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Contents

Introduction **i**

1 Elements of Nevanlinna's theory **1**

1.1 Basic definitions 1

1.2 Nevanlinna's theory 3

1.3 The growth and distribution of the values of an entire or meromorphic function 5

1.3.1 The exponent of convergence of zeros and poles of a meromorphic function 6

1.3.2 The (p, q)-order, Lower (p, q)-order, (p, q)-type and (p, q)-exponent of convergence of an entire or meromorphic function 8

1.4 Elements of the theory of Wiman-Valiron 10

1.5 Hadamard's factorization theorem 11

1.6 Linear measure and Logarithmic measure of sets 12

2 The (p, q)-order of solutions of some complex linear differential equations **13**

2.1 Introduction and main results 13

2.2 Preliminary Lemmas 17

2.3 Proof of Theorem 2.1.1 20

2.4 Proof of Theorem 2.1.2 24

2.5 Proof of Theorem 2.1.3 27

2.6 Proof of Theorem 2.1.4 28

Conclusion **33**

Bibliography **34**

Introduction

The study of meromorphic functions and their value distribution has played a pivotal role in the development of modern complex analysis. A landmark contribution to this field came in the 1920s from finnish mathematician Rolf Nevanlinna (1895–1980), who is credited with founding what is now known as Nevanlinna's Theory. Building on earlier work by mathematicians like Émile Picard, who proved that a non-constant entire function can omit at most one complex value, Nevanlinna revolutionized the subject by introducing a precise, quantitative framework to analyze how often and in what manner a meromorphic function takes certain values.

In his seminal doctoral dissertation "Zur Theorie der meromorphen Funktionen" (1925), Nevanlinna introduced three key tools: the characteristic function $T(r, f)$, the counting function $N(r, a)$, and the proximity function $m(r, a)$. These concepts became the building blocks for the First and Second Fundamental Theorems of Nevanlinna, which form the core of value distribution theory. A fundamental concept derived from the characteristic function is the order of growth of a meromorphic function—a measure of how rapidly the function increases. This notion of order is essential in the classification of meromorphic functions and plays a crucial role in analyzing the nature of solutions to complex differential equations.

While Nevanlinna laid the foundations, the theory was further expanded and refined by a number of influential mathematicians. Lars Ahlfors, a student of Nevanlinna, made significant contributions to value distribution and function theory, and was awarded the first Fields Medal in 1936. Others, such as W. K. Hayman, Paul Montel, and Shoshichi Kobayashi, extended the theory in various directions, including uniqueness theory, value-sharing results, and connections with complex differential geometry. Later developments by Ilpo Laine,

Kazuya Tohge, and R. G. Halburd deepened the connections between Nevanlinna Theory and differential equations, especially in the study of Painlevé-type equations and integrability conditions.

Nevanlinna's theory has since found profound applications across mathematics, including in Diophantine approximation, algebraic geometry, and mathematical logic. The Second Fundamental Theorem, in particular, has become a cornerstone in proving uniqueness theorems and understanding defect relations, which describe how a meromorphic function may "miss" or underrepresent certain values.

This research aims to explore key applications of Nevanlinna Theory to the analysis of meromorphic solutions of differential equations, with special attention to how the value distribution and growth, measured by order, offer deep insights into the behavior and structure of these solutions.

This work is composed of an introduction, and two chapters.

In the first chapter, we introduce some elementary definitions of the theory of Nevanlinna, and some other notations that we will need in the next chapter.
The second chapter is devoted to the results obtained in the article [5], where the authors have studied the growth of solutions of some complex linear differential equations.
Four differential equations were used to obtain some results on the (p,q)-order of the solutions.
First, we consider the differential equations

$$A_k(z)f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0,$$

$$A_k(z)f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = F(z),$$

where $A_j(z)$ $(j = 0, 1, \cdots, k)$ and $F(z)$ with $A_k(z)A_0(z) \not\equiv 0$ are entire functions and obtained some results when there exists some coefficient A_s $(0 \leq s \leq k)$, of maximal (p,q)-order and being a Lacunary serie.
For the differential equations

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0,$$

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = F(z),$$

where $A_j(z)$ $(j = 0, 1, \cdots, k-1)$ and $F(z)$ are entire functions, some results were generelized in [5] due to the authors in [11], [19].
We conclude this work with a conclusion.

