,3}}{(\frac{9}{4}i^2+\frac{3}{2}i)\Gamma(\frac{3}{2}i)} t^{\frac{3}{2}i+1} \right]. \end{aligned}$$

In control theory, the dynamical systems can be described and represented in different ways, where, one of them is the state-space representation as shown previously. Another way is the representation by a transfer function which will presented and detailed in the following section.

3 Transfer function

This section introduces the notion of the transfer function which is the relation between the input and the output of a linear dynamical system. It is a convenient representation of a linear time-invariant dynamical system. In mathematics, the transfer function is a function of complex variables, which is the ratio of the Laplace transform of output and input. It can be obtained by inspection or by simple algebraic manipulations of the differential equations that describe the systems and systems in very-high order. The general form of transfer function is written in the following form [8].

$$\mathrm{G}(s) = \mathrm{K}\frac{\mathrm{H}(s)}{s^{\alpha}(1 + a_1 s + \ldots + a_n s^n)},$$

where K is the gain, α is the class of system, and $\mathrm{H}(0) = 1$.

3.1 Transfer function representation of differential equation

Given the following differential equation, which describes the linear input-output system

$$a_0\frac{d^n y}{dt^n} + a_1\frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_n y = b_0\frac{d^m u}{dt^m} + b_1\frac{d^{m-1}u}{dt^{m-1}} + \ldots + b_m u, \tag{2.16}$$

where u is the input and y is the output. The transfer function G associated with (2.16) is identified as follows [4]

$$\begin{aligned}\mathrm{Y}(s) &= \mathrm{G}(s)\mathrm{U}(s),\\ &= \frac{\mathrm{N}(s)}{\mathrm{D}(s)}\mathrm{U}(s),\\ &= \frac{b_0 s^m + b_1 s^{m-1} + \ldots + b_m}{a_0 s^n + a_{n-1}s^{n-1} + \ldots + a_n}\mathrm{U}(s).\end{aligned}$$

The order of the transfer function is defined as the order of the denominator polynomial.

The most powerful ways to represent systems are the transfer function and the state-space forms. In the following, we describe how we get the transfer function from the state-space representation.

3.2 State-space representation to transfer function

Consider the state-space LTI dynamical system

$$\begin{cases} \dot{x}(t) = \mathrm{A}x(t) + \mathrm{B}u(t), \\ y(t) = \mathrm{C}x(t) + \mathrm{D}u(t), \end{cases} \tag{2.17}$$

where $x \in \mathbb{R}^m$ is the state vector, $u \in \mathbb{R}^q$ is the input vector, $y \in \mathbb{R}^p$ is the output vector and A, B, C, and D are real matrices of an appropriate dimensions with null initial conditions. Moreover, we assume that the pencil (Id, A) is regular.

Applying the Laplace transform, the system (2.17) becomes, for some $s \in \mathbb{C}$

$$\begin{cases} s\mathrm{X}(s) = \mathrm{AX}(s) + \mathrm{BU}(s), \\ \mathrm{Y}(s) = \mathrm{CX}(s) + \mathrm{DU}(s). \end{cases}$$

In order to find the relation between $\mathrm{Y}(s)$ and $\mathrm{U}(s)$, we need to remove $\mathrm{X}(s)$ from the output equation.

For this purpose, we start by solving the state equation

$$s\mathrm{X}(s) = \mathrm{AX}(s) + \mathrm{BU}(s) \Rightarrow \mathrm{X}(s) = (s\,\mathrm{Id} - \mathrm{A})^{-1}\mathrm{BU}(s),$$

since $\det(s\,\mathrm{Id} - \mathrm{A}) \neq 0$, for some $s \in \mathbb{C}$.

By substitution of the expression of $\mathrm{X}(s)$ into the output equation, we get

$$\begin{aligned} \mathrm{Y}(s) &= \mathrm{C}(s\,\mathrm{Id} - \mathrm{A})^{-1}\mathrm{BU}(s) + \mathrm{DU}(s), \\ &= \left(\mathrm{C}(s\,\mathrm{Id} - \mathrm{A})^{-1}\mathrm{B} + \mathrm{D}\right)\mathrm{U}(s). \end{aligned}$$

Finally, the transfer function which is the ratio of $\mathrm{Y}(s)$ to $\mathrm{U}(s)$ is given by

$$\begin{aligned} \mathrm{G}(s) &= \frac{\mathrm{Y}(s)}{\mathrm{U}(s)}, \\ &= \mathrm{C}\,(s\,\mathrm{Id} - \mathrm{A})^{-1}\,\mathrm{B} + \mathrm{D}. \end{aligned}$$

Example 3.1 [4] *Let us consider the response of a damped oscillator, which is described by its state-space system*

$$\begin{cases} \dot{x}(t) = \mathrm{A}x(t) + \mathrm{B}u(t), \\ y(t) = \mathrm{C}x(t), \end{cases} \tag{2.18}$$

where

$$\mathrm{A} = \begin{bmatrix} 0 & \omega_0 \\ -\omega_0 & -2\zeta\omega_0 \end{bmatrix}, \quad \mathrm{B} = \begin{bmatrix} 0 \\ k\omega_0 \end{bmatrix}, \text{ and } \mathrm{C} = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

with ω_0 *is the undamped angular frequency of the oscillator,* ψ *is the damping ratio,* k *is* *the force constant, and* $x(0) = 0$.

The transfer function G *which describes the system* (2.18) *is given by*

$$\begin{aligned} G(s) &= C\,(s\,\mathrm{Id} - A)^{-1}\,B, \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -\omega_0 \\ \omega_0 & s + 2\zeta\omega_0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ k\omega_0 \end{bmatrix}, \\ &= \frac{1}{s^2 + 2\zeta\omega_0 s + \omega_0^2} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & \omega_0 \\ -\omega_0 & s + 2\zeta\omega_0 \end{bmatrix} \begin{bmatrix} 0 \\ k\omega_0 \end{bmatrix}, \\ &= \frac{k\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}. \end{aligned}$$

3.3 Fractional-order transfer function

In the following, we will extend the concept of transfer function for fractional differential equations and fractional dynamical linear systems.

Let us consider the following fractional differential equation which represents the dynamics of a system

$$a_0 \mathbf{D}^{\alpha_0} y(t) + a_1 \mathbf{D}^{\alpha_1} y(t) + \cdots + a_n \mathbf{D}^{\alpha_n} y(t) = b_0 \mathbf{D}^{\beta_0} u(t) + \cdots + b_m \mathbf{D}^{\beta_m} u(t),$$

with null initial conditions.

Taking the Laplace transform of the above equation, the input-output fractional order transfer function model takes the form [48]

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^{\beta_m} + b_{m-1} s^{\beta_{m-1}} + \cdots + b_0 s^{\beta_0}}{a_n s^{\alpha_n} + a_{n-1} s^{\alpha_{n-1}} + \cdots + a_0 s^{\alpha_0}},$$

for some $s \in \mathbb{C}$.

In the above fractional differential equation, if orders of differentiation are integer multiples of a single base order i.e., $\alpha_k, \beta_k = k\alpha$, $\alpha \in \mathbb{R}_+^*$, the system will be termed as commensurate order and takes the following form [48]

$$\sum_{k=0}^{n} a_k \mathbf{D}^{k\alpha} y(t) = \sum_{k=0}^{m} b_k \mathbf{D}^{k\alpha} u(t),$$

thus, the fractional-order transfer function becomes [48]

$$G(s) = \frac{\sum\limits_{k=0}^{m} b_k s^{\alpha k}}{\sum\limits_{k=0}^{n} a_k s^{\alpha k}}. \tag{2.19}$$

where $(a_k, b_k) \in \mathbb{R}^2$, α is the commensurable differentiation order, and m and n are respectively numerator and denominator degrees, with $n > m$ for strictly causal systems.

3.4 State-space representation to fractional-order transfer function

Let us consider the LTI dynamical system

$$\begin{cases} \mathbf{D}^{\alpha} x(t) = \mathrm{A}x(t) + \mathrm{B}u(t), \\ y(t) = \mathrm{C}x(t) + \mathrm{D}u(t), \end{cases} \tag{2.20}$$

where $x \in \mathbb{R}^q$ is the state vector, $u \in \mathbb{R}^m$ is the input vector, $y \in \mathbb{R}^p$ is the output vector and $\mathrm{A} \in \mathbb{R}^{q\times q}$, $\mathrm{B} \in \mathbb{R}^{q\times m}$, $\mathrm{C} \in \mathbb{R}^{p\times q}$ and $\mathrm{D} \in \mathbb{R}^{p\times m}$ with null initial conditions; $\mathbf{D}^{\alpha}$ is called the Caputo fractional derivative [13], where $n-1 \leq \alpha < n$, for some $n \in \mathbb{N}^*$. We assume that the matrix $(s^{\alpha}\,\mathrm{Id} - \mathrm{A})$ is invertible for some $s \in \mathbb{C}$.

The fractional-order transfer function is the ratio of $\mathrm{Y}(s)$ to $\mathrm{U}(s)$ which is given by

$$\begin{aligned} \mathrm{G}(s) &= \frac{\mathrm{Y}(s)}{\mathrm{U}(s)}, \\ &= \mathrm{C}\left(s^{\alpha}\,\mathrm{Id} - \mathrm{A}\right)^{-1}\mathrm{B} + \mathrm{D}. \end{aligned}$$

Remark 3.2 *The fractional-order transfer function is unique.*

In the following, we will show how to extract the fractional-order transfer function from a state-space representation.

Example 3.3 *Let us consider the fractional linear system written in a state-space representation*

$$\begin{cases} \mathbf{D}^{\alpha} x(t) = \mathrm{A}x(t) + \mathrm{B}u(t), \\ y(t) = \mathrm{C}x(t) + \mathrm{D}u(t), \end{cases} \tag{2.21}$$

where

$$\mathrm{A} = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}, \quad \mathrm{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \mathrm{C} = \begin{pmatrix} 6 & 7 \end{pmatrix}, \textit{ and } \mathrm{D} = 2.$$

The corresponding fractional-order transfer function is given by the following expression for some $s \in \mathbb{C}$

$$\mathrm{G}(s) = \mathrm{C}\left(s^{\alpha}\,\mathrm{Id} - \mathrm{A}\right)^{-1}\mathrm{B} + \mathrm{D},$$

with

$$s^{\alpha}\,\mathrm{Id} - \mathrm{A} = \begin{pmatrix} s^{\alpha} & -1 \\ -2 & s^{\alpha} - 1 \end{pmatrix},$$

and

$$\det(s^{\alpha}\,\mathrm{Id} - \mathrm{A}) = s^{\alpha}\left(s^{\alpha} - 1\right) - 2.$$

Then, the matrix $(s^\alpha \mathrm{Id} - A)^{-1}$ *is given by*

$$(s^\alpha I - A)^{-1} = \begin{pmatrix} \dfrac{s^\alpha - 1}{s^\alpha (s^\alpha - 1) - 2} & \dfrac{1}{s^\alpha (s^\alpha - 1) - 2} \\ \\ \dfrac{2}{s^\alpha (s^\alpha - 1) - 2} & \dfrac{s^\alpha}{s^\alpha (s^\alpha - 1) - 2} \end{pmatrix}.$$

The fractional-order transfer function G *associated with the system* (2.21) *is*

$$G(s) \quad = \begin{pmatrix} 6 & 7 \end{pmatrix} \begin{pmatrix} \dfrac{s^\alpha - 1}{s^\alpha (s^\alpha - 1) - 2} & \dfrac{1}{s^\alpha (s^\alpha - 1) - 2} \\ \dfrac{2}{s^\alpha (s^\alpha - 1) - 2} & \dfrac{s^\alpha}{s^\alpha (s^\alpha - 1) - 2} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + 2,$$

thus

$$G(s) = \frac{2s^{2\alpha} + 5s^\alpha + 2}{s^\alpha (s^\alpha - 1) - 2}.$$

The commensurate fractional-order transfer function (2.19) can always be written in a factorized form using elementary fractional-order transfer functions of the first and the second kind respectively [45]

$$G_1(s) = \frac{K_1}{s^\alpha + \lambda_1},$$

and

$$G_2(s) = \frac{K_2}{(s^\alpha + \lambda_2)(s^\alpha + \overline{\lambda}_2)},$$

where $K_1, K_2 \in \mathbb{R}^*, \lambda_1 \in \mathbb{R}^+$, and $\lambda_2 \in \mathbb{C}^*$.

