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On Certain Quasilinear Elliptic Equations with Indefinite Terms

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1. Introduction

In this paper, we consider the existence of positive solutions of the following problem:

$$(\mathscr{P}_p) \qquad \begin{cases} -\Delta_p u = m(x)g(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded regular domain in $\boldsymbol{R}^N$ with a smooth boundary $\partial\Omega$, $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian with $1 < p < N$, m is a continuous function on $\bar{\Omega}$, which changes sign, and g is a continuous function on $\boldsymbol{R}$ which will be specified later.

In the case where m has a constant sign, many existence results have been obtained by various authors. They used in their works sub-super solutions, topological, and variational methods, and bifurcation approach.

To our knowledge the only problems with indefinite term that have been considered concern the Laplacian, see for example S. Alama et al. [1], C. Bandle et al. [3], H. Berestycki et al. [4], and T. Ouyang [8]. In their proof, these authors used the decomposition $H_0^1(\Omega) = \boldsymbol{R} \oplus V$ where $V = \{u \in H_0^1(\Omega) \mid \int_\Omega u\,dx = 0\}$.

In [6], the authors modified the semilinear problem in order to apply the Palais Smale Condition (P.S.).

Our purpose here is to generalize the results in [6] to the case $p \neq 2$. While the form of the results is the same, due to the nonlinear nonself-adjoint nature of the principal operator Δ_p, the approch of [6] must be modified and we employ the Moser's Iterative Scheme as in Ôtani [7]. We consider only weak solutions.

Suppose that there exists $q : p < q < Np/(N-p)$ such that:

g_1) $g(u) = o(u^{q-1})$ as u tends to 0^+;

g_2) $\exists R_0 > 0$ such that $ug(u) \geq q \int_0^u g(s)\,ds$ for $u \geq R_0$;

g_3) $\exists C > 0$, such that $g(u) \leq C(1 + u^{q-1})$.

2. Preliminaries and existence results

Let

$$G(u) = \begin{cases} \int_0^u g(s)\,ds & \text{for } u > 0 \\ 0 & \text{for } u \le 0 \end{cases}$$

Problem $(\mathscr{P}_p)$ corresponds to the Euler-Lagrange equation of the functional

$$I(u) := \frac{1}{p}\int_\Omega |\nabla u|^p\,dx - \int_\Omega m(x)G(u(x))\,dx$$

defined on $W_0^{1,p}(\Omega)$.

The functional I does not satisfy the (P.S.) since $ug(u) - qG(u)$ is not bounded. We proceed by modifying g so that the corresponding quantity will be bounded.

Let $R \ge R_0$ be fixed, and set

$$G_R(s) = \begin{cases} \int_0^s g(t)\,dt & \text{for } s \le R \\ A(R+1-s)^q + Bs^q & \text{for } R \le s \le R+1 \\ Bs^q & \text{for } s \ge R+1, \end{cases}$$

where $A := [qG(R) - Rg(R)]/[q(R+1)]$ (≤ 0), and $B := [qG(R) + g(R)]/[q(R+1)R^{q-1}]$ ($\le C$ independently of R). By construction G_R is C^1, non-negative and nondecreasing on $\boldsymbol{R}$.

Let

$$M_R := \max_{s\in[0,R]}(sg(s) - qG(s),) \qquad \text{and} \qquad g_R := G_R'.$$

From construction of G_R, we have: $g_R(u) = g(u)$ for $u \in [0, R]$, and $g_R(u) \le C_R u^{q-1}$ for $u \in [R, +\infty[$, where $C_R := (M_R/R^q) + Bq$. Consider the following modified problem,

$$(\mathscr{P}_p)_R \qquad \begin{cases} -\Delta_p u = m(x)g_R(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

which has an associated functional I_R defined on $W_0^{1,p}(\Omega)$ by:

$$I_R(u) := \frac{1}{p}\int_\Omega |\nabla u|^p\,dx - \int_\Omega m(x)G_R(u(x))\,dx.$$

We have:

$$sg_R(s) - qG_R(s) = sBqs^{q-1} - qBs^q = 0 \qquad \text{for } s \ge R+1;$$

and

$$sg_R(s) - qG_R(s) = qG(R) - RG(R)(R+1-s)^{q-1} \le M_R \qquad \text{for } R \le s \le R+1.$$

Hence, we obtain $0 \le \max_{s\in \mathbf{R}}(sg_R(s) - qG_R(s)) \le M_R$.