Chapter 1

Elements of Nevanlinna's theory

1.1 Basic definitions

Definition 1.1.1 ***(Entire function)*** *An entire function is a type of complex function that is holomorphic (complex differentiable) everywhere on the entire complex plane $\mathbb{C}$. In simpler terms, an entire function has no singularities or points of discontinuity anywhere in the complex plane .*

Example 1.1.1 *The functions e^z, $\sin(z)$, $\cos(z)$, z^n are entire functions.*

Definition 1.1.2 ***(Meromorphic function)*** *A meromorphic function is a function that is holomorphic everywhere in a given domain except for a set of isolated poles (singularities where the function tends to infinity).*
In practice, we can consider a meromorphic function as the quotient of two entire functions.

Example 1.1.2 *The functions $\dfrac{1}{z}$, $\dfrac{1}{\sin(z)}$, $\dfrac{z^2+3}{z-1}$ are meromorphic functions.*

Definition 1.1.3 ***(Lacunary serie)*** *A lacunary serie is a type of power series in which the sequence of exponents that has large gaps. More formally, a lacunary power series is of the form*

$$\sum_k a_{n_k} x^{n^k},$$

where $\{n_k\}$ is a strictly increasing sequence of integers that grows rapidly, meaning that the differences $n_{k+1} - n_k$ increase significantly.

Example 1.1.3 $f(z) = \sum_{k=0}^{\infty} z^{2^k} = z + z^2 + z^4 + z^8 + \cdots$ *is a lacunary serie.*

Definition 1.1.4 ***(The symbols O and o)***
Let $f(r)$ and $g(r)$ be complex functions defined over $[0,+\infty[$ with $g(r)$ a real positive function. We say that
(i) $f(r) = o(g(r))$ when $r \to +\infty$ if $\dfrac{f(r)}{g(r)} \to 0$ when $r \to +\infty$.
(ii) $f(r) = O(g(r))$ when $r \to +\infty$, if there exist constants c, r_0 such that

$$|f(r)| \leq cg(r), \forall r \geq r_0.$$

Example 1.1.4 *We have $r\sin(r) = O(r)$ and $\sin(r) = o(r)$ since $\dfrac{\sin(r)}{r} \to 0$ when $r \to +\infty$.*

Remark 1.1.1 *$f(r) = O(1)$ means that f is bounded.*

Definition 1.1.5 ***(Maximum modulus)*** *The maximum modulus of a complex function $f(z)$ is defined by*

$$M(r,f) = \max_{|z|=r} |f(z)|,$$

where $M(r,f)$ represents the maximum value of $|f(z)|$ on the circle of radius r centered at the origin in the complex plane.

Example 1.1.5 *Let $f(z) = z^2+1$. Let's compute the maximum modulus $M(r,f)$ on a circle of radius r, that is*

$$M(r,f) = \max_{|z|=r} |f(z)| = \max_{|z|=r} \left|z^2+1\right|$$

Let $z = re^{i\theta}$. Then $|z| = r$ and $f(z) = r^2e^{2i\theta}+1$.
Therefore

$$|f(z)| = \left|r^2e^{2i\theta}+1\right| = \sqrt{(r^2\cos(2\theta)+1)^2 + (r^2\sin(2\theta))^2}.$$

