Advanced Studies in Contemporary Mathematics 26 (2016), No. 4, pp. 607 - 625

A REVIEW ON WEAK SIMILARITIES RELATIONS OF BOUNDED OPERATORS

ABDELHALIM AZZOUZ *, GHOUTI DJELLOULI, AND BEKKAI MESSIRDI

Abstract. The relations of similarity, quasi-similarity, almost similarity, asymptotic similarity and compalence are discussed in this paper. Various spectral results on these notions are presented. In addition, we establish the link between invariant subspaces and hyperinvariant subspaces with quasi-similarity, almost similarity and compalence operator properties.

2010 Mathematics Subject Classification. 47A10, 47A11 and 47A15.

Keywords and phrases. Similarity and Quasi-similarity; Almost similarity; Asymptotic similarity; Compalence; Essential spectrum; Invariant and hyperinvariant subspaces.

1. Introduction

Let X and Y be nonempty sets and $S : X \longrightarrow Y$ is a bijective operator. Consider an operator $A : X \longrightarrow X$ and let y be a given element of X. Define $\Omega = \{x \in X : Ax = y\}$. Define $B : Y \longrightarrow Y$ by $B = S^{-1}AS$ and let $\Lambda = \{z \in Y : Bz = Sy\}$. Using only basic properties of the composition of operators it is easy to see that $\Omega = S^{-1}(\Lambda)$ and $\Lambda = S(\Omega)$. Therefore, finding the solution set Ω is equivalent to finding the set Λ. In other words, solving $Ax = y$ in the set X yields to to solving $Bz = Sy$ in the set Y. In some concrete situations, solve one of the two problems is much easier than the other. In practice, in order to solve $Ax = y$ one applies the transform S to get the transformed equation $Bz = Sy$, which is considered easier to be solved. Then one finds somehow the solution set Λ, and getting the set Ω by applying the inverse transform S^{-1} to each element of Λ. This made the foundation of the transform methods, like the similarity between operators on Banach spaces.

Unitarily equivalence is the natural concept of equivalence between Hilbert space operators. Similarity of operators is a weaker concept than unitary equivalence. Much of the study of similarity has been driven by a desire to characterize operators. In fact, it is easy to see that similarity preserves the spectrum and its various parts. Some other concepts related to operators may be studied via similarity. For instance, when we focus on the reflexivity of operators, we observe that reflexivity is the same for two similar operators; invariant subspaces of an operator also can be identified in terms of invariant subspaces of a similar operator.

Throughout this paper, H and K are Hilbert spaces and $\mathcal{B}(H,K)$ denotes the space of all bounded linear operators from H to K and $\mathcal{B}(H,H) = \mathcal{B}(H)$. A contraction on H is an operator $A \in \mathcal{B}(H)$ such that $\|Ax\| \leq \|x\|$ for all nonzero x in H. Let I and P_M denote respectively the identity operator on H and the orthogonal projection onto a subspace M of H. $\overline{M}$ denotes the closure of M in H. $N(A)$ and $R(A)$ are respectively the null space and the range of A.

For a bounded linear operator $A \in \mathcal{B}(H)$, $\rho(A)$, $\sigma(A)$, $\sigma_{ap}(A)$, $\sigma_p(A)$ and A^* denote the resolvent set, the spectrum, the approximate point spectrum, the point spectrum and the adjoint operator of A, respectively. Thus $\sigma_{ap}(A)$ consists of all $\lambda \in \mathbb{C}$ for which there exists a sequence of unit vectors $x_n \in H$ such that $(A - \lambda I)x_n \longrightarrow 0$ as $n \to \infty$. Further, let $\sigma_{su}(A) = \{\lambda \in \mathbb{C} \ : \ R(A - \lambda I) \neq H\}$ denote the surjective spectrum of A. It well known that $\sigma_{su}(A) = \sigma_{ap}(A^*)$ and $\sigma_{ap}(A) = \sigma_{su}(A^*)$. Moreover, $\sigma_{su}(A)$ is compact with $\partial\sigma(A) \subseteq \sigma_{su}(A) \subseteq \sigma(A) = \sigma_{su}(A) \cup \sigma_p(A)$. $r(A)$ denotes the spectral radius of A.

Also, let us recall that an operator $A \in \mathcal{B}(H)$ is Fredholm if its null space $N(A)$ is of finite dimension and its range $R(A)$ is of finite codimension in H; the range is then automatically closed. The essential spectrum $\sigma_e(A)$ of A is the set of complex numbers λ for which $(A - \lambda I)$ is not Fredholm and $\rho_e(A) = \mathbb{C} \setminus \sigma_e(A)$ is the essential resolvent set of A. Let π be the quotient map from $\mathcal{B}(H)$ onto $\mathcal{B}(H)/\mathcal{K}(H)$, where $\mathcal{K}(H)$ is the ideal of all compact operators on H. In particular, its well-known that $\sigma_e(A) = \sigma(\pi(A))$. The index of a Fredholm operator $A \in \mathcal{B}(H)$ is $ind(A) = \dim N(A) - \dim(H/R(A))$. $\mathcal{F}(H)$ denotes the set of all Fredholm operators on H. The operator A is Weyl if it is Fredholm of index zero.

Recall that $A \in \mathcal{B}(H)$ is called essentially normal if $(A^*A - AA^*) \in \mathcal{K}(H)$. $A \in \mathcal{B}(H)$ is hyponormal iff $AA^* \leq A^*A$ in the partial order for Hermitian operators, or equivalently iff $\|A^*x\| \leq \|Ax\|$ for all x in H. An operator A is said to be quasinormal if A commutes with A^*A. It is known that every quasinormal operator is hyponormal. An operator A on H is subnormal if there is a Hilbert space K containing H and a normal operator N on K such that $NH \subset H$ and $A = N_{|H}$.

Definition 1.1. *Let $A \in B(H)$ and $B \in \mathcal{B}(K)$. An operator $S \in \mathcal{B}(H, K)$ intertwines A and B (or A is intertwined to B through S) if $SA = BS$. If there is an $S \in \mathcal{B}(H, K)$ with dense range intertwining A to B, then A is densely intertwined to B. If S has dense range and is injective, then it is quasi-invertible (or a quasiaffinity). If a quasi-invertible S intertwines A to B, then A is a quasiaffine transform of B.*

A is similar to B (denoted by $A \overset{s}{\sim} B$) if there is an invertible operator $S : H \longrightarrow K$ such that $SA = BS$ or equivalently $A = S^{-1}BS$.

A and B are unitarily equivalent, denoted by $A \overset{u.e}{\sim} B$, if there exists a unitary transformation intertwining them. Then $A \overset{u.e}{\sim} B$ if there exists a unitary operator $W \in \mathcal{B}(H, K)$ such that $B = WA_1W^*$. Thus, Unitary equivalence is the special case of similarity through a (surjective) isometry.

A and B are approximately unitarily equivalent, denoted by $A \overset{a.u.e}{\sim} B$, if there exists a sequence $(W_n)_{n\in\mathbb{N}^*}$ of unitary operators in $\mathcal{B}(H, K)$, such that $B - W_nAW_n^* \in \mathcal{K}(K)$ for all $n \in \mathbb{N}^*$ and

$$\lim_{n\to\infty} \|B - W_nAW_n^*\| = 0$$

The operators A and B are quasi-similar, denoted by $A \overset{q.s}{\sim} B$, if there are quasi-affine transform of each other, equivalently, if there exists quasi-invertible operators $S \in \mathcal{B}(H, K)$ and $T \in B(K, H)$ such that $BS = SA$ and $AT = TB$.

A and B are said to be almost similar (denoted by $A \overset{a.s}{\sim} B$) if there exists an invertible operator S such that the following two conditions hold:

a) $A^*A = S^{-1}(B^*B)S$,

b) $A^* + A = S^{-1}(B^* + B)S$.

The operators A and B are compalent, denoted by $A \overset{c}{\sim} B$, if there exist a unitary operator $W \in \mathcal{B}(H,K)$ and a compact operator $K \in \mathcal{K}(K)$ such that

$$B = WAW^* + K$$

Similarity of some operators is discussed in the literature. Campbell-Wright has considered in [5] the problem of similarity of compact composition operators. For similarity of subnormal operators, one can see [8]. Shields has presented in [30] some conditions equivalent to the similarity of the multiplication operator M_z on two weighted Hardy spaces.

