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On the geometry of surfaces in manifolds with density

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A special and heartfelt tribute to the memory of my late father. I would have dearly loved for him to be present with me today to witness my graduation, but his spirit and values have been my greatest motivation. Though he is gone, this achievement is dedicated to his soul.

Abstract

This thesis explores the geometry of surfaces in pseudo-Riemmennian manifolds with density, an important extension of classical geometric analysis that provides a natural framework for studying variational problems, curvature, and minimal surfaces under weighted volume measures. A manifold with density is a Riemannian or pseudo-Riemannian manifold (M, g) endowed with a smooth weight function $\rho = e^{\varphi}$, which modifies the standard geometric structure and leads to generalized notions of curvature, mean curvature, and divergence. Such weighted structures arise naturally in probability theory, diffusion processes, and the study of geometric flows.

In this work, we investigate the geometric properties of surfaces in Euclidean space $\mathbb{R}^3$ and in Minkowski space $\mathbb{R}^3_1$, both endowed with a density function $\rho = e^{\varphi}$. After reviewing the necessary preliminaries on differential geometry and pseudo-Riemannian manifolds, we develop and analyze weighted curvature formulas, including the generalized Gauss curvature and mean curvature. We also examine Plateau-type problems and study the behavior of the weighted divergence operator in manifolds with density, supported by explicit examples.

The main contributions of this thesis focus on the classification and geometric analysis of weighted flat translation surfaces in Minkowski 3-space $\mathbb{R}^3_1$ and weighted flat radial surfaces in both Euclidean $\mathbb{R}^3$ and Minkowski $\mathbb{R}^3_1$ spaces. By deriving the corresponding flatness conditions and curvature relations, we extend previous results on linear densities and establish new characterizations of surfaces governed by weighted geometric structures. These results provide deeper insight into minimal and flat surfaces in weighted settings and offer a foundation for future research in differential geometry with density.

Résumé

Cette thèse porte sur la géométrie des surfaces dans les variét'es pseudo-Riemmannienes á densité, une extension importante de l'analyse géométrique classique qui offre un cadre naturel pour l'étude des problèmes variationnels, de la courbure et des surfaces minimales sous des mesures de volume pondérées. Une variété densité est une variété riemannienne ou pseudo-riemannienne (M, g) munie d'une fonction densité lisse

$$\rho = e^{\varphi},$$

qui modifie la structure géométrique standard et conduit à des notions généralisées de courbure, de courbure moyenne et de divergence. De telles structures pondérées apparaissent naturellement en théorie des probabilités, dans les processus de diffusion et dans l'étude des flots géométriques.

Dans ce travail, nous étudions les propriétés géométriques des surfaces dans l'espace euclidien $\mathbb{R}^3$ et dans l'espace de Minkowski $\mathbb{R}^3_1$, tous deux munis d'une densité $\rho = e^{\varphi}$. Après avoir rappelé les préliminaires nécessaires en géométrie différentielle et en géométrie pseudo-riemannienne, nous développons et analysons les formules de courbure pondérée, incluant la courbure de Gauss et la courbure moyenne généralisées. Nous examinons également des problèmes de type Plateau et étudions le comportement de l'opérateur de divergence pondéré dans les variétés à densité, en nous appuyant sur des exemples explicites.

Les principales contributions de cette thèse portent sur la classification et l'analyse géométrique des surfaces de translation plates pondérées dans l'espace de Minkowski $\mathbb{R}^3_1$ et des surfaces radiales plates pondérées dans les espaces euclidien $\mathbb{R}^3$ et Minkowskien $\mathbb{R}^3_1$. En dérivant les conditions de platitude correspondantes et les relations de courbure associées, nous étendons des résultats antérieurs relatifs aux densités linéaires et établissons de nouvelles caractérisations de surfaces gouvernées par des structures géométriques pondérées. Ces résultats offrent une meilleure compréhension des surfaces minimales et plates dans les contextes pondérés et constituent une base solide pour de futures recherches en géométrie différentielle avec densité.

الملخص

تتناول هذه الأطروحة هندسة الأسطح في متشعبات شبه ريمان ذات كثافة، وهي امتداد مهم للهندسة التفاضلية الكلاسيكية، وتوفر إطارًا طبيعيًا لدراسة مسائل التغير، والانحناء، والأسطح الدنيا تحت مقاييس حجم موزونة. المتشعب ذو الكثافة هو متشعب ريمان أو شبه ريمان (M,g) مزوّد بدالة كثافة ناعمة من الشكل $\rho=e^{\varphi}$ والتي تغيّر البنية الهندسية المعتادة وتؤدي إلى تعميم مفاهيم الانحناء، الانحناء المتوسط، والتباعد. هذه البنى الموزونة تظهر بشكل طبيعي في الاحتمالات، عمليات الانتشار، ودراسة التدفقات الهندسية.

في هذا العمل، ندرس الخصائص الهندسية للأسطح في الفضاء الإقليدي $\mathbb{R}^3$ وفي فضاء مينكوفسكي $\mathbb{R}^3_1$ ، وكلاهما مزوّد بدالة كثافة $\rho=e^{\varphi}$. بعد التذكير بالمفاهيم الأساسية في الهندسة التفاضلية والهندسة شبه الريمانية، نطوّر ونحلّل صيغ الانحناء الموزون، بما فيها الانحناء الغاوسي والانحناء المتوسط المعمّمين. كما ندرس مسائل من نوع بلاتو (Plateau) ونبحث سلوك مؤثر التباعد الموزون في المتشعبات ذات الكثافة، مع تقديم أمثلة صريحة.

تركّز النتائج الرئيسية لهذه الأطروحة على تصنيف ودراسة هندسية للأسطح الانتقالية المستوية الموزونة في فضاء مينكوفسكي $\mathbb{R}^3_1$ ، وكذلك الأسطح الشعاعية المستوية الموزونة في كل من الفضاء الإقليدي $\mathbb{R}^3$ وفضاء مينكوفسكي $\mathbb{R}^3_1$. من خلال استخراج شروط الاستواء (flatness conditions) وعلاقات الانحناء المناسبة، نوسّع نتائج سابقة تخص الكثافات الخطية ونقدّم توصيفات جديدة للأسطح التي تتحكم فيها بنى هندسية موزونة. هذه النتائج تعطي فهمًا أعمق للأسطح الدنيا والمستوية في السياقات الموزونة، وتشكل أساسًا لأبحاث لاحقة في الهندسة التفاضلية ذات الكثافة.

Introduction

Differential geometry is a fundamental branch of mathematics that studies the geometry of smooth surfaces and manifolds. It has significant applications in physics, engineering, and other areas of mathematics. In recent years, the study of manifolds with density has gained increasing attention. These manifolds modify the standard volume measure by a smooth weighting function, leading to new geometric phenomena and extensions of classical results. Manifolds with density are particularly important in variational problems, minimal surfaces, and the study of curvature, providing a natural framework for weighted geometric analysis.

A manifold with density is a Riemannian manifold $\mathcal{M}^n$ with positive density function e^{φ} used to weight volume and hyperarea (and sometimes lower-dimensional area and length). In terms of underlying Riemannian volume dV_0 and area dA_0, the new weighted volume and area are given by

$$dV = e^{\varphi}.dV_0$$

$$dA = e^{\varphi}.dA_0$$

One of the first examples of a manifold with density appeared in the realm of probability and statistics, Euclidean space with the Gaussian density $e^{-2\pi|x|}$ (see [37] for a detailed exposition in the context of isoperimetric problems).

For reasons coming from the study of diffusion processes, Bakry and Émery ([2]) defined a generalization of the Ricci tensor of Riemannian manifold $\mathcal{M}^n$ with density e^{φ} (or the $\infty-$Bakry-Émery-Ricci tensor) by

$$Ric_{\varphi}^{\infty} = Ric - Hess\varphi, \tag{1}$$

where Ric denotes the Ricci curvature of $\mathcal{M}^n$ and $Hess\varphi$ the Hessian of φ.

By Perelman (in [35], p.6), in a Riemannian manifold $\mathcal{M}^n$ with density e^{φ} in order for the Lichnerovicz formula to hold, the corresponding $\varphi-$scalar curvature is given by

$$S_{\varphi}^{\infty} = S - 2\Delta\varphi - \mid \nabla\varphi \mid^2, \tag{2}$$

where S denotes the scalar curvature of $\mathcal{M}^n$. Note that this is different than taking the trace of Ric_{φ}^{∞} which is $S - \Delta\varphi$.

Following Gromov ([22], p.213), the natural generalization of the mean curvature of hypersurfaces on a manifold with density e^{φ} is given by

$$H_\varphi = H - \frac{1}{n-1}\frac{d\varphi}{d\mathbf{N}}, \tag{3}$$

where H is the Riemannian mean curvature and $\mathbf{N}$ is the unit normal vector field of hypersurface. For a 2-dimensional smooth manifold with density e^φ, Corwin et al. ([18], p.6) define a generalized Gauss curvature

$$K_\varphi = K - \Delta\varphi, \tag{4}$$

and obtain a generalization of the Gauss-Bonnet formula for a smooth disc $\mathbf{D}$:

$$\int_{\mathbf{D}} \mathbf{K}_\varphi + \int_{\partial\mathbf{D}} \kappa_\varphi = 2\pi, \tag{5}$$

where κ_φ is the inward one-dimensional generalized mean curvature as (3) and the integrals are with respect to unweighted Riemannian area and arclength ([29], p.181).

Bayle [3] has derived the first and second variation formulae for the weighted volume functional (see also [29],[37]). From the first variation formula, it can be shown that an immersed submanifold $\mathcal{N}^{n-1}$ in $\mathcal{M}^n$ is minimal if and only if the generalized mean curvature H_φ vanishes ($H_\varphi = 0$).

Doan The Hieu and Nguyen Minh Hoang [28] classified ruled minimal surfaces in $\mathbb{R}^3$ with density $\Psi = e^z$. In [41], classified weighted minimal translation surfaces in Minkowski 3-space.

In [11], Belarbi and Belkhalfa have previously write the equation of minimal surfaces in $\mathbb{R}^3$ with linear density $\Psi = e^\varphi$ (in the case $\varphi(x,y,z) = x,\ \varphi(x,y,z) = y$ and $\varphi(x,y,z) = z$), and characterized some solutions of the equation of minimal graphs in $\mathbb{R}^3$ with linear density $\Psi = e^\varphi$.

In [12], Belarbi and Belkhalfa have studied the $\varphi-$Laplace-Beltrami operator of a non-parametric surface in $\mathbb{R}^3$ with density and we give prove that

$$\Delta_\varphi X = 2H_\varphi.\mathbf{N} + \nabla\varphi = 2H\mathbf{N} + (\nabla\varphi)^T$$

, where X is the vector position of a nonparametric surface $z = f(x^1, x^2)$ in $\mathbb{R}^3$ with density $\Psi = e^\varphi$, and $(\nabla\varphi)^T$ the component tangent of $\nabla\varphi$.

This thesis is devoted to the study of weighted surfaces in Euclidean and Minkowski spaces, focusing on the geometric properties induced by density functions. In particular, it investigates the behavior of flat translation and radial surfaces under these weighted geometries, and explores new curvature formulas, variational problems, and applications of the divergence operator in manifolds with density.

The thesis is organized into four chapters:

1. Preliminaries : Introduces the differential geometry of surfaces and pseudo-Riemannian manifolds, establishing the foundational tools used throughout the thesis. This chapter includes explicit examples to illustrate the concepts.

2. Manifolds with Density : Defines manifolds with density and presents new curvature formulas. The chapter also discusses Plateau's Problem in $\mathbb{R}^3$ with density and the divergence operator. Explicit examples are provided to clarify the theory.

3. Weighted Flat Translation Surfaces in Minkowski 3-Space with Density : Investigates flat translation surfaces under a density function in Minkowski space.

4. Weighted Flat Radial Surfaces in Euclidean and Minkowski 3-Space with Density : Studies radial surfaces under density in both Euclidean and Minkowski spaces.

By systematically exploring these topics, this thesis contributes to a deeper understanding of weighted geometric structures in both Euclidean and Lorentzian contexts, extending classical results and providing a rigorous foundation for further studies in differential geometry with density.

Chapter 1

Preliminaries

In this chapter, we recall and summarize the main mathematical concepts and notations that will be used throughout this work. We begin with a review of the *geometry of surfaces*, introducing parametric and implicit representations, the notions of regularity and tangent spaces, as well as the fundamental forms and curvature properties of surfaces.

Next, we present some essential notions from *differential geometry*, including topological and smooth manifolds, charts and atlases, tangent spaces, tangent bundles, and vector fields. These notions extend the study of surfaces to higher-dimensional smooth manifolds and provide the language for discussing differentiable structures in a rigorous way.

Finally, we introduce the theory of *pseudo-Riemannian manifolds*, which generalizes Riemannian geometry to include indefinite metrics. We define pseudo-Riemannian metrics, linear connections, and curvature tensors, and conclude with an overview of the Minkowski space as the fundamental example of a flat pseudo-Riemannian manifold. ([17], [19], [20], [23], [?], [31], [32], [33], and [36]).

1.1 Geometry of surfaces

1.1.1 Parametric Surfaces

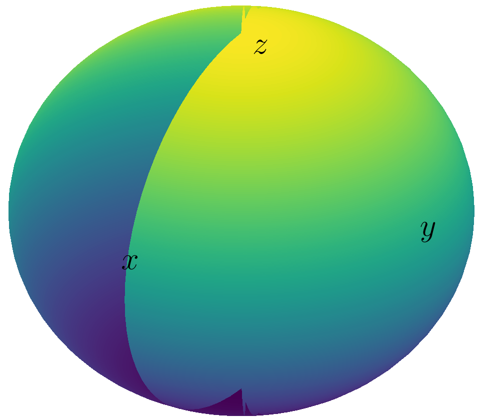
Definition 1. *A parametric surface in $\mathbb{R}^3$ is a vector-valued function*

$$\mathbb{X}(u,v) = \big(x(u,v),\, y(u,v),\, z(u,v)\big),$$

defined on a domain $D \subset \mathbb{R}^2$. The surface is the image of D under $\mathbb{X}$, that is, the set of points $\mathbb{X}(u,v) \in \mathbb{R}^3$.

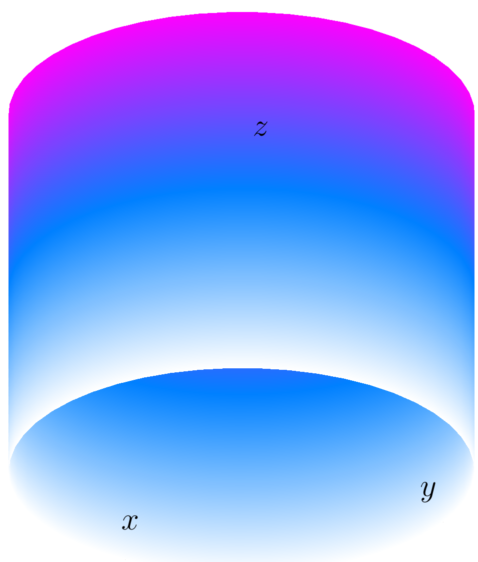
Example 1.

- *Sphere of radius r:*



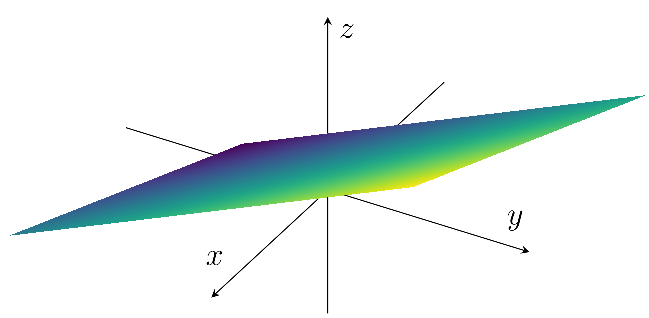
Sphere: $\mathbb{X}(u,v) = \big(r\sin u\cos v,\ r\sin u\sin v,\ r\cos u\big)$, *with* $u \in [0,\pi],\ v \in [0,2\pi]$

- *Cylinder of radius r and height h:*



: *Cylinder :*$\mathbf{X}(u,v) = \big(r\cos u,\ r\sin u,\ v\big)$, *with* $u \in [0,2\pi],\ v \in [0,h]$

- *Plane:*



Parametrization: $\mathbb{X}(u,v) = \big(u,\ v,\ au+bv+c\big)$, *with* $(u,v) \in \mathbb{R}^2$

1.1.2 Regular Surfaces

Definition 2. *A subset $S \subset \mathbb{R}^3$ is called a regular surface if for every point $p \in S$, there exists an open set $U \subset \mathbb{R}^2$, an open set $V \subset \mathbb{R}^3$, and a map*

$$\mathbb{X} : U \to V \cap S,$$

such that :

- *$\mathbb{X}$ is smooth,*
- *$\mathbb{X}$ is a homeomorphism onto its image,*
- *the differential $d\mathbb{X}_{(u,v)}$ is injective at every point of U.*

This map $\mathbb{X}$ is called a local parametrization of the surface.

Proposition 1. *Let $\mathbb{X} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a smooth map such that $\mathbb{X}_u \times \mathbb{X}_v \neq 0$ for all $(u, v) \in U$. Then the image $\mathbb{X}(U)$ is a regular surface.*

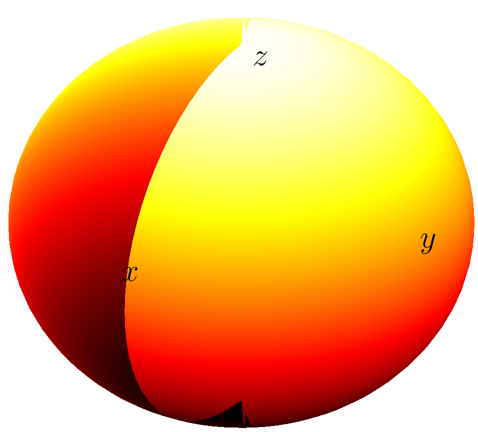
Sketch of proof. Since $\mathbb{X}_u \times \mathbb{X}_v \neq 0$, the differential $d\mathbb{X}$ is of rank 2, hence injective. Smoothness and the inverse function theorem imply that $\mathbb{X}$ is a local diffeomorphism onto its image.

Example 2.

- *The unit sphere:*

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\},$$

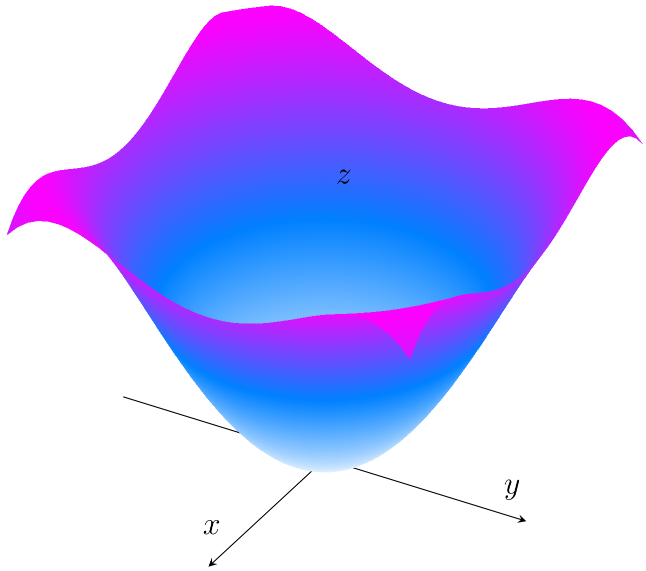
can be covered by local parametrizations using spherical coordinates.



Unit Sphere: $\mathbb{X}(u, v) = \big(\sin u \cos v,\, \sin u \sin v,\, \cos u\big)$, *with* $u \in [0, \pi]$, $v \in [0, 2\pi]$

- *The graph of a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ is a regular surface:*

$$\mathbb{X}(u, v) = (u, v, f(u, v))$$



Graph of function: $\mathbb{X}(u,v) = \big(u,\ v,\ f(u,v)\big)$, *Example:* $f(u,v) = \sin(u^2+v^2)$

Remark 1. *Not all subsets of $\mathbb{R}^3$ are regular surfaces. For example, the set*

$$S = \{(x,y,z) \in \mathbb{R}^3 \mid z = \sqrt{x^2+y^2}\},$$

fails to be regular at the origin because the function $\sqrt{x^2+y^2}$ is not differentiable at $(0,0)$.