Then our main result is

Theorem 1. *Assume $g_1)$–$g_3)$ hold and $M_R = o(R^{(p^*(p+q))/(p^2(p^*-q))})$ for R sufficiently large. Then Problem $(\mathscr{P}_p)$ has at least one nontrivial solution.*

We can use here the Mountain Pass Theorem [2].
In the following Lemma, we prove that the (P.S.) holds for I_R.

Lemma 1. *Under the hypothesis of Theorem* 1, *the* (P.S.) *is satisfied.*

Proof. Let $(u_n) \in W_0^{1,p}(\Omega)$ such that $I_R(u_n)$ is bounded and $I_R'(u_n) \to 0$ strongly in $W^{-1,p'}(\Omega)$ (dual space of $W_0^{1,p}(\Omega)$).

Claim 1. (u_n) is bounded in $W_0^{1,p}(\Omega)$. In fact, for any majorant M, we have

$$-M \le \frac{1}{p}\int_\Omega |\nabla u_n|^p\,dx - \int_\Omega m(x)G_R(u_n(x))\,dx \le M,$$

and for $\varepsilon \in (0,1)$, we have again

$$-\varepsilon \le \int_\Omega |\nabla u_n|^p\,dx - \int_\Omega m(x)u_n g_R(u_n)\,dx \le \varepsilon.$$

Then we obtain, the following inequalities

$$0 \le \left(\frac{q}{p} - 1\right)\int_\Omega |\nabla u_n|^p\,dx \le Mq + 1 + \int_\Omega m(x)[u_n g_R(u_n) - qG_R(u_n)]$$
$$\le Mq + 1 + M_R|m|_0(\text{meas }\Omega),$$

where $|m|_0 := \max_{x\in\bar{\Omega}}(|m(x)|)$. Hence (u_n) is bounded in $W_0^{1,p}(\Omega)$.

Claim 2. (u_n) converges strongly in $W_0^{1,p}(\Omega)$. Since (u_n) is bounded in $W_0^{1,p}(\Omega)$, there exists a subsequence denoted again by (u_n) which converges weakly in $W_0^{1,p}(\Omega)$ and strongly in the space $L^\gamma(\Omega)$ for any $1 < \gamma < Np/(N-p)$. From the definition of I_R', we write:

$$\int_\Omega (\operatorname{div}(|\nabla u_n|^{p-2}\nabla u_n) - \operatorname{div}(|\nabla u_l|^{p-2}\nabla u_l))(u_n - u_l)\,dx$$
$$= \langle I_R'(u_n) - I_R'(u_l), u_n - u_l\rangle + \int_\Omega m(x)(g_R(u_n) - g_R(u_l))(u_n - u_l)\,dx.$$

By assumption $\langle I_R'(u_n) - I_R'(u_l), u_n - u_l \rangle$ converges to 0 as n and l tend to $+\infty$. In what follows, C denotes a generic positive constant.