The most well known stability result of commensurate fractional-order transfer function which has been achieve by *Matignon* in [46] are presented in the following.

3.5 Stability of linear fractional-order continuous time systems

As in classical calculus, stability analysis is a central task in the study of fractional differential system and fractional control. As for linear time invariant integer order systems, it is well known that stability of a linear fractional-order system depends on the location of the system poles in the complex plane.

Conjecture 3.4 [46] *A fractional-order system defined by its fractional-order transfer function*

$$G(s) = \frac{Q(s)}{P(s)}, \quad \mathrm{Re}\,(s) \geq 0,$$

where

$$P(s) = \sum_{k=0}^{p} P_k s^{\alpha_k},$$

with $\alpha_{k+1} > \alpha_k \geq 0$, *and*

$$Q(s) = \sum_{l=0}^{q} q_l s^{\beta_l},$$

with $\beta_{l+1} > \beta_l \geq 0$ *are no longer polynomials.*

The system has the main property of

$$\textit{BIBO stability} \iff \exists M > 0, \|G(s)\| \leq M, \forall s, \quad \mathrm{Re}(s) \geq 0.$$

Moreover, in the case where no simplification occurs between P *and* Q, *that is all the* $P(s) = 0$ *not being roots of* $Q(s) = 0$, *the stability property then reads*

$$\textit{BIBO stability} \iff P(s) \neq 0, \forall s, \mathrm{Re}(s) \geq 0.$$

4 The $\mathcal{H}_2$-norm

The impulse response energy, known also as the $\mathcal{H}_2$-norm, of fractional-order transfer function makes a substantial contribution in the field of the dynamical systems theory and plays an important role in the field of dynamical systems and control. The $\mathcal{H}_2$-norm can be used to measure the precision of rational approximation of a fractional-order transfer function. In the following, we will give the definitions of the $\mathcal{H}_2$-norm of MIMO and SISO transfer functions respectively.

Definition 4.1 [65] *The $\mathcal{H}_2$-norm of multiple-input, multiple-output (MIMO) transfer function* G *is given by*

$$\|G\|_{\mathcal{H}_2} = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{tr}\left[G(j\omega)G^*(j\omega)\right] d\omega}. \tag{2.22}$$

Definition 4.2 [65] *The $\mathcal{H}_2$-norm of single-input, single-output (SISO) transfer function* G *is given by*

$$\|G\|_{\mathcal{H}_2} = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{+\infty} \left|G(j\omega)\right|^2 d\omega}. \tag{2.23}$$

Remark 4.3 [65] *Because of* $G(-j\omega) = G^*(j\omega)$,

$$\int_{-\infty}^{+\infty} G(j\omega)G^*(j\omega)d\omega = \int_{0}^{+\infty} G(j\omega)G^*(j\omega)d\omega.$$

Thus,

$$\|G\|_{\mathcal{H}_2} = \sqrt{\frac{1}{\pi}\int_{0}^{+\infty} G(j\omega)G^*(j\omega)d\omega}.$$

In the literature, there are some analytical formulas to compute the $\mathcal{H}_2$-norm of fractional-order transfer function [45, 65] which depend on an analytic computation using transfer function coefficients and differentiation orders. However, in the next chapters, we will present a new process for computing the $\mathscr{H}_2$-norm of fractional-order transfer function using matrix theory tools and integral transform.

5 Conclusion

In this chapter, we have presented the important concepts used in this thesis. Indeed, we have started by some basic notions of dynamical systems which are described by fractional differential equation or state-space representation. Then, we have recalled the fractional-order transfer function since its importance. Finally, the concept of the $\mathscr{H}_2$-norm, which is our main aim in this thesis has been introduced.

Chapter 3

The $\mathscr{H}_2$-norm of the fractional-order transfer function of first kind

1 Introduction

As said in the introduction, computing the impulse response energy, know also as the $\mathscr{H}_2$-norm, of a fractional-order transfer function is one of the most important problems in modelling and control of dynamical systems since it can be used to measure the precision of a rational approximation of a fractional-order transfer function and vice versa [3, 45, 64, 65]. More than that, the $\mathscr{H}_2$-norm is a useful measure for assessing the system's performance [45, 65].

In the state-of-the-art, numerous methods have been proposed to calculate the impulse response energy involving an analytic or an algebraic formulation [3, 45, 65]. However, the goal of this chapter is to provide an efficient method for computing the $\mathscr{H}_2$-norm for a fractional-order transfer function of the first kind using the state-space realization consisting of parameters which are extracted from the fractional-order transfer function, the parahermitian matrix and by combining some techniques that are transformation matrices and integral transforms. It must be emphasized that the transformation matrices used let the parahermitian matrix invariant [22, 66].

For the sake of simplicity, we will present the new approach to compute the $\mathscr{H}_2$-norm for the fractional-order transfer function of the first kind with real parameters and, then, we will improve it for those with complex parameters. The present chapter is concluded by numerical examples where the effectiveness of the proposed method is highlighted.

2 The $\mathscr{H}_2$-norm of the fractional-order transfer function of first kind with real parameters

This section is dedicated to present one of our main results that is the expression of the $\mathscr{H}_2$-norm for a fractional-order transfer function of the first kind with real parameters [36]. Nonetheless, we will start by recalling some preliminaries.

2.1 Preliminaries

A generalized fractional-order state-space model consisting of the parameters $\{a, b, c, \alpha\}$ can be represented as

$$\begin{cases} \mathbf{D}^{\alpha} x(t) = a x(t) + b u(t), \\ \quad y(t) = c x(t), \end{cases} \tag{3.1}$$

where x, u, $y \in \mathbb{R}^*$ are respectively the state, the input, and the output, $a \in \mathbb{R}^*_-$ and b, $c \in \mathbb{R}^*$ with a null initial condition. $\mathbf{D}^{\alpha}$, where $n-1 \leq \alpha < n$, for some $n \in \mathbb{N}^*$, is the α fractional-order derivation of the function x in the sense of the Caputo derivative, given by [13]

$$\mathbf{D}^{\alpha} x(t) = \frac{1}{\Gamma[n-\alpha]} \int_0^t \frac{x^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \quad x^{(n)}(\tau) = \frac{d^n x(\tau)}{d\tau^n}, \quad n \in \mathbb{N}^*.$$

For some $s \in \mathbb{C}$, we assume that the pencil $(1, a)$ is regular which is equivalent to

$$\left(s^{\alpha} - a\right) \neq 0.$$

From the system (3.1), the fractional-order transfer function G can be extracted. Indeed, by the means of the Laplace transform [55], the direct input-output relation of the system (3.1) can be written as, for some $s \in \mathbb{C}$

$$\mathrm{Y}(s) = c\left(s^{\alpha} - a\right)^{-1} b\,\mathrm{U}(s).$$

However,

$$\mathrm{Y}(s) = \mathrm{G}(s)\,\mathrm{U}(s),$$

then, the fractional-order transfer function associated with the system (3.1) is described as

$$\mathrm{G}(s) = c\left(s^{\alpha} - a\right)^{-1} b,$$

which has the generalized state-space realization consisting on $\{a, b, c, \alpha\}$, for some $s \in \mathbb{C}$.

The function G is the Schur complement of the corresponding system matrix $\mathrm{S_G}(s)$ as follows [22]

$$\mathrm{S_G}(s) = \left[\begin{array}{c|c} s^{\alpha} - a & b \\ \hline -c & 0 \end{array}\right],$$

with respect to its right block entry.

The fractional-order transfer function G is proper, thus, its conjugate transpose is

$$\mathrm{G}^*(s) = b\left(-s^{\alpha} - a\right)^{-1} c,$$

and it is the Schur complement of the corresponding system matrix $\mathrm{S_{G^*}}$

$$\mathrm{S_{G^*}}(s) = \left[\begin{array}{c|c} -s^{\alpha} - a & c \\ \hline -b & 0 \end{array}\right].$$

Using simple algebraic manipulation on the matrices $\mathrm{S_G}$ and $\mathrm{S_{G^*}}$, it follows

$$\begin{aligned} \mathrm{S}_{\phi}(s) &= \mathrm{S_G}(s)\,\mathrm{S_{G^*}}(s), \\ &= \left[\begin{array}{cc|c} 0 & -s^{\alpha} - a & c \\ s^{\alpha} - a & -b^2 & 0 \\ \hline -c & 0 & 0 \end{array}\right]. \end{aligned}$$

Note that the matrix S_{ϕ} is also known as the parahermitian matrix where its corresponding parahermitian transfer function is

$$\begin{aligned} \phi(s) &= \mathrm{G}(s)\,\mathrm{G}^*(s), \\ &= c\left(s^{\alpha} - a\right)^{-1} b^2 \left(-s^{\alpha} - a\right)^{-1} c. \end{aligned}$$

2.2 Main result

This part is devoted to present the first main result of this chapter. It is about computing the $\mathscr{H}_2$-norm of fractional-order transfer function of the first kind with real parameters using the state-space realization, parahermitian transfer matrices, matrices transformation, and integral transform. The obtained result is presented by the following theorem.

Theorem 2.1 [36] *Assuming that* $\frac{1}{2} < \alpha < 2$ *and* $\alpha \neq 1$*. Then, the* $\mathscr{H}_2$*-norm of the fractional-order transfer function* G*, with generalized state-space realization* $\{a, b, c, \alpha\}$*, where* $a \in \mathbb{R}^*_-$*,* $b, c \in \mathbb{R}^*$ *and* $\left(s^{\alpha} - a\right) \neq 0$ *for some* $s \in \mathbb{C}$*, is defined as*

$$\|\mathrm{G}\|^2_{\mathscr{H}_2} = -\frac{b^2 c^2 (-a)^{\frac{1}{\alpha} - 2} \cot\left(\alpha \frac{\pi}{2}\right)}{\alpha \sin\left(\frac{\pi}{\alpha}\right)}.$$

Proof. Let us consider the dynamical system which is written in the following form

$$\begin{cases} \mathbf{D}^{\alpha} x(t) = a x(t) + b u(t), \\ \quad y(t) = c x(t), \end{cases} \tag{3.2}$$

where x, u, $y \in \mathbb{R}^*$ are respectively the state, the input, and the output, $a \in \mathbb{R}^*_-$, and b, $c \in \mathbb{R}^*$ with a null initial condition. $\mathbf{D}^{\alpha}$, where $\frac{1}{2} < \alpha < 2$ with $\alpha \neq 1$, is the α fractional-order derivation of the function x in the sense of the Caputo derivative.