This gets a little messy, but we can simplify

$$|f(z)|^2 = \left|r^2e^{2i\theta}+1\right|^2 = r^4 + 2r^2\cos(2\theta) + 1.$$

So the maximum occurs when $\cos(2\theta) = 1$, that is $M(r,f)^2 = r^4+2r^2+1 = (r^2+1)^2$.
Hence

$$M(r,f) = r^2+1.$$

1.2 Nevanlinna's theory

Definition 1.2.1 ([16]) *For any real number $x \geq 0$, we define*

$$\log^+ x := \max(\log x, 0) = \begin{cases} \log x, & \text{if} \quad x > 1 \\ 0, & \text{if} \quad 0 \leq x \leq 1. \end{cases}$$

Lemma 1.2.1 ([16]) *We have the following properties of $\log^+ x$:*

(1) $\log\ x \leq \log^+\ x,\ x \geq 0.$

(2) $\log^+\ x \leq \log^+ y,\ \ 0 \leq x \leq y.$

(3) $\log\ x = \log^+ x - \log^+\left(\frac{1}{x}\right),\ \ x \geq 0.$

(4) $|\log\ x| = \log^+ x + \log^+\left(\frac{1}{x}\right),\ x \geq 0.$

(5) $\log^+\left(\prod_{j=1}^{n} x_j\right) \leq \sum_{j=1}^{n} \log^+ x_j,\ \ x_j \geq 0.$

(6) $\log^+\left(\sum_{j=1}^{n} x_j\right) \leq \sum_{j=1}^{n} \log^+ x_j + \log\ n,\ x_j \geq 0.$

Definition 1.2.2 ***(Counting function)*** ([16]) *Let f be a meromorphic function. For any complex number a, we denote by $n(t, a, f)$ the number of roots of the equation $f(z) = a$ located in the disk $|z| \leq t$. Each root is counted with its multiplicity, and we denote by $n(t, \infty, f)$ the number of poles of the function f in the disk $|z| \leq t$.*

We have

$$N(r, a, f) = N\left(r, \frac{1}{f-a}\right) = \int_0^r \frac{n(t, a, f) - n(0, a, f)}{t} dt + n(0, a, f) \log r, \quad \text{if } f \not\equiv a \in \mathbb{C}$$

and

$$N(r, \infty, f) = N(r, f) = \int_0^r \frac{n(t, \infty, f) - n(0, \infty, f)}{t} dt + n(0, \infty, f) \log r, \quad \text{if } a = \infty.$$

Definition 1.2.3 ***(Proximity function)*** ([16]) *Let f be a meromorphic function that is not constant on the complex plane. Then, we define the proximity function of f as*

$$m(r, a, f) = m\left(r, \frac{1}{f-a}\right) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{1}{|f(re^{i\theta}) - a|} d\theta, \ \text{if } f \not\equiv a \in \mathbb{C}$$

and

$$m(r, \infty, f) = m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left|f\left(re^{i\theta}\right)\right| d\theta.$$

Definition 1.2.4 ***(Characteristic function)*** ([16]) *We define the characteristic function of the function f (in the sense of R. Nevanlinna) as*

$$T(r,f)=N(r,f)+m\left(r,f\right).$$

Example 1.2.1 *Let $f\left(z\right)=e^{z}$. We have $n\left(t,\infty,f\right)=n\left(0,\infty,f\right)=0$ because $f\left(z\right)=e^{z}$ is an entire function, and so $N\left(r,f\right)=0$.*
Let's calculate $m\left(r,f\right)$. We have