Using Hölder's inequality and Sobolev's embeddings, we obtain:

$$\left| \int_\Omega m(x)(g_R(u_n) - g_R(u_l))(u_n - u_l)\, dx \right|$$
$$\leq C|m|_0 \int_\Omega [(|u_n|^{q-1} + 1)|u_n - u_l| + (|u_l|^{q-1} + 1)|u_n - u_l|]$$
$$\leq C(\|u_n - u_l\|_{L^{\mu p^*}} + \|u_n - u_l\|_{L^p}),$$

for some $\mu \in]0,1[$. Observe that for any $a, b \in \boldsymbol{R}^N$,

$$(*) \qquad |a-b|^p \leq C\{(|a|^{p-2}a - |b|^{p-2}b)(a-b)\}^{s/2}(|a|^p + |b|^p)^{1-s/2}$$

with

$$s = \begin{cases} p & \text{for } 1 < p \leq 2 \\ 2 & \text{for } p \geq 2. \end{cases}$$

By Hölder's inequality and (*), we obtain:

$$\int_\Omega |\nabla u_n - \nabla u_l|^p\, dx$$
$$\leq C\left\{\left(\int_\Omega |\nabla u_n|^{p-2}\nabla u_n - |\nabla u_l|^{p-2}\nabla u_l\right)(\nabla u_n - \nabla u_l)\right\}^{s/2}\left(\int_\Omega |\nabla u_n|^p + |\nabla u_l|^p\right)^{1-s/2}$$

From the above inequalities, (u_n) converges strongly in $W_0^{1,p}(\Omega)$. The lemma is thus proved.

The next lemma shows that I_R satisfies the geometric assumptions of the Mountain Pass Theorem [2].

Lemma 2. *Under the assumptions of Theorem* 1,

a) *there exists* $\rho, \delta > 0$ *such that* $\forall\, u : \|u\| = \rho$ *implies* $I_R(u) > \delta$;

b) *there exists* $u_0 \in W_0^{1,p}(\Omega)$, *nonnegative and* $\|u_0\| > R_0$ *such that* $I_R(u_0) \leq 0$.

Proof. a) From g_3) and the construction of G_R, we have

$$G_R(u) \leq Bu^q + Cu \qquad \forall u > 0.$$

Also from g_1) and the fact that $G_R = G$ in $[0, R]$, we obtain

$$\forall \varepsilon > 0 \qquad \exists \delta_\varepsilon > 0 : G_R(u) \leq \varepsilon u^q \qquad \forall u < \delta_\varepsilon.$$

Put $h(u) = \int_\Omega m(x) G_R(u(x))\, dx$. To have a) it is enough to prove that $h(u) =$

$o(\|u\|^p)$ when $\|u\|$ goes to zero.

$$|h(u)| \leq \int_\Omega |m|\,|G_R(u)| \leq C|m|_0[\varepsilon\|u\|^q + B\|u\|^q + C\|u\|^\gamma]$$

$$\leq C|m|_0\|u\|^p[\varepsilon\|u\|^{q-p} + B\|u\|^{q-p} + C\|u\|^{\gamma-p}],$$

for any $\gamma \in]p,p^*[$. Thus $h(u) = o(\|u\|^p)$ when $\|u\| \to 0$.

b) We have $G_R(u) \geq R_0^{-q}G(R_0)u^q$ $\forall u \geq R_0$. Thus we can choose $\varphi \in W_0^{1,p}(\Omega)$, nonnegative, and $\operatorname{supp}\varphi \subset \Omega^+ := \{x \in \Omega \mid m(x) > 0\}$.

Hence we have:

$$I_R = (t^{1/p}\varphi) = \frac{t}{p}\|\varphi\|^p - \int_\Omega m(x)G_R(t^{1/p}\varphi(x))\,dx \leq Ct - Ct^{q/p}.$$

Thus, for t_0 sufficiently large, $u_0 = t_0^{1/p}\varphi$ satisfies $\|u_0\| > R_0$ and $I_R(u_0) \leq 0$.

By the usual Mountain Pass Theorem, we know that there exists a critical point of I_R which we denote by u_R, and a corresponding critical value of I_R, denoted by c_R such that $c_R \geq \delta$. Since $(u_R)^+ := \max(u_R, 0) \in W_0^{1,p}(\Omega)$ is also a solution, we may assume $u_R \geq 0$. Positivity of u_R follows from Harnack's inequality (see J. Serrin [9]).