1.1.3 Tangent Space of a Surface

Definition 3. *Let $S \subset \mathbb{R}^3$ be a regular surface and let $p \in S$. The tangent space to S at the point p, denoted by T_pS, is the set of all tangent vectors to curves on S passing through p. More precisely,*

$$T_pS = \left\{\gamma'(0) \in \mathbb{R}^3 \mid \gamma : (-\varepsilon, \varepsilon) \to S,\ \gamma(0) = p,\ \gamma \text{ smooth}\right\}.$$

Proposition 2. *If $\mathbb{X} : U \subset \mathbb{R}^2 \to S$ is a local parametrization with $\mathbb{X}(u_0, v_0) = p$, then the partial derivatives $\mathbb{X}_u(u_0,v_0)$ and $\mathbb{X}_v(u_0,v_0)$ span the tangent space T_pS. That is,*

$$T_pS = \operatorname{span}\left\{\mathbb{X}_u(u_0,v_0),\ \mathbb{X}_v(u_0,v_0)\right\}.$$

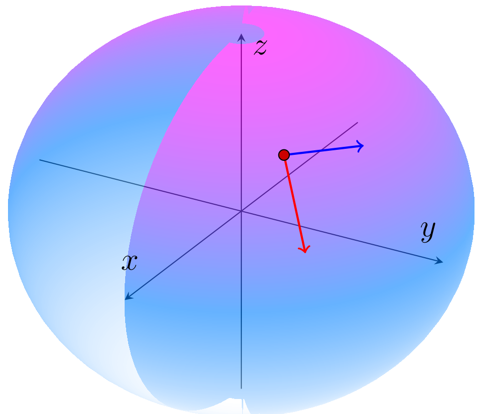
Example 3. *Consider the parametrization of the unit sphere:*

$$\mathbb{X}(u,v) = (\sin u \cos v, \sin u \sin v, \cos u),$$

where $0 < u < \pi$, $0 < v < 2\pi$. The tangent vectors at a point $\mathbb{X}(u_0,v_0)$ are given by

$$\mathbb{X}_u = (\cos u \cos v, \cos u \sin v, -\sin u), \quad \mathbb{X}_v = (-\sin u \sin v, \sin u \cos v, 0).$$

These vectors span the tangent plane at that point.



Tangent Plane to the Unit Sphere: $\mathbb{X}(u,v) = \big(\sin u \cos v,\, \sin u \sin v,\, \cos u\big)$,
$0 < u < \pi,\ 0 < v < 2\pi$

Tangent vectors at $\mathbb{X}(u_0, v_0)$*:*

$$\mathbb{X}_u = (\cos u \cos v,\, \cos u \sin v,\ -\sin u), \quad \mathbb{X}_v = (-\sin u \sin v,\, \sin u \cos v,\ 0).$$

Remark 2. *The tangent space T_pS is a 2-dimensional linear subspace of $\mathbb{R}^3$ and depends smoothly on the point $p \in S$. It provides the best linear approximation to the surface at the point p.*

1.1.4 Surfaces Defined by Implicit Equations

Definition 4. *A surface $S \subset \mathbb{R}^3$ is said to be defined implicitly if there exists a smooth function $F : \mathbb{R}^3 \to \mathbb{R}$ such that*

$$S = \left\{(x,y,z) \in \mathbb{R}^3 \mid F(x,y,z) = 0\right\},$$

and the gradient $\nabla F(p) \neq 0$ for all $p \in S$. The condition on the gradient ensures that S is a regular surface near each point, by the Implicit Function Theorem.

Proposition 3. *Let $S = \{F(x,y,z) = 0\}$ be a surface defined implicitly as above. Then the tangent space T_pS at a point $p \in S$ is given by*

$$T_pS = \left\{v \in \mathbb{R}^3 \mid \nabla F(p) \cdot v = 0\right\}.$$

In other words, the tangent space is the orthogonal complement of the gradient vector $\nabla F(p)$, which is normal to the surface at p.

Example 4. *Consider the unit sphere:*

$$S^2 = \left\{(x,y,z) \in \mathbb{R}^3 \mid F(x,y,z) = x^2 + y^2 + z^2 - 1 = 0\right\}.$$

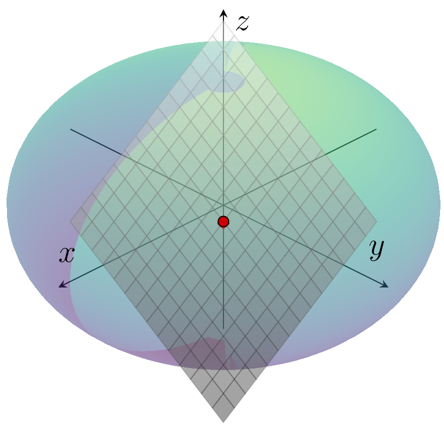
The gradient is

$$\nabla F(x,y,z) = (2x, 2y, 2z),$$

so at the point $p = (x_0, y_0, z_0)$*, the tangent space is*

$$T_pS^2 = \left\{ v \in \mathbb{R}^3 \mid x_0 v_1 + y_0 v_2 + z_0 v_3 = 0 \right\}.$$

This is a plane through the origin of $\mathbb{R}^3$ *orthogonal to the position vector* p*.*



Unit Sphere with Tangent Plane and Normal Vector
Point on the sphere: $p = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$
Gradient (normal vector): $\nabla F(p) = \left(\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}} \right)$
Tangent plane at p*:*

$$T_pS^2 = \left\{ v \in \mathbb{R}^3 \mid \frac{1}{\sqrt{3}} v_1 + \frac{1}{\sqrt{3}} v_2 + \frac{1}{\sqrt{3}} v_3 = 0 \right\}.$$

Let S *be the hyperboloid of one sheet:*

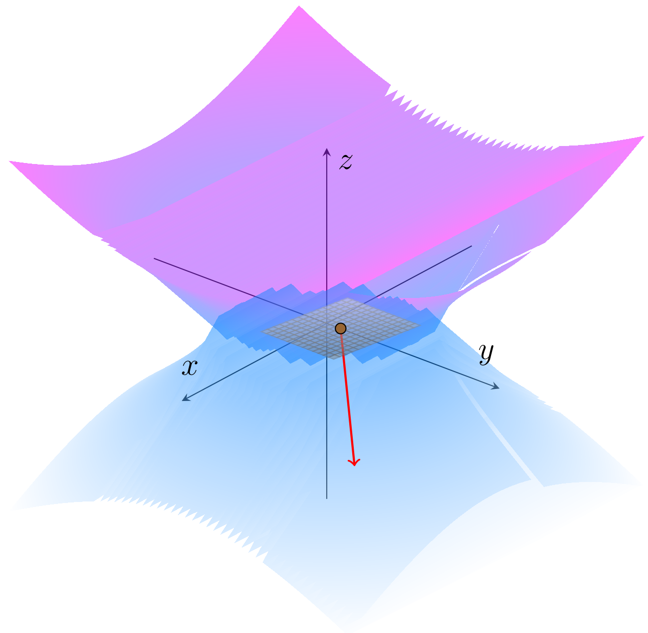
$$F(x, y, z) = x^2 + y^2 - z^2 - 1 = 0.$$

Then,

$$\nabla F(x, y, z) = (2x, 2y, -2z),$$

so the tangent space at $p = (x_0, y_0, z_0)$ *is*

$$T_pS = \left\{ v \in \mathbb{R}^3 \mid x_0 v_1 + y_0 v_2 - z_0 v_3 = 0 \right\}.$$



Hyperboloid and Tangent Plane at Point p
Point on the surface: $p = (1, 1, 1)$
Gradient at p: $\nabla F(p) = (2, 2, -2)$
Tangent plane at p:

$$T_pS = \left\{ v \in \mathbb{R}^3 \mid v_1 + v_2 - v_3 = 0 \right\}.$$

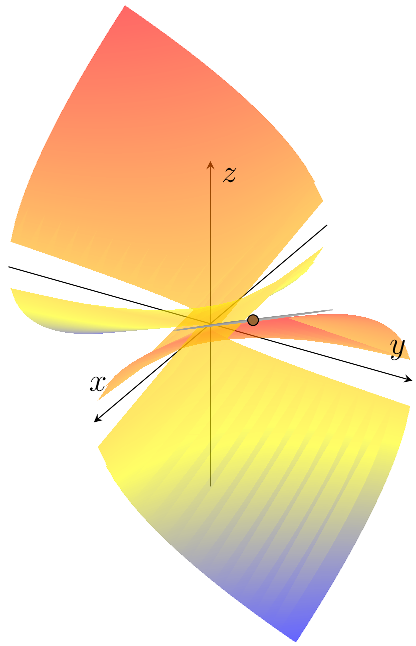
Remark 3. *Implicit surfaces are not always easily parametrized, but the description of the tangent space via the gradient is often simpler and geometrically meaningful. This approach is particularly useful in differential geometry and physics, especially when dealing with constraints defined by equations.*

Definition 5. *Let S be an implicit surface defined by an equation of the form $F(x, y, z) = 0$ for (x, y, z) in an open set $U \subset \mathbb{R}^3$. Consider a point $M_0 = (x_0, y_0, z_0)$ which is a regular point on the surface. Then, locally around M_0, the surface can be described by a parametrized patch. Therefore, it has a tangent space whose Cartesian equation is given by:*

$$\frac{\partial F}{\partial x}(M_0)(x - x_0) + \frac{\partial F}{\partial y}(M_0)(y - y_0) + \frac{\partial F}{\partial z}(M_0)(z - z_0) = 0.$$

Example 5. *The tangent planes to the cone defined by the equation $z^2 = xy$ are given by:*

$$xx_0 + yy_0 - 2zz_0 = 0.$$



Cone $z^2 = xy$ with Tangent Plane at p
Point on the cone: $p = (1, 1, 1)$ since $1^2 = 1 \cdot 1$
Tangent plane at p:

$$x \cdot 1 + y \cdot 1 - 2z \cdot 1 = 0 \quad \Rightarrow \quad x + y - 2z = 0$$

1.1.5 Normal Vector to a Surface

Definition 6. *Let $S \subset \mathbb{R}^3$ be a regular surface, and let $p \in S$. A normal vector to S at the point p is a vector $N_p \in \mathbb{R}^3$ such that $N_p \perp T_pS$, where T_pS is the tangent plane to the surface at p.*

If the surface is given as a regular parametrized surface $\mathbf{x}(u, v)$, then a normal vector at $p = X(u_0, v_0)$ is given by the cross product:

$$N = X_u \times X_v.$$

This vector is orthogonal to both tangent vectors X_u and X_v and hence to the tangent plane.

Proposition 4. *If S is a surface given implicitly by $F(x, y, z) = 0$, with $\nabla F(p) \neq 0$, then the gradient vector*

$$N_p = \nabla F(p)$$

is a normal vector to S at p.

Example 6.

- *Consider the unit sphere* $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$. *The gradient of the defining function* $F(x, y, z) = x^2 + y^2 + z^2 - 1$ *is*

$$\nabla F(x, y, z) = (2x, 2y, 2z),$$

so the normal vector at a point $(x, y, z) \in S^2$ *is* $N = (2x, 2y, 2z)$, *which is proportional to the position vector. Thus, the normal vector at any point on the unit sphere points radially outward.*

- *Let* S *be the surface given by the parametrization*

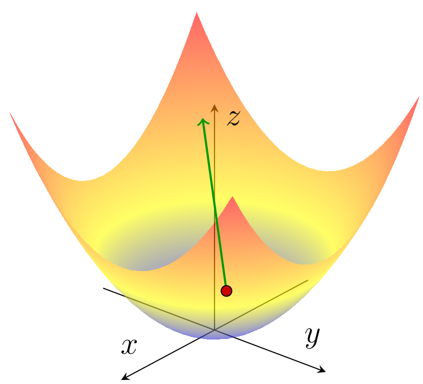
$$X(u, v) = (u, v, u^2 + v^2),$$

which describes a paraboloid. Then

$$X_u = (1, 0, 2u), \quad X_v = (0, 1, 2v),$$

and the normal vector is given by

$$N = X_u \times X_v = \begin{vmatrix} i & j & k \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix} = (-2u, -2v, 1).$$



Paraboloid with Normal Vector

Remark 4. *The normal vector is not uniquely determined; if* N *is a normal vector, then* $-N$ *is also a normal vector.*

However, in many applications (such as when defining orientations, surface integrals, or mean curvature), one usually makes a consistent choice of direction, called the unit normal vector:

$$\hat{N} = \frac{N}{\|N\|}.$$

Remark 5. *On a surface in* $\mathbb{R}^3$, *the normal vector field is crucial for defining geometric quantities such as the shape operator, the second fundamental form, and curvatures. It also plays a central role in physics, particularly in computing flux and in the formulation of boundary conditions for differential equations on surfaces.*

1.1.6 First Fundamental Form

Definition 7. *Let $S \subset \mathbb{R}^3$ be a regular surface, and let $X(u,v)$ be a local parametrization of S. The first fundamental form of the surface is the quadratic form on the tangent space T_pS at $p = \mathbf{x}(u,v)$, given by:*

$$I = E\,du^2 + 2F\,du\,dv + G\,dv^2,$$

where

$$E = \langle X_u, X_u \rangle, \quad F = \langle X_u, X_v \rangle, \quad G = \langle X_v, X_v \rangle,$$

and $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product in $\mathbb{R}^3$.

Interpretation. The first fundamental form encodes all the information about lengths, angles, and areas on the surface. Given a tangent vector $w = a\mathbf{x}_u + b\mathbf{x}_v \in T_pS$, its squared norm is:

$$\|w\|^2 = Ea^2 + 2Fab + Gb^2.$$

Proposition 5. *Let $\gamma(t) = \mathbf{x}(u(t), v(t))$ be a smooth curve on the surface. The length of γ between $t = a$ and $t = b$ is given by:*

$$L(\gamma) = \int_a^b \sqrt{E\,\dot{u}^2 + 2F\,\dot{u}\dot{v} + G\,\dot{v}^2}\,dt.$$

Thus, the first fundamental form allows us to measure arc length of curves lying on the surface.

Example 7.

- *Consider the unit sphere S^2 with the parametrization*

  $$\mathbf{x}(u,v) = (\sin u \cos v, \sin u \sin v, \cos u), \quad 0 < u < \pi,\ 0 < v < 2\pi.$$

  Then,

  $$\mathbf{x}_u = (\cos u \cos v, \cos u \sin v, -\sin u), \quad \mathbf{x}_v = (-\sin u \sin v, \sin u \cos v, 0),$$

  and computing the coefficients:

  $$E = 1, \quad F = 0, \quad G = \sin^2 u.$$

  So the first fundamental form is:

  $$I = du^2 + \sin^2 u\,dv^2.$$

- *For the surface of revolution given by*

  $$\mathbf{x}(u,v) = (f(u)\cos v, f(u)\sin v, u),$$

  we compute:

  $$\mathbf{x}_u = (f'(u)\cos v, f'(u)\sin v, 1), \quad \mathbf{x}_v = (-f(u)\sin v, f(u)\cos v, 0),$$

yielding

$$E = f'(u)^2 + 1, \quad F = 0, \quad G = f(u)^2.$$

So the first fundamental form is:

$$I = \left(1 + f'(u)^2\right)\, du^2 + f(u)^2\, dv^2.$$

Remark 6. *The first fundamental form plays a central role in intrinsic geometry: it allows the definition of geodesics, curvature, and other geometric objects that depend only on measurements made within the surface, not on its embedding in $\mathbb{R}^3$.*

Remark 7. *Two surfaces are said to be isometric if there is a diffeomorphism between them that preserves the first fundamental form. Thus, the first fundamental form is the foundation for comparing surface geometries.*

Theorem 1. *Let $S \subset \mathbb{R}^3$ be a regular surface. Then, for any point $p \in S$, the first fundamental form defines a positive definite quadratic form on T_pS.*

Proof. Let $\mathbf{x}(u, v)$ be a local parametrization of S near p. The tangent vectors $\mathbf{x}_u$ and $\mathbf{x}_v$ are linearly independent by the regularity of the surface. For any tangent vector $w = a\mathbf{x}_u + b\mathbf{x}_v$, we compute

$$I(w, w) = Ea^2 + 2Fab + Gb^2 = \langle w, w\rangle.$$

Since the dot product in $\mathbb{R}^3$ is positive definite and $w \neq 0$ implies $\langle w, w\rangle > 0$, the result follows. $\square$

Definition 8. *Let $S \subset \mathbb{R}^3$ be a regular surface with local parametrization $\mathbf{x}(u, v)$ defined over a domain $D \subset \mathbb{R}^2$. The area of the portion of the surface covered by the parametrization is:*

$$Area(S) = \iint_D \|\mathbf{x}_u \times \mathbf{x}_v\|\, du\, dv.$$

Using the first fundamental form, this becomes:

$$Area(S) = \iint_D \sqrt{EG - F^2}\, du\, dv.$$

Proposition 6. *The quantity $\sqrt{EG - F^2}$ is the area of the parallelogram spanned by the tangent vectors $\mathbf{x}_u$ and $\mathbf{x}_v$ at each point.*

Example 8. *Consider the unit sphere parametrized as above. We compute:*

$$Area(S^2) = \int_0^{2\pi}\int_0^{\pi} \sqrt{1 \cdot \sin^2 u - 0}\, du\, dv = \int_0^{2\pi}\int_0^{\pi} \sin u\, du\, dv = 4\pi.$$

Remark 8. *The area formula is intrinsic and depends only on the first fundamental form. Two isometric surfaces have equal area over corresponding regions.*

1.1.7 Second Fundamental Form

Definition 9 (Second Fundamental Form). *Let $S \subset \mathbb{R}^3$ be a regular surface with a local parametrization $\mathbf{x}(u, v)$, and let N be a unit normal vector field on S. The second fundamental form at a point $p \in S$ is the symmetric bilinear form on the tangent space T_pS defined by:*

$$\mathrm{II}(w, z) = -\langle dN_p(w), z\rangle = \langle D_w N, z\rangle,$$

where D denotes the standard directional derivative in $\mathbb{R}^3$, and $w, z \in T_pS$.

In coordinates, if $\mathbf{x}_u$ and $\mathbf{x}_v$ span the tangent plane, the second fundamental form becomes:

$$\mathrm{II} = l\, du^2 + 2m\, du\, dv + n\, dv^2,$$

where

$$l = \langle \mathbf{x}_{uu}, N\rangle, \quad m = \langle \mathbf{x}_{uv}, N\rangle, \quad n = \langle \mathbf{x}_{vv}, N\rangle.$$

Proposition 7. *The second fundamental form encodes the extrinsic curvature of the surface, that is, how the surface bends in the ambient space $\mathbb{R}^3$.*

Example 9. *For the unit sphere S^2 with parametrization:*

$$\mathbf{x}(u, v) = (\sin u \cos v, \sin u \sin v, \cos u),$$

the unit normal vector is $N = \mathbf{x}(u, v)$. Computation yields:

$$l = 1, \quad m = 0, \quad n = \sin^2 u,$$

and so the second fundamental form is:

$$\mathrm{II} = du^2 + \sin^2 u\, dv^2.$$

Note that in this case, the first and second fundamental forms coincide because the surface is totally umbilic.

Remark 9. *The second fundamental form depends on the choice of orientation (i.e., the sign of N), in contrast with the first fundamental form which is intrinsic.*

Proposition 8. *Let $\gamma(t)$ be a curve on S. Then the normal component of the acceleration vector $\ddot{\gamma}(t)$ is given by:*

$$\langle \ddot{\gamma}(t), N\rangle = \mathrm{II}(\dot{\gamma}, \dot{\gamma}),$$

which measures how the curve bends with respect to the surface.

Example 10. *Let S be the paraboloid given by $z = x^2 + y^2$, with parametrization:*

$$\mathbf{x}(u, v) = (u, v, u^2 + v^2).$$

Then,

$$\mathbf{x}_u = (1, 0, 2u), \quad \mathbf{x}_v = (0, 1, 2v),$$

and the normal vector is:

$$N = \frac{(-2u, -2v, 1)}{\sqrt{4u^2 + 4v^2 + 1}}.$$

We find:

$$l = \langle \mathbf{x}_{uu}, N \rangle = \langle (0,0,2), N \rangle = \frac{2}{\sqrt{4u^2 + 4v^2 + 1}},$$

$$m = \langle \mathbf{x}_{uv}, n \rangle = 0, \quad N = \langle \mathbf{x}_{vv}, N \rangle = \frac{2}{\sqrt{4u^2 + 4v^2 + 1}}.$$

Thus, the second fundamental form is:

$$\mathrm{II} = \frac{2}{\sqrt{4u^2 + 4v^2 + 1}}(du^2 + dv^2).$$

Remark 10. *The second fundamental form allows the computation of principal curvatures and directions, as well as the mean and Gaussian curvatures when combined with the first fundamental form.*

1.1.8 Curvatures of Surfaces

Definition 10 (Weingarten Map and Matrix)**.** *Let $S \subset \mathbb{R}^3$ be a regular surface with unit normal vector field N. The Weingarten map (also called the shape operator) at a point $p \in S$ is the linear map*

$$\begin{array}{rcl} W_p : T_pS & \to & T_pS, \\ v & \mapsto & W_p(v) = -dN_p(v), \end{array}$$

where dN_p is the differential of the Gauss map N. In a local parametrization $\mathbf{x}(u, v)$, the Weingarten map can be expressed as a matrix:

$$W = I^{-1} II,$$

where I and II are the matrices of the first and second fundamental forms, respectively.