For some $s \in \mathbb{C}$, the fractional-order transfer function associated with the system (3.2) is

$$\mathrm{G}(s) = c\left(s^{\alpha} - a\right)^{-1} b. \tag{3.3}$$

As the expression of the $\mathcal{H}_2$-norm $\big($formula (2.22)$\big)$ is written in the frequency domain, then, the fractional-order transfer function (3.3) turns into

$$\mathrm{G}(\tilde{\omega}) = c\left(j^{\alpha}\tilde{\omega} - a\right)^{-1} b, \tag{3.4}$$

with $s = j\omega$, $\tilde{\omega} = \omega^{\alpha}$, $j = e^{j\frac{\pi}{2}}$, and $j^2 = -1$. Its conjugate transpose function becomes

$$\mathrm{G}^*(\tilde{\omega}) = b\left(\overline{j}^{\alpha}\tilde{\omega} - a\right)^{-1} c.$$

The multiplication between the functions G and G^* gives the parahermitian transfer function ϕ as

$$\begin{aligned} \phi(\tilde{\omega}) &= \mathrm{G}(\tilde{\omega})\mathrm{G}^*(\tilde{\omega}), \\ &= c\left(j^{\alpha}\tilde{\omega} - a\right)^{-1} b^2 \left(\overline{j}^{\alpha}\tilde{\omega} - a\right)^{-1} c, \end{aligned}$$

that is, also, the Schur complement of the so-called system matrix S_{ϕ}

$$\mathrm{S}_{\phi}(\tilde{\omega}) = \left[\begin{array}{cc|c} 0 & \overline{j}^{\alpha}\tilde{\omega} - a & c \\ j^{\alpha}\tilde{\omega} - a & -b^2 & 0 \\ \hline -c & 0 & 0 \end{array}\right].$$

It is well known that the parahermitian matrix can be transformed under row and column matrices transformations that leave the system state-space realization $\{a, b, c, \alpha\}$ invariant [22, 66]. Therefore, the matrix $\mathrm{S}_{\phi}(\tilde{\omega})$ can be transformed into the following matrix

$$\mathrm{S}_{\tilde{\phi}}(\tilde{\omega}) = \left[\begin{array}{cc|c} 0 & \overline{j}^{\alpha}\tilde{\omega} - a & c \\ j^{\alpha}\tilde{\omega} - a & f & pc \\ \hline -c & -cp & 0 \end{array}\right],$$

where

$$f = \left(j^{\alpha}\tilde{\omega} - a\right) p + p\left(\bar{j}^{\alpha}\tilde{\omega} - a\right) - b^2,$$

with p given by

$$p = \frac{b^2}{2\left(\cos\left(\frac{\alpha\pi}{2}\right)\tilde{\omega} - a\right)},$$

is the solution of the equation $f = 0$. Thus, the matrix $\mathrm{S}_{\tilde{\phi}}(\tilde{\omega})$ becomes

$$\mathrm{S}_{\tilde{\phi}}(\tilde{\omega}) = \left[\begin{array}{cc|c} 0 & \bar{j}^{\alpha}\tilde{\omega} - a & c \\ j^{\alpha}\tilde{\omega} - a & 0 & \dfrac{c\,b^2}{2\left(\cos\left(\frac{\alpha\pi}{2}\right)\tilde{\omega} - a\right)} \\ \hline -c & -\dfrac{c\,b^2}{2\left(\cos\left(\frac{\alpha\pi}{2}\right)\tilde{\omega} - a\right)} & 0 \end{array}\right].$$

In this case, the Schur complement of the matrix $\mathrm{S}_{\tilde{\phi}}$ can be written as

$$\tilde{\phi}(\tilde{\omega}) = \begin{pmatrix} c & \dfrac{c\,b^2}{2\left(\cos\left(\frac{\alpha\pi}{2}\right)\tilde{\omega} - a\right)} \end{pmatrix} \begin{pmatrix} 0 & \left(j^{\alpha}\tilde{\omega} - a\right)^{-1} \\ \left(\bar{j}^{\alpha}\tilde{\omega} - a\right)^{-1} & 0 \end{pmatrix} \begin{pmatrix} c \\ \dfrac{c\,b^2}{2\left(\cos\left(\frac{\alpha\pi}{2}\right)\tilde{\omega} - a\right)} \end{pmatrix},$$

or even by

$$\tilde{\phi}(\tilde{\omega}) = \frac{c^2\,b^2}{2\left(\cos\left(\frac{\alpha\pi}{2}\right)\tilde{\omega} - a\right)} \left(\frac{1}{\bar{j}^{\alpha}\tilde{\omega} - a} + \frac{1}{j^{\alpha}\tilde{\omega} - a}\right).$$

Finally, by simple manipulations, yields

$$\tilde{\phi}(\tilde{\omega}) = -\frac{c^2\,b^2}{2\,j\,a\sin\left(\frac{\alpha\pi}{2}\right)} \left[\frac{1}{\tilde{\omega} - a\,e^{-\frac{\alpha\pi}{2}j}} - \frac{1}{\tilde{\omega} - a\,e^{\frac{\alpha\pi}{2}j}}\right].$$

Thus, the $\mathscr{H}_2$-norm of the fractional-order transfer function G is

$$\begin{aligned} \|\mathrm{G}\|^2_{\mathscr{H}_2} &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{G}(j\omega)\mathrm{G}^*(j\omega)\,d\omega, \\ &= \frac{1}{\alpha\pi}\int_0^{+\infty} \tilde{\omega}^{\frac{1}{\alpha}-1}\,\tilde{\phi}(\tilde{\omega})\,d\tilde{\omega}, \\ &= -\frac{c^2\,b^2}{2\pi\,j\,a\alpha\sin\left(\frac{\alpha\pi}{2}\right)} \left[\int_0^{+\infty} \left(\frac{1}{\tilde{\omega} - a\,e^{-\frac{\alpha\pi}{2}j}}\right)\tilde{\omega}^{\frac{1}{\alpha}-1}\,d\tilde{\omega}\right. \\ &\qquad \left. - \int_0^{+\infty} \left(\frac{1}{\tilde{\omega} - a\,e^{\frac{\alpha\pi}{2}j}}\right)\tilde{\omega}^{\frac{1}{\alpha}-1}\,d\tilde{\omega}\right]. \end{aligned} \tag{3.5}$$

Thanks to Mellin integral transform [17], we obtain

$$\begin{aligned}\|G\|^2_{\mathcal{H}_2} &= -\frac{c^2 b^2}{2\pi j a \alpha \sin\left(\frac{\alpha\pi}{2}\right)}\left[\left(\frac{\pi}{\sin\left(\frac{\pi}{\alpha}\right)}(-a)^{\frac{1}{\alpha}-1} e^{\left(\frac{\pi}{2}-\frac{\pi}{2}\alpha\right)j}\right)-\left(\frac{\pi}{\sin\left(\frac{\pi}{\alpha}\right)}(-a)^{\frac{1}{\alpha}-1} e^{-\left(\frac{\pi}{2}-\frac{\pi}{2}\alpha\right)j}\right)\right],\\ &= -\frac{b^2 c^2 (-a)^{\frac{1}{\alpha}-2}\cot\left(\frac{\pi}{2}\alpha\right)}{\alpha \sin\left(\frac{\pi}{\alpha}\right)},\end{aligned}$$

which achieves the proof of the theorem 2.1.

■

For $\alpha = 1$, we get

Proposition 2.2 *The $\mathcal{H}_2$-norm of the transfer function* G*, with generalized state-space realization* $\{a, b, c\}$*, where* $a \in \mathbb{R}^*_-$*,* $b, c \in \mathbb{R}^*$ *and* $(s-a) \neq 0$ *for some* $s \in \mathbb{C}$*, is defined as*

$$\|G\|^2_{\mathcal{H}_2} = -\frac{b^2 c^2}{2a}.$$

Proof. The expression of the $\mathcal{H}_2$-norm can be obtained by straightforward integration of the expression (3.5). In fact, for $\alpha = 1$ the expression (3.5) becomes

$$\begin{aligned}\|G\|^2_{\mathcal{H}_2} &= -\frac{c^2 b^2}{2j\pi a}\left[\int_0^{+\infty}\left(\frac{1}{\tilde{\omega}-ae^{-\frac{\pi}{2}j}}-\frac{1}{\tilde{\omega}-ae^{\frac{\pi}{2}j}}\right)d\tilde{\omega}\right],\\ &= -\frac{c^2 b^2}{2j\pi a}\left[\lim_{\tau\to+\infty}\left(\int_0^{\tau}\left(\frac{1}{\tilde{\omega}-ae^{-\frac{\pi}{2}j}}-\frac{1}{\tilde{\omega}-ae^{\frac{\pi}{2}j}}\right)d\tilde{\omega}\right)\right],\\ &= -\frac{c^2 b^2}{2j\pi a}\left[\lim_{\tau\to+\infty}\left(\ln\left|\frac{\tilde{\omega}-ae^{-\frac{\pi}{2}j}}{\tilde{\omega}-ae^{\frac{\pi}{2}j}}\right|\right)_0^{\tau}\right],\\ &= -\frac{c^2 b^2}{2a}.\end{aligned}$$

that completes the proof of the proposition 2.2.

■

Remark 2.3 *For* $\alpha = 1$*, the above technique can be applied for any transfer function represented by a regular differential linear system written in a state-space as*

$$\begin{cases}\dot{x}(t) = Ax(t) + Bu(t),\\ y(t) = Cx(t),\end{cases} \tag{3.6}$$

where $x \in \mathbb{R}^q$ *is the state vector,* $u \in \mathbb{R}^m$ *is the input vector,* $y \in \mathbb{R}^p$ *is the output vector and* $A \in \mathbb{R}^{q\times q}$*,* $B \in \mathbb{R}^{q\times m}$*, and* $C \in \mathbb{R}^{p\times q}$ *with a null initial condition and regular pencil.*

In this case, the $\mathcal{H}_2$-norm of the transfer function

$$G(s) = C\,(s\,\mathrm{Id} - A)^{-1}\,B,$$

with generalized state-space realization $\{\mathrm{A}, \mathrm{B}, \mathrm{C}\}$ *and* $\det(s\mathrm{Id}-\mathrm{A}) \neq 0$ *for some* $s \in \mathbb{C}$*, is defined as*

$$\|\mathrm{G}\|^2_{\mathscr{H}_2} = \mathrm{tr}\left(\mathrm{CPC}^{\mathrm{T}}\right).$$

The matrix P *is the solution of the Lyapunov equation*

$$\mathrm{AP} + \mathrm{PA}^{\mathrm{T}} + \mathrm{BB}^{\mathrm{T}} = 0,$$

where, A^{T}, B^{T} *and* C^{T} *are respectively the transpose of the matrices* A, B *and* C.

3 The $\mathscr{H}_2$-norm of the fractional-order transfer function of first kind with complex parameters

In this section, we mainly present the way to strengthen the technique presented in Section 2 when the fractional-order transfer function of the first kind has complex parameters. However, we start by recalling some important concepts.

3.1 Preliminaries

Let us consider the fractional-order transfer function of the first kind with complex parameters of the form

$$\mathrm{G}(s) = c\left(s^{\alpha} + a\right)^{-1} b,$$

associated with the following system

$$\left\{\begin{array}{rcl} \mathbf{D}^{\alpha} x(t) & = & -ax(t) + bu(t), \\ y(t) & = & cx(t), \end{array}\right. \tag{3.7}$$

where x is the state, u is the input, y is the output, $a, b, c \in \mathbb{C}^*$ and $\left(s^{\alpha} + a\right) \neq 0$ for some $s \in \mathbb{C}$. $\mathbf{D}^{\alpha}$ is the Caputo fractional derivative with $\alpha \in \left]\frac{1}{2}, 2\right[$.