We prove now that u_R is also solution of $(\mathscr{P}_p)$.

3. Existence results

3.1. A priori estimate of u_R

We have the following Lemma.

Lemma 3. *There exists a constant* $C_1 > 0$ *such that*

$$\|u_R\|^p \leq C_1(1 + M_R|m|_0)$$

Proof. We have

$$c_R \leq \max I_R(t^{1/p}\varphi), \qquad I_R(t^{1/p}\varphi) \leq Ct - Ct^{q/p},$$

so c_R is bounded that is $c_R \leq C$ for all $R > R_0$.

We can write

$$qc_R = qI_R(u_R) - I_R'(u_R)u_R,$$

that is

$$qc_R = \left(\frac{q}{p} - 1\right)\|u_R\|^p + \int_\Omega m(x)[u_R g(u_R) - qG_R]\,dx.$$

Hence

$$\|u_R\|^p \leq C(1 + M_R|m|_0). \tag{1}$$

3.2. Bootstrap argument

We adapt the Moser's iteration scheme used by Ôtani [7].

Proposition 1. *If a weak solution u of $(\mathscr{P}_p)_R$ belongs to $L^q(\Omega)$ with $p < q < p^*$, then $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.*

Lemma 4. *Let u be a weak solution as in Proposition 1. Fix two sequences of numbers (q_k) and (C_k) by*

$$q_{k+1} = q_k^* \frac{p^*}{p}, \qquad q_k^* = q_k - q + p; \tag{2}$$

$$C_{k+1} = K^{p/q_k^*}(q_k - q + 1)^{-1/q_k^*}\left(\frac{q_k^*}{p}\right)^{p/q_k^*}(|m|_0 C_R)^{1/q_k^*} C_k^{q_k/q_k^*} \qquad (k \in \boldsymbol{N}) \tag{3}$$

and $C_1 = \|u\|_{q_1}$, with $q_1 = q$. Then u belongs to $L^{q_k}(\Omega)$ for all $k \in \boldsymbol{N}^*$, and satisfies

$$\|u\|_{L^{q_k}} \leq C_k \qquad \text{for all } k \in \boldsymbol{N}^*. \tag{4}$$

Proof of Lemma 4. We prove (4) by induction. It is obvious for $k = 1$. Suppose that (4) holds for k. Let $\xi_n, n \in \boldsymbol{N}$, be C^1 functions such that:

$$\xi_n(s) = \begin{cases} s & \text{if } s \leq n \\ n+1 & \text{if } s \geq n+2 \\ 0 \leq \xi_n'(s) \leq 1 & \text{for all } s \in \boldsymbol{R}^+. \end{cases}$$

Put $u_n = \xi_n(u)$ then $m(x)u_n^{q-1} \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ for all $q \geq 2$. Then we can multiply $(\mathscr{P}_p)_R$ by $u_n^{q_k - q + 1}$ and integrate over Ω to get

$$\begin{aligned} \int_\Omega -\Delta_p u u_n^{q_k-q+1} &= \int_\Omega m(x) g_R(u) u_n^{q_k-q+1}\,dx \\ &\leq |m|_0 C_R \int_\Omega u^{q-1} u_n^{q_k-q+1}\,dx \\ &\leq |m|_0 C_R \int_\Omega u^{q_k}\,dx \leq |m|_0 C_R \|u\|_{q_k}^{q_k}. \end{aligned} \tag{5}$$