Definition 11 (Principal Curvatures and Directions)**.** *The eigenvalues κ_1, κ_2 of the Weingarten matrix W are called the principal curvatures at a point $p \in S$. The corresponding eigenvectors give the principal directions in the tangent plane T_pS.*

Definition 12 (Mean and Gaussian Curvature)**.** *Let κ_1, κ_2 be the principal curvatures at a point $p \in S$. The Gaussian curvature is defined by*

$$K = \kappa_1 \kappa_2,$$

and the mean curvature is defined by

$$H = \frac{1}{2}(\kappa_1 + \kappa_2).$$

Proposition 9. *The Gaussian curvature K and mean curvature H are invariant under isometries of the ambient space $\mathbb{R}^3$. In particular, K is an intrinsic invariant (it depends only on the first fundamental form), while H is extrinsic.*

Example 11. *Let $S \subset \mathbb{R}^3$ be the unit sphere given by $X(u, v) = (\sin u \cos v, \sin u \sin v, \cos u)$. Then, the normal vector at each point is $N = X(u, v)$, and the shape operator is the identity map. Thus:*

$$\kappa_1 = \kappa_2 = 1, \quad K = 1, \quad H = 1.$$

So every point on the sphere has constant positive curvature.

Example 12. *Let $S \subset \mathbb{R}^3$ be the hyperbolic paraboloid $z = x^2 - y^2$. A parametrization is*

$$X(u, v) = (u, v, u^2 - v^2).$$

The first and second fundamental forms can be computed from the partial derivatives. After a lengthy computation, one finds that the Gaussian curvature is negative at all points except at the origin, which is a saddle point. Hence, the surface has negative curvature almost everywhere.

Remark 11.

- *If $K > 0$, the surface is locally convex and the point is called elliptic;*
- *If $K < 0$, the surface has a saddle shape and the point is called hyperbolic;*
- *If $K = 0$, the surface is locally cylindrical and the point is called parabolic.*

Proposition 10. *A point $p \in S$ is umbilic if $\kappa_1 = \kappa_2$. At such points, the shape operator is a scalar multiple of the identity.*

For example, every point on a sphere is umbilic.

1.1.9 Umbilic Points and Lines of Curvature

Definition 13 (Umbilic Point)**.** *A point $p \in S$ is called an umbilic point if the principal curvatures satisfy $\kappa_1(p) = \kappa_2(p)$. Equivalently, the Weingarten map W_p is a scalar multiple of the identity at p.*

Proposition 11. *At an umbilic point p, every direction in the tangent plane T_pS is a principal direction. Equivalently, for any unit tangent vector v,*

$$W_p(v) = \kappa\, v,$$

where $\kappa = \kappa_1 = \kappa_2$.

Proof. If $W_p = \kappa\, I$, then clearly $W_p(v) = \kappa v$ for all $v \in T_pS$, so all directions are eigenvectors with the same eigenvalue. □

Example 13. *On the unit sphere S^2, every point is umbilic because $\kappa_1 = \kappa_2 = 1$ everywhere. Thus, the sphere has no distinguished principal directions; it exhibits perfect symmetry in every tangent direction.*

Definition 14 (Lines of Curvature)**.** *A regular curve $\gamma(t) \subset S$ is called a line of curvature if its tangent vector $\dot{\gamma}(t)$ at each point is a principal direction of the Weingarten map, i.e.,*

$$W_{\gamma(t)}(\dot{\gamma}(t)) = \kappa_i(t)\,\dot{\gamma}(t)$$

for either κ_1 or κ_2.

Proposition 12. *Away from umbilic points, lines of curvature form two orthogonal families of curves on the surface. These foliations are determined by the principal directions corresponding to κ_1 and κ_2.*

Example 14. *On a torus of revolution, the two principal curvature directions correspond to the meridians (circles through the hole) and the parallels (circles around the tube). These curves intersect orthogonally and follow the lines of curvature. Indeed, the parametrization of a torus is*

$$X(\theta,\phi) = \begin{pmatrix} (R + r\cos\theta)\cos\phi \\ (R + r\cos\theta)\sin\phi \\ r\sin\theta \end{pmatrix}, \qquad \theta,\phi \in [0, 2\pi),$$

its first fundamental form (I) is given by

$$I = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} r^2 & 0 \\ 0 & (R + r\cos\theta)^2 \end{pmatrix},$$

and the second fundamental form (II) is

$$II = \begin{pmatrix} l & m \\ m & n \end{pmatrix} = \begin{pmatrix} r & 0 \\ 0 & (R + r\cos\theta)\cos\theta \end{pmatrix}.$$

Then, the Weingarten Matrix is

$$\begin{aligned} W &= I^{-1} II \\ &= \begin{pmatrix} \frac{1}{r} & 0 \\ 0 & \frac{\cos\theta}{R + r\cos\theta} \end{pmatrix}. \end{aligned}$$

Finally, the principal curvatures are

$$\kappa_1 = \frac{1}{r}, \quad \kappa_2 = \frac{\cos\theta}{R + r\cos\theta}.$$

Remark 12. *Lines of curvature are important in applications such as stress analysis and computer graphics, because they naturally follow the principal bending directions of a surface.*

1.1.10 Minimal and Flat Surfaces

Minimal and flat surfaces are two important classes of surfaces in differential geometry, each characterized by specific curvature conditions. These surfaces appear naturally in both theoretical contexts and practical applications such as physics, architecture, and material science.

Definition 15 (Minimal Surface)**.** *A regular surface $S \subset \mathbb{R}^3$ is called a minimal surface if its mean curvature vanishes identically:*

$$H = 0 \quad \textit{everywhere on } S.$$

Equivalently, a minimal surface is a critical point of the area functional with respect to compactly supported variations.

1.1.11 First Variation of Area - Minimal Surfaces

Let $X : \overline{\Omega} \to \mathbb{R}^3$ be a regular surface of class C^2, with its spherical image (also known as the Gauss map)

$$N : \overline{\Omega} \to \mathbb{R}^3 \quad \text{defined by} \quad N = \frac{1}{W} X_u \wedge X_v, \quad W = \sqrt{EG - F^2} = \sqrt{g},$$

and denote by $g_{\alpha\beta}$ and $b_{\alpha\beta}$ (or E, F, G and l, m, n, respectively) the coefficients of the first and second fundamental forms. Moreover, H stands for the mean curvature of X.

We write:

$$w = (u, v), \quad u^1 = u, \quad u^2 = v, \quad X_{,\alpha} = \frac{\partial X}{\partial u^\alpha},$$

and $\Gamma^\gamma_{\alpha\beta}$ denote the Christoffel symbols of the second kind for X.

We now consider a variation of X, that is, a mapping:

$$Z : \overline{\Omega} \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^3, \quad \varepsilon_0 > 0,$$

of class C^2, with the property that:

$$Z(w, 0) = X(w), \quad \text{for all } w \in \overline{\Omega}.$$

This map will be interpreted as a family of surfaces $Z(w, \varepsilon)$, $w \in \overline{\Omega}$, which vary X, and in which X is embedded.

By Taylor expansion, we can write:

$$Z(w, \varepsilon) = X(w) + \varepsilon Y(w) + \varepsilon^2 R(w, \varepsilon).$$

with a continuous remainder term $\varepsilon^2 R(w, \varepsilon)$ of square order, i.e.,

$$R(w, \varepsilon) = O(1) \quad \text{as } \varepsilon \to 0.$$

The vector field

$$Y(w) = \frac{\partial}{\partial \varepsilon} Z(w, \varepsilon)\Big|_{\varepsilon=0} \in C^1(\overline{\Omega}, \mathbb{R}^3)$$

is called the first variation of the family of surfaces $Z(\cdot, \varepsilon)$.

We can write:

$$Y(w) = \eta^\beta(w) X_{,\beta}(w) + \lambda(w) N(w)$$

with functions η^1, η^2, λ of class $C^1(\overline{\Omega})$. Then:

$$Z_{,\alpha} = X_{,\alpha} + \varepsilon \left[\eta^\beta_{,\alpha} X_{,\beta} + \eta^\beta X_{,\alpha\beta} + \lambda_{,\alpha} N + \lambda N_{,\alpha}\right] + \varepsilon^2 R_{,\alpha}.$$

By virtue of the Gauss equations:

$$X_{,\alpha\beta} = \Gamma^\gamma_{\alpha\beta} X_{,\gamma} + b_{\alpha\beta} N,$$

and the Weingarten equations:

$$N_{,\alpha} = -b^\beta_\alpha X_{,\beta}, \quad \text{with } b^\beta_\alpha = b_{\alpha\gamma} g^{\beta\gamma},$$

we obtain that:

$$Z_{,\alpha} = X_{,\alpha} + \varepsilon\,[\xi^{\gamma}_{\alpha} X_{,\gamma} + \nu_{\alpha} N] + \varepsilon^2 R_{,\alpha},$$

where we have set:

$$\xi^{\gamma}_{\alpha} = \eta^{\gamma}_{,\alpha} + \Gamma^{\gamma}_{\alpha\beta}\eta^{\beta} - b_{\alpha\beta} g^{\beta\gamma}\lambda, \quad \nu_{\alpha} = b_{\alpha\beta}\eta^{\beta} + \lambda_{,\alpha}.$$

Then, indicating the ε^2-terms by $\cdots$, we find:

$$|Z_u|^2 = E + 2\varepsilon(\xi^1_1 E + \xi^2_1 F) + \cdots,$$

$$|Z_v|^2 = G + 2\varepsilon(\xi^1_2 F + \xi^2_2 G) + \cdots,$$

$$\langle Z_u, Z_v\rangle = F + \varepsilon[\xi^1_2 E + (\xi^1_1 + \xi^2_2)F + \xi^2_1 G] + \cdots,$$

whence:

$$|Z_u|^2 |Z_v|^2 - \langle Z_u, Z_v\rangle^2 = W^2\left[1 + 2\varepsilon(\xi^1_1 + \xi^2_2) + \cdots\right].$$

We, moreover, have:

$$\xi^1_1 + \xi^2_2 = \eta^1_u + \eta^2_v - \lambda b_{\alpha\beta} g^{\alpha\beta} + \Gamma^{\alpha}_{\alpha\beta}\eta^{\beta}.$$

Since:

$$b_{\alpha\beta} g^{\alpha\beta} = 2H, \quad \Gamma^{\alpha}_{\alpha\beta} = \frac{1}{2g} g_{,\beta} = \frac{1}{W} W_{,\beta}.$$

We infer that:

$$\xi^1_1 + \xi^2_2 = \frac{1}{W}\left\{(\eta^1 W)_u + (\eta^2 W)_v\right\} - 2H\lambda.$$

On account of $\sqrt{1+x} = 1 + \frac{x}{2} + O(x^2)$ for $|x| \ll 1$, we see that:

$$\left(|Z_u|^2 |Z_v|^2 - \langle Z_u, Z_v\rangle^2\right)^{1/2} = W + \varepsilon\left[(\eta^1 W)_u + (\eta^2 W)_v - 2HW\lambda\right] + \cdots.$$

Then we can conclude that the first variation:

$$\delta A_{\Omega}(X, Y) := \frac{d}{d\varepsilon} A_{\Omega}(Z(\cdot, \varepsilon))\Big|_{\varepsilon=0}$$

of the area functional $A_{\Omega}(X)$ on Ω at X in the direction of a vector field $Y = \eta^{\alpha} X_{\alpha} + \lambda N$ is given by:

$$\delta A_{\Omega}(X, Y) = \int_{\Omega}\left[(\eta^1 W)_u + (\eta^2 W)_v - 2HW\lambda\right] du\, dv.$$

Performing an integration by parts, it follows that:

$$\delta A_{\Omega}(X, Y) = \int_{\partial\Omega} W(\eta^1\, dv - \eta^2\, du) - 2\int_{\Omega} \lambda HW\, du\, dv.$$

This, in particular, implies that:

$$\delta A_{\Omega}(X, Y) = -2\int_X \langle Y, N\rangle HW\, du\, dv = -2\int_X \langle Y, N\rangle H\, dA,$$

for all $Y \in C^{\infty}_c(\Omega, \mathbb{R}^3)$. Since $\lambda = \langle Y, N\rangle$ can be chosen as an arbitrary function of class $C^{\infty}_c(\Omega)$, the fundamental theorem of the calculus of variations yields:

Theorem 2. *The first variation $\delta A_\Omega(X,Y)$ of A_Ω at X vanishes for all vector fields $Y \in C_c^\infty(\Omega,\mathbb{R}^3)$ if and only if the mean curvature H of X is identically zero.*

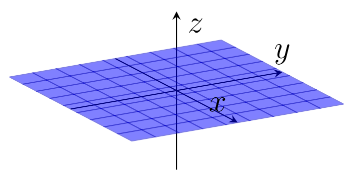
Proposition 13. *A regular surface $S \subset \mathbb{R}^3$ is minimal if and only if it satisfies the minimal surface equation:*

$$\left(1+q^2\right)r - 2pqs + \left(1+p^2\right)t = 0,$$

where $p = z_x$, $q = z_y$, $r = z_{xx}$, $s = z_{xy}$, $t = z_{yy}$, and $z = z(x,y)$ is a local representation of the surface as a graph over the xy-plane.

Example 15. *The following are classical examples of minimal surfaces:*

- ***Plane:*** *Every plane in $\mathbb{R}^3$ is trivially minimal, since it is totally geodesic and flat.*



Plane $z = 0$

- ***Catenoid:*** *the catenoid is the only non-planar minimal surface of revolution in $\mathbb{R}^3$. Consider a surface of revolution around the z-axis, parametrized by*

$$X(u,v) = (r(u)\cos v, r(u)\sin v, u), \quad u \in I,\ v \in [0,2\pi),$$

where $r(u)$ is a smooth positive function. A straightforward computation of the mean curvature H in terms of $r(u)$ and its derivatives gives

$$H = \frac{r''(u)(1+r'(u)^2) - r(u)r'(u)^2 - r(u)}{2r(u)(1+r'(u)^2)^{3/2}}.$$

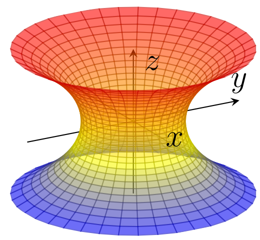
Imposing the minimality condition $H = 0$ leads to the second-order ODE

$$r(u)r''(u) - (1+r'(u)^2) = 0.$$

Solving this ODE, one finds the general solution

$$r(u) = a\cosh\left(\frac{u}{a} + b\right), \quad a > 0,\ b \in \mathbb{R},$$

which is precisely the generating curve of the catenoid. Therefore, up to rigid motions, the catenoid is the only non-planar minimal surface of revolution.



Catenoid

- ***Helicoid:*** *The helicoid is the only non-planar minimal ruled surface in* $\mathbb{R}^3$*. Consider a ruled surface in* $\mathbb{R}^3$ *parametrized by*

$$X(u,v) = c(u) + v\,d(u),$$

where $c(u)$ *is a curve and* $d(u)$ *is a unit vector field along* $c(u)$*. A straightforward computation of the mean curvature* H *gives*

$$H = \frac{(c' \times d)\cdot d'}{2\,|c' \times d|}.$$

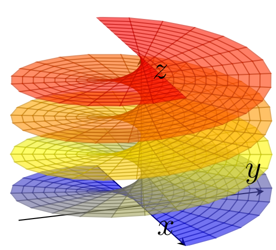
Imposing the minimality condition $H = 0$ *leads to the differential equation*

$$(c'(u) \times d(u)) \cdot d'(u) = 0,$$

which forces the director vector $d(u)$ *to rotate uniformly around a fixed axis. Solving this equation, one finds, up to rigid motions, the parametrization*

$$X(u,v) = (v\cos u, v\sin u, au), \quad a \in \mathbb{R},$$

which is exactly the helicoid. Therefore, the helicoid is the unique non-planar minimal ruled surface in $\mathbb{R}^3$*.*



Helicoid with $c = 0.3$

- ***Scherk's surface:*** *Scherk's first surface is a doubly periodic minimal translation surface discovered by Heinrich Scherk. It can be written explicitly as:*

$$z(x,y) = \log\left(\frac{\cos y}{\cos x}\right)$$

for x, y near zero. It is a classical example of a minimal surface that is not a surface of revolution.

Consider a surface in $\mathbb{R}^3$ given as a graph of a function

$$z = f(x, y).$$

For a graph, the mean curvature is

$$H = \frac{(1 + f_y^2)f_{xx} - 2f_x f_y f_{xy} + (1 + f_x^2)f_{yy}}{2(1 + f_x^2 + f_y^2)^{3/2}}.$$

Imposing the minimality condition $H = 0$ leads to the minimal surface equation :

$$(1 + f_y^2)f_{xx} - 2f_x f_y f_{xy} + (1 + f_x^2)f_{yy} = 0.$$

Searching for solutions of the form

$$f(x, y) = g(x) + h(y),$$

the PDE reduces to

$$(1 + h'^2)g'' + (1 + g'^2)h'' = 0.$$

Solving this ODE system, one obtains, up to translations and rotations, Scherk's first surface

$$z = \frac{1}{a} \ln \frac{\cos(ay)}{\cos(ax)}, \quad a \in \mathbb{R} \setminus \{0\}.$$

Thus, Scherk's surface is a classical example of a nontrivial minimal surface in $\mathbb{R}^3$ that is periodic and can be represented as a sum of functions of x and y.

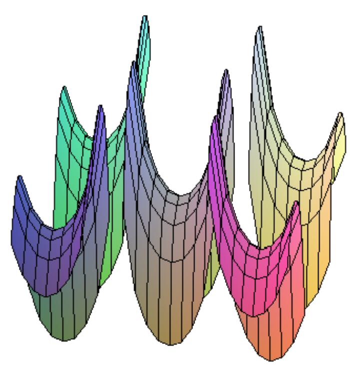


Figure 1.1: Scherk's minimal surface: $z = \frac{1}{4} \log\left(\frac{\cos 4y}{\cos 4x}\right)$

Remark 13. *Minimal surfaces are characterized by their property of having locally minimal area. In physical terms, they model soap films stretched across wireframes. Their geometry is tightly controlled by the vanishing of the mean curvature.*

Definition 16 (Flat Surface)**.** *A regular surface $S \subset \mathbb{R}^3$ is called a flat surface if its Gaussian curvature vanishes identically:*

$$K = 0 \quad \textit{everywhere on } S.$$

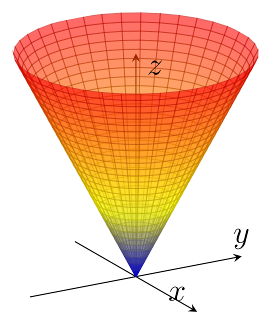
Proposition 14. *A surface with vanishing Gaussian curvature is locally isometric to a region of the Euclidean plane. That is, flatness is an intrinsic property of the surface.*

Example 16. *Examples of flat surfaces include:*

- *Planes: Every plane in $\mathbb{R}^3$ is trivially flat and minimal.*
- *Cylinders: The standard circular cylinder*

$$X(u, v) = (a \cos u, a \sin u, v)$$

  has zero Gaussian curvature but non-zero mean curvature.
- *Cones: Cones (except at the vertex) are also flat surfaces.*



Cone $z = \sqrt{x^2 + y^2}$

Remark 14. *Unlike minimal surfaces, flat surfaces do not necessarily minimize area. However, they are important in applications such as developable surfaces in manufacturing, where bending a flat sheet without stretching is desirable.*

Proposition 15. *Let $S \subset \mathbb{R}^3$ be a developable surface (i.e., a surface that can be flattened onto a plane without distortion). Then S is a flat surface with $K = 0$.*

Definition 17 (Developable Surface)**.** *A surface is called developable if it can be unfolded onto a plane without stretching, meaning there exists a local isometry from the surface to $\mathbb{R}^2$.*

Example 17. *All cylinders, cones, and tangent developables (formed by moving a line along a space curve) are developable surfaces.*

Remark 15. *All developable surfaces are flat, but not all flat surfaces are developable. For instance, the helicoid is minimal and hence not developable, despite having interesting curvature properties.*

1.2 Differential Geometry

1.2.1 Topological Spaces

Definition 18 (Topological Space)**.** *A topological space is an ordered pair (X, τ) consisting of:*

- *a set X*
- *a collection $\tau \subseteq \mathcal{P}(X)$ of subsets of X (called open sets) satisfying:*
    1. *$\emptyset, X \in \tau$*
    2. *Any union of elements of τ belongs to τ*
    3. *Any finite intersection of elements of τ belongs to τ*

The collection τ is called a topology on X.