The parahermitian transfer function associated with the system (3.7) is

$$\begin{array}{rcl} \phi(s) & = & \mathrm{G}(s)\mathrm{G}^*(s), \\ & = & |b|^2 |c|^2 \left(s^{\alpha} + a\right)^{-1} \left(\overline{s}^{\alpha} + \overline{a}\right)^{-1}, \end{array}$$

and its corresponding parahermitian matrix is given by

$$\mathrm{S}_{\phi}(s) = \left[\begin{array}{cc|c} 0 & \overline{s}^{\alpha} + \overline{a} & \overline{c} \\ s^{\alpha} + a & -|b|^2 & 0 \\ \hline -c & 0 & 0 \end{array}\right].$$

Let us recall that the $\mathscr{H}_2$-norm of such system is defined as [65]

$$\|G\|^2_{\mathscr{H}_2} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} G(j\omega)G^*(j\omega)\,d\omega,$$

where $s = j\omega$, with $j = e^{j\frac{\pi}{2}}$ and $j^2 = -1$.

Note that, for $G(-j\omega) = G^*(j\omega)$, we get [65]

$$\int_{-\infty}^{0} G(j\omega)G^*(j\omega)d\omega = \int_{0}^{+\infty} G(j\omega)G^*(j\omega)d\omega,$$

then,

$$\|G\|^2_{\mathscr{H}_2} = \frac{1}{\pi}\int_{0}^{+\infty} G(j\omega)G^*(j\omega)d\omega,$$

which will be used in the next subsection.

3.2 Main result

The second main result of the present chapter is presented in the following. The objective is to adjust the proposed approach for calculating the $\mathscr{H}_2$-norm for a fractional-order transfer function of the first kind with complex parameters.

Theorem 3.1 *Let us suppose that, for some $s \in \mathbb{C}$*

$$G(s) = c\left(s^\alpha + a\right)^{-1} b,$$

where $a, b, c \in \mathbb{C}^*$ with $\arg(a) = \theta$ and $\theta \neq k\pi + \frac{\pi}{2}\alpha, k \in \mathbb{Z}$, $(s^\alpha + a) \neq 0$, and $\frac{1}{2} < \alpha < 2$, is a proper, causal and stable elementary fractional-order transfer function. Then, the $\mathscr{H}_2$-norm of the fractional-order transfer function G, is given according to α by

$$\|G\|^2_{\mathscr{H}_2} = \begin{cases} \dfrac{|b|^2|c|^2|a|^{\frac{1}{\alpha}-2}}{\alpha\sin\left(\dfrac{\pi}{\alpha}\right)}\left[\cos\left(\dfrac{\theta}{\alpha}\right)\cot\left(\theta-\dfrac{\pi}{2}\alpha\right)+\sin\left(\dfrac{\theta}{\alpha}\right)\right] & \text{if } \ \dfrac{1}{2}<\alpha<2 \text{ and } \alpha\neq 1, \\[2ex] \dfrac{|c|^2\,|b|^2}{\pi\,|a|\sin\left(\theta-\dfrac{\pi}{2}\right)}\left(\theta-\dfrac{\pi}{2}\right) & \text{if } \ \alpha = 1. \end{cases}$$

Proof. Let us consider the state-space representation

$$\begin{cases} \mathbf{D}^\alpha x(t) &= -ax(t) + bu(t), \\ y(t) &= cx(t), \end{cases} \tag{3.8}$$

where x is the state, u is the input, y is the output, a, b, $c \in \mathbb{C}^*$ and $\left(s^\alpha + a\right) \neq 0$ for some $s \in \mathbb{C}$. $\mathbf{D}^\alpha$ is the Caputo fractional derivative with $\alpha \in \left]\frac{1}{2}, 2\right[$.

The fractional-order transfer function associated with the system (3.8) is

$$\text{for some } s \in \mathbb{C} : \mathrm{G}(s) = c\left(s^{\alpha} + a\right)^{-1} b,$$

furthermore, the so-called parahermitian transfer function becomes

$$\phi(s) = |c|^2 |b|^2 \left(s^{\alpha} + a\right)^{-1} \left(\overline{s}^{\alpha} + \overline{a}\right)^{-1},$$

that is the Schur complement of the matrix S_{ϕ}

$$\mathrm{S}_{\phi}(s) = \left[\begin{array}{cc|c} 0 & \left(\overline{s}^{\alpha} + \overline{a}\right) & \overline{c} \\ (s^{\alpha} + a) & -|b|^2 & 0 \\ \hline -c & 0 & 0 \end{array}\right].$$

As reported by *Genin et al* in [22], the matrix S_{ϕ} can be transformed into the matrix $\mathrm{S}_{\tilde{\phi}}$ using transformation matrices without shifting the properties of the original matrix. Thus, for particular transformation matrices, yields

$$\mathrm{S}_{\tilde{\phi}}(s) = \left[\begin{array}{cc|c} 0 & \left(\overline{s}^{\alpha} + \overline{a}\right) & \overline{c} \\ (s^{\alpha} + a) & 0 & \dfrac{\overline{c}|b|^2}{2\,\mathrm{Re}\,(s^{\alpha} + a)} \\ \hline -c & \dfrac{-c|b|^2}{2\,\mathrm{Re}\,(s^{\alpha} + a)} & 0 \end{array}\right],$$

where its Schur complement is given by

$$\tilde{\phi}(s) = \begin{pmatrix} c & \dfrac{c\,|b|^2}{2\,\mathrm{Re}\,(s^{\alpha} + a)} \end{pmatrix} \begin{pmatrix} 0 & (s^{\alpha} + a)^{-1} \\ (\overline{s}^{\alpha} + \overline{a})^{-1} & 0 \end{pmatrix} \begin{pmatrix} \overline{c} \\ \dfrac{|b|^2\,\overline{c}}{2\,\mathrm{Re}\,(s^{\alpha} + a)} \end{pmatrix}.$$

In the frequency domain the parahermitian function $\tilde{\phi}$ is equivalent to

$$\tilde{\phi}(\tilde{\omega}) = \frac{|c|^2 |b|^2}{2j|a| \sin\left(\theta - \dfrac{\pi}{2}\alpha\right)} \left[\left(\tilde{\omega} + \overline{a} e^{\frac{\pi}{2}\alpha j}\right)^{-1} - \left(\tilde{\omega} + a e^{-\frac{\pi}{2}\alpha j}\right)^{-1}\right],$$

where $s = j\omega$, $s^{\alpha} = j^{\alpha}\tilde{\omega}$, with $\tilde{\omega} = \omega^{\alpha}$, and $\theta = \arg(a)$.

Before performing the computation of the $\mathscr{H}_2$-norm, it must be emphasized that two cases can be distinguished $\dfrac{1}{\alpha} \in \mathbb{N}$ and $\dfrac{1}{\alpha} \notin \mathbb{N}$, where $\dfrac{1}{2} < \alpha < 2$.

First, consider the case $\frac{1}{\alpha} \notin \mathbb{N}$. The $\mathcal{H}_2$-norm of the fractional-order transfer function G is

$$\begin{aligned}
\|G\|^2_{\mathcal{H}_2} &= \frac{1}{\pi}\int_0^{+\infty} G(j\omega)G^*(-j\omega)d\omega,\\
&= \frac{1}{\alpha\pi}\int_0^{+\infty} \tilde{\phi}(\tilde{\omega})\,\tilde{\omega}^{\frac{1}{\alpha}-1}d\tilde{\omega},\\
&= \frac{|c|^2|b|^2}{2j\alpha\pi|a|\sin\left(\theta-\frac{\pi}{2}\alpha\right)}\int_0^{+\infty}\left[\left(\tilde{\omega}+|a|e^{-(\theta-\frac{\pi}{2}\alpha)j}\right)^{-1}-\left(\tilde{\omega}+|a|e^{(\theta-\frac{\pi}{2}\alpha)j}\right)^{-1}\right]\tilde{\omega}^{\frac{1}{\alpha}-1}d\tilde{\omega}.
\end{aligned}$$

Finally, using an integral transformation [19], we get

$$\|G\|^2_{\mathcal{H}_2} = \frac{|b|^2|c|^2|a|^{\frac{1}{\alpha}-2}}{\alpha\sin\left(\frac{\pi}{\alpha}\right)}\left[\cos\left(\frac{\theta}{\alpha}\right)\cot\left(\theta-\frac{\pi}{2}\alpha\right)+\sin\left(\frac{\theta}{\alpha}\right)\right].$$

Secondly, when $\frac{1}{\alpha} \in \mathbb{N}$, which means that $\alpha = 1$, the $\mathcal{H}_2$-norm of the fractional-order transfer function G becomes

$$\begin{aligned}
\|G\|^2_{\mathcal{H}_2} &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} G(j\omega)G^*(-j\omega)d\omega,\\
&= \frac{1}{\pi}\int_0^{+\infty} \tilde{\phi}(\tilde{\omega})\,d\tilde{\omega},\\
&= \frac{|c|^2|b|^2}{2j\pi|a|\sin\left(\theta-\frac{\pi}{2}\right)}\int_0^{+\infty}\left[\left(\tilde{\omega}+|a|e^{-(\theta-\frac{\pi}{2})j}\right)^{-1}-\left(\tilde{\omega}+|a|e^{(\theta-\frac{\pi}{2})j}\right)^{-1}\right]d\tilde{\omega},\\
&= \frac{|c|^2|b|^2}{\pi|a|\sin\left(\theta-\frac{\pi}{2}\right)}\left(\theta-\frac{\pi}{2}\right),
\end{aligned}$$

due to simple manipulations [68].

Finally, the proof of theorem 3.1 is completed.

■

Remark 3.2 *If the parameters a, b and c are real, then, we get the same results presented by theorem 2.1 and proposition 2.2.*

4 Numerical examples

The goal of this section is to demonstrate the arguments of our work in a visual way. The algorithm has been tested for different examples, and compared to the existed methods in the state-of-art [3, 45, 64, 65]. The first two examples are taken from the references [45, 64] to validate our method and the third one is consecrated to the $\mathcal{H}_2$-norm of a fractional-order transfer of the first kind with complex parameters.

Note that all academic examples, showing the effectiveness and the simplicity of our methods, have been performed using a MATLAB code.

Example 4.1 *Consider the system* (3.6) *with*

$$A = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 4 & -3 \\ 1 & -3 & -1 & -3 \\ 0 & 4 & 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ -1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

The corresponding transfer matrix is

$$G(s) = \begin{bmatrix} \dfrac{-s^3-3s^2-13s+5}{s^4+4s^3+36s^2+92s+43} & \dfrac{-6s^2-22s-16}{s^4+4s^3+36s^2+92s+43} \\ \\ \dfrac{-s^3-2s^2-31s-48}{s^4+4s^3+36s^2+92s+43} & \dfrac{6s+16}{s^4+4s^3+36s^2+92s+43} \end{bmatrix}.$$

Thus, according to remark 2.3, *the* $\mathcal{H}_2$*-norm of the fractional-order transfer matrix* G *is*

$$\|G\|_{\mathcal{H}_2} = 1.1751,$$

which is the same result when using the method presented in [64].

Example 4.2 *Consider the fractional-order transfer function*

$$G(s) = \frac{k}{s^\alpha + \lambda},$$

associated with the following system

$$\begin{cases} \mathbf{D}^\alpha x(t) &= -\lambda\, x(t) + k\, u(t), \\ y(t) &= x(t), \end{cases}$$

with $\frac{1}{2} < \alpha < 2$, $\lambda \in \mathbb{R}_+^*$, *and* $k \in \mathbb{R}^*$.

The fractional-order transfer function G *satisfies the conditions of theorem* 2.1. *Thus, the* $\mathcal{H}_2$*-norm of the fractional-order transfer function* G *is given by*

$$\|G\|_{\mathcal{H}_2}^2 = \begin{cases} -\dfrac{k^2 \lambda^{\frac{1}{\alpha}-2} \cot\left(\alpha \frac{\pi}{2}\right)}{\alpha \sin\left(\frac{\pi}{\alpha}\right)} & \text{if} \quad \frac{1}{2} < \alpha < 2 \text{ and } \alpha \neq 1, \\ \\ \dfrac{k^2}{2\lambda} & \text{if} \quad \alpha = 1. \end{cases}$$

For $\alpha \in \left]\frac{1}{2}, 2\right[$, *the obtained results are similar to the ones in* [45]. *For simplicity, if we take* $k = 1$ *and* $\lambda = 2$, *the comparison between both methods is plotted versus* α *in figure* 3.1.