Here we have, by Sobolev's embeddings,

$$\begin{aligned} \int_\Omega -\Delta_p u u_n^{q_k-q+1} &= (q_k - q + 1)\int_\Omega |\nabla u|^p \xi_n'(u) u_n^{q_k-q}\,dx \\ &\geq (q_k - q + 1)\int_\Omega |\nabla u_n|^p u_n^{q_k-q}\,dx \end{aligned} \tag{6}$$

$$\geq (q_k - q + 1)\left(\frac{p}{q_k^*}\right)^p \int_\Omega |\nabla(u_n^{q_k^*/p})|^p \, dx$$
$$\geq K^{-p}(q_k - q + 1)\left(\frac{p}{q_k^*}\right)^p \|u_n^{q_k^*/p}\|_{p^*}^p.$$

Then, combining (5) and (6), we deduce

$$\|u_n^{q_k^*/p}\|_{L^{p^*}}^p = \|u_n\|_{L^{q_{k+1}}}^{q_k^*} \leq K^p(q_k - q + 1)^{-1}\left(\frac{q_k^*}{p}\right)^p |m|_0 C_R C_k^{q_k}.$$

Hence, by letting n tend to $+\infty$, we obtain (4) with $k+1$.

Proof of Proposition 1. Put $E_k = q_k \ln C_k$, then in view of (3) and (4), we find

$$E_{k+1} = p^*(\ln K - p^{-1}\ln(q_k - q + 1) + \ln q_k^* - \ln p) + a\ln(|m|_0 C_R) + aE_k$$
$$E_{k+1} \leq r_k + a\ln(|m|_0 C_R) + aE_k,$$

where $r_k := p^* \ln K q_k^*$ and $a := p^*/p > 1$. Then, we obtain

(7) $$E_k \leq r_{k-1} + ar_{k-2} + \cdots + a^{k-2} r_1 + (a + a^2 + \cdots + a^{k-1})\ln(|m|_0 C_R) + a^{k-1}E_1.$$

Since

(8) $$q_k = a^{k-1}\left(q - \frac{p^*(q-p)}{p^*-p}\right) + \frac{p^*(q-p)}{p^*-p} = a^{k-1}\left(\frac{p(p^*-q)}{p^*-p}\right) + \frac{p^*(q-p)}{p^*-p},$$

we obtain

$$r_k = p^* \ln K\left[a^{k-1}\left(\frac{p(p^*-q)}{p^*-p}\right) + \frac{p^*(q-p)}{p^*-p} - (q-p)\right].$$

Therefore,

(9) $$r_k \leq p^* \ln K a^{k-1} p \leq (k-1)p^* \ln a + b \qquad \text{where } b := p^* \ln Kp.$$

From (7)–(9), we deduce

$$E_k \leq a^{k-1}E_1 + \frac{a(a^{k-1}-1)}{a-1}\ln(|m|_0 C_R) + \{b(a-1) + p^* \ln a\}\frac{(a^{k-1}-1)}{(a-1)^2}.$$

Consequently,

(10) $$\|u\|_\infty \leq \lim_{k\to\infty} \sup \|u\|_{q_k} \leq \|u\|_q^{(q(p^*-p))/(p(p^*-q))}(|m|_0 C_R)^{p^*/(p(p^*-q))} \exp\frac{\{b(a-1) + p^* \ln a\}p}{(p^*-p)(p^*-q)}.$$

Then $u \in L^\infty(\Omega)$.

Proof of Theorem 1. From (1), (10), and Sobolev's embeddings, we get

$$\|u\|_\infty \le C(1 + M_R|m|_0)^{(q(p^*-p))/(p^2(p^*-q))}$$
$$\times \left(\frac{M_R}{R^q} + Bq\right)^{p^*/(p(p^*-q))} (|m|_0)^{p^*/(p(p^*-q))} \exp \frac{\{b(a-1) + p^* \ln a\}p}{(p^*-p)(p^*-q)}.$$

If we take $M_R = o(R^{(p^*(p+q))/(p^2(p^*-q))})$, for R sufficiently large, we deduce $\|u\|_\infty < R$. Hence u is also a solution of $(\mathscr{P}_p)$, and from the result of DiBenedetto [5], u enjoys $C^{1+\alpha}(\Omega)$-regularity.

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