If A is a quasiaffine transform of B and B is a quasiaffine transform of A, then A and B are quasi-similar. If an invertible operator S (with a bounded inverse) intertwines A to B (so that S^{-1} intertwines B to A), then A and B are similar. Unitary equivalence is the special case of similarity through a (surjective) isometry: operators A and B are unitarily equivalent if there exists a unitary transformation S intertwining them. Two operators are considered the "same" if they are unitarily equivalent since they have the same properties of invertibility, normality, spectral picture (norm, spectrum, spectral radius).

It is not difficult to show that the similarity and quasi-similarity are equivalence relations and both are weaker than unitary equivalence. Moreover, it is easy to see that similarity preserves the spectrum and the various parts of the spectrum, ie similar operators have the same spectrum, the same point spectrum, the same approximate point spectrum, the same compression spectrum and hence the same spectral radius.

Lemma 1.2. *Suppose that A and B are similar bounded operators on a Hilbert space H, then A and B have the same:*

1) Spectrum,

2) Point spectrum,

3) Approximate point spectrum.

Note that the same cannot be said about quasi-similarity. In particular, if two normal operators are quasi-similar, then they are unitarily equivalent. The following result gives us conditions under when an operator A happens to be unitarily equivalent to its adjoint A^*.

Proposition 1.3. *A is unitarily equivalent to its adjoint if and only if A is the product of a symmetry and a self-adjoint operator.*

Proof. If $A = SB$ where $S = S^* = S^{-1}$ is a symmetry and B is self-adjoint, then $AS = SBS = S(SB)^* = SA^*$, so that $SAS = A^*$ and A is unitarily equivalent to its adjoint.

Conversely, suppose $AU = UA^*$ where U is unitary. Then A commutes with U^2. Let $e^{i\theta} dE_\theta$ be the spectral representation of U^2. If $V = e^{i\theta/2} dE_\theta$, then V is a unitary operator, $V^2 = U^2$ and V commutes with every operator that commutes with U^2. It follows that V commutes with U and A. Therefore, $S = V^{-1}U$ is a symmetry and $AS = SA^*$. Hence $A = S(AS)$ is the product of a symmetry and a self adjoint operator. □

Corollary 1.4. *A unitary operator U is similar to its inverse if and only if U is the product of two symmetries.*

Let $A \in \mathcal{B}(H)$ be a contraction. If A is unitarily equivalent to a unitary operator, then A is normal.

The Fuglede-Putnam theorem help us to obtain the relationship between similarity of normal operators and their adjoints.

Theorem 1.5. *(Fuglede-Putnam theorem) Assume that $A, B, C \in \mathcal{B}(H)$, where A and B are normal, and $AC = CB$. Then $A^*C = CB^*$.*

As generalization of Fuglede-Putnam theorem, it seems that if A, B and C be operators on H such $AC = CB$ implies $A^*C = CB^*$, if C is either one-one or has dense range, then A and B are normal operators.

The following result shows that unitary equivalence preserves normality of operators.

Proposition 1.6. *1) If A is a normal operator on H and $B \in \mathcal{B}(H)$ is unitarily equivalent to A, then B is normal.*

2) If $A, B \in \mathcal{B}(H)$ are similar, then A^* and B^* are similar.

3) Two similar normal operators are unitarily equivalent and conversely.

Theorem 1.7. *An invertible operator $A \in B(H)$ is a product of two selfadjoint operators if and only if A is similar to A^*.*

Proof. Suppose A is invertible with $A = BC$, $B = B^*$ and $C = C^*$. Since A is invertible, then $I = AA^{-1} = (BC)(C^{-1}B^{-1})$. This shows that B and C are invertible and hence CB is invertible. $A^* = CB = CIB = CAA^{-1}B = CA(BC)^{-1}B = CAC^{-1}B^{-1}B = CAC^{-1}$. This shows that $A \overset{s}{\sim} A^*$.

Conversely, suppose that A is invertible and $A \overset{s}{\sim} A^*$. Since A is invertible and by the polar decomposition theorem, A has a unique polar decomposition $A = UP$, where U is unitary (not necessarily selfadjoint) and $P = (A^*A)^{1/2}$ is a positive selfadjoint operator. We use the similarity of A and A^* to show that U must indeed be self-adjoint. $A \overset{s}{\sim} A^*$ implies that $UP = S^{-1}(UP)^*S$. Without loss of generality, let $S = I$. In that case $U = U^*$ which proves that U is selfadjoint. □

Proposition 1.8. *([29], Corollary 3.3). Let H and K be separable infinite dimensional complex Hilbert spaces. If $A \in \mathcal{B}(H)$ and $B \in \mathcal{B}(K)$ have disjoint essential spectra and $S \in B(H, K)$ intertwines A and B, then S is compact.*

Fialkow showed in [10] that, the essential spectra of arbitrary quasi-similar operators on a separable Hilbert space always have non-empty intersection. Moreover, Herrero [14] showed that, for every pair $A \in \mathcal{B}(H)$ and $B \in \mathcal{B}(K)$ of quasi-similar operators, each connected component of $\sigma_e(A)$ touches $\sigma_e(B)$, and vice versa. This result now has an attractive sheaf-theoretic proof (see [9]).

Quasi-similarity preserves the existence of non-trivial hyperinvariants subspaces. But it is a weak equivalence relation that doesn't preserve much spectral properties, such as the spectrum. This happens only in some special cases, for instance if A and B are quasi-similar hyponormal operators, see [7]); or whenever A and B have totally disconnected spectra, see [[17], Corollary 2.5]. Therefore, it is not quite surprising that, if A and B are intertwined by a quasi-affinity, the preservation of "certain" spectral properties from A to B requires some other additional conditions. Sz. Nagy and Foias [25] gave the first examples of quasi-similar operators with different spectra. Herrero [14] showed recently that if A and B are quasi-similar bounded operators on H, then every component of $\sigma_e(A)$ meets $\sigma_e(B)$ and vice versa. The key to this theorem is the result on n-multicyclic operators. Various properties are preserved by quasi-similarity, such as being triangularizable or being n-multicyclic, other spectral properties be presented in this paper. A description of the set of operators quasi-similar to the unilateral shift is also an interesting problem.

This article concerns the spectral theory of many classes of operators defined by means of some weak similarity notions such as quasi-similarity, almost similarity, asymptotic similarity and compalence. Particular emphasis is given to the Fredholm theory and local spectral theory of these classes of operators. Our main interest concerns the preservation of certain spectral properties : spectrum, essential spectrum, invariant and hyperinvariant subspaces for quasi-affine transform of operators. Precisely, several results are presented, such as those touching conservation of Fredholm character and index, equality of spectra of quasi-similar, selfadjoint, hyponormal, essentially normal and polaroid operators and also partial isometries.

The relevant definitions and background material will be collected in Section 1. In section 2, we collect some basic properties about similarity of linear operators in Hilbert spaces. We study some basic spectral results about quasi-similarity of operators in section 3, and where concepts and techniques from local spectral theory are used.

In section 4, we introduce the notion of almost similar operators and we investigate some results of hermitian, normal and partially isometric operators under almost similarity.

In section 5, we discuss some properties of asymptotic similarity. In particular, the spectrum, surjectivity spectrum, approximate point spectrum, single valued extension, Dunford's property and Bishop's property are preserved under asymptotic similarity.

In section 6, we introduce the concept of compalent operators. Compalence is different from the notions of weak similarity previously established in this paper, we indicate nevertheless that the compalence is an equivalence relation and the unitary equivalence implies compalence. Compalence preserves the Fredholmness, the essential spectrum and the index.

The weak notions of similarity allows to describe the problem of invariant and hyperinvariant subspaces under similarity, quasi-similarity and compalence. Similarity preserves invariant subspaces, furthermore if A and B are quasi-similar operators and if B has a nontrivial hyperinvariant subspace, we establish, in section 7, that A has a nontrivial hyperinvariant subspace. If in addition, A is normal, then the lattice of hyperinvariant subspaces for B contains a sublattice which is lattice isomorphic to the lattice of spectral projections for A.