Example 18 (Standard Topologies)**.**

1. *Discrete Topology: $\tau = \mathcal{P}(X)$ (every subset is open),*
2. *Indiscrete Topology: $\tau = \{\emptyset, X\}$,*
3. *Euclidean Topology: On $\mathbb{R}^n$, τ consists of all unions of open balls $B_r(x) = \{y \in \mathbb{R}^n \mid \|y - x\| < r\}$.*

Definition 19 (Basis for a Topology)**.** *A basis for a topology on X is a collection $\mathcal{B} \subseteq \mathcal{P}(X)$ such that:*

1. *$\forall x \in X$, $\exists B \in \mathcal{B}$ with $x \in B$,*
2. *If $x \in B_1 \cap B_2$ for $B_1, B_2 \in \mathcal{B}$, then $\exists B_3 \in \mathcal{B}$ with $x \in B_3 \subseteq B_1 \cap B_2$.*

The topology τ generated by $\mathcal{B}$ consists of all unions of basis elements.

Example 19 (Basis Examples)**.**

- *Open balls form a basis for $\mathbb{R}^n$'s standard topology,*
- *Open rectangles $(a, b) \times (c, d)$ form a basis for $\mathbb{R}^2$,*
- *All singletons $\{x\}$ form a basis for the discrete topology.*

Definition 20 (Hausdorff Space)**.** *A topological space (X, τ) is Hausdorff if for every $x, y \in X$ with $x \neq y$, there exist disjoint open sets $U, V \in \tau$ such that $x \in U$ and $y \in V$.*

Proposition 16 (Metric Spaces are Hausdorff)**.** *Every metric space (X, d) equipped with its metric topology is Hausdorff.*

Proof. Given $x \neq y$, let $r = d(x, y)/2 > 0$. Then $B_r(x)$ and $B_r(y)$ are disjoint open neighborhoods. $\square$

Definition 21 (Second-Countable Space). *A topological space is second-countable if its topology has a countable basis.*

Example 20 (Manifold-Relevant Examples).

- $\mathbb{R}^n$ *is second-countable (rational balls form countable basis),*
- *The long line is Hausdorff but not second-countable,*
- *Every smooth manifold is second-countable by definition.*

1.2.2 Topological Surfaces

Definition 22 (Topological Surface). *A topological surface is a 2-dimensional topological manifold. Specifically, it is a Hausdorff, second-countable space M where every point $p \in M$ has an open neighborhood $U \subseteq M$ homeomorphic to an open subset of $\mathbb{R}^2$.*

Example 21 (Classical Surfaces).

1. *Sphere:* $\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ *with subspace topology,*
2. *Torus:* $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ *with product topology,*
3. *Real Projective Plane:* $\mathbb{RP}^2 = \mathbb{S}^2 / \sim$ *where* $x \sim -x$ *with quotient topology*
4. *Klein Bottle: The non-orientable surface obtained by identifying a square's sides via* $aba^{-1}b$.

Theorem 3 (Classification of Compact Surfaces). *Every connected, compact topological surface is homeomorphic to exactly one of:*

- *The sphere* $\mathbb{S}^2$,
- *A connected sum of torus* $\mathbb{T}^2 \# \cdots \# \mathbb{T}^2$ *(orientable case),*
- *A connected sum of projective planes* $\mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2$ *(non-orientable case).*

Definition 23 (Triangulable Surface). *A surface M is triangulable if it admits a decomposition into finitely many triangles (homeomorphic images of 2-simplices) where any two triangles intersect only along edges or vertices.*

Proposition 17 (Smooth Surfaces are Triangulable). *Every compact smooth surface admits a triangulation. Moreover, any two triangulations have a common refinement.*

Example 22 (Constructing Surfaces).

- *Connected Sum: For surfaces M_1, M_2, remove disks $D_i \subset M_i$ and glue along boundary circles ∂D_i,*
- *Quotient Space: The Möbius strip as* $[0, 1] \times [0, 1]/(0, y) \sim (1, 1 - y)$,
- *Suspension: The double cone over* $\mathbb{S}^1$ *(not a surface at cone points).*

Definition 24 (Orientability). *A surface M is orientable if it contains no subset homeomorphic to a Möbius strip. Otherwise, it is non-orientable.*

Theorem 4 (Euler Characteristic). *For any triangulation of a compact surface M with V vertices, E edges, and F faces:*

$$\chi(M) = V - E + F,$$

is a topological invariant satisfying:

- $\chi(\mathbb{S}^2) = 2$,
- $\chi(\mathbb{T}^2) = 0$,
- $\chi(\mathbb{RP}^2) = 1$,
- $\chi(M_1 \# M_2) = \chi(M_1) + \chi(M_2) - 2$.

Remark 16 (Genus-Orientability Relationship). *For closed surfaces:*

- *Orientable genus g surfaces have $\chi = 2 - 2g$,*
- *Non-orientable genus k surfaces have $\chi = 2 - k$.*

1.2.3 Charts

Definition 25 (Chart). *A chart on a topological manifold M is a pair (U, φ) where:*

- *$U \subset M$ is an open subset,*
- *$\varphi : U \to \varphi(U) \subset \mathbb{R}^n$ is a homeomorphism to an open subset of $\mathbb{R}^n$.*

The component functions $\varphi = (x^1, \dots, x^n)$ are called local coordinates on U.

Example 23 (Standard Charts).

1. *Euclidean Space: For $M = \mathbb{R}^n$, the trivial chart $(\mathbb{R}^n, id_{\mathbb{R}^n})$,*
2. *Sphere: On $\mathbb{S}^2 \subset \mathbb{R}^3$, the stereographic chart (U, φ) where:*

$$U = \mathbb{S}^2 \setminus \{(0,0,1)\}, \quad \varphi(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z} \right),$$

3. *Graph of a Function: For $f : \mathbb{R}^n \to \mathbb{R}$ smooth, the graph $\Gamma(f) \subset \mathbb{R}^{n+1}$ has a global chart:*

$$\varphi(x^1, \dots, x^n, f(x^1, \dots, x^n)) = (x^1, \dots, x^n).$$

Definition 26 (Compatible Charts). *Two charts (U_1, φ_1) and (U_2, φ_2) on M are smoothly compatible if either:*

- *$U_1 \cap U_2 = \emptyset$, or*
- *The transition map $\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$ is a diffeomorphism*

Example 24 (Non-Compatible Charts)**.** *On* $\mathbb{R}$*, consider:*

- $\varphi_1(x) = x$ *(standard chart),*
- $\varphi_2(x) = x^3$ *(cubic chart).*

These charts are not smoothly compatible since $\varphi_2 \circ \varphi_1^{-1}(y) = y^{1/3}$ *is not differentiable at* 0*.*

Proposition 18 (Local Representation)**.** *Let* (U, φ) *be a chart on* M *with* $\varphi = (x^1, \dots, x^n)$*. For any function* $f : M \to \mathbb{R}$*, the coordinate representation:*

$$f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R},$$

is well-defined on $\mathbb{R}^n$*. We say* f *is smooth at* $p \in U$ *if* $f \circ \varphi^{-1}$ *is smooth at* $\varphi(p)$*.*

Example 25 (Polar Chart)**.** *On* $M = \mathbb{R}^2 \setminus \{(0,0)\}$*, the polar coordinate chart* (U, φ) *where:*

$$U = \mathbb{R}^2 \setminus \{\textit{non-positive } x\textit{-axis}\}, \quad \varphi(r, \theta) = (r\cos\theta, r\sin\theta),$$

with inverse $\varphi^{-1}(x, y) = \left(\sqrt{x^2 + y^2}, \arctan\left(\frac{y}{x}\right)\right)$*. This chart is not compatible with the standard Cartesian chart at* $\theta = \pi$*.*

1.2.4 Atlas

Definition 27 (Atlas)**.** *An atlas* $\mathscr{A}$ *for* M *is a collection of charts* $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in I}$ *such that:*

$$\bigcup_{\alpha \in I} U_\alpha = M.$$

Two charts (U, ϕ) *and* (V, ψ) *are compatible if either* $U \cap V = \emptyset$ *or the transition map:*

$$\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V),$$

is a diffeomorphism between open subsets of $\mathbb{R}^n$*.*

Example 26 (Standard Atlases)**.**

1. *Euclidean Space: The single chart* $(\mathbb{R}^n, id_{\mathbb{R}^n})$ *forms an atlas,*
2. *Sphere* $\mathbb{S}^n$*: The stereographic atlas with two charts:*
   - $U_1 = \mathbb{S}^n \setminus \{N\}$ *(north pole removed),*
   - $U_2 = \mathbb{S}^n \setminus \{S\}$ *(south pole removed),*

   with transition maps given by inversion through the equatorial plane,
3. *Torus* $\mathbb{T}^2$*: The atlas generated by angular coordinates:*

   $$(\theta, \phi) \in (0, 2\pi) \times (0, 2\pi)$$

   with periodic extensions,

4. *Real Projective Space* $\mathbb{RP}^n$*: The atlas of homogeneous coordinates:*

$$[x^0 : \cdots : x^n] \mapsto \left(\frac{x^0}{x^i}, \ldots, \frac{\widehat{x^i}}{x^i}, \ldots, \frac{x^n}{x^i} \right) \quad \text{on } U_i = \{x^i \neq 0\}.$$

Definition 28 (Smooth Atlas)**.** *An atlas is called smooth if all its charts are pairwise compatible. A smooth atlas is maximal if it is not properly contained in any larger smooth atlas.*

Theorem 5 (Equivalence of Atlases)**.** *Two smooth atlases $\mathscr{A}_1$ and $\mathscr{A}_2$ on M are equivalent (define the same smooth structure) if $\mathscr{A}_1 \cup \mathscr{A}_2$ is again a smooth atlas.*

Example 27 (Non-Equivalent Atlases)**.** *On $\mathbb{R}$, consider:*

1. *The standard atlas $\{(\mathbb{R}, id)\}$,*
2. *The atlas $\{(\mathbb{R}, \phi)\}$ where $\phi(x) = x^3$,*

These define different smooth structures, though the resulting manifolds are diffeomorphic.

Definition 29 (Complex Atlas)**.** *A complex atlas of dimension n is a collection of charts where transition maps are biholomorphic maps between open sets in $\mathbb{C}^n$. This defines a complex manifold.*

Example 28 (Complex Atlases)**.**

- *Riemann Sphere: $\mathbb{CP}^1$ with stereographic projections,*
- *Complex Torus: $\mathbb{C}/\Lambda$ for lattice Λ,*
- *Hopf Surface: $(\mathbb{C}^2 \setminus \{0\})/\sim$ where $(z, w) \sim (2z, 2w)$.*

Remark 17. *The minimal number of charts needed to cover a manifold is a topological invariant:*

- *$\mathbb{S}^n$ can be covered by 2 charts (stereographic projection),*
- *$\mathbb{RP}^n$ requires at least $n + 1$ charts (homogeneous coordinates),*
- *Every smooth manifold admits an atlas with countably many charts.*

1.2.5 Smooth Manifolds

Definition 30 (Smooth Manifold)**.** *An n-dimensional smooth manifold is a topological manifold M equipped with a $\mathcal{C}^\infty$ atlas $\mathscr{A} = \{(U_\alpha, \phi_\alpha)\}_{\alpha \in I}$ such that for any two charts (U_α, ϕ_α) and (U_β, ϕ_β) with $U_\alpha \cap U_\beta \neq \emptyset$, the transition map:*

$$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

is a diffeomorphism between open subsets of $\mathbb{R}^n$.

Example 29 (Standard Smooth Structures).

1. *Euclidean Space:* $\mathbb{R}^n$ *with the identity chart* $(\mathbb{R}^n, id)$*,*
2. *Sphere:* $\mathbb{S}^n$ *with stereographic projection charts,*
3. *Torus:* $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ *with product charts,*
4. *Matrix Groups:* $GL(n, \mathbb{R})$*,* $SL(n, \mathbb{R})$ *as open submanifolds of* $\mathbb{R}^{n^2}$*.*

Theorem 6 (Smooth Extension). *Let* M *be a smooth manifold and* $A \subset M$ *a closed subset. For any smooth function* $f : A \to \mathbb{R}$*, there exists a smooth function* $\tilde{f} : M \to \mathbb{R}$ *such that* $\tilde{f}|_A = f$*.*

Definition 31 (Smooth Map). *A continuous map* $f : M \to N$ *between smooth manifolds is smooth if for every* $p \in M$*, there exist charts* (U, ϕ) *about* p *and* (V, ψ) *about* $f(p)$ *such that:*

$$\psi \circ f \circ \phi^{-1} : \phi(U \cap f^{-1}(V)) \to \psi(V),$$

is smooth as a map between Euclidean spaces.

Example 30 (Smooth Maps).

- *The antipodal map* $A : \mathbb{S}^n \to \mathbb{S}^n$*,* $A(x) = -x$ *is smooth,*
- *Matrix multiplication* $GL(n, \mathbb{R}) \times GL(n, \mathbb{R}) \to GL(n, \mathbb{R})$ *is smooth,*
- *Projection* $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$ *is smooth.*

Proposition 19 (Smoothness is Local). *A map* $f : M \to N$ *is smooth if and only if for each* $p \in M$*, there exists an open neighborhood* U *of* p *such that* $f|_U$ *is smooth.*

Theorem 7 (Whitney Approximation). *Any continuous map between smooth manifolds is homotopic to a smooth map. Moreover, if two smooth maps are continuously homotopic, they are smoothly homotopic.*

1.2.6 Tangent Spaces

Definition 32 (Tangent Space). *Let* M *be a smooth* n*-manifold and* $p \in M$*. The tangent space* T_pM *is the set of all derivations at* p*:*

$$T_pM := \{v : C^\infty(M) \to \mathbb{R} \mid v \text{ is linear and satisfies } v(fg) = v(f)g(p) + f(p)v(g)\}.$$

This forms an n*-dimensional vector space called the tangent space to* M *at* p*.*

Proposition 20 (Basis for T_pM). *Given a chart* $(U, \varphi) = (U, x^1, \ldots, x^n)$ *about* p*, the operators*

$$\left.\frac{\partial}{\partial x^i}\right|_p : f \mapsto \left.\frac{\partial (f \circ \varphi^{-1})}{\partial r^i}\right|_{\varphi(p)},$$

form a basis for T_pM*, where* (r^i) *are standard coordinates on* $\mathbb{R}^n$*.*

Example 31 (Euclidean Space). *For $M = \mathbb{R}^n$ and $p \in \mathbb{R}^n$, we have the natural identification:*

$$T_p\mathbb{R}^n \cong \mathbb{R}^n \quad \textit{via} \quad v \mapsto (v(r^1), \dots, v(r^n)),$$

where r^i are the standard coordinate functions.

Theorem 8 (Dimension Theorem). *For any smooth n-manifold M and point $p \in M$:*

$$\dim T_pM = n.$$

Definition 33 (Differential). *For a smooth map $F : M \to N$ between manifolds, the differential $dF_p : T_pM \to T_{F(p)}N$ is defined by:*

$$(dF_p(v))(f) := v(f \circ F),$$

for $v \in T_pM$ and $f \in C^\infty(N)$.

Example 32 (Tangent Space to Spheres). *For $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, the tangent space at p is:*

$$T_p\mathbb{S}^n = \{v \in \mathbb{R}^{n+1} \mid v \cdot p = 0\}.$$

where $\cdot$ denotes the Euclidean dot product.

Proposition 21 (Chain Rule for Manifolds). *For smooth maps $F : M \to N$ and $G : N \to P$, we have:*

$$d(G \circ F)_p = dG_{F(p)} \circ dF_p.$$

Theorem 9 (Velocity Vector Interpretation). *For any $v \in T_pM$, there exists a smooth curve $\gamma : (-\epsilon, \epsilon) \to M$ with $\gamma(0) = p$ and $\gamma'(0) = v$, where:*

$$\gamma'(0)(f) := \left.\frac{d}{dt}\right|_{t=0} (f \circ \gamma)(t),$$

Example 33 (Matrix Lie Groups). *For $G = \mathrm{GL}(n, \mathbb{R}) \subset \mathbb{R}^{n^2}$, the tangent space at I is:*

$$T_I\mathrm{GL}(n, \mathbb{R}) \cong \mathfrak{gl}(n, \mathbb{R}).$$

the space of all $n \times n$ real matrices.

Remark 18 (Geometric Interpretation). *The tangent space T_pM can be visualized as:*

- *The set of all possible velocity vectors of curves through p,*
- *The "best linear approximation" to M at p,*
- *The space of directional derivatives at p.*

1.2.7 Tangent Bundle

Definition 34 (Tangent Bundle). *Let M be a smooth n-manifold. The tangent bundle TM is the disjoint union of all tangent spaces of M:*

$$TM := \bigsqcup_{p\in M} T_pM = \{(p,v) \mid p \in M, v \in T_pM\},$$

equipped with:

- *The projection map $\pi : TM \to M$ defined by $\pi(p,v) = p$,*
- *A natural smooth structure making TM a $2n$-dimensional manifold.*

Theorem 10 (Manifold Structure). *Given any smooth atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in A}$ for M, the tangent bundle TM admits a smooth structure via the induced atlas:*

$$\{\widetilde{\varphi}_\alpha : \pi^{-1}(U_\alpha) \to \varphi_\alpha(U_\alpha) \times \mathbb{R}^n\},$$

where for $v = \sum_{i=1}^n v^i \left.\frac{\partial}{\partial x^i}\right|_p$, the local coordinates are:

$$\widetilde{\varphi}_\alpha(p,v) = (\varphi_\alpha(p), v^1, \ldots, v^n).$$

Example 34 (Trivial Bundles).

- *For $M = \mathbb{R}^n$, the tangent bundle is trivial:*
$$T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n,$$
- *The cylinder $\mathbb{S}^1 \times \mathbb{R}$ is naturally isomorphic to $T\mathbb{S}^1$.*

Example 35 (Non-Trivial Bundles).

- *The tangent bundle $T\mathbb{S}^2$ is non-trivial (by the Hairy Ball Theorem),*
- *The Möbius strip can be viewed as a non-trivial line bundle over $\mathbb{S}^1$.*

Proposition 22 (Fiberwise Vector Space Structure). *For each $p \in M$, the fiber $\pi^{-1}(p) = T_pM$ has a canonical n-dimensional vector space structure:*

- *$(p,v) + (p,w) := (p, v+w)$,*
- *$\lambda \cdot (p,v) := (p, \lambda v)$ for $\lambda \in \mathbb{R}$.*

Definition 35 (Bundle Coordinates). *The local coordinates $(x^1, \ldots, x^n, v^1, \ldots, v^n)$ on TM induced from a chart (U, φ) on M are called bundle coordinates. The transformation rules are:*

$$\widetilde{v}^i = \sum_{j=1}^n \frac{\partial \widetilde{x}^i}{\partial x^j} v^j,$$

where $\widetilde{x}^i$ are the new coordinates on M.

Remark 19 (Physical Interpretation). *The tangent bundle represents:*

- *The configuration space (base manifold M),*
- *The velocity space (fibers T_pM),*
- *The state space for classical mechanical systems.*

1.2.8 Vector Fields

Definition 36 (Vector Field). *A vector field X on a smooth manifold M is a smooth section of the tangent bundle, i.e., a smooth map $X : M \to TM$ satisfying:*

$$\pi \circ X = id_M,$$

where $\pi : TM \to M$ is the canonical projection. Equivalently, X assigns to each $p \in M$ a tangent vector $X_p \in T_pM$ in a smooth manner.

Proposition 23 (Local Representation). *Given local coordinates $(U, (x^i))$ on M, every vector field X can be expressed as:*

$$X = \sum_{i=1}^{n} X^i \frac{\partial}{\partial x^i},$$

where $X^i : U \to \mathbb{R}$ are smooth functions called the components of X.

Example 36 (Canonical Vector Fields).