$$\begin{aligned}
m\left(r,f\right) &= \frac{1}{2\pi}\int_{0}^{2\pi}\log^{+}\left|f\left(re^{i\theta}\right)\right|d\theta\\
&= \frac{1}{2\pi}\int_{0}^{2\pi}\log^{+}\left(e^{r\cos\theta}\right)d\theta\\
&= \frac{1}{2\pi}\int_{0}^{\frac{\pi}{2}}\log^{+}\left(e^{r\cos\theta}\right)d\theta+\frac{1}{2\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\log^{+}\left(e^{r\cos\theta}\right)d\theta+\frac{1}{2\pi}\int_{\frac{3\pi}{2}}^{2\pi}\log^{+}\left(e^{r\cos\theta}\right)d\theta\\
&= \frac{1}{2\pi}\int_{0}^{\frac{\pi}{2}}r\cos\theta d\theta+\frac{1}{2\pi}\int_{\frac{3\pi}{2}}^{2\pi}r\cos\theta d\theta\\
&= \frac{1}{2\pi}\left[r\sin\theta\right]_{0}^{\frac{\pi}{2}}+\frac{1}{2\pi}\left[r\sin\theta\right]_{\frac{3\pi}{2}}^{2\pi}\\
&= \frac{1}{2\pi}\left(r\sin\frac{\pi}{2}-r\sin 0\right)+\frac{1}{2\pi}\left(r\sin 2\pi-r\sin\frac{3\pi}{2}\right)\\
&= \frac{r}{2\pi}+\frac{r}{2\pi}\\
&= \frac{r}{\pi}.
\end{aligned}$$

Hence

$$T(r,f)=N(r,f)+m\left(r,f\right)=\frac{r}{\pi}$$

Example 1.2.2 *Let $g\left(z\right)=z+1$. We have $n\left(t,\infty,g\right)=n\left(0,\infty,g\right)=0$ because $g\left(z\right)=z+1$ is an entire function, and so $N\left(r,g\right)=0$ and*

$$\begin{aligned} m(r,g) &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|g\left(re^{i\theta}\right)\right| d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|re^{i\theta}+1\right| d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \log \left|re^{i\theta}(1+o(1))\right| d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \log r(1+o(1))d\theta \\ &= \log r(1+o(1)) \\ &= \log r + C,\ C = Cte. \end{aligned}$$

Hence

$$T(r,g) = N(r,g) + m(r,g) = \log r + C.$$

1.3 The growth and distribution of the values of an entire or meromorphic function

Definition 1.3.1 ***(Order of growth)*** ([10], [16]) *Let f be a meromorphic function. The order of growth of the function f is given by*

$$\sigma(f) = \overline{\lim_{r\to+\infty}} \frac{\log T(r,f)}{\log r}.$$

If f is an entire function, then the order of f is given by

$$\sigma(f) = \overline{\lim_{r\to+\infty}} \frac{\log T(r,f)}{\log r} = \overline{\lim_{r\to+\infty}} \frac{\log\log M(r,f)}{\log r}.$$

Example 1.3.1 *Let $f(z) = e^z$. We have f is an entire function and therefore*

$$T(r,f) = \frac{r}{\pi}.$$

Then we get

$$\sigma(f) = \overline{\lim_{r\to+\infty}} \frac{\log\left(\frac{r}{\pi}\right)}{\log r} = \overline{\lim_{r\to+\infty}} \frac{\log r - \log\pi}{\log r} = 1.$$

Example 1.3.2 *Let $g(z) = z + 1$. We have g is an entire function and therefore*

$$T(r, g) = \log r + C.$$

Since $T(r, g) \sim \log r$ as $r \to +\infty$, then we have

$$\sigma(g) = \overline{\lim_{r\to+\infty}} \frac{\log T(r, g)}{\log r} = \overline{\lim_{r\to+\infty}} \frac{\log(\log r)}{\log r} = 0.$$

Proposition 1.3.1 ([1], [10], [16]) *Let f and g be meromorphic functions. Then*
(1) $\sigma(af) = \sigma(f)$, $a \in \mathbb{C}^*$.
(2) $\sigma(f + g) \leq \max\{\sigma(f), \sigma(g)\}$.
(3) $\sigma(fg) \leq \max\{\sigma(f), \sigma(g)\}$.
In particular, if $\sigma(f) < \sigma(g)$, then $\sigma(f + g) = \sigma(fg) = \sigma(g)$.