2. Similarity of operators

In this section we collect some basic definitions and facts about similarity of linear operators in Hilbert spaces. We discuss several generalizations of this notion in the next sections. The following result expresses the nature of similar operators.

Theorem 2.1. *([24]) Let $A \in \mathcal{B}(H)$ be invertible. Then the following assertions are equivalent:*

1) A and A^{-1} are similar to a contraction.

2) A is similar to a unitary operator.

3) $\sup_{n\in\mathbb{Z}} \|A^n\| < \infty$.

Proof. The implications 2) $\Longrightarrow$ 1) $\Longrightarrow$ 3) are trivial. It remains to show 3) $\Longrightarrow$ 2). Suppose 3) and note $C = \sup_{n\in\mathbb{Z}} \|A^n\|$. Define a new norm on H by:

$$\{x\}^2 = \lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} \left\|A^k x\right\|^2 \quad \text{for all } x \in H \tag{1}$$

We have obviously, $\frac{1}{C}\|x\| \le \{x\} \le C\|x\|$ and $\{Ax\} \le \{x\}$ for all $x \in H$. □

Remark 2.2. *It is important to note that if A is invertible, that A and A^{-1} are both contractions is equivalent to say that A is unitary.*

A classical problem (the "Halmos problem") asks for a characterization of the operators which are similar to a contraction. Two important variants have appeared. One proposed by Dixmier in 1950 asks whether the similarity property for all uniformly bounded representations on a group G is equivalent to the amenability of G. The other one, proposed by Kadison in 1995, is about representations of a C^*-algebra. In particular, if A is similar to a contraction, then A satisfy the Von Neumann's inequality:

$$\|p(A)\| \le K \max_{|z|\le 1} |p(z)| \tag{2}$$

with $K = \|U\| \left\|U^{-1}\right\|$ and p is a polynomial. The question is now: if A is an operator on a Hilbert space such that (2) holds for any polynomial p, is A similar to a contraction? This question, due to Halmos, has been studied extensively, but still open. Some partial and interesting results have been obtained but the conclusions hold under a slightly stronger assumption on A. We mention a quantitative result of Bourgain in this direction [2]. If A is defined on a finite-dimensional Hilbert space H, with $dimH = n$, and if A satisfies (2), there exists an operator U such that $U^{-1}TU$ is a contraction, and:

$$\|U\| \left\|U^{-1}\right\| \le K^4 \log n$$

A subspace M of H is invariant under $A \in \mathcal{B}(H)$ if $A(M) \subset M$. If M is an invariant linear manifold for A, then its closure $\overline{M}$ is an invariant subspace for A. M is said to reduce A if both M and $M^\perp$ are invariant under A.

Proposition 2.3. *Let $A \in \mathcal{B}(H)$ and $B \in \mathcal{B}(K)$ be normal operators. If $S \in \mathcal{B}(H, K)$ intertwines A to B, then*

1) $N(S)$ reduces A and $\overline{R(S)}$ reduces B so that $A_{|N(S)^\perp} \in \mathcal{B}(N(S)^\perp)$ and $B_{|\overline{R(S)}} \in \mathcal{B}(\overline{R(S)})$. Moreover,

2) $A_{|N(S)^\perp}$ and $B_{|\overline{R(S)}}$ are unitarily equivalent .

A special case of Proposition 2.2 says that if a quasi-invertible bounded linear transformation intertwines two normal operators, then these normal operators are unitarily equivalent . This happens, in particular, when S is invertible.

Proposition 2.4. *Two similar normal operators are unitarily equivalent.*

We consider now natural extensions of certain spectral results, where similarity is replaced by weaker conditions, and where concepts and techniques from local spectral theory are used.

Definition 2.5. *An operator $A \in \mathcal{B}(H)$ is said to have Bishop's property (β), if, for each open subset U of $\mathbb{C}$ and every sequence of analytic functions $f_n : U \longrightarrow H$ for which $(A - \lambda I)f_n(\lambda) \longrightarrow 0$ as $n \to \infty$, locally uniformly on U, it follows that $f_n(\lambda) \longrightarrow 0$ as $n \to \infty$, again locally uniformly on U.*

We will concentrate on proving the following result.

Theorem 2.6. *Suppose that $A \in \mathcal{B}(H)$ and $B \in \mathcal{B}(K)$ both have property (β) and are densely similar (i.e. that $S \in \mathcal{B}(H,K)$ and $T \in \mathcal{B}(K,H)$ are operators with dense ranges for which $BS = SA$ and $AT = TB$). Then $\sigma(A) = \sigma(B)$, $\sigma_e(A) = \sigma_e(B)$ and $ind(A - \lambda I) = ind(B - \lambda I)$ for all $\lambda \in \rho_e(A)$.*

Proof. Since A and B have property (β) and are densely similar, $\sigma_e(A) = \sigma_e(B)$ and $\sigma(A) = \sigma(B)$ [[21], Theorem 3.7.15]. □

Example 2.7. *Let A be the unilateral right shift on $l^2(\mathbb{N})$ and let B be the unilateral weighted right shift on $l^2(\mathbb{N})$ with weight sequence $(\frac{1}{n+1})_{n\in\mathbb{N}}$. Then A has property (β), $\sigma(A) = \overline{D}$, and $\sigma_e(A) =$ torus, while B is compact with $\sigma(B) = \sigma_e(B) = \{0\}$. Thus $\sigma_e(A)$ and $\sigma_e(B)$ are disjoint. However, the pair (A, B) is intertwined by a quasi-affinity, namely the multiplication operator S on $l^2(\mathbb{N})$ given by $Sx = \left(\frac{x_n}{n!}\right)_{n\in\mathbb{N}}$, $x = (x_n)_n \in l^2(\mathbb{N})$. It is easy to see that S is injective, has dense range, and that $BS = SA$.*

3. Quasi-similarity of operators

Quasi-similarity was introduced by Nagy and Foias [25], in their theory on infinite dimensional analogue of the Jordan form for certain classes of contractions as a means of studying their invariant subspace structures. In finite dimensional spaces, quasi-similarity is the same thing as similarity but in infinite dimensional spaces it is a much weaker relation. It is easily verified that quasi-similarity is an equivalence relation and also that A^* is quasisimilar to B^* whenever A is quasi-similar to B and that similar operators are, of course, quasi-similar but not conversely. The following lemma links similarity of operators with quasi-similarity. Here we shall see what happens if the conditions on the intertwining operator are weakened, for example to that of being a quasi-similarity.

Lemma 3.1. *$A \in \mathcal{B}(H)$ and $B \in \mathcal{B}(K)$ are similar operators, then they are quasi-similar.*

Proof. There exists an invertible operator $S \in \mathcal{B}(H,K)$ such that $SA = BS$. This implies that $S^{-1}B = AS^{-1}$, where $S^{-1} \in \mathcal{B}(K,H)$ which implies that $A \overset{s}{\sim} B$. □

The converse of the above lemma is false. Indeed, the example 2 of [34] shows that two quasi-similar quasi-normal operators need not be similar even if both operators are pure (ie. if M is a reducing subspace of a pure operator X and $X_{|M}$ is normal, then $M = \{0\}$).

Let A and B be quasi-similar hyponormal operators on infinite dimensional Hilbert spaces, Clary proved in [7] that $\sigma_e(A) = \sigma_e(B)$. It is also showed that there are several cases where $\sigma_e(A) = \sigma_e(B)$, for example if A and B are biquasitriangular, if A and B are both weighted shifts, if A and B are partial isometries or if A and B are quasinormal [34].