- *On $\mathbb{R}^n$, the coordinate fields $\dfrac{\partial}{\partial r^i}$ form a global frame,*
- *On $\mathbb{S}^2$, the azimuthal field $X = -x^2 \dfrac{\partial}{\partial x^1} + x^1 \dfrac{\partial}{\partial x^2}$ vanishes at the poles,*
- *On a Lie group G, left-invariant fields satisfy $X_{gh} = (dL_g)_h(X_h)$.*

Theorem 11 (Existence of Vector Fields). *A compact manifold M admits a nowhere-zero vector field if and only if $\chi(M) = 0$, where χ is the Euler characteristic.*

Definition 37 (Lie Bracket). *The Lie bracket of two vector fields $X, Y \in \mathfrak{X}(M)$ is the vector field $[X, Y]$ defined by its action on functions:*

$$[X, Y](f) := X(Y(f)) - Y(X(f)).$$

In local coordinates, if $X = X^i \partial_i$ and $Y = Y^j \partial_j$, then:

$$[X, Y] = \left(X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j} \right) \frac{\partial}{\partial x^i}.$$

Proposition 24 (Properties of the Lie Bracket). *The Lie bracket satisfies:*

- *Bilinearity: $[aX + bY, Z] = a[X, Z] + b[Y, Z]$,*
- *Antisymmetry: $[X, Y] = -[Y, X]$,*
- *Jacobi Identity: $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$,*
- *Leibniz Rule: $[X, fY] = f[X, Y] + X(f)Y$.*

Example 37 (Lie Algebra Structure). *The space $\mathfrak{X}(M)$ of all smooth vector fields forms an infinite-dimensional Lie algebra under the bracket $[\cdot, \cdot]$. For matrix Lie groups, this recovers the standard Lie algebra structure.*

Remark 20 (Geometric Interpretation). *Vector fields can be viewed as:*

- *Differential operators acting on functions,*
- *Infinitesimal generators of flows.*
- *First-order PDEs (integral curves satisfy $\dot{\gamma}(t) = X_{\gamma(t)}$),*

1.3 Pseudo-Reimannian Manifolds

1.3.1 Pseudo-Reimannian Metric

Pseudo-Riemannian Metric Tensor

Definition 38. *A pseudo-Riemannian metric tensor g on a manifold M is a symmetric, non-degenerate $(0,2)$-tensor field of constant index. That is, to each point $x \in M$, the tensor g assigns a scalar product g_x on the tangent space T_xM, and the index of g_x is the same for all $x \in M$. It is common to denote this scalar product by $\langle\cdot,\cdot\rangle$ or occasionally by $h(\cdot,\cdot)$, so that $g(v,w) = \langle v,w\rangle$.*

Pseudo-Riemannian Manifold

Definition 39. *A pseudo-Riemannian manifold of dimension n is a smooth n-dimensional manifold equipped with a pseudo-Riemannian metric tensor g. The constant index s, with $0 \leq s \leq n$, is called the index of the manifold. When $s = 0$, M is called a Riemannian manifold, in which case each g_x defines a positive definite inner product on T_xM.*

Corollary 1. *A pseudo-Riemannian manifold (or metric) is also referred to as a semi-Riemannian manifold (or metric).*

In particular, a pseudo-Riemannian metric on an even-dimensional manifold is called neutral if its index equals $\frac{1}{2}\dim M$.

If the index of M is one, the manifold is called a Lorentz manifold, and the corresponding metric is called Lorentzian.

A manifold of dimension ≥ 2 admits a Lorentzian metric if and only if it admits a one-dimensional distribution.

Proposition 25. *At each point x in the Euclidean n-space $\mathbb{E}^n$, there exists a canonical linear isomorphism from $\mathbb{E}^n$ onto $T_x\mathbb{E}^n$. In terms of natural coordinates on $\mathbb{E}^n$, it sends a vector v to $v_x = \sum_j v^j\partial_j$. The inner product on $\mathbb{E}^n$ gives rise to a metric tensor on $\mathbb{E}^n$ with*

$$\langle v_x, w_x\rangle = \sum_{j=1}^{n} v^j w^j, \tag{1.1}$$

where $v = \sum_{j=1}^n v^j\partial_j$ and $w = \sum_{j=1}^n w^j\partial_j$.

For an integer $s \in [0,n]$, if we change the first s plus signs in (1.1) to minus signs, then we obtain a metric tensor of the form

$$\langle v_x, w_x \rangle = -\sum_{j=1}^{s} v^j w^j + \sum_{k=s+1}^{n} v^k w^k, \tag{1.2}$$

which has index s. The resulting pseudo-Euclidean space is denoted by $\mathbb{E}^n_s$.

If $s = 0$, $\mathbb{E}^n_s$ reduces to the Euclidean n-space $\mathbb{E}^n$. The space $\mathbb{E}^n_1$ is called the Minkowski n-space.

1.3.2 Linear Connections

Affine Connection

Definition 40. *An affine connection ∇ on a manifold M is a function*

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

such that for all vector fields $X, Y \in \mathfrak{X}(M)$ and smooth functions $f \in C^\infty(M)$, the following properties hold:

1. *$\nabla_X Y$ is $C^\infty(M)$-linear in X,*
2. *$\nabla_X Y$ is $\mathbb{R}$-linear in Y,*
3. *$\nabla_X(fY) = (Xf)Y + f\nabla_X Y$.*

The expression $\nabla_X Y$ is called the covariant derivative of Y with respect to X.

Levi-Civita Connection

Theorem 12. *On a pseudo-Riemannian manifold M, there exists a unique affine connection ∇ such that:*

1. *∇ is torsion-free, i.e.,*
$$[Y, Z] = \nabla_Y Z - \nabla_Z Y,$$
2. *The metric is compatible with the connection:*
$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle,$$
for all $X, Y, Z \in \mathfrak{X}(M)$.

This unique affine connection ∇ is called the Levi-Civita connection of M, and it is characterized by the Koszul formula:

$$\begin{aligned} 2\langle \nabla_Y Z, X\rangle = {} & Y\langle Z, X\rangle + Z\langle X, Y\rangle - X\langle Y, Z\rangle \\ & - \langle Y, [Z, X]\rangle + \langle Z, [X, Y]\rangle + \langle X, [Y, Z]\rangle. \end{aligned}$$

Torsion Tensor

Definition 41. *Let M be a smooth manifold, and ∇ be a connection on the tangent bundle TM, then the torsion of ∇ is a tensor field of type $(1,2)$ defined by:*

$$\begin{aligned} T : \mathfrak{X}(M) \times \mathfrak{X}(M) &\longrightarrow \mathfrak{X}(M) \\ (X,Y) &\longmapsto T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y], \end{aligned}$$

where $[.,.]$ is the Lie bracket on $\mathfrak{X}(M)$. The connection ∇ on the tangent bundle TM is said to be torsion-free if the corresponding torsion T vanishes i.e.:

$$[X,Y] = \nabla_X Y - \nabla_Y X, \ \ \forall \ X,Y \in \mathfrak{X}(M).$$

Remark 21. $T(X,Y) = -T(Y,X)$*, for all* $X,Y \in \mathfrak{X}(M)$ *(T is antisymmetric).*

Geodesics

Definition 42. *A geodesic on a smooth manifold M with an affine connection ∇ is defined as a curve $\gamma(t)$ such that parallel transport along the curve preserves the tangent vector to the curve. This condition can be expressed by the equation*

$$\nabla_{\dot{\gamma}} \dot{\gamma} = 0, \tag{1.3}$$

at each point along the curve, where $\dot{\gamma}$ denotes the derivative of γ with respect to t.

More precisely, to define the covariant derivative $\nabla_{\dot{\gamma}}\dot{\gamma}$, it is necessary to first extend $\dot{\gamma}$ to a continuously differentiable vector field in an open set containing the image of γ. However, the resulting value of (1.3) *is independent of the choice of extension.*

Using local coordinates (x^μ) on M, the geodesic equation becomes

$$\frac{d^2\gamma^\lambda}{dt^2} + \Gamma^\lambda_{\mu\nu} \frac{d\gamma^\mu}{dt} \frac{d\gamma^\nu}{dt} = 0, \tag{1.4}$$

where $\gamma^\mu(t) = x^\mu \circ \gamma(t)$ are the coordinate functions of the curve $\gamma(t)$, and $\Gamma^\lambda_{\mu\nu}$ are the Christoffel symbols of the connection ∇.

This is a second-order ordinary differential equation for the coordinate functions of $\gamma(t)$, and it has a unique solution given an initial position and initial velocity. Therefore, from the point of view of classical mechanics, geodesics can be interpreted as the trajectories of free particles moving in the manifold.

1.3.3 Curvatures

Riemann Curvature Tensor

Gauss' *"theorema egregium"* shows that the Gauss curvature, defined as the product of the two principal curvatures of a surface in Euclidean 3-space $\mathbb{E}^3$, is an **isometric invariant** of the surface itself. This remarkable result led G. Riemann to the development of **Riemannian geometry**, whose most important feature is the generalization of Gauss curvature to arbitrary Riemannian manifolds. No significant modifications are needed when extending this theory from Riemannian to pseudo-Riemannian manifolds.

For a pseudo-Riemannian manifold M with Levi-Civita connection ∇, the function

$$R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

is a $(1,3)$-tensor field, called the Riemann curvature tensor. Sometimes, we also write

$$R(X,Y;Z,W) = \langle R(X,Y)Z, W \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the metric tensor on M.

Proposition 26. *Let (M,g) be a semi-Riemannian manifold. For all $X,Y,Z,W \in \mathfrak{X}(M)$ we have:*

1. $R(X,Y)Z = -R(Y,X)Z$ *(antisymmetric).*
2. $g(R(X,Y)Z, W) = -g(R(X,Y)W, Z)$.
3. $g(R(X,Y)Z, W) = g(R(Z,W)X, Y)$.
4. *R verified Bianchi's identity algebraic:*

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0.$$

5. *R verified Bianchi's identity differential:*

$$(\nabla_X R)(Y,Z) + (\nabla_Y R)(Z,X) + (\nabla_Z R)(X,Y) = 0.$$

Sectional Curvature

Definition 43. *Let (M,g) be a pseudo-Riemannian manifold and $\pi \subset T_pM$ be a 2-dimensional subspace spanned by two linearly independent vectors $X,Y \in T_pM$. The sectional curvature $K(\pi)$ at the plane π is defined by:*

$$K(\pi) = \frac{g(R(X,Y)Y,X)}{g(X,X)g(Y,Y) - g(X,Y)^2},$$

where R is the Riemann curvature tensor and g is the metric tensor.

Ricci Curvature

Definition 44. *Let (M,g) be a pseudo-Riemannian manifold with Riemann curvature tensor R. The Ricci curvature is the symmetric (0,2)-tensor* Ric *defined by taking the trace of the Riemann curvature tensor over the first and third indices:*

$$Ric(X,Y) = \mathrm{tr}(Z \mapsto R(Z,X)Y),$$

for all vector fields $X,Y \in \mathfrak{X}(M)$, or in local coordinates:

$$Ric_{ij} = R^k_{\ ikj}.$$

1.3.4 Minkowski Space $\mathbb{R}^3_1$

Definition 45. *The space $\mathbb{R}^3_1$ is defined as a space to be the usual three-dimensional $\mathbb{R}-$vector space consisting of vectors $\{(x_1, x_2, x_3)/x_1, x_2, x_3 \in \mathbb{R}\}$, but endowed with the inner product*

$$< \xi, \zeta >_{\mathbb{R}^3_1} = -\xi_1\zeta_1 + \xi_2\zeta_2 + \xi_3\zeta_3.$$

This space is called the Minkowski space or Lorentz space.

Tangent vectors are defined precisely as in the case of Euclidean space $\mathbb{R}^3$. A vector ξ is said to be :
• space-like, if $< \xi, \xi >_{\mathbb{R}^3_1} > 0$,
• time-like, if $< \xi, \xi >_{\mathbb{R}^3_1} < 0$,
• light-like or isotropic or a null vector, if $< \xi, \xi >_{\mathbb{R}^3_1} = 0$, but $\xi \neq 0$.

Lemma 1. *A regular surface element is defined as an immersion $X : U \to \mathbb{R}^3_1$, exactly as in $\mathbb{R}^3$. A regular surface element $X : U \to \mathbb{R}^3_1$ is called:*
• *space-like, in case the first fundamental form is positive definite, and if and only if, at every point $p = X(u)$ there is a time-like vector $\xi \neq 0$ which is perpendicular, with respect to the inner product $< .,. >_{\mathbb{R}^3_1}$ in Minkowski space, to the tangent plane of the surface at the point p.*
• *time-like, in case the first fundamental is indefinite, and if and only if, at every point $p = X(u)$ there is a space-like vector $\xi \neq 0$ which is perpendicular, with respect to the inner product $< .,. >_{\mathbb{R}^3_1}$ in Minkowski space, to the tangent plane of the surface at the point p.*
• *isotropic, in case the first fundamental form has rank 1, and if and only if, at every point $p = X(u)$ there is a isotropic vector $\xi \neq 0$ which is perpendicular, with respect to the inner product $< .,. >_{\mathbb{R}^3_1}$ in Minkowski space, to the tangent plane of the surface at the point p.*

Translation Surface in a Minkowski 3-Space

Definition 46. *A translation surface in a Minkowski 3-space is a surface that is parametrized by either*
• *$X(s, t) = (s, t, f(s) + g(t))$, if L is timelike,*
• *$X(s, t) = (f(s) + g(t), s, t)$, if L is spacelike,*
• *$X(s, t) = (s + t, g(t), f(s) + t)$, if L is lightlike,*
with the intersection L of the two planes that contain the curves that generate the surface.

Chapter 2

Manifolds with Density

In this chapter, we introduce the concept of *manifolds with density*, which generalize classical manifolds by incorporating a smooth weight function that influences geometric and analytic quantities. We define the main notions, study the corresponding curvature formulas, and examine *Plateau's problem in* $\mathbb{R}^3$ *with density.* Finally, we present the divergence operator adapted to this setting. These tools will be useful for the analysis of geometric and variational problems in weighted manifolds.

2.1 Manifold with Density

Definition 47. *A manifold with density is a Riemannian manifold M^n with a positive density function $\Psi(x)$ used to weight volume and hyperarea (and sometimes lower-dimensional area and length). In terms of the underlying Riemannian volume dV_0 and area dA_0, the new weighted volume and area are given by:*

$$dV = \Psi\, dV_0, \qquad \textit{and } dA = \Psi\, dA_0.$$

Such a density is not equivalent to scaling the metric conformally by a factor $\lambda(x)$ since in that case volume and area would scale by different powers of λ.

Manifolds with density, a special case of the "mm spaces" of Gromov or the earlier "spaces of homogeneous type" (see [Coifman and Weiss, pp. 587, 591]), long have arisen on an ad hoc basis in mathematics. Quotients of Riemannian manifolds are manifolds with density.

Example 38. $\mathbb{R}^3$ *modulo rotation about the z-axis is the half-plane:*

$$H = \{(x, z) : x \geq 0\}.$$

2.2 Curvatures in Manifolds with Density

Let M^n be a smooth Riemannian manifold equipped with a smooth positive density e^{ψ}. Several notions of curvature can be extended naturally to this setting. We summarize the generalizations of Ricci curvature, Gauss curvature, and mean curvature below.

2.2.1 Generalized Ricci Curvature

In general dimensions, a useful and elegant extension of Ricci curvature to manifolds with density is given by the formula:

$$Ric_\psi = Ric - Hess\psi,$$

where Ric is the usual Ricci curvature tensor, and Hess ψ is the Hessian of the density function ψ. This definition, attributed to Lichnerowicz, Bakry-Émery, and Bakry-Ledoux, plays a central role in the generalization of comparison theorems such as the Lévy-Gromov inequality. It also appears in second variation formulas and Perelman's work on the Poincaré Conjecture.

In particular, for Gaussian space $\mathbb{G}^n(1)$ with density $\Phi = e^\varphi$, this curvature simplifies to:

$$Ric_\varphi = 0 - Hess\varphi = I.$$

Note that, unlike in classical Riemannian geometry, in two dimensions the generalized Gauss curvature defined below is not necessarily half the trace of Ric_ψ.

2.2.2 Generalized Gauss Curvature

For a two-dimensional manifold with density e^ψ, Corwin et al. define the generalized Gauss curvature as:

$$K_\psi = G - \Delta\psi,$$

where G is the classical Gauss curvature. This modification leads to a weighted version of the Gauss-Bonnet Theorem. For a smooth disc R with boundary ∂R, the generalized Gauss-Bonnet formula becomes:

$$\int_R K_\psi + \int_{\partial R} \kappa_\psi = 2\pi,$$

where κ_ψ is the inward one-dimensional generalized mean curvature.

Examples of manifolds with $G_\psi = 0$ (e.g., certain punctured planes with density) have been studied in detail by Carroll et al. Other generalizations of Gauss curvature, involving terms like $|\nabla\psi|^2$, are also used to derive asymptotic formulas for areas and perimeters of small geodesic discs (see Corwin et al., Propositions 5.8 and 5.9).

2.2.3 Generalized Mean Curvature

For a smooth hypersurface in M^n, the generalized mean curvature H_ψ reflects how the surface evolves with respect to the weighted area. It is defined as:

$$H_\psi = H - \frac{1}{n-1}\frac{d\psi}{dn},$$

where H is the classical mean curvature, and $\dfrac{d\psi}{dn}$ is the derivative of ψ in the direction of the unit normal vector to the surface.

This quantity arises in the first variation formula for area and is constant for any surface that minimizes perimeter under a fixed volume constraint. This is because any variation that decreases H_ψ locally would reduce the weighted perimeter while preserving volume.

As in classical differential geometry, the quantity $-(n-1)H_\psi$ can be interpreted as the derivative $\dfrac{dP}{dV}$.

2.3 Plateau's Problem in $\mathbb{R}^3$ with Density

We shall now study a family of surface S_t parameterized by $X_t : U \to \mathbb{R}^3$ in $\mathbb{R}^3$ with density e^{ϕ_t}, where U is a subset of $\mathbb{R}^2$ independent of t, and t lies in some open interval $(-\varepsilon, \varepsilon)$, for some $\varepsilon > 0$.

Let $S = S_0$ and $e^{\phi_0} = e^{\phi}$. The family is required to be smooth, in the sense that the map $(u, v, t) \mapsto X_t(u, v)$ from the open subset $\{(u, v, t) \,|\, (u, v) \in U,\, t \in (-\varepsilon, \varepsilon)\}$ of $\mathbb{R}^3$ to $\mathbb{R}^3$ is smooth. The surface variation of the family is the function $\eta : U \to \mathbb{R}^3$ given by

$$\eta = \left.\frac{\partial X_t}{\partial t}\right|_{t=0}.$$

Let γ be a simple closed curve that is contained, along with its interior $\operatorname{int}(\gamma)$, in U. Then γ corresponds to a closed curve $\gamma_t = X_t \circ \gamma$ in the surface S_t, and we define the ϕ_t-area function $A_{\phi_t}(t)$ in $\mathbb{R}^3$ with density e^{ϕ_t} to be the area of the surface S_t inside γ_t in $\mathbb{R}^3$ with density e^{ϕ_t}:

$$A_{\phi_t}(t) = \int_{\operatorname{int}(\gamma)} e^{\phi_t}\, dA_{X_t}.$$

Theorem 13. *With the above notation, assume that the surface variation η_t vanishes along the boundary curve γ. Then,*

$$\left.\frac{\partial A_{\phi_t}(t)}{\partial t}\right|_{t=0} = A'_\phi(0) = -2\int_{int(\gamma)} H_\phi \cdot \eta \cdot N \cdot e^{\phi} \cdot (EG - F^2)^{\frac{1}{2}}\, du\, dv, \tag{2.1}$$

where $H_\phi = H - \frac{1}{2}\nabla\phi \cdot N$ is the ϕ-mean curvature of S in $\mathbb{R}^3$ with density e^ϕ, E, F and G are the coefficients of its first fundamental form, and N is the standard unit normal of S.

Proof. Let $\eta_t = \dfrac{\partial X_t}{\partial t}$, so that $\eta_0 = \eta$, and let N_t be the standard unit normal of S_t.