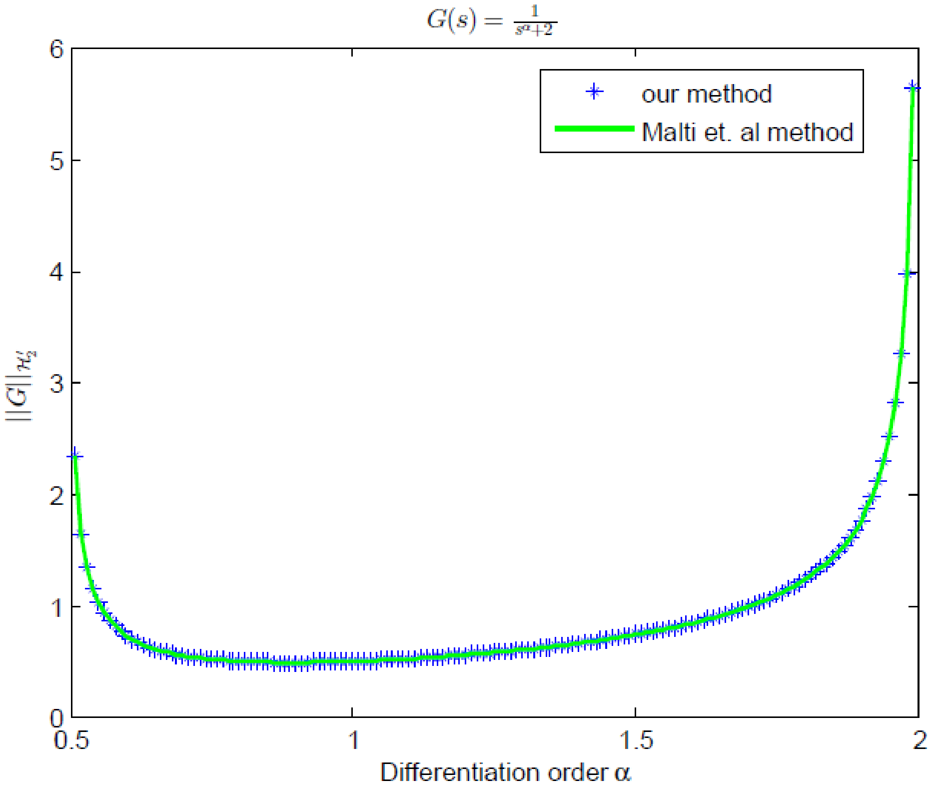

Figure 3.1: Comparison of the values of the $\mathcal{H}_2$-norm between our method and the method presented in [45] for $G(s) = \dfrac{1}{s^\alpha + 2}$ with $\dfrac{1}{2} < \alpha < 2$.

Example 4.3 *Let us consider in frequency domain, for* $\frac{1}{2} < \alpha < 2$, *the fractional-order transfer function*

$$G(s) = \frac{e^{\frac{\pi}{2}j}}{s^\alpha + 2e^{\frac{\pi}{12}j}}, \tag{3.9}$$

associated with the following fractional system

$$\begin{cases} \mathbf{D}^\alpha x(t) &= -2e^{\frac{\pi}{12}j}x(t) + u(t), \\ y(t) &= e^{\frac{\pi}{2}j}x(t). \end{cases} \tag{3.10}$$

According to theorem 3.1, *the* $\mathcal{H}_2$*-norm of the fractional-order transfer function* G *is* *given by*

$$\|G\|^2_{\mathscr{H}_2} = \begin{cases} \dfrac{2^{\frac{1}{\alpha}-2}}{\alpha \sin\left(\dfrac{\pi}{\alpha}\right)} \left[\cos\left(\dfrac{\pi}{12\alpha}\right)\cot\left(\dfrac{\pi}{12} - \dfrac{\pi}{2}\alpha\right) + \sin\left(\dfrac{\pi}{12\alpha}\right)\right] & \text{if} \quad \frac{1}{2} < \alpha < 2 \quad \text{and} \quad \alpha \neq 1, \\ \dfrac{5}{24 \sin\left(\dfrac{5\pi}{12}\right)} & \text{if} \quad \alpha = 1, \end{cases}$$

and presented by the following figure.

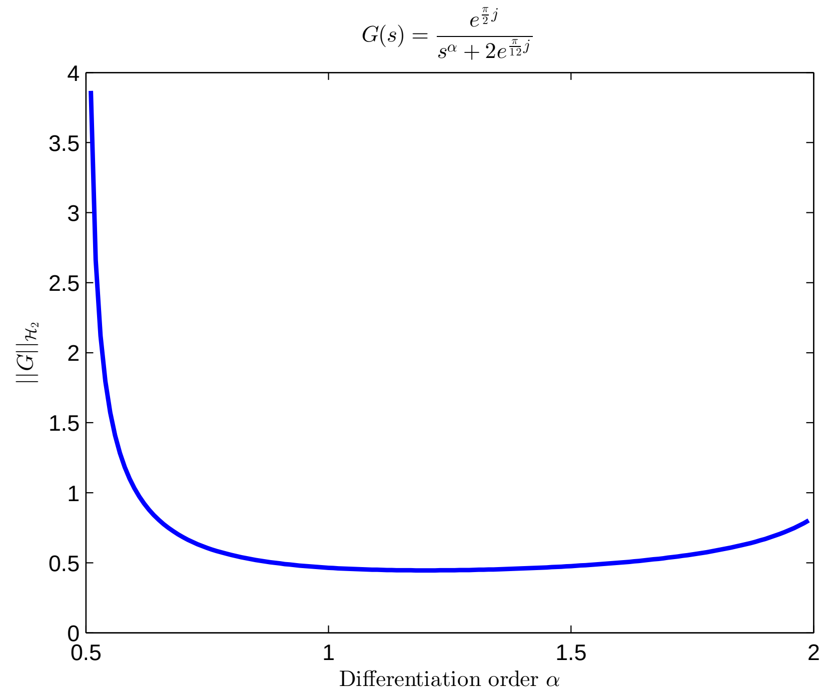

Figure 3.2: The $\mathscr{H}_2$-norm of $G(s) = \dfrac{e^{\frac{\pi}{2}j}}{s^\alpha + 2e^{\frac{\pi}{12}j}}$ for $\dfrac{1}{2} < \alpha < 2$.

5 Conclusion

In this chapter, we have established a novel method for computing the $\mathscr{H}_2$-norm of fractional-order transfer function of the first kind based on the use of the state-space realization, the parahermitian matrix, the transformation matrices satisfying some conditions mentioned above, and integral transform.

As a first step, the proposed method is developed in order to compute the $\mathscr{H}_2$-norm for a fractional-order transfer function of the first kind with real parameters, and in the second step, we have extended the same approach in the case where the state, the input and the output are complex parameters.

Finally, numerical examples were presented to pointed out the forcefulness of our proposed approach, and compared to the state-of-the-art methods.

Chapter 4

The $\mathscr{H}_2$-norm of the fractional-order transfer function of second kind

1 Introduction

The goal of this chapter is to extend the method proposed in the previous chapter to the fractional-order transfer function of the second kind of the form

$$\text{for some } s \in \mathbb{C} : \mathrm{G}(s) = \frac{c}{(s^\alpha + a)(s^\alpha + \overline{a})},$$

where $a, c \in \mathbb{C}^*$.

We will start by recalling some mathematical concepts used in this chapter, then, we will expand the new approach for computing the $\mathscr{H}_2$-norm for the fractional-order transfer function of the second kind. In order to show the performance of the proposed method, we will present an academic example where the obtained result will be compared to the existing methods.

2 Mathematical background

In the field of dynamical systems, it is well known that the transfer function, which is formulated through the ratio of the Laplace transform of the output and the input with zero initial conditions, characterizes the behaviour of the linear system whereof it is extracted.

Among the existing transfer functions, the following fractional-order transfer function of the second kind appears

$$\mathrm{G}(s) = \frac{c}{\left(s^\alpha + a\right)\left(s^\alpha + \overline{a}\right)}, \tag{4.1}$$

for some $s \in \mathbb{C}$ and $a, c \in \mathbb{C}^*$.

The fractional-order transfer function (4.1) can be expressed as a state-space realization as

$$\begin{cases} \mathbf{D}^{\alpha} x(t) &= \mathrm{A}x(t) + \mathrm{B}u(t), \\ y(t) &= \mathrm{C}x(t), \end{cases} \tag{4.2}$$

where

$$\mathrm{A} = \begin{pmatrix} -\left(a+\overline{a}\right) & -a\overline{a} \\ 1 & 0 \end{pmatrix}, \quad \mathrm{B} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{ and } \quad \mathrm{C} = \begin{pmatrix} 0 & c \end{pmatrix},$$

with $a, c \in \mathbb{C}^*$, $x \in \mathbb{R}^2$ is the state vector, $u \in \mathbb{R}$ is the input and $y \in \mathbb{R}$ is the output. $\mathbf{D}^{\alpha}$ is called the Caputo fractional derivative [13], where $n-1 \leq \alpha < n,$ for some $n \in \mathbb{N}^*$.

In the frequency domain, the fractional-order transfer function (4.1) turns into

$$\mathrm{G}(j\omega) = c\left(j^{\alpha}\omega^{\alpha} + a\right)^{-1}\left(j^{\alpha}\omega^{\alpha} + \overline{a}\right)^{-1}, \tag{4.3}$$

and its fractional-order complex-conjugate transpose transfer function becomes

$$\mathrm{G}^*(j\omega) = \overline{c}\left(\overline{j}^{\alpha}\omega^{\alpha} + \overline{a}\right)^{-1}\left(\overline{j}^{\alpha}\omega^{\alpha} + a\right)^{-1}, \tag{4.4}$$

where $s = j\omega$ with $j = e^{\frac{\pi}{2}j}$.

Let us recall that one of the most advantages of the transfer function is its simple algebraic expression which allows us the analysis and the description of the systems. In the field of dynamical systems and control theory [65], the transfer function is used to compute the $\mathscr{H}_2$-norm, which corresponds also to the impulse response energy. Its expression is presented in the section 4 of the chapter 2.

3 Main results

As mentioned in the introduction of this chapter, the present section generalizes the method illustrated in the previous chapter for fractional-order transfer function of the second kind. For this purpose let $\{\mathrm{A}, \mathrm{B}, \mathrm{C}, \alpha\}$ with

$$\mathrm{A} = \begin{pmatrix} -\left(a+\overline{a}\right) & -a\overline{a} \\ 1 & 0 \end{pmatrix}, \quad \mathrm{B} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathrm{C} = \begin{pmatrix} 0 & c \end{pmatrix}, \quad \text{and} \quad \frac{1}{4} < \alpha < 2,$$

be a state-space representation of a linear system and let, for some $s \in \mathbb{C}$

$$\mathrm{G}(s) = c\left(s^{\alpha} + a\right)^{-1}\left(s^{\alpha} + \overline{a}\right)^{-1},$$

be its fractional-order transfer function, where $a, c \in \mathbb{C}^*$ and $\det(s^{\alpha}\,\mathrm{Id} - \mathrm{A}) \neq 0$.