Clary obtained in [7] (see also [22]) the following result:

Theorem 3.2. *Quasi-similar hyponormal operators have equal spectra.*

Proof. If $A \in \mathcal{B}(H)$ and $B \in \mathcal{B}(K)$ are quasi-similar hyponormal operators, then for any complex number λ, $(A-\lambda I)$ and $(B-\lambda I)$ are also quasi-similar and hyponormal. Let us remark that if A is invertible, B is hyponormal and $S \in \mathcal{B}(H, K)$ has a dense range and satisfies $SA = BS$, then B is also invertible. Thus, $(A-\lambda I)$ and $(B-\lambda I)$ are both invertible or both non-invertible, which implies that $\sigma(A) = \sigma(B)$. □

Theorem 3.3. *Let $A, B, S \in \mathcal{B}(H)$ such that $AS = SB$ where A and B satisfy Fuglede-Putnam theorem and S is a quasiaffinity. Then, $\sigma(A) = \sigma(B)$, $\sigma(AA^*) = \sigma(BB^*)$ and $\sigma(A^*A) = \sigma(B^*B)$.*

Proof. Note that $AS = SB$ implies $A^*S = SB^*$ and thus $BS^* = S^*A$. Since S is a quasiaffinity it follows that A and B are quasi-similar. It now follows from Theorem 3.2 that A and B are unitarily equivalent normal operators. Hence $\sigma(A) = \sigma(B)$.

Also using the equations $AS = SB$, $A^*S = SB^*$, $BS^* = S^*A$ and $B^*S^* = S^*A^*$ we have

$$\begin{aligned} A^*AS &= A^*SB = SB^*B \\ B^*BS^* &= B^*S^*A = S^*A^*A \\ AA^*S &= ASB^* = SBB^* \\ BB^*S^* &= BS^*A^* = S^*AA^* \end{aligned}$$

From the first two equations, A^*A and B^*B are quasi-similar normal positive operators. Hence $\sigma(A^*A) = \sigma(B^*B)$. We have also $\sigma(AA^*) = \sigma(BB^*)$, since AA^* and BB^* are quasi-similar normal operators from the last two equations. □

Clary asked whether this extends to the essential spectra. This problem received considerable attention. The final solution, due to Putinar [28] and, independently, to Yang [33], revealed that the assumption of hyponormality is not really needed: all that matters is that hyponormal operators have the Bishop's property (β). From the result of Theorem 3.3 the following corollary is immediate:

Corollary 3.4. *Let $A, B, S \in \mathcal{B}(H)$ such that $AS = SB$ where A and B satisfy Fuglede-Putnam theorem and S is a quasiaffinity. Then, $\sigma_e(A) = \sigma_e(B)$, $\sigma_e(AA^*) = \sigma_e(BB^*)$ and $\sigma_e(A^*A) = \sigma_e(B^*B)$.*

The following result appears in [35]:

Theorem 3.5. *Suppose that A and B are hyponormal operators and there exist quasiaffinities S and T such that $SA = BS$ and $AT = TB$. If either S or T is compact, then $\sigma_e(A) = \sigma_e(B)$.*

Definition 3.6. *We say that $A \in \mathcal{B}(H)$ has the single valued extension property (SVEP) at $\lambda_0 \in \mathbb{C}$, if for an arbitrary open neighborhood U of λ_0, $f = 0$ is the only analytic function $f : U \longrightarrow H$ such that $(A-\lambda I)f(\lambda) = 0$ for all $\lambda \in U$. We will say that A has the SVEP if A has this property at every $\lambda \in \mathbb{C}$.*

For $A \in \mathcal{B}(H)$ having the single valued extension property and for $x \in H$ we can consider the set $\rho_A(x)$ of elements $\lambda_0 \in \mathbb{C}$ such that there exists an analytic function $f(\lambda)$ defined in a neighborhood of λ_0, with values in H, which verifies $(A-\lambda I)f(\lambda) = x$. We denote $\sigma_A(x) = \mathbb{C} \backslash \rho_A(x)$ and $H_A(F) = \{x \in H \ : \ \sigma_A(x) \subseteq F\}$ where $F \subset \mathbb{C}$.

A satisfies Dunford's condition (C) if for every set $F \subset \mathbb{C}$ the linear manifold $H_A(F)$ is closed in H.

If A has the single valued extension property, then $\sigma(A) = \sigma_{su}(A)$, and if A^* has the single valued extension property, then $\sigma(A) = \sigma_{ap}(A)$.

When we say $A \in \mathcal{B}(H)$ satisfies condition (C), we are also asserting that it has the single valued extension property. Indeed, Bishop's property (β) implies Dunford's property (C), and property (C) implies the single valued extension property.

If A is quasi-similar to B and A has some nice properties one can often show that $\sigma(A) \subset \sigma(B)$ or a similar result on local spectrums of operators A and B.

Proposition 3.7. *([31]) Let $A, B, S \in \mathcal{B}(H)$ where $AS = SB$. Assume that S is quasi-invertible and A has a single valued extension property. Then B has a singled valued extension property and $\sigma_A(Sx) \subset \sigma_B(x)$ for all $x \in H$.*

Proof. Fix y in H. Assume that $f_1(.)$ and $f_2(.)$ are two analytic extensions of $(B - \lambda I)^{-1}y$ with common domain Ω. Then $(A-\lambda I)Sf_i(\lambda) = S(B-\lambda I)f_i(\lambda) = Sy$ for all $\lambda \in \Omega$ and $i = 1, 2$. Thus $Sf_i(\lambda)$ is an analytic extension of $(A-\lambda I)^{-1}Sy$ for $i = 1, 2$. By the single valued extension property for A, it follows that $Sf_1(\lambda) = Sf_2(\lambda)$ for $\lambda \in \Omega$, whence the first conclusion follows.

For $w \in H$, let $\widetilde{w}(.)$ be the analytic extension of $(B - \lambda I)^{-1}w$. Then, $(A - \lambda I)S\widetilde{w}(\lambda) = S(B - \lambda I)\widetilde{w}(\lambda) = Sw$. Thus, $S\widetilde{w}(.)$ is an analytic extension of $(A - \lambda I)^{-1}Sw$ which implies $\sigma_A(Sw) \subset \sigma_B(w)$. □

Theorem 3.8. *Let $A, B, S \in \mathcal{B}(H)$ where S is quasi-invertible and $AS = SB$. If A satisfies the condiction (C), then $\sigma(A) \subset \sigma(B)$.*

Proof. It follows from precedent proposition that $\sigma_A(Sw) \subset \sigma_B(w)$ for all $w \in H$. But $\sigma_B(w)$ is a compact and non-empty subset ot the complex plane and so $\sigma_B(w) \subset \sigma(B)$. Also, $\{Sw : w \in H\}$ is dense in H and hence by condition (C), $\sigma_A(w) \subset \sigma(B)$ for all $w \in H$. Thus $\sigma(A) \subset \sigma(B)$. □

The following result shows that, quasi-similar operators satisfying Dunford's condition have the same spectrum.

Corollary 3.9. *Let $A, B \in \mathcal{B}(H)$ be quasi-similar and satisfy the condition (C), then $\sigma(A) = \sigma(B)$.*

We also note that quasi-similarity preserves Fredholm property. Since invertible operators are Fredholm operators of index zero, it follows that the Fredholm property as well as the index are preserved under similarity. Does this hold for quasi-similarities also? Williams [35] showed the following result:

Theorem 3.10. *Suppose that A and B are quasi-similar hyponormal operators. Then A is a Weyl operator if and only if B is a Weyl operator.*

Definition 3.11. *An operator $A \in \mathcal{B}(H)$ is said to be polaroid if every isolated point of the spectrum $\sigma(A)$ is a pole of the resolvent of A. A is said to be hereditarily polaroid if every part of A is polaroid.*

Note that (see [1]), A is polaroid $\Leftrightarrow$ A^* is polaroid, and the property of being hereditarily polaroid is similarity invariant, but is not preserved by a quasi-affinity. Every hereditarily polaroid operator has SVEP, see [[1], Theorem 2.8].

Theorem 3.12. *Let $A \in \mathcal{B}(H)$ and $B \in \mathcal{B}(K)$ be quasi-similar.*

1) If both A and B have property (β), then A is polaroid if and only if B is polaroid.

2) If both A and B satisfy the property (C) then A is polaroid if and only if B is polaroid.

Foias and Pearcy [13], studied the pertinent question : is every quasi-nilpotent operator are quasi-similar to a compact operator? The next result indicates that the answer might be affirmative.

Theorem 3.13. *([13]) Every nilpotent operator in $B(H)$ is quasi-similar to a compact operator. Furthermore, there exist quasi-nilpotent operators in $B(H)$ that does not commute with any nonzero compact operator, and hence are not quasi-similar to any compact operator.*

4. Almost similarity of operators

Recently, the class of almost similar operators interested many authors. Almost similarity was first introduced by Jibril [18], he proved various results that relate almost similarity and other classes of operators, including isometries, normal operators, unitary operators, compact operators and characterization of θ-operators. Nzimbi et al. have classified in [26] the operators for which almost similarity implies similarity. We investigate in this section some results of operators which are almost similar.