There are smooth functions α_t, β_t and δ_t of (u, v, t) such that

$$\eta_t = \alpha_t N_t + \beta_t X_{t,u} + \delta_t X_{t,v},$$

so that $\alpha = \alpha_0 = \eta \cdot N$. We have

$$A_{\phi_t}(t) = \int_{\operatorname{int}(\gamma)} e^{\phi_t} \|X_{t,u} \wedge X_{t,v}\|\, du\, dv = \int_{\operatorname{int}(\gamma)} e^{\phi_t} N_t \cdot (X_{t,u} \wedge X_{t,v})\, du\, dv,$$

so

$$\frac{\partial A_{\phi_t}(t)}{\partial t} = \int_{\operatorname{int}(\gamma)} \frac{\partial}{\partial t}\left(e^{\phi_t} N_t \cdot (X_{t,u} \wedge X_{t,v})\right) du\, dv. \tag{2.2}$$

Now,

$$\frac{\partial}{\partial t}\left(e^{\phi_t} N_t \cdot (X_{t,u} \wedge X_{t,v})\right) = \frac{\partial e^{\phi_t}}{\partial t} \cdot N_t \cdot (X_{t,u} \wedge X_{t,v}) + \frac{\partial N_t}{\partial t} \cdot e^{\phi_t} \cdot (X_{t,u} \wedge X_{t,v})$$

$$+e^{\phi_t} N_t \cdot \left(\frac{\partial X_{t,u}}{\partial t} \wedge X_{t,v} \right) + e^{\phi_t} N_t \cdot \left(X_{t,u} \wedge \frac{\partial X_{t,v}}{\partial t} \right).$$

Since N is the unit vector,

$$e^{\phi_t} \cdot \frac{\partial N_t}{\partial t} \cdot (X_{t,u} \wedge X_{t,v}) = e^{\phi_t} \cdot \frac{\partial N_t}{\partial t} \cdot N_t \cdot \|X_{t,u} \wedge X_{t,v}\| = 0.$$

On the other hand,

$$e^{\phi_t} \cdot N_t \cdot \left(\frac{\partial X_{t,u}}{\partial t} \wedge X_{t,v} \right) = e^{\phi_t} \cdot \frac{(X_{t,u} \wedge X_{t,v}) \cdot (\frac{\partial X_{t,u}}{\partial t} \wedge X_{t,v})}{\|X_{t,u} \wedge X_{t,v}\|},$$

$$= e^{\phi_t} \cdot \frac{(X_{t,u} \cdot \frac{\partial X_{t,u}}{\partial t})(X_{t,v} \cdot X_{t,v}) - (X_{t,u} \cdot X_{t,v})(X_{t,v} \cdot \frac{\partial X_{t,u}}{\partial t})}{\|X_{t,u} \wedge X_{t,v}\|},$$

$$= e^{\phi_t} \cdot \frac{G_t(X_{t,u} \cdot \frac{\partial X_{t,u}}{\partial t}) - F_t(X_{t,v} \cdot \frac{\partial X_{t,u}}{\partial t})}{(E_t G_t - (F_t)^2)^{1/2}}.$$

Similarly,

$$e^{\phi_t} \cdot N_t \cdot \left(X_{t,u} \wedge \frac{\partial X_{t,v}}{\partial t} \right) = e^{\phi_t} \cdot \frac{E_t(X_{t,v} \cdot \frac{\partial X_{t,v}}{\partial t}) - F_t(X_{t,u} \cdot \frac{\partial X_{t,v}}{\partial t})}{(E_t G_t - (F_t)^2)^{1/2}}.$$

Using the definition of the function ϕ,

$$\frac{\partial e^{\phi_t}}{\partial t} = \frac{\partial e^{\phi(X(u,v),t)}}{\partial t} = \eta_t \cdot \nabla \phi_t \cdot e^{\phi(X(u,v),t)}.$$

Substituting these results into (2.1), we get

$$\begin{aligned} \frac{\partial}{\partial t} \left(e^{\phi_t} N_t \cdot (X_{t,u} \wedge X_{t,v}) \right) &= \eta_t \cdot \nabla \phi_t \cdot e^{\phi_t} \cdot N_t \cdot (X_{t,u} \wedge X_{t,v}) \\ &\quad + e^{\phi_t} \cdot \frac{E_t(X_{t,v} \cdot \frac{\partial X_{t,v}}{\partial t}) - F_t(X_{t,u} \cdot \frac{\partial X_{t,v}}{\partial t})}{(E_t G_t - (F_t)^2)^{1/2}} \\ &\quad + e^{\phi_t} \cdot \frac{-F_t(X_{t,v} \cdot \frac{\partial X_{t,u}}{\partial t}) + G_t(X_{t,u} \cdot \frac{\partial X_{t,u}}{\partial t})}{(E_t G_t - (F_t)^2)^{1/2}}. \end{aligned} \tag{2.3}$$

Now,

$$\frac{\partial X_{t,u}}{\partial t} = \frac{\partial \eta_t}{\partial u} = \alpha_{t,u} N_t + \beta_{t,u} X_{t,u} + \delta_{t,u} X_{t,v} + \alpha_t \frac{\partial N_t}{\partial u} + \beta_t \frac{\partial X_{t,u}}{\partial u} + \delta_t \frac{\partial X_{t,v}}{\partial u}.$$

Therefore,

$$X_{t,u} \cdot \frac{\partial X_{t,u}}{\partial t} = E_t \beta_{t,u} + F_t \delta_{t,u} + (X_{t,u} \cdot \frac{\partial N_t}{\partial u})\alpha_t + (X_{t,u} \cdot X_{t,uu})\beta_t + (X_{t,u} \cdot X_{t,vu})\delta_t.$$

Since $X_{t,u} \cdot \frac{\partial N_t}{\partial u} = -X_{t,uu} \cdot N_t = -L_t$, $X_{t,u} \cdot X_{t,uu} = \frac{1}{2} E_{t,u}$, and $X_{t,u} \cdot X_{t,uv} = \frac{1}{2} E_{t,v}$, we get

$$X_{t,u} \cdot \frac{\partial X_{t,u}}{\partial t} = E_t \beta_{t,u} + F_t \delta_{t,u} - L_t \alpha_t + \frac{1}{2} E_{t,u} \beta_t + \frac{1}{2} E_{t,v} \delta_t.$$

Similarly,

$$X_{t,v}\cdot\frac{\partial X_{t,u}}{\partial t}=F_t\beta_{t,u}+G_t\delta_{t,u}-M_t\alpha_t+(F_{t,u}-\tfrac{1}{2}E_{t,v})\beta_t+\tfrac{1}{2}G_{t,u}\delta_t.$$

$$X_{t,u}\cdot\frac{\partial X_{t,v}}{\partial t}=E_t\beta_{t,v}+F_t\delta_{t,v}-M_t\alpha_t+\tfrac{1}{2}E_{t,v}\beta_t+(F_{t,v}-\tfrac{1}{2}G_{t,u})\delta_t.$$

$$X_{t,v}\cdot\frac{\partial X_{t,v}}{\partial t}=F_t\beta_{t,v}+G_t\delta_{t,v}-N_t\alpha_t+\tfrac{1}{2}G_{t,u}\beta_t+\tfrac{1}{2}G_{t,v}\delta_t.$$

Substituting these last four equations into the right-hand side of (2.3), simplifying, and using the formula for the mean curvature H_t of S, we find that

$$\frac{\partial}{\partial t}\left(e^{\phi_t}N_t\cdot(X_{t,u}\wedge X_{t,v})\right)=\eta_t\cdot\nabla\phi_t\cdot e^{\phi_t}\cdot N_t\cdot(X_{t,u}\wedge X_{t,v})$$

$$+\left[\beta_t\cdot e^{\phi_t}\cdot(E_tG_t-(F_t)^2)^{1/2}\right]_u+\left[\delta_t\cdot e^{\phi_t}\cdot(E_tG_t-(F_t)^2)^{1/2}\right]_v-2\alpha_tH_te^{\phi_t}(E_tG_t-(F_t)^2)^{1/2}.$$

And if $t=0$, we have

$$\frac{\partial}{\partial t}\left(e^{\phi_t}N_t\cdot(X_{t,u}\wedge X_{t,v})\right)\bigg|_{t=0}=\eta\cdot\nabla\phi\cdot e^{\phi}(EG-F^2)^{1/2}$$

$$+\left[\beta_0\cdot e^{\phi}(EG-F^2)^{1/2}\right]_u+\left[\delta_0\cdot e^{\phi}(EG-F^2)^{1/2}\right]_v.$$

$$-2\,\eta\cdot N\cdot H\cdot e^{\phi}(EG-F^2)^{1/2}=\left[\beta_0\cdot e^{\phi}(EG-F^2)^{1/2}\right]_u+\left[\delta_0\cdot e^{\phi}(EG-F^2)^{1/2}\right]_v$$

$$-2\left(H-\tfrac{1}{2}\nabla\phi\cdot N\right)\cdot\eta\cdot N\cdot e^{\phi}(EG-F^2)^{1/2}.$$

Comparing with (2.2), we see that we must prove that

$$\int_{\mathrm{int}(\gamma)}\left[\left(\beta_0\cdot e^{\phi}(EG-F^2)^{1/2}\right)_u+\left(\delta_0\cdot e^{\phi}(EG-F^2)^{1/2}\right)_v\right]\,du\,dv=0.$$

But by Green's theorem, this integral is equal to

$$\int_\gamma e^{\phi}(EG-F^2)^{1/2}(\beta_0\,dv-\delta_0\,du),$$

and this obviously vanishes because $\beta_0=\delta_0=0$ along the boundary curve γ. □

Definition 48. *A minimal surface in $\mathbb{R}^3$ with density e^ϕ is a surface whose ϕ–mean curvature is zero everywhere.*

Example 39. *The surface S in $\mathbb{R}^3$ with linear density e^x defined by the parametrization:*

$$X:(x,y)\to\left(x,\ y,\ -\frac{a^2}{\sqrt{1+a^2}}\arcsin\left(\beta e^{-\frac{1+a^2}{a^2}x}\right)+ay+b+\gamma\right),$$

where $(x,y)\in\mathbb{R}^2$, and $a,b,\beta\in\mathbb{R}^*$ is minimal.

2.4 The Divergence Operator in Manifolds with Density

Let $(\mathcal{M}^n, g)$ be a Riemannian manifold equipped with the Riemannian metric g. For any smooth function f on $\mathcal{M}$, the gradient ∇f is a vector field on $\mathcal{M}$, which in local coordinates $x_1, x_2, \ldots, x_n$ has the form

$$(\nabla f)^i = g^{ij} \frac{\partial f}{\partial x_j},$$

where summation is assumed over repeated indices. For any smooth vector field F on $\mathcal{M}$, the divergence $\operatorname{div} F$ is a scalar function on $\mathbb{R}$, which is given in local coordinates by

$$\operatorname{div} F = \frac{1}{\det g_{ij}} \frac{\partial}{\partial x_i} \left(\det g_{ij} F^i \right).$$

Let ν be the Riemannian volume on $\mathcal{M}$, that is

$$\nu = \det g_{ij} \, dx_1 \ldots dx_n.$$

By the divergence theorem, for any smooth function f and a smooth vector field F, such that either f or F has compact support,

$$\int_{\mathcal{M}} f \, \operatorname{div} F \, d\nu = - \int_{\mathcal{M}} \langle \nabla f, F \rangle \, d\nu.$$

where $\langle \cdot, \cdot \rangle = g(\cdot, \cdot)$. In particular, if $F = \nabla \psi$ for a function ψ then we obtain

$$\int_{\mathcal{M}} f \, \operatorname{div} \nabla \psi \, d\nu = - \int_{\mathcal{M}} \langle \nabla f, \nabla \psi \rangle \, d\nu. \tag{2.4}$$

provided one of the functions f, ψ has compact support. The operator

$$\Delta := \operatorname{div} \circ \nabla$$

is called the Laplace (or Laplace-Beltrami) operator of the Riemannian manifold $\mathcal{M}$. From (2.4) we obtain the Green formulas

$$\int_{\mathcal{M}} f \Delta \psi \, d\nu = - \int_{\mathcal{M}} \langle \nabla f, \nabla \psi \rangle \, d\nu = \int_{\mathcal{M}} \psi \Delta f \, d\nu.$$

Let now μ be another measure on $\mathcal{M}$ defined by

$$d\mu = e^{\varphi} d\nu.$$

where φ is a smooth function on $\mathcal{M}$. A triple $(\mathcal{M}^n, g, \mu)$ is called a weighted manifold or manifold with density. The associative divergence div_μ is defined by

$$\operatorname{div}_\mu F = \frac{1}{e^{\varphi} \det g_{ij}} \frac{\partial}{\partial x_i} \left(e^{\varphi} \det g_{ij} F^i \right),$$

and the Laplace-Beltrami operator Δ_μ of $(\mathcal{M}^n, g, \mu)$ is defined by

$$\Delta_\mu \cdot := \operatorname{div}_\mu \circ \nabla \cdot = \frac{1}{e^{\varphi}} \operatorname{div} \left(e^{\varphi} \nabla \cdot \right) = \Delta \cdot + \nabla \varphi \nabla \cdot \, .$$

It is easy to see that the Green formulas hold with respect to the measure μ, that is,

$$\int_{\mathcal{M}} f \Delta_\mu \psi \, d\mu = -\int_{\mathcal{M}} \langle \nabla f, \nabla \psi \rangle \, d\mu = \int_{\mathcal{M}} \psi \Delta_\mu f \, d\mu.$$

provided f or ψ belongs to $C_0^\infty(\mathcal{M})$.

Theorem 14. *Let S be a surface in $\mathcal{M}^3$ with density $\Psi = e^\varphi$, we have*

$$\operatorname{div}_\varphi N = -2H_\varphi.$$

where H_φ is the φ-mean curvature of a surface S and N is the unit normal vector field of a surface S.

Proof. By definition we have

$$\begin{aligned}
\operatorname{div}_\varphi N &= \frac{1}{e^\varphi} \operatorname{div}(e^\varphi N) + \langle \nabla \varphi, N \rangle \\
&= \operatorname{div} N + N \cdot \nabla \varphi \\
&= \sum_{i=1}^{2} \langle \nabla_{e_i} N, e_i \rangle + N \cdot \nabla \varphi \\
&= \sum_{i=1}^{2} \big(\nabla_{e_i} \langle e_i, N \rangle - \langle \nabla_{e_i} e_i, N \rangle \big) + N \cdot \nabla \varphi \\
&= -2 \langle HN, N \rangle + N \cdot \nabla \varphi \\
&= -2 \left(H - \frac{1}{2} \nabla \varphi \cdot N \right) \\
&= -2H_\varphi,
\end{aligned}$$

where we have used that $\langle e_i, N \rangle = 0$ and the definition of the mean curvature vector. $\square$

Chapter 3

Weighted Flat Translation Surfaces in Minkowski 3-Space with Density

In this chapter we gives a classifications of all weighted flat translation surfaces in Minkowski space with radial density $e^{-a(x^2+y^2+z^2)+c}$, where $a > 0$ and $c \in \mathbb{R}$.

3.1 Weighted Flat Timelike Translation Surfaces in Minkowski 3-Space with Density

In this section we study weighted flat timelike translation surfaces Σ in Minkowski 3-space $\mathbb{R}^3_1$ which are parameterized by

$$X(s,t) = (s, t, f(s) + g(t)),\ (s,t) \in \mathbb{R}^2,$$

where f and g are reals functions $\mathcal{C}^2(\mathbb{R})$, have an orthogonal pair of vector fields on (Σ), namely,

$$e_1 := X_s = (1, 0, f'(s)).$$

and

$$e_2 := X_t = (0, 1, g'(t)).$$

The coefficients of the first fundamental form are:

$$E =< e_1, e_1 >_{\mathbb{R}^3_1}= 1 - f'^2,\ F =< e_1, e_2 >_{\mathbb{R}^3_1}= -f'g',\ G =< e_2, e_2 >_{\mathbb{R}^3_1}= 1 - g'^2.$$

As a unit normal field we can take

$$N = \frac{-1}{\sqrt{|-1 + f'^2 + g'^2|}}(f', g', 1)$$

The coefficients of the second fundamental form are

$$l =< X_{ss}, N >_{\mathbb{R}^3_1}= \frac{f''}{\sqrt{|-1 + f'^2 + g'^2|}}$$

$$m =< X_{st}, N >_{\mathbb{R}^3_1}= 0$$

$$n =< X_{tt}, N >_{\mathbb{R}^3_1}= \frac{g''}{\sqrt{|-1+f'^2+g'^2|}}.$$

Let K be the Gauss curvature of Σ,

$$K = \frac{ln - m^2}{EG - F^2} = \frac{f''g''}{(-1+f'^2+g'^2)^2}. \tag{3.1}$$

The weighted Gauss curvature of Σ,

$$K_\varphi = K - \Delta\varphi, \tag{3.2}$$

where $\Delta\varphi$ is the Laplacian of the function φ in the Minkowski 3-space. We have $\varphi(x, y, z) = -a(x^2 + y^2 + z^2) + c$, thus the Laplacian of φ is

$$\Delta\varphi = \frac{1}{\sqrt{detg_{ij}}}\frac{\partial}{\partial x_i}\left(\sqrt{detg_{ij}}.(\nabla\varphi)^i\right) = -2a. \tag{3.3}$$

Then

$$K_\varphi = \frac{f''g''}{(-1+f'^2+g'^2)^2} + 2a = \frac{f''g'' + 2a(-1+f'^2+g'^2)^2}{(-1+f'^2+g'^2)^2}. \tag{3.4}$$

Thus Σ is weighted flat timelike translation surface in Minkowski 3-space with density e^φ if and only if

$$K_\varphi = 0,$$

that is, if and only if

$$f''g'' + 2a(-1+f'^2+g'^2)^2 = 0 \tag{3.5}$$

to classify weighted flat timelike translation surfaces must solve the equation (3.5)

• If $f' = \alpha \in [-1, 1]$.

We replace $f(s) = \alpha s + \alpha_1$, in (3.5) gives

$$g'^2 = 1 - \alpha^2,$$

and we get $g(t) = \pm\sqrt{1-\alpha^2}t + \alpha_2$. In this case Σ is a timelike plane.

• If $g' = \beta \in [-1, 1]$.

We replace $g(t) = \beta t + \beta_1$, in (3.5) become

$$f'^2 = 1 - \beta^2,$$

and $f(s) = \pm\sqrt{1-\beta^2}s + \beta_2$. In this case Σ is a timelike plane.

• If f' and g' are not constants smooth functions. In this case, we take derivation of the equation (3.5) by s and t respectively,

$$f'''g''' + 16af'f''g'g'' = 0. \tag{3.6}$$

We can writes equation (3.6) as

$$\frac{f'''}{f'f''} = -\frac{16ag'g''}{g'''} = \lambda, \tag{3.7}$$

where λ is real constant. Solving the equation (3.7) with respect to the variable s, first integration gives

$$f'' = \frac{\lambda}{2} f'^2 + \beta, \tag{3.8}$$

where β is a real constant.

- Now if $\beta = 0$, then

$$f'' = \frac{\lambda}{2} (f')^2 \Rightarrow \frac{f''}{(f')^2} = \frac{\lambda}{2}.$$

Observe that

$$\frac{d}{ds}\left(-\frac{1}{f'}\right) = \frac{f''}{(f')^2},$$

so

$$\frac{d}{ds}\left(-\frac{1}{f'}\right) = \frac{\lambda}{2}.$$

Integrating both sides gives

$$\int \frac{d}{ds}\left(-\frac{1}{f'}\right) ds = \frac{\lambda}{2}\int ds \Rightarrow -\frac{1}{f'} = \frac{\lambda}{2}s + \alpha \Rightarrow f' = -\frac{1}{\frac{\lambda}{2}s+\alpha}.$$

Integrating again,

$$\int f' ds = -\int \frac{1}{\frac{\lambda}{2}s+\alpha} ds = -\frac{2}{\lambda}\int \frac{1}{\frac{\lambda}{2}s+\alpha} d\left(\frac{\lambda}{2}s+\alpha\right)$$

finally, the solutions of equation (3.8) are

$$f(s) = -\frac{2}{\lambda}\ln\left|\frac{\lambda}{2}s+\alpha\right| + \alpha_1. \tag{3.9}$$

Replacing the function f given in (3.9) in the equation (3.5) gives

$$\frac{a\lambda^4}{8}(g'^2-1)^2 s^4 - \frac{a\lambda^3\alpha_1}{4}(g'^2-1)^2 s^3 + \left[\frac{a\lambda^2\alpha_1^2}{2}(g'^2-1)^2 + \frac{\lambda^2}{4}(\frac{\lambda}{2}g'' + 4a(g'^2-1))\right] s^2$$
$$-\left[a\lambda\alpha_1^3(g'^2-1)^2 + \lambda\alpha_1^2(\frac{\lambda}{2}g'' + 4a(g'^2-1))\right] s + \left[2a\alpha_1^4(g'^2-1)^2 + \alpha_1^2(\frac{\lambda}{2}g'' + 4a(g'^2-1)) + 2a\right] = 0. \tag{3.10}$$

The equation (3.10) is a polynomial on the variable s and thus the coefficients must vanish. It follows that the function g satisfies :

$$\begin{cases} g'^2 - 1 = 0, \\ \frac{a\lambda^2\alpha_1^2}{2}(g'^2-1)^2 + \frac{\lambda^2}{4}(\frac{\lambda}{2}g'' + 4a(g'^2-1)) = 0, \\ a\lambda\alpha_1^3(g'^2-1)^2 + \lambda\alpha_1^2(\frac{\lambda}{2}g'' + 4a(g'^2-1)) = 0, \\ 2a\alpha_1^4(g'^2-1)^2 + \alpha_1^2(\frac{\lambda}{2}g'' + 4a(g'^2-1)) + 2a = 0. \end{cases} \tag{3.11}$$

Thus, $g' = \pm 1$ and it is a contradiction.