Example 1.3.3 *Let $f(z) = \dfrac{e^z + z}{4}$. Then*

$$\sigma(f) = \sigma\left(\frac{e^z + z}{4}\right) = \sigma\left(\frac{e^z}{4} + \frac{z}{4}\right) \leq \max\left\{\sigma\left(\frac{e^z}{4}\right), \sigma\left(\frac{z}{4}\right)\right\}.$$

Since $\sigma\left(\dfrac{z}{4}\right) = 0 < 1 = \sigma\left(\dfrac{e^z}{4}\right)$, then $\sigma(f) = 1$.

Definition 1.3.2 ***(The type)*** ([10]) *Let f be a meromorphic function with $0 < \sigma(f) < +\infty$. Then the type $\tau(f)$ of f is given by*

$$\tau(f) = \overline{\lim_{r\to+\infty}} \frac{T(r, f)}{r^{\sigma(f)}}.$$

If f is an entire function, then the type of f is given by

$$\tau_M(f) = \overline{\lim_{r\to+\infty}} \frac{\log M(r, f)}{r^{\sigma(f)}}.$$

Example 1.3.4 *Let $f(z) = \dfrac{e^z}{z}$. We have $T(r, f) = \dfrac{r}{\pi} + O(\log r)$ and therefore $\sigma(f) = 1$ and $\tau(f) = \dfrac{1}{\pi}$.*

1.3.1 The exponent of convergence of zeros and poles of a meromorphic function

Definition 1.3.3 ([16]) *Let f be a meromorphic function. The exponent of convergence of the zeros of f is defined by*

$$\lambda(f) = \overline{\lim_{r\to+\infty}} \frac{\log N\left(r, \frac{1}{f}\right)}{\log r}$$

and the exponent of convergence of the distinct zeros of f by

$$\overline{\lambda}(f) = \overline{\lim_{r\to+\infty}} \frac{\log \overline{N}\left(r, \frac{1}{f}\right)}{\log r},$$

where

$$\overline{N}\left(r, \frac{1}{f}\right) = \int_0^r \frac{\overline{n}\left(t, \frac{1}{f}\right) - \overline{n}\left(0, \frac{1}{f}\right)}{t} dt + \overline{n}\left(0, \frac{1}{f}\right) \log r$$

and $\overline{n}\left(t, \frac{1}{f}\right)$ denotes the number of distinct zeros of the function f lying within the disk of radius $|z| \leq t$.

Example 1.3.5 *Let $g(z) = z + 1$. Then we have*

$$\overline{N}\left(r, \frac{1}{g}\right) = \log r + O(1).$$

So

$$\overline{\lambda}(g) = \overline{\lim_{r\to+\infty}} \frac{\log \overline{N}\left(r, \frac{1}{g}\right)}{\log r} = 0.$$

Definition 1.3.4 ([16]) *We define the exponent of convergence of the sequence of the poles of the meromorphic function f by*

$$\lambda\left(\frac{1}{f}\right) = \overline{\lim_{r\to+\infty}} \frac{\log N(r, f)}{\log r}.$$

Example 1.3.6 *Let $f(z) = \frac{e^z}{z}$. We have $z = 0$ is a simple pole and therefore*

$$\begin{aligned} N(r, f) &= \int_0^r \frac{n(t, \infty, f) - n(0, \infty, f)}{t} dt + n(0, \infty, f) \log r \\ &= \int_0^r \frac{1 - 1}{t} dt + \log r \\ &= \log r \end{aligned}$$

Then

$$\lambda\left(\frac{1}{f}\right) = \overline{\lim_{r\to+\infty}} \frac{\log N(r, f)}{\log r} = 0.$$

1.3.2 The (p,q)-order, Lower (p,q)-order, (p,q)-type and (p,q)-exponent of convergence of an entire or meromorphic function

For $r \in [0,+\infty[$, we define $\exp_1 r := e^r$ and $\exp_{p+1} r := \exp(\exp_p r)$, $p \in \mathbb{N}$. For all r sufficiently large, we define $\log_1 r := \log r$ and $\log_{p+1} r := \log(\log_p r)$, $p \in \mathbb{N}$. We also denote $\exp_0 r = r = \log_0 r$, $\log_{-1} r = \exp_1 r$ and $\exp_{-1} r = \log_1 r$. For the unity of notations, we present here the definition of (p,q)-order where p and q are integers with $p \geq q \geq 1$.