Proposition 4.1. *If $A, B \in \mathcal{B}(H)$ such that $A \overset{a.s}{\sim} B$ and B is hermitian, then A is hermitian.*

Proof. Since $A \overset{a.s}{\sim} B$, there exists an invertible operator S such that $4A^*A = S^{-1}(4B^*B)S$ and $A^* + A = S^{-1}(B^* + B)S$. On squaring both sides of the second identity, we obtain,

$$(A^* + A)^2 = S^{-1}(B^*B)^2S \tag{3}$$

Since B is hermitian, we have that $(B^* + B)^2 = 4B^2 = 4B^*B$. Substituting this in (3) we get $S^{-1}(4B^*B)S = (A^* + A)^2$ and then $4A^*A = (A^* + A)^2$ which shows that A is hermitian, by the fact a bounded operator A is hermitian if and only if $(A^* + A)^2 \geq 4A^*A$. □

Remark that if $A, B \in \mathcal{B}(H)$ such that A and B are unitarily equivalent, then $A \overset{a.s}{\sim} B$. The converse is not true, indeed the following result gives a condition under which almost similarity of operators implies similarity.

Proposition 4.2. *Let $A, B \in \mathcal{B}(H)$ such that A and B are unitarely equivalent, then $A \overset{a.s}{\sim} B$.*

Proof. By assumption, there exists a unitary operator W such that $A = WBW^*$ which implies that $A^* = WB^*W^*$. Thus,

$$\begin{aligned} A^*A &= WB^*BW^* = WB^*BW^{-1} \\ A^* + A &= W(B^* + B)W^* = W(B^* + B)W^{-1} \end{aligned}$$

□

Corollary 4.3. *Let $A, B \in \mathcal{B}(H)$ where H is a finite dimensional Hilbert space such that A and B are quasi-similar, then $A \overset{a.s}{\sim} B$.*

This corollary gives a condition under which quasi-similarity implies almost similarity i.e. only if the quasiaffinities are unitary and are equal.

Proposition 4.4. *If $A, B \in \mathcal{B}(H)$ such that $A \overset{a.s}{\sim} B$ and A is hermitian, then A and B are unitarily equivalent.*

Proof. By assumption there exists an invertible operator N such that $A^* + A = S^{-1}(B^* + B)S$. Since A is hermitian and $A \overset{a.s}{\sim} B$, by the above theorem, B is hermitian, which implies that $A = S^{-1}BS$. Then A and B are similar and since both operators are normal (both A and B are hermitian), they are unitarily equivalent. □

Two quasi-similar operators having equivalent quasiaffinities on a finite dimensional Hilbert space which are unitary are also almost similar. We study now projections and partially isometric operators under almost similarity.

Definition 4.5. *An operator $A \in \mathcal{B}(H)$ is said to be partially isometric if A^*A is an orthogonal projection. Equivalently, $AA^*A = A$, i.e $(A^*A)^2 = A^*A$ and $(A^*A)^* = A^*A$.*

Theorem 4.6. *Let $A, B \in \mathcal{B}(H)$ such that $A \overset{a.s}{\sim} B$.*
1) Then $(A + \lambda I) \overset{a.s}{\sim} (B + \lambda I)$ for all real λ.
2) If A is an orthogonal projection, then so is B.
3) If A is partially isometric then so is B.
4) if A is compact, then so is B.

Proof. Let S be an invertible operator such that $A^*A = S^{-1}B^*BS$ and $A^* + A = S^{-1}(B^* + B)S$.

1) Since $A \overset{a.s}{\sim} B$, $A^* + A + 2\lambda I = S^{-1}B^*S + S^{-1}BS + 2\lambda I$. Thus, $(A^* + \lambda I) + (A + \lambda I) = S^{-1}[(B^* + \lambda I) + (B + \lambda I)]S$. Then

$$(A + \lambda I)^*(A + \lambda I) = S^{-1}(B + \lambda I)^*(B + \lambda I)S$$

which gives the result.

2) Since A is an orthogonal projection, it is hermitian, and this implies that B is also hermitian. $A \overset{a.s}{\sim} B$ implies that $A^2 = AA = S^{-1}B^2S$ and $2A = S^{-1}2BS$. Thus, $A = S^{-1}BS$. This implies that $S^{-1}B^2S = S^{-1}BS$ which proves that B is an orthogonal projection.

3) Since A is partially isometric, A^*A is an orthogonal projection, which implies that $S^{-1}B^*BSS^{-1}B^*BS = S^{-1}B^*BS$. Thus we have $S^{-1}B^*BB^*BS = S^{-1}B^*BS$ and $(B^*B)^2 = B^*B$. This shows that B^*B is also an orthogonal projection, which implies that B is partially isometric.

4) Since A is compact, then A^*A is compact and conversely, which implies that B^*B is compact. Thus B is also compact, because of the equivalence between the compactness of an operator and the compactness of the product of this operator by its adjoint. □

Corollary 4.7. *Let $A, B \in \mathcal{B}(H)$.*
1) If A and B are hermitian operators such $A \overset{a.s}{\sim} B$, then $\sigma_p(A) = \sigma_p(B)$.
2) For hermitian or projection operators, if $A \overset{a.s}{\sim} B$, then they have equal spectrum.
3) If A is normal, then $A \overset{a.s}{\sim} A^*$. The converse is false in general, for example consider $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Remark 4.8. *1) The almost similarity is an equivalence relation on $\mathcal{B}(H)$.*
2) If $A \in \mathcal{B}(H)$, then

$$A \overset{a.s}{\sim} 0 \implies A = 0$$
$$A \overset{a.s}{\sim} I \implies A = I$$

3) If $A \overset{a.s}{\sim} B$ and B is isometric ($B^*B = I$), then A is isometric.

We can directly prove the following result:

Proposition 4.9. *If A is unitary and $B \in \mathcal{B}(H)$ such that $A \overset{a.s}{\sim} B$. Then either B is an isometry or a unitary operator.*

If B is assumed to be hermitian in the above proposition, then B is unitary. This leads us to the following conjecture.

Conjecture 4.10. *Two operators A and B in $\mathcal{B}(H)$ are similar if and only if both A and B are hermitian and $A \overset{a.s}{\sim} B$.*

Theorem 4.11. *If $A \in \mathcal{B}(H)$ is normal, then $A \overset{a.s}{\sim} A^*$.*

Remark 4.12. *The converse of this theorem is not true in general. Consider $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $N = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. By matrix computation, $A^*A = N^{-1}(AA^*)N$ and $A^* + A = I^{-1}(A^* + A)I$. That is $A \overset{a.s}{\sim} A^*$ but A is not normal.*

Theorem 4.13. *Let $A, B \in \mathcal{B}(H)$. Suppose that $A \overset{a.s}{\sim} B$. Then A is quasi-similar to B if and only if A and B are orthogonal projections.*

Proof. Since $A \overset{a.s}{\sim} B$, there exists an invertible operator S such that $A^*A = S^{-1}B^*BS$ and $A^* + A = S^{-1}(B^* + B)S$. From the first identity we have $SA = BS$, since $A^*A = A$ and $B^*B = B$. A and B are projections imply that $A^* = A$ and $B^* = B$. Therefore the second identity yields, $2A = S^{-1}2BS$, that is, $A = S^{-1}BS$ which implies that, $SA = BS$, hence the result. □

Corollary 4.14. *If $A, B \in \mathcal{B}(H)$ are normal where H is a finite dimensional Hilbert space such that A and B are quasi-similar, then $A \overset{a.s}{\sim} B$.*

5. Asymptotic similarity of operators

For given operators $A \in \mathcal{B}(H)$ and $B \in \mathcal{B}(K)$, we consider the corresponding commutator $C(A, B)(T) = AT - TB$ for all $T \in \mathcal{B}(H, K)$. Clearly, for all $n \in \mathbb{N}$ and all $T \in \mathcal{B}(H, K)$ we have:

$$C(A, B)^n(T) = C(A, B)^{n-1}(AT - TB) = \sum_{k=0}^{n} C_n^k (-1)^k A^{n-k} T B^k$$

An operator $T \in \mathcal{B}(H, K)$ is said to intertwine A and B asymptotically if $\|C(A, B)^n(T)\|^{1/n} \longrightarrow 0$ as $n \to \infty$. This condition has been investigated by Colojoara and Foias [6] and Vasilescu [32] in the context of decomposable operators. Remark that an operator $T \in \mathcal{B}(H, K)$ intertwines A and B asymptotically if and only if its adjoint $T^* \in \mathcal{B}(K, H)$ intertwines B^* and A^* asymptotically.