• In the case that $\beta \neq 0$, we integrate the equation (3.8) with respect to s, and we get

$$f(s) = \begin{cases} \frac{-2}{\lambda} \ln|\cos\left(\frac{\lambda}{2}\sqrt{\frac{2\beta}{\lambda}}s\right) + \beta_1\sqrt{\frac{2\beta}{\lambda}}| + \beta_2, if \frac{\beta}{\lambda} > o, \\ \sqrt{\frac{-2\beta}{\lambda}}s - \frac{2}{\lambda}\ln|1 - e^{\lambda\sqrt{\frac{-2\beta}{\lambda}}s+\beta_1}| + \beta_2, if \frac{\beta}{\lambda} < 0 \end{cases} \tag{3.12}$$

By replacing f in the equation (3.5), we have:

If $\frac{\beta}{\lambda} > 0$

$$\frac{8a\beta^2}{\lambda^2}\tan^4\left(\frac{\lambda}{2}\sqrt{\frac{2\beta}{\lambda}}s + \beta_1\sqrt{\frac{2\beta}{\lambda}}\right) + \left[\beta g'' + \frac{4a\beta}{\lambda}(g'^2 - 1)\right] tan^2\left(\frac{\lambda}{2}\sqrt{\frac{2\beta}{\lambda}}s + \beta_1\sqrt{\frac{2\beta}{\lambda}}\right) + \left[2a(g'^2 - 1)^2 + g''\right] = 0 \tag{3.13}$$

The equation (3.13) is a polynomial on the function $\tan\left(\frac{\lambda}{2}\sqrt{\frac{2\beta}{\lambda}}s + \beta_1\sqrt{\frac{2\beta}{\lambda}}\right)$ and thus the coefficients must vanish. It follows that the function g satisfies :

$$\begin{cases} \frac{8a\beta^2}{\lambda^2} = 0, \\ \beta g'' + \frac{4a\beta}{\lambda}(g'^2 - 1) = 0, \\ 2a(g'^2 - 1)^2 + g'' = 0. \end{cases} \tag{3.14}$$

Thus, $\beta = 0$, $g' = \pm 1$ and it is a contradiction.

And if $\frac{\beta}{\lambda} < 0$, we have

$$2a\left[1 - g'^2 + \frac{2\beta}{\lambda}\right]^2 e^{4\lambda\sqrt{\frac{-2\beta}{\lambda}}s+4\beta_1} + \left[-\beta g'' - 8a(1 - g'^2)^2 + \frac{32a\beta^2}{\lambda^2}\right] e^{3\lambda\sqrt{\frac{-2\beta}{\lambda}}s+3\beta_1} + \left[2\beta g'' - \frac{-16a\beta}{\lambda}(1 - g'^2) + 12a(1 - g'^2)^2 + \frac{48a\beta^2}{\lambda^2}\right] e^{2\lambda\sqrt{\frac{-2\beta}{\lambda}}s+2\beta_1} + \left[-\beta g'' - 8a(1 - g'^2)^2 + \frac{32a\beta^2}{\lambda^2}\right] e^{\lambda\sqrt{\frac{-2\beta}{\lambda}}s+\beta_1} + 2a\left[1 - g'^2 + \frac{2\beta}{\lambda}\right]^2 = 0 \tag{3.15}$$

The equation (3.15) is a polynomial on the function $e^{\lambda\sqrt{\frac{-2\beta}{\lambda}}s+\beta_1}$ and thus the coefficients must vanish. It follows that the function g satisfies :

$$\begin{cases} 1 - g'^2 + \frac{2\beta}{\lambda} = 0, \\ -\beta g'' - 8a(1 - g'^2)^2 + \frac{32a\beta^2}{\lambda^2} = 0, \\ 2\beta g'' - \frac{-16a\beta}{\lambda}(1 - g'^2) + 12a(1 - g'^2)^2 + \frac{48a\beta^2}{\lambda^2} = 0. \end{cases} \tag{3.16}$$

Thus, $\beta = 0$, $g' = \pm 1$ and it is a contradiction.

Thus we have the following theorem.

Theorem 15. *([14]) Let Σ be a timelike translation surface in Minkowski 3-space with density $e^{-a(x^2+y^2+z^2)+c}$, parametrized by*

$$X(s,t) = (s, t, f(s) + g(t)),\ (s,t) \in \mathbb{R}^2,$$

Then Σ is weighted flat timelike translation surface in Minkowski 3-space with density $e^{-a(x^2+y^2+z^2)+c}$ if and only if

- $X(s,t) = (s, t, \alpha s \pm \sqrt{1-\alpha^2}t + \alpha_1),\ \alpha \in [-1,1],\ \alpha_1 \in \mathbb{R},$ *or*
- $X(s,t) = (s, t, \beta t \pm \sqrt{1-\beta^2}s + \beta_1),\ \beta \in [-1,1],\ \beta_1 \in \mathbb{R}.$

3.2 Weighted Flat Spacelike Translation Surfaces in Minkowski 3-Space with Density

In this section we study weighted flat spacelike translation surfaces Σ in Minkowski 3-space $\mathbb{R}^3_1$ which are parameterized by

$$X(s,t) = (f(s) + g(t), s, t),\ (s,t) \in \mathbb{R}^2,$$

where f and g are reals functions $\mathcal{C}^2(\mathbb{R})$, have an orthogonal pair of vector fields on (Σ), namely,

$$e_1 := X_s = (f'(s), 1, 0).$$

and

$$e_2 := X_t = (g'(t), 0, 1).$$

The coefficients of the first fundamental form are:

$$E =< e_1, e_1 >_{\mathbb{R}^3_1} = 1 + f'^2,\ F =< e_1, e_2 >_{\mathbb{R}^3_1} = f'g',\ G =< e_2, e_2 >_{\mathbb{R}^3_1} = -1 + g'^2.$$

As a unit normal field we can take

$$N = \frac{1}{\sqrt{|1 + f'^2 - g'^2|}}(1, -f', -g')$$

The coefficients of the second fundamental form are

$$l =< X_{ss}, N >_{\mathbb{R}^3_1} = \frac{f''}{\sqrt{|1 + f'^2 - g'^2|}}$$

$$m =< X_{st}, N >_{\mathbb{R}^3_1} = 0$$

$$n =< X_{tt}, N >_{\mathbb{R}^3_1} = \frac{g''}{\sqrt{|1 + f'^2 - g'^2|s}}.$$

Let K be the Gauss curvature of Σ,

$$K = \frac{ln - m^2}{EG - F^2} = \frac{f''g''}{(1 + f'^2 - g'^2)^2}. \tag{3.17}$$

According to (3.3) and (3.17)the weighted Gaussian curvature of Σ is given by

$$K_\varphi = K - \Delta\varphi = \frac{f''g''}{(1+f'^2-g'^2)^2} + 2a = \frac{f''g'' + 2a(1+f'^2-g'^2)^2}{(1+f'^2-g'^2)^2}. \tag{3.18}$$

Thus Σ is weighted flat spacelike translation surface in Minkowski 3-space with density e^φ if and only if

$$K_\varphi = 0,$$

that is, if and only if

$$f''g'' + 2a(1+f'^2-g'^2)^2 = 0 \tag{3.19}$$

to classify weighted flat spacelike translation surfaces must solve the equation (3.19).

• If $f' = \alpha \in \mathbb{R}$.
We replace $f(s) = \alpha s + \alpha_1$, in (3.19) gives

$$g'^2 = 1 - \alpha^2,$$

and we get $g(t) = \pm\sqrt{1+\alpha^2}t + \alpha_2$. In this case Σ is a spacelike plane.

• If $g' = \beta \in \left]-\infty, -1\right[\cup \left]1, +\infty\right[$.
We replace $g(t) = \beta t + \beta_1$, in (3.19) become

$$f'^2 = -1 + \beta^2,$$

and $f(s) = \pm\sqrt{-1+\beta^2}s + \beta_2$. In this case Σ is a timelike plane.
• If f' and g' are not constants smooth functions. In this case, we take derivation of the equation (3.19) by s and t respectively,

$$f'''g''' - 16af'f''g'g'' = 0. \tag{3.20}$$

We can writes equation (3.20) as

$$\frac{f'''}{f'f''} = \frac{16ag'g''}{g'''} = \lambda, \tag{3.21}$$

where λ is real constant. Solving the equation (3.21) with respect to the variable s, according to (3.8) and and (3.21) the function f is given by (3.12).

By replacing f in the equation (3.19), we have:
If $\frac{\beta}{\lambda} > 0$

$$\frac{8a\beta^2}{\lambda^2}\tan^4\left(\frac{\lambda}{2}\sqrt{\frac{2\beta}{\lambda}}s + \beta_1\sqrt{\frac{2\beta}{\lambda}}\right) + \left[\beta g'' + \frac{4a\beta}{\lambda}(1-g'^2)\right] tan^2\left(\frac{\lambda}{2}\sqrt{\frac{2\beta}{\lambda}}s + \beta_1\sqrt{\frac{2\beta}{\lambda}}\right)$$
$$+ \left[2a(1-g'^2)^2 + g''\right] = 0 \tag{3.22}$$

The equation (3.22) is a polynomial on the function $\tan\left(\frac{\lambda}{2}\sqrt{\frac{2\beta}{\lambda}}s+\beta_1\sqrt{\frac{2\beta}{\lambda}}\right)$ and thus the coefficients must vanish. It follows that the function g satisfies :

$$\begin{cases} \frac{8a\beta^2}{\lambda^2}=0, \\ \beta g''+\frac{4a\beta}{\lambda}(1-g'^2)=0, \\ 2a(1-g'^2)^2+g''=0. \end{cases} \tag{3.23}$$

Thus, $\beta=0$, $g'=\pm 1$ and it is a contradiction.

And if $\frac{\beta}{\lambda}<0$, we have

$$2a\left[-1+g'^2+\frac{2\beta}{\lambda}\right]^2 e^{4\lambda\sqrt{\frac{-2\beta}{\lambda}}s+4\beta_1}+\left[-\beta g''-8a(-1+g'^2)^2+\frac{32a\beta^2}{\lambda^2}\right]e^{3\lambda\sqrt{\frac{-2\beta}{\lambda}}s+3\beta_1}$$
$$+\left[2\beta g''-\frac{-16a\beta}{\lambda}(1-g'^2)+12a(-1+g'^2)^2+\frac{48a\beta^2}{\lambda^2}\right]e^{2\lambda\sqrt{\frac{-2\beta}{\lambda}}s+2\beta_1}+$$
$$\left[-\beta g''-8a(1-g'^2)^2+\frac{32a\beta^2}{\lambda^2}\right]e^{\lambda\sqrt{\frac{-2\beta}{\lambda}}s+\beta_1}+2a\left[-1+g'^2+\frac{2\beta}{\lambda}\right]^2=0 \tag{3.24}$$

The equation (3.24) is a polynomial on the function $e^{\lambda\sqrt{\frac{-2\beta}{\lambda}}s+\beta_1}$ and thus the coefficients must vanish. It follows that the function g satisfies :

$$\begin{cases} -1+g'^2+\frac{2\beta}{\lambda}=0, \\ -\beta g''-8a(-1+g'^2)^2+\frac{32a\beta^2}{\lambda^2}=0, \\ 2\beta g''-\frac{-16a\beta}{\lambda}(-1+g'^2)+12a(1-g'^2)^2+\frac{48a\beta^2}{\lambda^2}=0. \end{cases} \tag{3.25}$$

Thus, $\beta=0$, $g'=\pm 1$ and it is a contradiction.

Thus we have the following theorem.

Theorem 16. *([14]) Let Σ be a spacelike translation surface in Minkowski 3-space with density $e^{-a(x^2+y^2+z^2)+c}$, parametrized by*

$$X(s,t)=(f(s)+g(t),s,t),\ (s,t)\in\mathbb{R}^2,$$

Then Σ is weighted flat timelike translation surface in Minkowski 3-space with density $e^{-a(x^2+y^2+z^2)+c}$ if and only if

- $X(s,t)=(\alpha s\pm\sqrt{1-\alpha^2}t+\alpha_1,s,t),\ \alpha,\alpha_1\in\mathbb{R},$ *or*
- $X(s,t)=(\beta t\pm\sqrt{1-\beta^2}s+\beta_1,s,t),\ \beta\in\,]-\infty,-1[\,\cup\,]1,+\infty[\,,\ \beta_1\in\mathbb{R}.$

3.3 Weighted Flat Lightlike Translation Surfaces in Minkowski 3-Space with Density

In this section we study weighted flat lightlike translation surfaces Σ in Minkowski 3-space $\mathbb{R}^3_1$ which are parameterized by

$$X(s,t)=(s+t,g(t),f(s)+t),\ (s,t)\in\mathbb{R}^2,$$

where f and g are reals functions $\mathcal{C}^2(\mathbb{R})$, have an orthogonal pair of vector fields on (Σ), namely,

$$e_1 := X_s = (1, 0, f'(s)).$$

and

$$e_2 := X_t = (1, g'(t), 1).$$

The coefficients of the first fundamental form are:

$$E =< e_1, e_1 >_{\mathbb{R}^3_1} = 1 - f'^2,\ F =< e_1, e_2 >_{\mathbb{R}^3_1} = 1 - f',\ G =< e_2, e_2 >_{\mathbb{R}^3_1} = g'^2.$$

As a unit normal field we can take

$$N = \frac{1}{\sqrt{|f'^2 g'^2 + (f'-1)^2 - g'^2|}}(-f'g', f'-1, -g')$$

The coefficients of the second fundamental form are

$$l =< X_{ss}, N >_{\mathbb{R}^3_1} = \frac{f''g''}{\sqrt{|f'^2 g'^2 + (f'-1)^2 - g'^2|}}$$

$$m =< X_{st}, N >_{\mathbb{R}^3_1} = 0$$

$$n =< X_{tt}, N >_{\mathbb{R}^3_1} = \frac{g''(f'-1)}{\sqrt{|f'^2 g'^2 + (f'-1)^2 - g'^2|}}.$$

Let K be the Gauss curvature of Σ,

$$K = \frac{ln - m^2}{EG - F^2} = \frac{f''g''g'(f'-1)}{(f'^2 g'^2 + (f'-1)^2 - g'^2)^2}. \tag{3.26}$$

According to (3.3) and (3.26)the weighted Gaussian curvature of Σ is given by

$$K_\varphi = K - \Delta\varphi = \frac{f''g''g'(f'-1)}{(f'^2 g'^2 + (f'-1)^2 - g'^2)^2} + 2a = \frac{f''g''g'(f'-1) + 2a\left((f'-1) + g'^2(f'^2-1)\right)^2}{(f'^2 g'^2 + (f'-1)^2 - g'^2)^2}. \tag{3.27}$$

Since the surface is non-degenerate, $f' \neq 1$ for all s.

Thus Σ is weighted flat lightlike translation surface in Minkowski 3-space with density e^φ if and only if

$$K_\varphi = 0,$$

that is, if and only if

$$f''g''g'(f'-1) + 2a\left((f'-1)^2 + g'^2(f'^2-1)\right)^2 = 0. \tag{3.28}$$

to classify weighted flat lightlike translation surfaces must solve the equation (3.28).

- If $f' = \alpha \in]-1, 1[$. It is a trivial solution of (3.28), $f(s) = \alpha s + \alpha_1$, $g(t) = \pm\sqrt{\frac{1-\alpha}{1+\alpha}}t + \alpha_2$, $\alpha_1, \alpha_2 \in \mathbb{R}$, in this case the surface is lightlike space.
- If $g' = \beta \in \mathbb{R}$. It is a trivial solution of (3.28), $g(t) = \beta t + \beta_1$, $f(s) = \frac{1-\beta^2}{1+\beta^2}s + \beta_2$, $\beta_1, \beta_2 \in \mathbb{R}$, in this case the surface is lightlike space.

• If f' is non-constant smooth function. We divide (3.28) by $(f'-1)(f'+1)^2$ and we take derivatives with s and t respectively.
Then one obtains

$$\left(\frac{f''}{(f'-1)(f'+1)^2}\right)'(g''g')' + 4a\left(\frac{f'-1}{f'+1}\right)'(g'^2)' = 0. \tag{3.29}$$

Suppose $g' = 0$. From (3.28) $f' = 1$, a contradiction. Therefore, there exists $\lambda \in \mathbb{R}$ such that

$$-\frac{4a\left(\frac{f'-1}{f'+1}\right)'}{\left(\frac{f''}{(f'-1)(f'+1)^2}\right)'} = \frac{(g''g')}{(g'^2)'} = \lambda. \tag{3.30}$$

• If $\lambda = 0$, then we have $g'^2 = 2\beta$ for some non-zero constant β. From (3.28) $f' = \frac{1-2\beta}{1+2\beta}$, a contradiction.
• If $\lambda \neq 0$, in this case we have

$$g''g' = \lambda g'^2 + \lambda_1,\ \ \lambda_1 \in \mathbb{R}. \tag{3.31}$$

We can write equation (3.31) by

$$\frac{2g'g''}{g'^2 + \frac{\lambda_1}{\lambda}} = 2\lambda, \tag{3.32}$$

and its solution is given by

$$g'^2 = \kappa e^{2\lambda t} - \frac{\lambda_1}{\lambda}, \kappa \in \mathbb{R}^{*,+}. \tag{3.33}$$

Submitting it into (3.28) and (3.31), (3.33), the result is polynomial on $e^{2\lambda t}$and thus coefficients must vanish. It follows that f satisfies then next three differential equations:

$$\begin{cases} f' + 1 = 0 = 0, \\ 2a(f'^2 - 1) - \frac{4a\lambda_1}{\lambda}(f'+1)^2 + \lambda f'' = 0, \\ (\lambda_1 + \lambda_1\lambda)f'' + (f'-1)^2 + \frac{2a\lambda_1^2}{\lambda^2}(f'+1)^2 + 2a\lambda_1(f'^2-1) = 0. \end{cases} \tag{3.34}$$

Thus we conclude that $f' = 1$, a contradiction again.
Thus we have the following theorem.

Theorem 17. *([14]) Let Σ be a lightlike translation surface in Minkowski 3-space with density $e^{-a(x^2+y^2+z^2)+c}$, parametrized by*

$$X(s,t) = (s+t, g(t), f(s)+t),\ \ (s,t) \in \mathbb{R}^2,$$

Then Σ is weighted flat lightlike translation surface in Minkowski 3-space with density $e^{-a(x^2+y^2+z^2)+c}$ if and only if

- $X(s,t) = (s+t, \pm\sqrt{\frac{1-\alpha}{1+\alpha}}t + \alpha_2, \alpha s + \alpha_1),\ \ \alpha \in \,]-1,1[\,, \alpha_1, \alpha_2 \in \mathbb{R},$ *or*
- $X(s,t) = (s+t, \beta t + \beta_1, \frac{1-\beta^2}{1+\beta^2}s + \beta_2 + t),\ \ \beta,\ \beta_1, \beta_2 \in \mathbb{R}.$

Chapter 4

Weighted Flat Radial Surfaces in Euclidean 3-Space and Minkowski 3-Space with Density

4.1 Weighted Flat Radial Surfaces in Euclidean 3-Space with Density

In this section we gives a classifications of all weighted flat radial surfaces in Euclidean 3-space with radial density $e^{\frac{a}{6}(x^2+y^2+z^2)+c}$, where $a > 0$ and $c \in \mathbb{R}$.