From the fractional-order transfer function G and its complex-conjugate transpose G^*

result the parahermitain transfer function ϕ that is

$$\begin{aligned}\phi(s) &= G(s)G^*(s),\\ &= c\left(s^\alpha+a\right)^{-1}\left(s^\alpha+\overline{a}\right)^{-1}\left(\overline{s}^\alpha+\overline{a}\right)^{-1}\left(\overline{s}^\alpha+a\right)^{-1}\overline{c}.\end{aligned}$$

The function ϕ is called the Schur complement [57] of the so-called system matrix S_ϕ, given as follows

$$S_\phi(s) = \left[\begin{array}{ll|l} 0 & \left(\overline{s}^\alpha+\overline{a}\right)\left(\overline{s}^\alpha+a\right) & \overline{c}\\ \left(s^\alpha+a\right)\left(s^\alpha+\overline{a}\right) & -1 & 0\\ \hline -c & 0 & 0\end{array}\right].$$

Let us recall that the first key idea behind the proposed method is the use of the transformations matrices. Hence, the matrix S_ϕ can be transformed to $S_{\tilde{\phi}}$ without shifting its characterization [22] as

$$S_{\tilde{\phi}}(s) = \left[\begin{array}{ll|l} 0 & \left(\overline{s}^\alpha+\overline{a}\right)\left(\overline{s}^\alpha+a\right) & \overline{c}\\ \left(s^\alpha+a\right)\left(s^\alpha+\overline{a}\right) & p\left(s^\alpha+a\right)\left(s^\alpha+\overline{a}\right)+p\left(\overline{s}^\alpha+\overline{a}\right)\left(\overline{s}^\alpha+a\right)-1 & p\overline{c}\\ \hline -c & -cp & 0\end{array}\right],$$

where

$$p = \frac{1}{2\,\mathrm{Re}\left(\left(s^\alpha+a\right)\left(s^\alpha+\overline{a}\right)\right)},$$

is the solution of the equation

$$p\left(s^\alpha+a\right)\left(s^\alpha+\overline{a}\right)+p\left(\overline{s}^\alpha+\overline{a}\right)\left(\overline{s}^\alpha+a\right)-1=0.$$

Thus, the matrix $S_{\tilde{\phi}}$ becomes

$$S_{\tilde{\phi}}(s) = \left[\begin{array}{ll|l} 0 & \left(\overline{s}^\alpha+\overline{a}\right)\left(\overline{s}^\alpha+a\right) & \overline{c}\\ \left(s^\alpha+a\right)\left(s^\alpha+\overline{a}\right) & 0 & \dfrac{\overline{c}}{2\,\mathrm{Re}\left(\left(s^\alpha+a\right)\left(s^\alpha+\overline{a}\right)\right)}\\ \hline -c & \dfrac{-c}{2\,\mathrm{Re}\left(\left(s^\alpha+a\right)\left(s^\alpha+\overline{a}\right)\right)} & 0\end{array}\right],$$

its Schur complement is the function $\tilde{\phi}$ that can be written, in the frequency domain, as

$$\tilde{\phi}(\tilde{\omega}) = \frac{|c|^2}{\left(\tilde{\omega}+\overline{a}e^{\frac{\pi}{2}\alpha j}\right)\left(\tilde{\omega}+ae^{-\frac{\pi}{2}\alpha j}\right)\left(\tilde{\omega}+\overline{a}e^{-\frac{\pi}{2}\alpha j}\right)\left(\tilde{\omega}+ae^{\frac{\pi}{2}\alpha j}\right)},$$

where $s=j\omega$, $s^\alpha = j^\alpha\tilde{\omega}$, with $\tilde{\omega}=\omega^\alpha$.

For $\frac{1}{4}<\alpha<2$, two cases can be distinguished: $\frac{1}{\alpha}\in\mathbb{N}$ and $\frac{1}{\alpha}\notin\mathbb{N}$.

On the one hand, for $\frac{1}{\alpha} \notin \mathbb{N}$, the function $\tilde{\phi}$ becomes

$$\begin{aligned}\tilde{\phi}(\tilde{\omega}) =& \frac{|c|^2}{8j|a|^3 \sin\theta \sin\left(\frac{\pi}{2}\alpha\right)\sin\left(\theta-\frac{\pi}{2}\alpha\right)}\left[\frac{e^{\left(\theta-\frac{\pi}{2}\alpha\right)j}}{\left(\tilde{\omega}+\overline{a}e^{\frac{\pi}{2}\alpha j}\right)}-\frac{e^{-\left(\theta-\frac{\pi}{2}\alpha\right)j}}{\left(\tilde{\omega}+ae^{-\frac{\pi}{2}\alpha j}\right)}\right]\\ &+\frac{|c|^2}{8j|a|^3 \sin\theta \sin\left(\frac{\pi}{2}\alpha\right)\sin\left(\theta+\frac{\pi}{2}\alpha\right)}\left[\frac{e^{-\left(\theta+\frac{\pi}{2}\alpha\right)j}}{\left(\tilde{\omega}+ae^{\frac{\pi}{2}\alpha j}\right)}-\frac{e^{\left(\theta+\frac{\pi}{2}\alpha\right)j}}{\left(\tilde{\omega}+\overline{a}e^{-\frac{\pi}{2}\alpha j}\right)}\right],\end{aligned}$$

with $\theta = \arg(a)$.

On the other hand, $\frac{1}{\alpha} \in \mathbb{N}$ means that $\alpha \in \left\{1, \frac{1}{2}, \frac{1}{3}\right\}$, then,

- For $\alpha = 1$:

$$\tilde{\phi}(\tilde{\omega}) = \frac{|c|^2}{8|a|^3 \sin\theta\cos\theta}\left[\frac{e^{\theta j}}{\left(\tilde{\omega}+\overline{a}e^{\frac{\pi}{2}j}\right)}-\frac{e^{-\theta j}}{\left(\tilde{\omega}+ae^{\frac{\pi}{2}j}\right)}+\frac{e^{-\theta j}}{\left(\tilde{\omega}+ae^{-\frac{\pi}{2}j}\right)}-\frac{e^{\theta j}}{\left(\tilde{\omega}+\overline{a}e^{-\frac{\pi}{2}j}\right)}\right].$$

- For $\alpha = \frac{1}{2}$:

$$\begin{aligned}\tilde{\phi}(\tilde{\omega}) =& \frac{|c|^2}{4\sqrt{2}j|a|^3 \sin\theta \sin\left(\theta-\frac{\pi}{4}\right)}\left[\frac{e^{\left(\theta-\frac{\pi}{4}\right)j}}{\left(\tilde{\omega}+\overline{a}e^{\frac{\pi}{4}j}\right)}-\frac{e^{-\left(\theta-\frac{\pi}{4}\right)j}}{\left(\tilde{\omega}+ae^{-\frac{\pi}{4}j}\right)}\right]\\ &+\frac{|c|^2}{4\sqrt{2}j|a|^3 \sin\theta \sin\left(\theta+\frac{\pi}{4}\right)}\left[\frac{e^{-\left(\theta+\frac{\pi}{4}\right)j}}{\left(\tilde{\omega}+ae^{\frac{\pi}{4}j}\right)}-\frac{e^{\left(\theta+\frac{\pi}{4}\right)j}}{\left(\tilde{\omega}+\overline{a}e^{-\frac{\pi}{4}j}\right)}\right].\end{aligned}$$

- For $\alpha = \frac{1}{3}$:

$$\begin{aligned}\tilde{\phi}(\tilde{\omega}) =& \frac{|c|^2}{4j|a|^3 \sin\theta \sin\left(\theta-\frac{\pi}{6}\right)}\left[\frac{e^{\left(\theta-\frac{\pi}{6}\right)j}}{\left(\tilde{\omega}+\overline{a}e^{\frac{\pi}{6}j}\right)}-\frac{e^{-\left(\theta-\frac{\pi}{6}\right)j}}{\left(\tilde{\omega}+ae^{-\frac{\pi}{6}j}\right)}\right]\\ &+\frac{|c|^2}{4j|a|^3 \sin\theta \sin\left(\theta+\frac{\pi}{6}\right)}\left[\frac{e^{-\left(\theta+\frac{\pi}{6}\right)j}}{\left(\tilde{\omega}+ae^{\frac{\pi}{6}j}\right)}-\frac{e^{\left(\theta+\frac{\pi}{6}\right)j}}{\left(\tilde{\omega}+\overline{a}e^{-\frac{\pi}{6}j}\right)}\right],\end{aligned}$$

where $\theta = \arg(a)$.

Finally, the $\mathscr{H}_2$-norm of the fractional-order transfer function G associated with the state-space

$$\left\{\begin{array}{rcl}\mathbf{D}^\alpha x(t) &=& \begin{pmatrix}-\left(a+\overline{a}\right) & -a\overline{a}\\ 1 & 0\end{pmatrix}x(t)+\begin{pmatrix}1\\0\end{pmatrix}u(t),\\ y(t) &=& \begin{pmatrix}c & 0\end{pmatrix}x(t),\end{array}\right.$$

where $a, c \in \mathbb{C}^*$ is

$$\begin{aligned}\|G\|^2_{\mathcal{H}_2} &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} G(j\omega)G^*(-j\omega)\,d\omega,\\ &= \frac{1}{\alpha\pi}\int_{0}^{+\infty} \tilde{\phi}(\tilde{\omega})\,\tilde{\omega}^{\frac{1}{\alpha}-1}\,d\tilde{\omega}.\end{aligned}$$

By the use of simplification techniques, tools, and integral transform [19, 68], the expressions of the $\mathcal{H}_2$-norm are obtained and presented in the following theorem that presents the third main result of the thesis.

Theorem 3.1 [37] *Assuming that* $\frac{1}{4} < \alpha < 2$. *Then, the* $\mathcal{H}_2$*-norm of the proper, causal and stable elementary fractional-order transfer function* G, *with generalized state-space realization* $\{A, B, C, \alpha\}$ *where*

$$A = \begin{pmatrix} -(a+\overline{a}) & -a\overline{a} \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \textit{and} \quad C = \begin{pmatrix} 0 & c \end{pmatrix},$$

with $\det(s^\alpha \mathrm{Id} - A) \neq 0$ *for some* $s \in \mathbb{C}$, $a, c \in \mathbb{C}^*$, *and let* $\theta = \arg(a)$ *such that* $|\theta| < \pi - \alpha\frac{\pi}{2}$, *is defined as*

$$\|G\|^2_{\mathcal{H}_2} = \begin{cases} \dfrac{|c|^2|a|^{\frac{1}{\alpha}-4}}{4\alpha\sin\theta\sin\left(\frac{\pi}{2}\alpha\right)\sin\left(\frac{\pi}{\alpha}\right)} & \text{if } \frac{1}{4}<\alpha<2 \\ & \text{and } \alpha\neq 1,\ \alpha\neq\dfrac{1}{2},\ \alpha\neq\dfrac{1}{3}, \\ \dfrac{|c|^2}{4|a|^3\cos\theta} & \text{if } \alpha = 1, \\ \dfrac{|c|^2}{\sqrt{2}\pi|a|^2\sin\theta}\left(\dfrac{\left(\frac{\pi}{4}-\theta\right)}{\sin\left(\theta-\frac{\pi}{4}\right)}+\dfrac{\left(\frac{\pi}{4}+\theta\right)}{\sin\left(\theta+\frac{\pi}{4}\right)}\right) & \text{if } \alpha=\dfrac{1}{2}, \\ \dfrac{|c|^2}{2|a|\pi\sin\theta}\left(\left(\theta-\dfrac{\pi}{6}\right)\cot\left(\theta-\dfrac{\pi}{6}\right)-\left(\theta+\dfrac{\pi}{6}\right)\cot\left(\theta+\dfrac{\pi}{6}\right)\right) & \text{if } \alpha=\dfrac{1}{3}. \end{cases}$$

4 Numerical example

This section is devoted to numerical validation. The proposed method has been tested and compared to the existing method [45]. The validation and the performance of the proposed method are illustrated through the example presented in the following.