We shall call the operators $A \in \mathcal{B}(H)$ and $B \in \mathcal{B}(K)$ asymptotically similar if there exists a bijection $T \in \mathcal{B}(H, K)$ such that T intertwines A and B asymptotically and its inverse T^{-1} intertwines B and A asymptotically. When $H = K$ is finite dimensional, A and B are asymptotically similar if and only if they are similar by [[15], Theorem 2.1]. However, when H is infinite dimensional, these two notions are quite diffrent (cf. [15]). In [[11], Proposition 2.16], Fialkow showed that quasi-similar cyclic subnormal operators are asymptotically similar.

Asymptotic similarity generalizes slightly the notion of quasi-nilpotent equivalence where, in the above definition, $A = B$ and $T = I$ is the identity operator on H (cf. [6]). It is easily seen that A and B are asymptotically similar if and only if B and $T^{-1}AT$ are quasi-nilpotent equivalent. In particular, it follows that asymptotic similarity is an equivalence relation.

Here we shall discuss some properties of asymptotic similarity and their results to more general classes of operators. As a consequence of Rosenblum's theorem (cf. [29]), we have:

Theorem 5.1. *If there exists a non-zero operator $T \in \mathcal{B}(H,K)$ which intertwines A and B asymptotically, then $\sigma_{su}(B) \cap \sigma_{ap}(A) \neq \emptyset$. Furthermore if T is injective then $\sigma_p(B) \subseteq \sigma_{ap}(A)$ or if T is surjective then $\sigma_{su}(A) \subseteq \sigma_{su}(B)$; in particular $\sigma(A) \subseteq \sigma(B)$ when A has the single valued extension property.*

Among the interesting properties of the asymptotic similarity of operators, the following results must always be mentioned:

Theorem 5.2. *(*[21]*) The following are preserved under asymptotic similarity : spectrum, surjectivity spectrum, approximate point spectrum, single valued extension, Dunford's property (C) and Bishop's property (β).*

Proof. Let A and B be asymptotically similar, then there exists a bijection $T \in \mathcal{B}(H,K)$ such that T intertwines A and B asymptotically and its inverse T^{-1} intertwines B and A asymptotically. By the previous theorem, we obtain that $\sigma(A) = \sigma(B)$ and $\sigma_{su}(A) = \sigma_{su}(B)$. Moreover, $\sigma_{ap}(B) = \sigma_{su}(B^*) = \sigma_{su}(A^*) = \sigma_{ap}(A)$. We have $TH_B(F) = H_B(F)$ for all closed subset F of $\mathbb{C}$. This shows that property (C) carries over from B to A and the same is true for the single valued extension property. The same is true for the Bishop's property (β). □

In [[12], Theorem 4.8], it was shown that quasi-similar bilateral weighted shifts have equal spectra, [11] extend this result to approximate point spectra and essential spectra, in particular it is shown that quasi-similar bilateral weighted shifts are asymptotically similar.

6. Compalent operators

Recall that two operators A and B are compalent, $A \overset{c}{\sim} B$, if A is unitarely equivalent to B modulo $\mathcal{K}(H)$ or equivalently $\pi(A) = \pi(B)$. We refer the reader to Pearcy [27] for more about compalence. The spectral picture of $A \in \mathcal{B}(H)$, denoted $SP(A)$, is the structure consisting of the set $\sigma_e(A)$, the collection of holes and pseudoholes in $\sigma_e(A)$, and the indices associated with these holes and pseudoholes. The spectral picture $SP(A)$ determines whether an operator is compalent to another operator [4]. Compalence is different from the concepts previously established in this paper, we indicate nevertheless that the unitary equivalence implies compalence.

Theorem 6.1. *Let $A, B \in \mathcal{B}(H)$.*

1) Compalence is an equivalence relation on $\mathcal{B}(H,K)$.

2) If $(A - B) \in K(H)$, *then* $A \overset{c}{\sim} B$.

3) $A \overset{c}{\sim} B \Longleftrightarrow A^* \overset{c}{\sim} B^*$.

4) If $A \in \mathcal{F}(H)$ *and* $A \overset{c}{\sim} B$ *then* $B \in \mathcal{F}(H)$ *and* $ind(A) = ind(B)$.

5) $A \overset{c}{\sim} B$ *then* $(A - \lambda I) \overset{c}{\sim} (B - \lambda I)$ *and* $(\lambda A) \overset{c}{\sim} (\lambda B)$ *for all* $\lambda \in \mathbb{C}$.

6) If $A \overset{c}{\sim} B$ $((U^*AU - B) \in \mathcal{K}(H)$, U *unitary operator) then for all* $T \in \mathcal{B}(H)$ *which commutes with* U, $A + T \overset{c}{\sim} B + T$.

7) If $A \overset{c}{\sim} B$ *then* $\rho_e(A) = \rho_e(B)$ *and* $ind(A - \lambda I) = ind(B - \lambda I)$ *for all* $\lambda \in \rho_e(A)$.

Proof. It is simple to establish from the defintion the assertions 1), 2), 3), 5) and 6). By using Theorem 3.1 of [20], we have easily the result 4). 7) is a direct consequence of the first assertion of 5) and the stability of the Fredholm character and the index under compact perturbations. Indeed,

$$\begin{aligned} \rho_e(A) &= \rho_e(U(B+K)U^*) = \rho_e(B) \\ ind(A - \lambda I) &= ind(UBU^* - \lambda I) = ind(U(B - \lambda I)U^*) \\ &= ind(B - \lambda I) \ \text{ for all } \lambda \in \rho_e(A) \end{aligned}$$

□

Compalence was characterized by the following theorem in the class of essentially normal operators due to Brown, Douglas, and Fillmore [3]:

Theorem 6.2. *If* A *and* B *are essentially normal operators then*

$$A \overset{c}{\sim} B \Longleftrightarrow SP(A) = SP(B)$$

Corollary 6.3. *Selfadjoint operators are compalent if and only if their essential spectra coincide.*

By the beautiful Brown-Douglas-Fillmore theorem [4], we have:

Proposition 6.4. *Let* $A \in \mathcal{B}(H)$ *be Weyl operator and* $A \in \mathcal{B}(H)$. *If* AB *and* BA *are essentially normal then* AB *and* BA *are compalent.*

Proof. If AB and BA are essentially normal then neither of them have any pseudo-holes, so that $\sigma(AB) = \sigma(BA)$. Now the result follows from Brown-Douglas-Fillmore theorem (if S and T are essentially normal then S and T are compalent if and only if $\sigma(S) = \sigma(T)$). □

7. Invariant subspaces with respect quasi-similarity, almost similarity and compalence

A natural method for constructing an invariant subspace for an operator on Hilbert space is to find a second known operator which is similar in some weak sense to the given operator and then to use this second operator and the weak similarity to construct the desired subspace. To begin, let us recall the definition of lattice of all invariant subspaces and the notion of hyperinvariant subspaces for a bounded linear operator. A lattice $\mathcal{P}$ is a partially ordered set such that every pair of elements of $\mathcal{P}$ has a supremum (least upper bound) and an infimum (greatest lower bound) in $\mathcal{P}$. Note that, the set of all invariant subspaces for $A \in \mathcal{B}(H)$ is a lattice. Lat(A) will denote the lattice of all invariant subspaces of A, that is,

$$Lat(A) = \{M \subseteq H : A(M) \subseteq M\}$$

If $A \in \mathcal{B}(H)$, we denote by $\{A\}^{'}$ the commutant of A, i.e.