Definition 1.3.5 ([17], [18]) *The (p,q)-order of a meromorphic function f is defined by*

$$\sigma_{(p,q)}(f) = \overline{\lim_{r\to+\infty}} \frac{\log_p T(r,f)}{\log_q r},$$

where $T(r,f)$ is the characteristic function of Nevanlinna of the function f. If f is an entire function, then

$$\sigma_{(p,q)}(f) = \overline{\lim_{r\to+\infty}} \frac{\log_p T(r,f)}{\log_q r} = \overline{\lim_{r\to+\infty}} \frac{\log_{p+1} M(r,f)}{\log_q r},$$

where $M(r,f)$ is the maximum modulus of f in the circle $|z| = r$.

Example 1.3.7 *Let $f(z) = e^z$ is an entire function. We have $T(r,f) = \frac{r}{\pi}$. Then*

$$\sigma_{(1,1)}(f) = \overline{\lim_{r\to+\infty}} \frac{\log_1 T(r,f)}{\log_1 r} = \overline{\lim_{r\to+\infty}} \frac{\log_1 \frac{r}{\pi}}{\log_1 r} = 1.$$

So $\sigma_{(1,1)}(e^z) = 1$.

Definition 1.3.6 ([17], [18]) *The lower (p,q)-order of a meromorphic function f is defined by*

$$\mu_{(p,q)}(f) = \underline{\lim_{r\to+\infty}} \frac{\log_p T(r,f)}{\log_q r}.$$

If f is an entire function, then

$$\mu_{(p,q)}(f) = \underline{\lim_{r\to+\infty}} \frac{\log_{p+1} M(r,f)}{\log_q r}.$$

Example 1.3.8 *Let $h(z) = \cos z$. We have $T(r,\cos z) = \frac{2r}{\pi} + O(1)$ and therefore*

$$\mu_{(1,1)}(h) = \underline{\lim_{r\to+\infty}} \frac{\log_1 T(r,h)}{\log_1 r} = \underline{\lim_{r\to+\infty}} \frac{\log_1 \left(\frac{2r}{\pi} + O(1)\right)}{\log_1 r} = 1,$$

So $\mu_{(1,1)}(\cos z) = 1$.

Definition 1.3.7 ([17], [18]) *The (p,q)-type of a meromorphic funtion f with $0<\sigma_{(p,q)}(f)<+\infty$ is defined by*

$$\tau_{(p,q)}(f)=\overline{\lim_{r\to+\infty}}\frac{\log_{p-1}T(r,f)}{\left(\log_{q-1}r\right)^{\sigma_{(p,q)}(f)}}.$$

If f is an entire function, then

$$\tau_{M,(p,q)}(f)=\overline{\lim_{r\to+\infty}}\frac{\log_{p}M(r,f)}{\left(\log_{q-1}r\right)^{\sigma_{(p,q)}(f)}}.$$

Example 1.3.9 *Let $f(z)=e^z$. We have $\log M(r,e^r)=r$ and $\log_0 r=r$. Then*

$$\tau_{M,(1,1)}(f)=\overline{\lim_{r\to+\infty}}\frac{r}{r^1}=1.$$

So

$$\tau_{M,(1,1)}(e^z)=1.$$

Definition 1.3.8 ([17], [18]) *The (p,q)-exponent of convergence of zeros of a meromorphic function f is defined by*

$$\lambda_{(p,q)}(f)=\overline{\lim_{r\to+\infty}}\frac{\log_p N\left(r,\frac{1}{f}\right)}{\log_q r}$$

and the (p,q)-exponent of convergence of distinct zeros of a meromorphic function f is defined by

$$\overline{\lambda}_{(p,q)}(f)=\overline{\lim_{r\to+\infty}}\frac{\log_p \overline{N}\left(r,\frac{1}{f}\right)}{\log_q r},$$

where $N\left(r,\frac{1}{f}\right)$ $\left(\overline{N}\left(r,\frac{1}{f}\right)\right)$ is the integrated counting function of zeros (distinct zeros) of f in $\{z:|z|\leq r\}$.