Let $\Omega = \left\{(x, y) \in \mathbb{R}^2, x^2 + y^2 < \frac{1}{a}\right\}$, and consider the radial surface

$$X : \Omega \to \mathbb{R}^3$$

$$X(x, y) = (x, y, h(\sqrt{x^2 + y^2}))$$

we have

$$\begin{aligned}
X_x &= (1, 0, \frac{x}{\sqrt{x^2 + y^2}}h') \\
X_{xx} &= (0, 0, \frac{y^2}{(x^2 + y^2)^{3/2}}h' + \frac{x^2}{x^2 + y^2}h'') \\
X_y &= (0, 1, \frac{y}{\sqrt{x^2 + y^2}}h') \\
X_{yy} &= (0, 0, \frac{x^2}{(x^2 + y^2)^{3/2}}h' + \frac{y^2}{x^2 + y^2}h'') \\
X_{xy} = X_{yx} &= (0, 0, \frac{-xy}{(x^2 + y^2)^{3/2}}h' + \frac{xy}{x^2 + y^2}h''),
\end{aligned}$$

and

$$\begin{aligned}
X_x \wedge X_y &= (\frac{-x}{\sqrt{x^2 + y^2}}h', \frac{-y}{\sqrt{x^2 + y^2}}h', 1) \\
\|X_x \wedge X_y\| &= \sqrt{1 + h'^2},
\end{aligned}$$

hence the unit normal vector is

$$N = \frac{1}{\sqrt{1+h'^2}}\left(\frac{-x}{\sqrt{x^2+y^2}}h', \frac{-y}{\sqrt{x^2+y^2}}h', 1\right).$$

The coefficients of the first fudamental form are

$$E = 1 + \frac{x^2}{x^2+y^2}h'^2$$
$$F = \frac{xy}{x^2+y^2}h'^2$$
$$G = 1 + \frac{y^2}{x^2+y^2}h'^2,$$

and the coefficients of the second fudamental form are

$$l = \frac{1}{\sqrt{1+h'^2}}\left(\frac{y^2}{(x^2+y^2)^{3/2}}h' + \frac{x^2}{x^2+y^2}h''\right)$$
$$m = \frac{1}{\sqrt{1+h'^2}}\left(\frac{-xy}{(x^2+y^2)^{3/2}}h' + \frac{xy}{x^2+y^2}h''\right)$$
$$n = \frac{1}{\sqrt{1+h'^2}}\left(\frac{x^2}{(x^2+y^2)^{3/2}}h' + \frac{y^2}{x^2+y^2}h''\right),$$

so the gauss curvature is

$$K = \frac{ln - m^2}{EG - F^2} = \frac{1}{(1+h'^2)^2}\frac{1}{\sqrt{x^2+y^2}}h'h''.$$

We define the function $\phi(x, y, z) = \frac{a}{6}(x^2 + y^2 + z^2)$ with $a > 0$, then $\Delta\phi = a$.

The gauss curvature with density e^{ϕ} is given by

$$K_\phi = K - \Delta\phi = \frac{1}{(1+h'^2)^2}\frac{1}{\sqrt{x^2+y^2}}h'h'' - a,$$

we want to classify the flat radial surfaces $z = h(\sqrt{x^2+y^2})$ with density e^{ϕ}, we have to solve the equation

$$K_\phi = 0, \tag{4.1}$$

let us put $\sqrt{x^2+y^2} = t$, so the equation (4.1) becoms

$$\frac{1}{(1+h'^2)^2}\frac{1}{t}h'h'' = a,$$

multiplying through by t, we simplify the expression.

$$\frac{h'h''}{(1+h'^2)^2} = at,$$

By the chain rule, $h'h''$ can be rewritten as $\frac{1}{2}(h'^2)'$, so this leads to a simpler representation

$$\frac{(h'^2)'}{(1+h'^2)^2} = 2at.$$

Integrating with respect to t, we obtain this expression, where b is the integration constant

$$\frac{-1}{1+h'^2} + b = at^2. \tag{4.2}$$

Consequently, wa conclude from (4.2) that $b > at^2$, then $t^2 < \frac{b}{a}$, and because of that we choose Ω as the domain of definition.

If we consider that $b \leq at^2 + 1$, then

$$h' = \pm\sqrt{\frac{1}{b-at^2} - 1},$$

we assume $b \in]0,1]$ to ensure the existance of the square root.

we want to simplify the integral

$$I = \pm \int \sqrt{\frac{1}{b-at^2} - 1}\, dt,$$

that is

$$I = \pm \int \sqrt{\frac{at^2-b+1}{b-at^2}}\, dt, \tag{4.3}$$

putting $t = \sqrt{\frac{b}{a}}\cos(\theta)$, with $\theta \in]0, \frac{\pi}{2}]$, $dt = -\sqrt{\frac{b}{a}}\sin(\theta)d\theta$ in (4.3) we get

$$\begin{aligned} I &= \pm \int \sqrt{\frac{b\cos^2(\theta)-b+1}{b-b\cos^2(\theta)}}\sqrt{\frac{b}{a}}\sin(\theta)\, d\theta, \\ &= \pm \int \sqrt{\frac{-b\sin^2(\theta)+1}{b\sin^2(\theta)}}\sqrt{\frac{b}{a}}\sin(\theta)\, d\theta, \end{aligned}$$

then

$$I = \pm\frac{1}{\sqrt{a}} \int \sqrt{1-b\sin^2(\theta)}\, d\theta, \tag{4.4}$$

We consider two cases for the integral:

1. **Case 1**: $b = 1$

$$I = \pm\frac{1}{\sqrt{a}}\sin(\theta) + c,$$

that is

$$h(t) = \pm\frac{1}{\sqrt{a}}\sin(\cos^{-1}(\sqrt{a}t)) + c,$$

thus

$$X(x,y) = (x, y, \pm\frac{1}{\sqrt{a}}\sin(\cos^{-1}(\sqrt{a}\sqrt{x^2+y^2})) + c),$$

2. **Case 2**: $0 < b < 1$
   in this case the integral in (4.4) is a standard elliptic integral of the second kind, denoted $E(\theta, k)$ where $k = \sqrt{b}$, thus we can write

$$I = \pm\frac{1}{\sqrt{a}}E(\theta|b) + c,$$

   Recall the substitution $t = \sqrt{\frac{b}{a}}\cos(\theta)$, so

$$h(t) = \pm\frac{1}{\sqrt{a}}E(\cos^{-1}(\sqrt{\frac{a}{b}}\sqrt{x^2+y^2}|b) + c,$$

   that is

$$X(x,y) = (x, y, \pm\frac{1}{\sqrt{a}}E(\cos^{-1}(\sqrt{\frac{a}{b}}\sqrt{x^2+y^2}|b) + c).$$

Thus we have the following theorem.

Theorem 18. *([15]) Let Σ be a radial surface in $\mathbb{R}^3$ with density $e^{\frac{a}{6}(x^2+y^2+z^2)+c}$, parametrized by*

$$X : \Omega \to \mathbb{R}^3$$
$$X(x,y) = (x, y, h(\sqrt{x^2+y^2})),$$

with $\Omega = \left\{(x,y) \in \mathbb{R}^2, x^2+y^2 < \frac{1}{a}\right\}$.

Then Σ is weighted flat radial surface in $\mathbb{R}^3$ with density $e^{\frac{a}{6}(x^2+y^2+z^2)+c}$ if and only if

- $X(x,y) = (x, y, \pm\frac{1}{\sqrt{a}}\sin(\cos^{-1}(\sqrt{a}\sqrt{x^2+y^2})) + c)$, $c \in \mathbb{R}$ *or*
- $X(x,y) = (x, y, \pm\frac{1}{\sqrt{a}}E(\cos^{-1}(\sqrt{\frac{a}{b}}\sqrt{x^2+y^2}|b) + c)$, $1 > b > 0$, $c \in \mathbb{R}$.

4.2 Weighted Flat Radial Surfaces in Minkowski 3-Space with Density

In this section we gives a classifications of all weighted flat radial surfaces in Minkowski 3-space with radial density $e^{\frac{a}{2}(x^2+y^2+z^2)+c}$, where $a > 0$ and $c \in \mathbb{R}$.

Let

$$X : \mathbb{R}^2 \to \mathbb{R}^3_1$$
$$X(x,y) = (x, y, h(\sqrt{x^2+y^2})),$$

be a radial surface in $\mathbb{R}^3_3$. Thus

$$\begin{aligned}
X_x &= (1, 0, \frac{x}{\sqrt{x^2+y^2}} h') \\
X_{xx} &= (0, 0, \frac{y^2}{(x^2+y^2)^{3/2}} h' + \frac{x^2}{x^2+y^2} h'') \\
X_y &= (0, 1, \frac{y}{\sqrt{x^2+y^2}} h') \\
X_{yy} &= (0, 0, \frac{x^2}{(x^2+y^2)^{3/2}} h' + \frac{y^2}{x^2+y^2} h'') \\
X_{xy} = X_{yx} &= (0, 0, \frac{-xy}{(x^2+y^2)^{3/2}} h' + \frac{xy}{x^2+y^2} h''),
\end{aligned}$$

and the unit normal vector is

$$N = \frac{1}{\sqrt{|h'^2 - 1|}} (\frac{-x}{\sqrt{x^2+y^2}} h', \frac{-y}{\sqrt{x^2+y^2}} h', -1).$$

The coefficients of the first fudamental form are

$$\begin{aligned}
E &= 1 - \frac{x^2}{x^2+y^2} h'^2 \\
F &= -\frac{xy}{x^2+y^2} h'^2 \\
G &= 1 - \frac{y^2}{x^2+y^2} h'^2,
\end{aligned}$$

and the coefficients of the second fudamental form are

$$\begin{aligned}
l &= \frac{1}{\sqrt{|h'^2 - 1|}} (\frac{-y^2}{(x^2+y^2)^{3/2}} h' - \frac{x^2}{x^2+y^2} h'') \\
m &= \frac{1}{\sqrt{|h'^2 - 1|}} (\frac{xy}{(x^2+y^2)^{3/2}} h' - \frac{xy}{x^2+y^2} h'') \\
n &= \frac{1}{\sqrt{|h'^2 - 1|}} (\frac{-x^2}{(x^2+y^2)^{3/2}} h' - \frac{y^2}{x^2+y^2} h'').
\end{aligned}$$

We define the function $\phi(x, y, z) = \frac{a}{2}(x^2 + y^2 + z^2)$ with $a > 0$, then

$$\Delta\phi = \frac{1}{\sqrt{detg_{ij}}} \frac{\partial}{\partial x_i} \left(\sqrt{detg_{ij}}.(\nabla\phi)^i \right) = a.$$

And the gauss curvature with density e^ϕ is given by

$$K_\phi = \frac{ln - m^2}{EG - F^2} - a.$$

We will discuss two cases based on the sign of $h'^2 - 1$:

1. **case 1** : $h'^2 - 1 < 0$, then

$$K_\phi = \frac{1}{(h'^2-1)^2}\frac{1}{\sqrt{x^2+y^2}}h'h'' - a,$$

to classify the flat radial surfaces in this case we have to solve

$$\frac{1}{(h'^2-1)^2}\frac{1}{\sqrt{x^2+y^2}}h'h'' = a. \tag{4.5}$$

2. **case 2** : $h'^2 - 1 > 0$, then

$$K_\phi = -\frac{1}{(h'^2-1)^2}\frac{1}{\sqrt{x^2+y^2}}h'h'' - a,$$

to classify the flat radial surfaces in this case we have to solve

$$-\frac{1}{(h'^2-1)^2}\frac{1}{\sqrt{x^2+y^2}}h'h'' = a. \tag{4.6}$$

Let us solve the equation (4.5). If we put $\sqrt{x^2+y^2} = t$, then (4.5) will be

$$\frac{(h'^2)'}{(h'^2-1)^2} = 2at,$$

leads to

$$\frac{-1}{h'^2-1} + b = at^2. \tag{4.7}$$

Thus, we can conclude from equation (4.7) that $b < at^2$, then

$$h' = \pm\sqrt{\frac{1}{b-at^2} + 1},$$

with $b \le at^2 - 1$. Thus, we assume $b \le -1$ to ensure the existence of the square root.

We want to simplify the integral

$$I = \pm\int\sqrt{\frac{1}{b-at^2}+1}\,dt, \tag{4.8}$$

for this purpose, let us consider $c = -b$, then we get in (4.8)

$$\begin{aligned} I &= \pm\int\sqrt{\frac{1}{-c-at^2}+1}\,dt\ , \\ &= \pm\int\sqrt{\frac{-c-at^2+1}{-c-at^2}}\,dt\ , \\ &= \pm\int\sqrt{\frac{c+at^2-1}{c+at^2}}\,dt, \end{aligned}$$

with $c \geq 1$.

Putting $t = \sqrt{\frac{c}{a}}\sinh(\theta)$, $dt = \sqrt{\frac{c}{a}}\cosh(\theta)$, with $\theta \geq 0$, we get

$$\begin{aligned} I &= \pm\int\sqrt{\frac{c + c\sinh^2(\theta) - 1}{c + c\sinh^2(\theta)}}\sqrt{\frac{c}{a}}\cosh(\theta)\, d\theta\ , \\ &= \pm\int\sqrt{\frac{c\cosh^2(\theta) - 1}{c\cosh^2(\theta)}}\sqrt{\frac{c}{a}}\cosh(\theta)\, d\theta\ , \\ &= \pm\int\frac{\sqrt{c\cosh^2(\theta) - 1}}{\sqrt{c}\cosh(\theta)}\sqrt{\frac{c}{a}}\cosh(\theta)\, d\theta, \end{aligned}$$

then

$$I = \pm\frac{1}{\sqrt{a}}\int\sqrt{c\cosh^2(\theta) - 1}\, d\theta. \tag{4.9}$$

1. **If** $c = 1$

$$\begin{aligned} I &= \pm\frac{1}{\sqrt{a}}\int\sqrt{\cosh^2(\theta) - 1}\, d\theta \\ &= \pm\frac{1}{\sqrt{a}}\cosh(\theta) + c_1, \end{aligned}$$

and

$$X(x, y) = (x, y, \pm\sqrt{x^2 + y^2} + c_1), c_1 \in \mathbb{R}.$$

2. **If** $c > 1$ then, (4.9) is an elliptic integral of the second kind, and it can given by

$$I = \pm\frac{1}{\sqrt{a}}\frac{i\sqrt{c\cosh(2\theta) + c - 2}E\left(i\theta\,\middle|\,\frac{c}{c-1}\right)}{\sqrt{\frac{c\cosh(2\theta)+c-2}{c-1}}} + c_1,$$

then

$$X(x, y) = (x, y, z(x, y)),$$

where

$$z(x, y) = \pm\frac{1}{\sqrt{a}}\frac{i\sqrt{c\cosh(2\sinh^{-1}(\sqrt{\frac{a}{c}}\sqrt{x^2 + y^2})) + c - 2}E\left(i\sinh^{-1}(\sqrt{\frac{a}{c}}\sqrt{x^2 + y^2})\,\middle|\,\frac{c}{c-1}\right)}{\sqrt{\frac{c\cosh(2\sinh^{-1}(\sqrt{\frac{a}{c}}\sqrt{x^2+y^2}))+c-2}{c-1}}} + c_1.$$

Now we solve the equation (4.6). For $\sqrt{x^2 + y^2} = t$, then (4.6) will be

$$-\frac{(h'^2)'}{(h'^2 - 1)^2} = 2at,$$

leads to

$$\frac{1}{h'^2 - 1} + b = at^2. \tag{4.10}$$

Thus, we can conclude from equation (4.10) that $b < at^2$, then

$$h' = \pm\sqrt{\frac{1}{at^2 - b} + 1},$$

with $b < at^2$. Thus we assume $b < 0$.

We want to simplify the integral

$$I = \pm \int \sqrt{\frac{1}{at^2 - b} + 1}\, dt, \tag{4.11}$$

consider $c = -b$, then we get in (4.11)

$$\begin{aligned} I &= \pm \int \sqrt{\frac{1}{at^2 + c} + 1}\, dt \\ &= \pm \int \sqrt{\frac{at^2 + c + 1}{at^2 + c}}\, dt, \end{aligned}$$

with $c > 0$.
putting $t = \sqrt{\frac{c}{a}}\sinh(\theta)$, $dt = \sqrt{\frac{c}{a}}\cosh(\theta)$, with $\theta \geq 0$, we get

$$\begin{aligned} I &= \pm \int \sqrt{\frac{c\sinh^2(\theta) + c + 1}{c\sinh^2(\theta) + c}}\sqrt{\frac{c}{a}}\cosh(\theta)\, d\theta\ , \\ &= \pm \int \sqrt{\frac{c\cosh^2(\theta) + 1}{c\cosh^2(\theta)}}\sqrt{\frac{c}{a}}\cosh(\theta)\, d\theta\ , \\ &= \pm \int \frac{\sqrt{c\cosh^2(\theta) + 1}}{\sqrt{c}\cosh(\theta)}\sqrt{\frac{c}{a}}\cosh(\theta)\, d\theta, \end{aligned}$$

then

$$I = \pm\frac{1}{\sqrt{a}} \int \sqrt{c\cosh^2(\theta) + 1}\, d\theta, \tag{4.12}$$

it is an elliptic integral of the second kind and it can given by

$$I = \pm\frac{1}{\sqrt{a}} \frac{i\sqrt{c\cosh(2\theta) + c + 2}E\left(i\theta \,\middle|\, \frac{c}{c+1}\right)}{\sqrt{\frac{c\cosh(2\theta)+c+2}{c+1}}},$$

then

$$X(x, y) = (x, y, w(x, y)),$$

where

$$w(x, y) = \pm\frac{1}{\sqrt{a}} \frac{i\sqrt{c\cosh(2\sinh^{-1}(\sqrt{\frac{a}{c}}\sqrt{x^2 + y^2}) + c + 2}E\left(i\sinh^{-1}(\sqrt{\frac{a}{c}}\sqrt{x^2 + y^2} \,\middle|\, \frac{c}{c+1}\right)}{\sqrt{\frac{c\cosh(2\sinh^{-1}(\sqrt{\frac{a}{c}}\sqrt{x^2+y^2})+c+2}{c+1}}}. \tag{4.13}$$

Thus, we have the following theorem.

Theorem 19. *([15]) Let Σ be a radial surface in Minkowski 3-space with density $e^{\frac{a}{2}(x^2+y^2+z^2)+c}$, parametrized by*

$$X : \mathbb{R}^2 \to \mathbb{R}^3_1$$
$$X(x,y) = (x, y, h(\sqrt{x^2+y^2})),$$

Then Σ is weighted flat radial surface in Euclidien 3-space with density $e^{\frac{a}{6}(x^2+y^2+z^2)+c}$ if and only if

- $X(x,y) = (x, y, \pm\sqrt{x^2+y^2} + c_1), c_1 \in \mathbb{R}$ *or*

- $X(x,y) = (x, y, \pm\frac{1}{\sqrt{a}} \frac{i\sqrt{c\cosh(2\sinh^{-1}(\sqrt{\frac{a}{c}}\sqrt{x^2+y^2}))+c-2}E\left(i\sinh^{-1}(\sqrt{\frac{a}{c}}\sqrt{x^2+y^2})\middle|\frac{c}{c-1}\right)}{\sqrt{\frac{c\cosh(2\sinh^{-1}(\sqrt{\frac{a}{c}}\sqrt{x^2+y^2}))+c-2}{c-1}}} + c_1),$

$c > 1, c_1 \in \mathbb{R}$ *or*

- $X(x,y) = (x, y, \pm\frac{1}{\sqrt{a}} \frac{i\sqrt{c\cosh(2\sinh^{-1}(\sqrt{\frac{a}{c}}\sqrt{x^2+y^2})+c+2}E\left(i\sinh^{-1}(\sqrt{\frac{a}{c}}\sqrt{x^2+y^2}\middle|\frac{c}{c+1}\right)}{\sqrt{\frac{c\cosh(2\sinh^{-1}(\sqrt{\frac{a}{c}}\sqrt{x^2+y^2})+c+2}{c+1}}} + c_1),$

$c > 0, c_1 \in \mathbb{R}$.

Conclusion

Contribution and Methodology

The core of our contribution lies in the geometric characterization and classification of specific classes of surfaces within the framework of **weighted manifolds**. Specifically, we focus our investigation on the following two primary axes:

I. Classification of Weighted Flat Translation Surfaces

We provide a complete classification of translation surfaces in the **Minkowski 3-space** $\mathbb{R}^3_1$ that exhibit a "weighted flat" nature. In this context, the flatness is investigated with respect to the **weighted Laplacian operator** (also known as the drift Laplacian or f-Laplacian). The surfaces are studied under the influence of a radial density function given by:

$$\varphi(x, y, z) = e^{-a(x^2+y^2+z^2)+c}, \quad \text{where } a > 0,\ c \in \mathbb{R} \tag{4.14}$$

This density introduces a non-trivial weight to the manifold, altering the traditional notion of curvature and requiring an in-depth analysis of the interaction between the translation generator curves and the density's gradient.

II. Study of Weighted Flat Radial Surfaces

In addition to translation surfaces, we extend our investigation to the class of **radial surfaces** (surfaces of revolution) parameterized by:

$$X(x, y) = \left(x, y, h\left(\sqrt{x^2 + y^2}\right)\right) \tag{4.15}$$

Our work establishes a comparative classification between the **Euclidean space** $\mathbb{R}^3$ and the **Minkowski 3-space** $\mathbb{R}^3_1$. By employing the same radial density, we solve the resulting non-linear differential equations to identify all possible profile functions $h(r)$ that satisfy the weighted flatness condition. This comparative analysis highlights how the signature of the metric (Riemannian vs. Pseudo-Riemannian) fundamentally influences the existence and geometric form of such surfaces.

Perspectives

The results presented in Chapters 3 and 4 open several compelling avenues for future investigation, specifically regarding the study of alternative surfaces in diverse spaces with varying densities.

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