$$\{A\}^{'} = \{B \in \mathcal{B}(H) : AB = BA\}$$

A subspace $M \subset H$ is said to be a nontrivial hyperinvariant subspace for a fixed operator in $A \in \mathcal{B}(H)$ if $0 \neq M \neq H$ and $BM \subset M$ for each B in $\{A\}^{'}$. Obviously, hyperinvariant subspaces are invariant and if A is not a quasiaffinity then either its null space or the closure of its range will be in $Lat(A)$. We observe:

Proposition 7.1. *If $A, B \in \mathcal{B}(H)$ are such that BA is a quasiaffinity, then*

$$Lat(BA) \quad nontrivial \Longrightarrow Lat(AB) \quad nontrivial$$

Proof. By assumption A is one-one and B has dense range. We claim that if $N \in Lat(BA)$ is nontrivial then $M = \overline{AN} \Longrightarrow M \in Lat(AB)$ with $\{0\} \neq M \neq X$. The invariance of M is clear; $M \neq \{0\}$ is because A is one-one and N is nonzero; $M \neq X$ is because B is dense and $N \neq X$. □

Hoover [16] studied hyperinvariant subspaces and proved that if A and B are quasi-similar operators acting on the Hilbert spaces, and if B has an hyperinvariant subspace, then so does A. If in addition, A is normal, then the lattice of hyperinvariant subspaces for B contains a sublattice which is isomorphic to the lattice of spectral projections for A. Similar results for invariance and hyperinvariance have been studied by Kubrusly [19].

Remark 7.2. *1) If $(M_\alpha)_\alpha$ is a family of invariant (hyperinvariant) subspaces for A, then $\bigcap_\alpha M_\alpha$ and $\bigcup_\alpha M_\alpha$ are also invariant (hyperinvariant) subspaces for A.*

2) If U is a unitary operator on H, then M is hyperinvariant for U if and only if P_M (the orthogonal projection of H into M) commutes with all bounded operators which commute with U. Furthermore, if M is hyperinvariant for U then its orthogonal $M^\perp$ is also hyperinvariant for U.

If $A \in \mathcal{B}(H)$ and $x \in H$, set

$$Ax = \{Cx \in H : C \in \{A\}'\} = \bigcup_{C \in \{A\}'} \{Cx\} \subseteq H$$

Then, $A0 = \{0\}$ and $x \in Ax$ for all $x \in H$. Hence $\overline{Ax} \neq \{0\}$ for every nonzero x in H. Therefore, if A has no nontrivial hyperinvariant subspace, then $\overline{Ax} = H$ for every $x \neq 0$.

Lemma 7.3. *For all $x \in H$; $\overline{\{Ax\}}$ is a hyperinvariant subspace for A in H.*

Theorem 7.4. *Let $A \in \mathcal{B}(H)$, $B \in \mathcal{B}(K)$, $S \in \mathcal{B}(H, K)$ and $T \in \mathcal{B}(K, H)$ be such that $SA = BS$ and $TB = AT$. Suppose that M is a nontrivial hyperinvariant subspace for B. If $\overline{R(S)} = K$ and $N(T) \cap M = \{0\}$, then $T(M) \neq \{0\}$ and for each nonzero $x \in T(M)$, $\overline{\{Ax\}}$ is a nontrivial subspace of H hyperinvariant for A.*

Proof. $\{0\} \neq \overline{\{Ax\}} \neq H$ for every $0 \neq x \in T(M) \neq \{0\}$. Note that $SCT \in \{B\}'$ for every $C \in \{A\}'$. Since M is hyperinvariant for B, it is hyperinvariant for SCT whenever $C \in \{A\}'$. Take an arbitrary $x \in T(M)$ such that $x = Tu$ for some $u \in M \subset K$ and take an arbitrary $y = Ax = CTu$ for some $C \in \{A\}'$. Since $u \in M$ and M is SCT-invariant, it follows that Sy must be in M. Therefore $S(Ax) \subseteq M$ and as M is closed and properly included in K, $S(\overline{\{Ax\}}) \subseteq \overline{M} = M \neq K$ for every

$x \in T(M)$ because S is continuous. Therefore, if $\overline{\{Ax\}} = H$, then $\overline{R(S)} = \overline{S(H)} = \overline{S(\overline{\{Ax\}})} \subseteq M \neq K$ and hence $\overline{R(S)} = K$ implies $Ax \neq H$ for every $x \in T(M)$. But $\overline{\{Ax\}}$ is a hyperinvariant subspace for A and since $Ax \neq 0$ whenever $x \neq 0$ in $T(M)$. Thus in order to show that $\overline{\{Ax\}}$ is a nontrivial hyperinvariant for A, we must ensure that there exists an $0 \neq x \in T(M)$ such that if $T(M) = \{0\}$ (i.e. if $M \subset N(T)$) then $N(T) \cap M = M \neq \{0\}$, since M is nonzero. Thus $N(T) \cap M = \{0\}$ implies that $T(M) \neq \{0\}$. Hence $\overline{R(S)} = K$ and $N(T) \cap M = \{0\}$, then $\overline{\{Ax\}}$ is a non-trivial subspace of H hyperinvariant for A for every non-zero $T(M)$ ($\overline{\{Ax\}} = 0$ if and only if $x = 0$). □

In view of the above proposition, if $SA = BS$ abd $TB = AT$ with $\overline{R(S)} = K$ and $N(T) = \{0\}$, then there exists $x \in T(M)$ such that $\overline{\{Ax\}}$ is a nontrivial subspace for A.Thus we have the following corollary:

Corollary 7.5. *Let $A \in \mathcal{B}(H)$ and $B \in \mathcal{B}(K)$ be quasi-similar operators. If B has a nontrivial hyperinvariant subspace, then A has a nontrivial hyperinvariant subspace. If in addition, A is normal, then the lattice of hyperinvariant subspaces for B contains a sublattice which is lattice isomorphic to the lattice of spectral projections for A.*

Proof. Let $V : H \longrightarrow K$ and $W : K \longrightarrow H$ be quasi-affinities of A and B. That is, $BV = VA$ and $AW = WB$. Let N be a nontrivial invariant subspace for B. Define

$$M = \bigcup \left\{ XWH : X \in (A)' \right\}$$

Clearly, M is B-hyperinvariant and $M \neq \{0\}$ because $WN \subset M$. Moreover, $M \neq H$ because

$$VM = V \left\{ \bigcup \left\{ XWH : X \in (A)' \right\} \right\} \subset V \left\{ YN : Y \in (B)' \right\} \subset N \neq K$$

Thus M is nontrivial. □

Proposition 7.6. *([25]) Let $A \in \mathcal{B}(H)$ quasi-similar to a unitary operator U on H. To every subspace M of H, which is hyperinvariant for U, there is a subspace $q(M)$ of H hyperinvariant for A, so that we have the following properties :*

1) $q(\{0\}) = \{0\}$.
2) $q(H) = H$.
3) $q(M_1) \subset q(M_2)$ if $M_1 \subset M_2$.
4) $q(M_1) \neq q(M_2)$ if $M_1 \neq M_2$.
5) $\bigcap_\alpha q(M_\alpha) = \{0\}$ if $\bigcap_\alpha M_\alpha = \{0\}$.
6) $\bigcup_\alpha q(M_\alpha) = q(M)$ if $\bigcup_\alpha M_\alpha = M$.

Also recall the following fundamental result ([25]).

Theorem 7.7. *Let $A \in \mathcal{B}(H)$ a power-bounded operator with $\dim H > 1$, such that neither A^n nor A^{*n} converge strongly to 0 as $n \to \infty$. Then either $A = \lambda I$ with $|\lambda| = 1$, or ther exists a non-trivial subspace of H, hyperinvariant for A.*

Proof. We distinguish three cases :

1) There exists a nonzero vector $a \in H$ such that $A^n a \longrightarrow 0$ as $n \to \infty$. Let us pose $M = \{x \in H : A^n x \longrightarrow 0 \text{ as } n \to \infty\}$. It is easy to see that M is a subspace of

H hyperinvariant for A. $a \in M$, hence $M \neq \{0\}$. On the other hand, $M \subsetneq H$, since the contrary would imply $A^n \longrightarrow 0$ as $n \to \infty$.