Example 1.3.10 *Let $f(z)=\sin z$. The zeros of f are $z_n=n\pi$, for all $n\in\mathbb{Z}$. Then $n(t,0,f)\sim\frac{2t}{\pi}$, as $t\to+\infty$ and therefore $N\left(r,\frac{1}{f}\right)=\frac{2r}{\pi}+O(1)$.*
Hence

$$\begin{aligned}\lambda_{(1,1)}(f) &= \overline{\lim_{r\to+\infty}}\frac{\log N\left(r,\frac{1}{f}\right)}{\log r}\\ &= \overline{\lim_{r\to+\infty}}\frac{\log\left(\frac{2r}{\pi}+O(1)\right)}{\log r}\\ &= \overline{\lim_{r\to+\infty}}\frac{\log\frac{2r}{\pi}\left(1+\frac{O(1)}{\frac{2r}{\pi}}\right)}{\log r}\\ &= 1.\end{aligned}$$

Now, let $g(z) = (1+z)^3 \sin z$. *The zeros of* g *are* $\begin{cases} z = -1, & (\textit{with multiplicity } 3) \\ z = n\pi, & (\textit{distinct, infinite}). \end{cases}$
But the distinct zeros $\{1, \pi, 2\pi, 3\pi, \ldots\}$ *are of same growth as* $\sin z$ *and so* $\lambda_{(1,1)}(g) = 1$.

Definition 1.3.9 ([17], [18]) *The lower* (p,q)*-exponent of convergence of zeros of a meromorphic function* f *is defined by*

$$\underline{\lambda}_{(p,q)}(f) = \varliminf_{r\to+\infty} \frac{\log_p N\left(r, \frac{1}{f}\right)}{\log_q r},$$

and the lower (p,q)*-exponent of convergence of distinct zeros of a meromorphic function* f *is defined by*

$$\overline{\lambda}_{(p,q)}(f) = \varliminf_{r\to+\infty} \frac{\log_p \overline{N}\left(r, \frac{1}{f}\right)}{\log_q r}.$$

Example 1.3.11 *Let* $f(z) = \sin(\pi z)$. *The zeros of* f *are* $z = n \in \mathbb{Z}$ *and* $N\left(r, \frac{1}{f}\right) = 2r + O(1)$.
So for $p = 1$ *and* $q = 1$, *we get*

$$\underline{\lambda}_{(1,1)}(f) = \varliminf_{r\to+\infty} \frac{\log(2r + O(1))}{\log r} = 1.$$

For the function f, *all zeros are distinct and so* $\overline{N}\left(r, \frac{1}{f}\right) = N\left(r, \frac{1}{f}\right)$, *which gives* $\overline{\lambda}_{(1,1)}(f) = 1$.

Definition 1.3.10 ([17], [18]) *The* (p,q)*-exponent of convergence of poles of a meromorphic funtion* f *is defined by*

$$\lambda_{(p,q)}\left(\frac{1}{f}\right) = \varlimsup_{r\to+\infty} \frac{\log_p N(r,f)}{\log_q r}.$$

Example 1.3.12 *Let* $f(z) = \tan(z)$. *The poles of* $f(z)$ *are* $z = \frac{\pi}{2} + n\pi$, *for all integers* n. *Then* $N(r,f) = \frac{2r}{\pi}$ *and*

$$\lambda_{(p,q)}\left(\frac{1}{f}\right) = \varlimsup_{r\to+\infty} \frac{\log_p \frac{2r}{\pi}}{\log_q r} = \begin{cases} 0, & \textit{if } p > q, \\ 1, & \textit{if } p = q. \end{cases}$$

1.4 Elements of the theory of Wiman-Valiron

The theory of Wiman-Valiron is primarily used to study the local behavior of entire functions based on their series expansion.