Example 4.1 *Let us consider, for* $\frac{1}{4} < \alpha < 2$, *the fractional-order transfer function*

$$\textit{for some } s \in \mathbb{C} : G(s) = \frac{1}{\left(s^\alpha + 2e^{\frac{\pi}{12}j}\right)\left(s^\alpha + 2e^{-\frac{\pi}{12}j}\right)}. \tag{4.5}$$

Using a MATLAB *command, the fractional-order state-space model associated with the fractional-order transfer function* (4.5) *is*

$$\begin{cases} \mathbf{D}^{\alpha}x(t) &= \begin{pmatrix} -(\sqrt{6}+\sqrt{2}) & -4 \\ 1 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t), \\ y(t) &= \begin{pmatrix} 0 & 1 \end{pmatrix} x(t). \end{cases}$$

By the application of theorem 3.1*, the different values of the* $\mathcal{H}_2$*-norm of the fractional-order transfer function* G*, for* α *between* $\frac{1}{4}$ *and* 2*, are presented in the figure* 4.1.

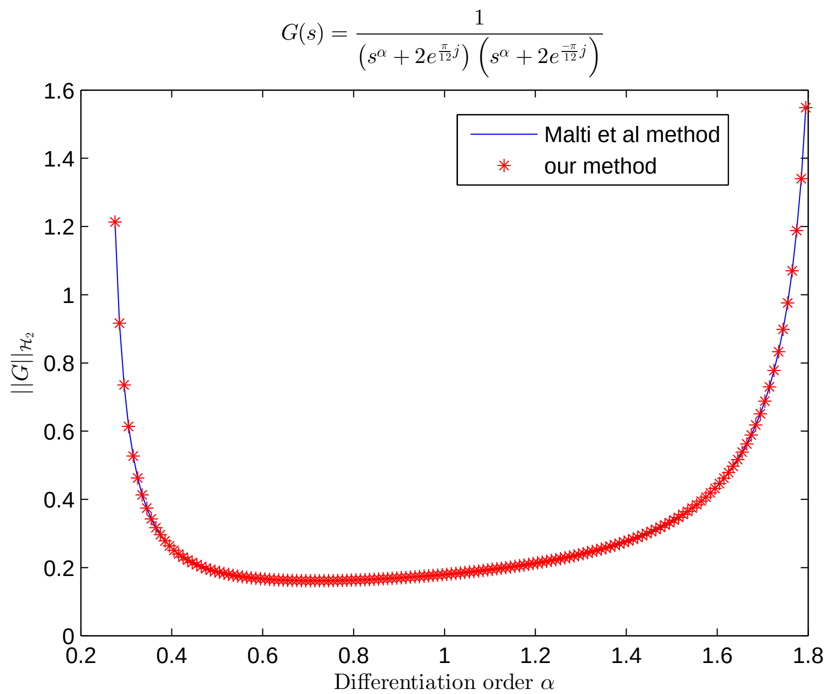

Figure 4.1: Comparison of the values of the $\mathcal{H}_2$-norm between our method and the method presented in [45] for $G(s) = \dfrac{1}{\left(s^{\alpha}+2e^{\frac{\pi}{12}j}\right)\left(s^{\alpha}+2e^{-\frac{\pi}{12}j}\right)}$ with $\frac{1}{4} < \alpha < 2$.

Note that the obtained results are the same one as in [45].

5 Conclusion

The method proposed in chapter 3 has been extended, in this chapter, to the fractional-order transfer function of the second kind. Indeed, it concerns the computation of the $\mathcal{H}_2$-norm by means of matrix transformation satisfying some conditions mentioned above, integral transform, and others tools. The chapter ends with numerical validation illustrating the performance of the proposed technique.

Conclusion

Mathematical models are used in the natural sciences such as physics, biology, earth sciences, chemistry, and engineering disciplines. On the one hand, Mathematical models can take many forms, and one of them is dynamical systems. On the other hand, these dynamical systems can be presented by a state-space representation or a transfer function which is defined as the ratio of Laplace transform of the output to the Laplace transform of the input where all the initial conditions are zero.

Therefore, the main objective of this thesis is the computation of the $\mathcal{H}_2$-norm for fractional-order transfer function of the first and second kind with real and complex parameters in the sense Caputo fractional derivative. The purpose approach is based on the use of the state-space realization, the parahermitian transfer matrices, matrices transformation, and integral transform including some conditions set out.

Nevertheless, all mathematical background needed to achieve to our results have been recalled in the first two chapters such fractional calculus, Laplace transform, Mellin Transform, Schur complement, dynamical systems, the fractional-order transfer function, and the the $\mathcal{H}_2$-norm. Chapter 3 is organized in two parts; the first part is devoted to explaining and establishing the approach to compute the $\mathcal{H}_2$-norm for a fractional-order transfer function of the first kind with real parameters, whereas the second part extends the new method for calculating the impulse response energy for fractional-order transfer function of the first kind with complex parameters. Chapter 4 regards the extension of the propose method for computing the $\mathcal{H}_2$-norm for a fractional-order transfer function of the second kind.

Based on the results presented in this thesis, several perspectives should be considered:

- Our result can be generalized and extended with regard to the general formula of the $\mathcal{H}_2$-norm for any fractional-order transfer function associated with standard or singular linear fractional systems.
- The reduction of the models via the $\mathcal{H}_2$-norm and $\mathcal{H}_\infty$-norm.
- The study of the relationship between the $\mathcal{H}_2$-norm and $\mathcal{H}_\infty$-norm for different fractional-order transfer functions.

References

[1] **Agrawal, O. P.** (2008), *A quadratic numerical scheme for fractional optimal control problems*, Journal of Dynamic Systems, Measurement, and Control, **130**(1), 011010.1 – 011010.6. https://doi.org/10.1115/1.2814055 1

[2] **Arena, P., Caponetto, R., and Porto, D.** (2000), *Nonlinear Noninteger Order Circuits and Systems - An introduction*, World Scientific, Singapore. 1

[3] **Aström, K. J.** (1970), *Introduction to Stochastic Control Theory*, Academic Press, New York. 2, 3, 41, 50

[4] **Aström, K. J., and Murray, R. M.** (2008), *Feedback Systems: An Introduction for Scientists and Engineers*, Princeton University Press, New Jersey. vi, 1, 25, 34, 35

[5] **Bagley, R. L., and Calico, R. A.** (1991), *Fractional order state equations for the control of viscoelastically damped structures*, Journal of Guidance, Control, and Dynamics, **14**(2), 304 – 311. https://doi.org/10.2514/3.20641 1

[6] **Bensoussan, A.** (2011), *Estimation and Control of Dynamical Systems*, Springer International Publishing AG, Switzerland. 1

[7] **Birnir, B.** (2008), *Dynamical Systems Theory*, Center for Complex and Nonlinear Dynamics and Department of Mathematics University of California, Santa Barbara. 23

[8] **Boreman, G. D.** (2001), *Modulation Transfer Function in Optical and Electro-Optical Systems*, The Society of Photo-Optical Instrumentation Engineers. 34

[9] **Bouagada, D.** (2007), *Systèmes Différentiels Singuliers Positifs et LMIs*, Doctoral Thesis, Université d'es-sénia, Algérie, presented on December 19th, 2007. 26, 27, 28

[10] **Bouagada, D., and Van Dooren, P.** (2012), *State space solution of implicit fractional continuous time systems*, Fractional Calculus and Applied Analysis, **15**(3), 356 – 361. https://doi.org/10.2478/s13540-012-0026-z 31

[11] **Caponetto, R., Dongola, G., Fortuna, L., and Petras, I.** (2010), *Fractional Order Systems: Modeling and Control Applications*, World Scientific Series on Nonlinear Science, Series A, 72, World Scientific, Singapore. 5, 7

[12] **Caputo, M.** (1967), *Linear models of dissipation whose Q is almost frequency independent-II*, Geophysical Journal International, **13**(5), 529 – 539. https://doi.org/10.1111/j.1365-246X.1967.tb02303.x 12

[13] **Caputo, M.** (1969), *Elasticità e Dissipazione*, Zanichelli, Bologna. 37, 42, 56

[14] **Das, S.** (2011), *Functional Fractional Calculus*, Springer-Verlag Berlin Heidelberg. 8

[15] **Debnath, L., and Bhatta, D.** (2007), *Integral Transforms and their Applications*, Chapman and Hall/CRC. 13, 14, 15, 16, 19

[16] **DeLarminat, P.** (2007), *Analysis and Control of Linear Systems*, Control Systems, Robotics and Manufactuning Series. ISTE Ltd, London, 2007. 1, 2, 24

[17] **Doetsch, G.** (1950), *Handbuch der Laplace-Transformation*, Birkhåuser, Basel. 46

[18] **Edwards, H. M.** (1974), *Riemann's Zeta Function*, Academic press, New York and London. 17

[19] **Erdélyi, A., Oberhettinger, F., and Tricomi, F. G.** (1954), *Tables of Integral Transforms I*, McGraw-Hill Book Company, New York. 3, 13, 50, 59

[20] **Fadiga, I., Farges, C., Sabatier, J., and Moze, M.** (2011), *On computation of $\mathscr{H}_\infty$-norm for commensurate fractional order systems*, 50$^{\text{th}}$ IEEE Conference on Decision and Control and European Control Conference, Orlando, FL, USA, 8231 – 8236. https://doi.org/10.1109/CDC.2011.6160942 vi, 1, 7, 29

[21] **Galkowski, K.** (2001), *State-Space Realizations of Linear 2-D Systems with Extensions to the General* nD $(n > 2)$ *case*, Lecture Notes in Control and Information Sciences 263. 24

[22] **Genin, Y., Hachez, Y., Nesterov, Y., Stefan, R., Van Dooren, P., and Xu, S.** (2002), *Positivity and linear matrix inequalities*, Eur. J. Control, **8**(3), 275 – 298. https://doi.org/10.3166/ejc.8.275-298 3, 20, 41, 43, 44, 49, 57

[23] **Graf, U.** (2010), *Introduction to Hyperfunctions and their Integral Transforms: An Applied and Computational Approach*, Springer Science and Business Media. 17, 18

[24] **Hartley, T. T., and Lorenzo, C. F.** (2002), *Dynamics and control of initialized fractional-order systems*, Nonlinear Dynamics, **29**(1-4), 201 – 233. https://doi.org/10.1023/A:1016534921583 1

[25] **Hogben, L.** (2013), *Handbook of Linear Algebra*, Chapman and Hall/CRC. 20

[26] **Ichise, M., Nagayanagi, Y., and Kojima, T.** (1971), *An analog simulation of non-integer order transfer functions for analysis of electrod processes*, Journal of Electroanalytical Chemistry and Interfacial Electrochemistry, **33**(2), 253 – 265. https://doi.org/10.1016/S0022-0728(71)80115-8 1

[27] **Ionescu, C., Lopes, A., Copot, D., Machado, J. A. T., and Bates, J. H. T.** (2017), *The role of fractional calculus in modelling biological phenomena*, Communications in Nonlinear Science and Numerical Simulation, **51**, 141 – 159. https://doi.org/10.1016/j.cnsns.2017.04.001 1, 5

[28] **Kaczorek, T.** (2002), *Positive* 1D *and* 2D *Systems*, Communication and Control Engineering Series, Springer, London. 28

[29] **Kaczorek, T.** (2011), *Selected Problems of Fractional Systems Theory*, Lecture Notes in Control and Information Sciences, Springer-Verlag Berlin Heidelberg. vi, 6, 29, 30

[30] **Kaczorek, T., and Rogowski, K.** (2015), *Fractional Linear Systems and Electrical Circuits*, Studies in Systems, Decision and Control, 13, Springer International Publishing, Switzerland. vi, 1, 5, 8, 11, 16, 17, 30, 31