2) There exists a nonzero vector $a \in H$ such that $A^{*n}a \longrightarrow 0$ as $n \to \infty$. If we set $N = H \ominus \{x \in H : A^{*n}x \longrightarrow 0 \text{ as } n \to \infty\}$, then N is again a non-trivial subspace of H, hyperinvariant for A.

3) There exists a nonzero vector $a \in H$ for which $A^n a \longrightarrow 0$ or $A^{*n}a \longrightarrow 0$ as $n \to \infty$. By virtue of [[25], Proposition 5.3], A is quasi-similar to a unitary operator U on H and as $\dim H > 1$, the spectral measure of U has values different from 0 and I unless U has a one-point spectrum $\{\lambda\}$, $|\lambda| = 1$. Thus, $U = \lambda I$ which implies that $A = \lambda I$. Now if A is not of this form it follows from Corollary 5.2 of [25] that there exists a nontrivial subspace of H, hyperinvariant for A. □

Finally remember the following result due to Kubursly [19] about the similarity with a shift operator:

Theorem 7.8. *Let A be an operator on a Hilbert space H. If $r(A) < 1$, then A is similar to a part of the canonical backward unilateral shift on l^2.*

Similar operators have isomorphic lattices of invariant and hyperinvariant subspaces.

Since if $A \overset{a.s}{\sim} B$ and A is hermitian, then A and B are unitarily equivalent, we can directly deduce:

Proposition 7.9. *If two operators $A, B \in \mathcal{B}(H)$ are almost similar and A is hermitian, then if one has a nontrivial invariant (hyperinvariant) subspace, then so has the other.*

Lomonosov showed in [23] that compact perturbations of selfadjoint operators on an infinite dimensional Hilbert space have nontrivial closed real invariant subspaces. In the following we extend this result to compalent selafadjoint operators.

Theorem 7.10. *Let $A \in \mathcal{B}(H)$. If A is compalent to a bounded linear selfadjoint operator on H, then A has a nontrivial closed real invariant subspace.*

Proof. There is $B \in \mathcal{B}(H)$ selfadjoint and some unitary operator U and a compact operator K on H such that $A = U^*BU + K$. The result follows from [[23], Theorem 1] since U^*BU is selfadjoint on H. □

Remark 7.11. *When N is a normal operator and K is a compact operator, we don't know in general if $N + K$ has a non trivial hyperinvariant subspace.*

References

[1] P. Aiena, E. Aponte, E. Bazan, Weyl type theorems for left and right polaroid operators, Integral Equations Oper. Theory, 66 (2010), 1-20.

[2] J. Bourgain, On the similarity problem for polynomially bounded operators on Hilbert Spaces, Israel J. of Maths, 54 (2) (1986), 227-241.

[3] L. Brown, R.G. Douglas, P. Fillmore, Unitary equivalence modulo the compact operators and extensions of C^*-algebras, Proc. Conf. Operator Theory, Lecture notes in Math., 345 (1973), Springer, Berlin, 58-128.

[4] L. Brown, R.G. Douglas, P. Fillmore, Extensions of C^*-algebras and K-homology, Ann. of Math., 105 (1977), 265-324.

[5] R. K. Campbell–Wright, Similar compact composition operators, Acta Sci. Math. (Szeged), 58 (1993), 473-495.

[6] I. Colojoara, C. Foias, Theory of generalized spectral operators, Gordon and Breach, NewYork, 1968.

[7] S. Clary, Equality of spectra of quastsimilar operators, Proc. Amer. Math. Soc., 53 (1975), 88-90.

[8] J.B. Conway, The theory of subnormal operators, Amer. Math. Soc. Surveys and Monographs, 36, 1991.

[9] J. Eschmeier, On the essential spectrum of Banach-space operators, Proc. Edinburgh Math. Soc., 43 (2000), 511-528.

[10] L.A. Fialkow, A note on quasisimilarity of operators, Acta Sci. Math. (Szeged), 39 (1977), 67-85.

[11] L.A. Fialkow, Quasisimilarity and closures of similarity orbits of operators, J. Operator Theory, 14 (1985), 215-238.

[12] L.A. Fialkow, A note on quasisimilarity of operators II, Pacific J. Math., 70 (1970), 151-162.

[13] C. Foias and C. Pearcy, A model for Quasinilpotent Operators, Michigan Math. J., 21 (1974), 399-404.

[14] D.A. Herrero, On the essential spectra of quasi-similar operators, Canadian J. Math., 40 (1988), 1436-57.

[15] D.A. Herrero, Approximation of Hilbert space operators, Research Note in Math., 224, 1989.

[16] T.B. Hoover, Hyperinvariant Subspaces for n-normal operators, Acta Sci. Math., 32 (1972), 109-119.

[17] S. Grabiner, Spectral consequences of the existence of interwining operators, Comment. Math. Prace Mat. 22 (1981), 227-238.

[18] A.A. Jibril, On almost similar Operators, Arabian J. Sci. Engrg., 21 (1996), 443-449.

[19] C.S. Kubrusly, An Introduction to Models and Decomposition in Operator Theory, Birkhauser, 1997.

[20] J.Ph. Labrousse, B. Mercier, Equivalence compacte entre deux opérateurs sur un espace de Hilbert, Math. Nach., 133 (1987), 91-105.

[21] K B. Laursen, M M. Neumann, An introduction to Local spectral theory, in London Mathematical Society Monograph, New series, Vol. 20, Clarendon Press, Oxford, 2000.

[22] M.S. Lee, A note On Quasi-Similar Quasi-Hyponormal Operators, J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math., 2 (1995), 91-95.

[23] V.I. Lomonosov, On real invariant subspaces of bounded operators with compact imaginary part, Proceedings of the A.M.S., 115 (3) (1992), 775-777.

[24] B. Sz.-Nagy, On uniformly bounded linear transformations on Hilbert space, Acta Sci. Math. (Szeged), 11 (1946), 152-157.

[25] B. Sz.-Nagy, C. Foias, Harmonic analysis for operators on Hilbert space, North-Holland, 1970.

[26] B.M. Nzimbi, G.P. Pokhariyal, and J.M. Khalagai, A note on Similarity, almost similarity and Equivalence of Operators, Far East Mathematics Journal, 28 (2) (2008), 305-317.

[27] C.M.Pearcy, Some recent developments in operator theory, Conference Board Math. Sci., Reg. Conf. Ser. in Math. 36, 1978.

[28] M. Putinar, Quasi-similarity of tuples with Bishop's property (β), Integral Equations Operator Theory, 15 (1992), 1047-1052.

[29] M. Rosenblum, On the operator equation $BX - XA = Q$, DukeMath .J ., 23 (1956), 263-269.

[30] A. L. Shields, Weighted shift operators and anaytic function theory, Math. Surveys. 13, Amer. Math. Soc., Providence, 1974.

[31] J.G. Stampfli, B.L. Wadhwa, Quasisimilarity of operators, Pro. of the Royal Irish Academy.Section A; Mathematical and physical sciences, 81A (1) (1986), 109-119.

[32] F.-H.Vasilescu, Analytic functional calculus and spectral decompositions, Editura Academiei and D. Reidel Publishing Company, Bucuresti and Dordrecht, 1982.

[33] L. Yang, Quasisimilarity of hyponormal and subdecomposable operators, J. Functional Analysis, 112 (1993)., 204-217.

[34] L.R. William, Equality of essential spectra of quasisimilar quasinormal operators, J. Operator Theory, 3 (1980), 57-69.

[35] L.R. Williams, Equality of essential spectra of certain quasisimilar seminormal operators, Proc. Amer. Math. Soc. 78 (1980), 203-209.

Department of Mathematic, Faculty of Sciences. University Tahar Moulay of Saida., P. Box 138, Nasr, 20000. Saida, Algeria.
E-mail address: abdelhalim.azzouz.cus@gmail.com (* Corresponding author)

Department of Mathematic, Faculty of Sciences. University Tahar Moulay of Saida. Algeria
E-mail address: gdjellouli@yahoo.fr

Department of Mathematics, Faculy of Exact and and Applicable Sciences. University of Oran 1, Algeria., Laboratory of Fundamental and Applicable Mathematics of Oran (LMFAO).
E-mail address: messirdi.bekkai@univ-oran.dz