[31] **Kaisserli, Z., and Bouagada, D.** (2021), *Application of the Sumudu transform to solve regular fractional continuous-time linear systems*, Kragujevac Journal of Mathematics, **45**(2), 267 – 274. https://doi.org/10.46793/KgJMat2102.267K vi, 7, 29, 33

[32] **Kilbas, A. A., Srivastava, H. M., and Trujillo, J. J.**(2006), *Theory and Applications of Fractional Differential Equations*, **204** of North-Holland Mathematics Studies. Elsevier, Amsterdam. 8, 9, 11

[33] **Kiliçman, A., and Omran, M.** (2016), *Note on Fractional Mellin Transform and Applications*, Springer Plus. https://doi.org/10.1186/s40064-016-1711-x 19

[34] **Kuhfitting, P. K. F.** (1978), *Introduction to the Laplace Transform*, Springer Science and Business Media, New York. 13

[35] **Kuznetsov, Y. A.** (1995), *Elements of Applied Bifurcation Theory*, Applied Mathematical Sciences Springer Science and Business Media, New York. 24

[36] **Lakeb, A., Kaisserli, Z., and Bouagada, D.** (2023), *Computing the $\mathcal{H}_2$-norm of a fractional order system using the state-space linear model*, Kragujevac Journal of Mathematics, **47**(4), 531-538. 42, 43

[37] **Lakeb, A., Kaisserli, Z., and Bouagada, D.**, *The impulse response energy for the first kind and the second kind fractional-order transfer functions with complex parameters*. Submitted. 59

[38] **Layek, G. C.** (2015), *An Introduction to Dynamical Systems and Chaos*, Springer, India. 1, 23

[39] **Levine, W. S.** (1996), *The Control Handbook*, Jaico Publishing House. 24

[40] **Lin, S.-D., and Lu, C.-H.** (2013), *Laplace Transform for Solving Some Families of Fractional Differential Equations and its Applications*, Advances in Difference Equations. https://doi.org/10.1186/1687-1847-2013-137 16

[41] **Ljung, L., and Glad, T.** (1994), *Modeling of Dynamic Systems*, PTR Prentice Hall, Englewood Cliffs. New Jersey. 1

[42] **Luenberger, D. G.** (1979), *Introduction to Dynamic Systems: Theory, Models, and Applications*, John Wiley and Sons, Inc. 23

[43] **Magin, R. L.** (2012), *Fractional calculus in bioengineering: A tool to model complex dynamics*, Proceedings of the 13th International Carpathian Control Conference (ICCC). https://doi.org/10.1109/CarpathianCC.2012.6228688 vi, 1, 2

[44] **Mainardi, F.** (2010), *Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models*, World scientific. 10

[45] **Malti, R., Aoun, M., Levron, F., and Oustaloup, A.** (2011), *Analytical computation of the $\mathscr{H}_2$-norm of fractional commensurate transfer functions*, Automatica, **47**, 2425 – 2432. https://doi.org/10.1016/j.automatica.2011.08.021 vi, 2, 3, 23, 38, 39, 41, 50, 52, 59, 60

[46] **Matignon, D.** (1998), *Stability propperties for generalized fractional differential systems*, ESAIM Proceedings Fractional Differential Systems, Models Methods and Applications, 145 – 158. http://dx.doi.org/10.1051/proc:1998004 38

[47] **Miller, K. S., and Ross, B.** (1993), *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley and Sons Inc., New York. 1, 9

[48] **Monje, C. A., Chen, Y. Q., Vinagre, B. M., Xue, D., and Feliu, V.** (2010), *Fractional-Order Systems and Controls: Fundamentals and Applications*, Advances in Industrial Control, Springer-Verlag, London. 1, 29, 36

[49] **Murio, D. A.** (2009), *Stable numerical evaluation of Grünwald-Letnikov fractional derivatives applied to a fractional IHCP*, Inverse Problems in Science and Engineering, **17**(2), 229 – 243. https://doi.org/10.1080/17415970802082872 11

[50] **Najim, K.** (2006), *Control of Continuous Linear Systems*, ISTE Ltd. 2, 24, 26

[51] **Oldham, K. B., and Spanier, J.** (1974), *The Fractional Calculus*, Academic Press, New York-London. 5

[52] **Padula, F., and Visioli, A.** (2015), *Advances in Robust Fractional Control*, Springer International Publishing, Switzerland. 2, 29

[53] **Parada, F. J. V., Tapia, J. A. O., and Ramirez, J. A.** (2007), *Effective medium equations for fractional Fick's law in porous media*, Physica A, **373**, 339 – 353. https://doi.org/10.1016/j.physa.2006.06.007 1

[54] **Petráš, I.** (2011), *Fractional-Order Nonlinear Systems*, Nonlinear Physical Science, Higher Education Press, Beijin and Springer-Verlag Berlin Heidelberg. 5, 7, 9, 10, 11, 12, 16

[55] **Podlubny, I.** (1998), *Fractional Differential Equations*, Mathematics in Science and Engineering, 198, Academic Press, New York. 42

[56] **Podlubny, I.** (1999), *Fractional Differential Equations*, Academic Press, San Diego. 5, 16, 17, 18, 19, 20

[57] **Puntanen, S., Styan, G. P. H., and Zhang, F.** (2005), *The Schur Complement and Its Applications*, Numerical Methods and Algorithms, 4, Springer International Publishing, United States. 57

[58] **Ross, B.** (1975), *A brief history and exposition of fundamental theory of the fractional Calculus*, Lecture Notes in Mathematics, **457**, 1 – 36, Springer Vergla, New-York. 5

[59] **Sabatier, J., Lanusse, P., Melchior, P., and Oustaloup, A.** (2015), *Fractional Order Differentiation and Robust Control Design. Crone, $\mathscr{H}_\infty$ and Motion Control*, Intelligent Systems, Control and Automation: Science and ENgineering, 77, Springer International Publishing. 2

[60] **Samko, S. G., Kilbas, A. A., and Marichev, O. I.** (1993), *Fractional Integrals and Derivatives: Theory and Applications*, Gorden and Breach Science Publisher, Amsterdam, the Netherlands. 11

[61] **Schiff, J. L.** (1999), *The Laplace transform: Theory and Applications*, Springer-Verlag New York, Inc. 13

[62] **Skaar, S. B., Michel, A. N., and Miller, R. K.** (1988), *Stability of viscoelastic control systems*, IEEE Transactions on Automatic Control, **33**(4), 348 – 357. https://doi.org/10.1109/9.192189 1

[63] **Torvik, P. J., and Bagley, R. L.** (1984), *On the appearance of the fractional derivative in the behavior of real materials*, Journal of applied mechanics, **51**(2), 294 – 298. https://doi.org/10.1115/1.3167615 1

[64] **Toscano, R.** (2013), *Structured Controllers for Uncertain Systems: A Stochastic Optimization Approach*, Advances in Industrial Control, Springer-Verlag, London. 2, 23, 41, 50, 51

[65] **Valério, D., and Da Costa, J. S.** (2013), *An Introduction to Fractional Control*, The Institution of Engineering and Technology, London, United Kingdom. 1, 2, 3, 23, 29, 39, 41, 48, 50, 56

[66] **Verghese, G., Van Dooren, P., and Kailath, T.** (1979), *Properties of the system matrix of a generalized state-space system*, Internat. J. Control, **30**(2) (1979), 235 – 243. https://doi.org/10.1080/00207177908922771 41, 44

[67] **Wang, J. C.** (1987), *Realizations of generalized Warburg impedance with RC ladder networks and transmission lines*, Journal of The Electrochemical Society, **134**, 1915 – 1920. 5

[68] **Yosida, K.** (1980), *Functional Analysis*, Sixth Edition, Springer-Verlag, Berlin Heidelberg New York. 3, 50, 59

[69] **Zadeh, L. A., and Desoer, C. A.** (1963), *Linear System Theory: The State Space Approach*, The McGraw-Hill Book Company, Inc. 1, 23

[70] **Zhang, F.** (2005), *The Schur Complement and its Applications*, Springer Science and Business Media. 20, 21

التحكم وحساب طاقة الاستجابة النبضية للنماذج المفردة و / أو الكسرية

الملخص : الغرض من هذه الأطروحة هو تقديم نهج جديد لحساب طاقة الاستجابة النبضية لفئة من النماذج الخطية الكسرية . تعتمد هذه الطريقة الجديدة على استخدام تحقيق فضاء الحالة المكونة من معاملات التي يتم استخلاصها من دالة وظيفة نقل الترتيب الجزئي المرتبطة بها، ثم، من تحويل مصفوفة بارهيرميتية باستخدام تقنيات مصفوفلت التحويل غير تلك المقترحة في المراجع والتي تجعلها محافظة على خصائصها .أخيرًا ، التعبير العام للطاقة الاستجابة النبضية مشتق بفضل بعض المفاهيم و الشروط المحددة .النتائج العددية لكل من وظائف التحويل الجزئي من الدرجة الأولى والثانية مع معاملات حقيقية ومركبة توضح فعالية الطريقة المقترحة من خلال النتائج الواعدة، يفتح هذا النهج آفاقًا كبيرة حول التطورات النظرية التي يمكنها التعامل مع العديد من الخوارزميات التي تستجيب للاحتياجات المتنوعة.

الكلمات المفتاحية : تكملة شور، تفاضل من نوع كابوتو ، طاقة الاستجابة النبضية، دالة التحويل، مصفوفات التحويل، تمثيل فضاء الحالة لنموذج كسري خطي.

Contrôle et calcul de la norme $\mathcal{H}_2$ pour des modèles singuliers et/ou fractionnaires

Résumé : Cette thèse porte sur le développement d'une nouvelle approche pour le calcul de l'énergie de la réponse impulsionnelle d'une certaine classe de modèles linéaires fractionnaires. Cette dernière repose sur l'utilisation de la réalisation de l'espace d'états constituée de paramètres qui sont extraits de la fonction de transfert d'ordre fractionnaire associée, puis d'une transformation de la matrice parahermitienne moyennant des techniques matricielles autres que celles proposées dans la littérature. En dernier, l'expression générale de la norme $\mathcal{H}_2$ est dérivée grâce à quelques conditions et transformations intégrales. Cette nouvelle approche a été testée avec sucées sur des fonctions de transfert fractionnaires du premier et second types à coefficients réels et complexes. Le calcul de la norme $\mathcal{H}_2$ ouvre par ailleurs de nouvelles perspectives théoriques lesquelles devraient permettre la mise au point de nombreux algorithmes répondant à divers besoins.

Mots−Clés : Complément de Schur, Dérivée fractionnaire au sens de Caputo, Énergie de la réponse impulsionnelle, Norme $\mathcal{H}_2$, Fonction de transfert, Matrice de transformation, Représentation d'état d'un modèle fractionnaire linéaire.

Control and Calculation of the $\mathcal{H}_2$-norm for Singular and/or Fractional Models

Abstract : This work emphasizes the development of a new approach for computing the impulse response energy, known, also, as the $\mathcal{H}_2$−norm, for a certain class of fractional linear models. The main concept of this new approach is the use of the state−space realization consisting of parameters that are extracted from the fractional−order transfer function and then a transformation other than those proposed in the literature of the parahermitian matrix which let it invariant. Finally, the general expression of the $\mathcal{H}_2$−norm is derived thanks to some concepts and some conditions set out. Numerical results on both first−order and second−order fractional transfer functions with real and complex parameters illustrate the performance of the proposed approach. Through the promising results, this approach opens great perspectives on theoretical developments that can handle numerous algorithms responding to diverse needs.

Key Words : Schur complement, Caputo fractional derivative, Impulse response energy, $\mathcal{H}_2$−norm, Fractional transfer function, Matrix transformation, State−space model.