Definition 1.4.1 ***(Maximal term)*** ([16]) *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *be an entire function. The maximal term is defined as*

$$\mu(r) = \mu(r, f) = \max\{|a_n| r^n, n \in \mathbb{N}\}.$$

Definition 1.4.2 ***(Central index)*** ([16]) *The central index of* f *is defined by*

$$\nu(r) = \nu(r, f) = \max\{m : \mu(r, f) = |a_m| r^m\}.$$

Example 1.4.1 *Let* $P(z) = a_n z^n + \cdots + a_0$, $a_n \neq 0$. *Then* $\mu(r) = \max\{|a_j| r^j;\ j = 0, \cdots, n\} = |a_n| r^n$, r *large enough, and therefore*

$$\nu_P(r) = \max\{m : |a_m| r^m = |a_n| r^n\} = n.$$

Proposition 1.4.1 ([16]) *Let* f *be an entire function of order* $\sigma(f)$. *Then*

$$\sigma(f) = \overline{\lim_{r \to +\infty}} \frac{\log \nu_f(r)}{\log r} = \overline{\lim_{r \to +\infty}} \frac{\log \log M(r, f)}{\log r}.$$

1.5 Hadamard's factorization theorem

Definition 1.5.1 ***(Canonical product)*** ([16]) *Let* f *be a transcendental meromorphic function and let* $z_1, z_2, ..$ *its zeros with* $0 < |z_1| \leq |z_2| \leq ...$ *Let* p *be the smallest integer such that the series*

$$\sum_{n=1}^{\infty} \frac{1}{|z_n|^{p+1}}$$

converges. We call

$$\begin{aligned} E(u, 0) &= (1 - u), \\ E(u, p) &= (1 - u) \exp\left(u + \frac{u^2}{2} + ... + \frac{u^p}{p}\right), \quad p = 1, 2, ... \end{aligned}$$

principal factors. The infinite product

$$P(z) = \prod_{n=1}^{\infty} E\left(\frac{z}{z_n}, p\right),$$

converges uniformly in each bounded domain in $\mathbb{C}$ *and* $P(z)$ *is called the canonical product of* f *formed by the zeros of* f. *The integer* p *is called the genus of the canonical product.*

Theorem 1.5.1 ([16]) *Let f be a meromorphic function of a finite order $\sigma(f)$ and let $\{a_1,a_2,...\}$ and $\{b_1,b_2,...\}$ be the zeros and the poles of f in $\mathbb{C}\setminus\{0\}$, respectively. Suppose that f can be represented as*

$$f(z) = c_k z^k + c_{k+1} z^{k+1} + ..., (c_k \neq 0),$$

in the neighborhood of $z = 0$. Then

$$f(z) = z^k e^{Q(z)} \frac{P_1(z)}{P_2(z)},$$

with $Q(z)$ is a polynomial with degree less or equal to $\sigma(f)$ and $P_1(z)$ and $P_2(z)$ are the canonical products of f formed by the zeros and poles of f.

1.6 Linear measure and Logarithmic measure of sets

Definition 1.6.1 ***(Linear measure)*** ([16]) *The linear measure of a set $E \subset [0,+\infty[$ is defined by*

$$m(E) = \int_E dt.$$

Example 1.6.1 *Let $E = [0, 2\pi]$. We have*

$$m(E) = \int_{[0,2\pi]} dt = 2\pi.$$

Definition 1.6.2 ***(Logarithmic measure)*** ([16]) *The logarithmic measure of a set $H \subset [1,+\infty[$ is defined by*

$$m_l(H) = \int_H \frac{dt}{t}.$$

Example 1.6.2 *Let $H = [1, e^7]$. We have*

$$m_l(H) = \int_{[1,e^7]} \frac{dt}{t} = 7.$$

Chapter 2

The (p,q)-order of solutions of some complex linear differential equations

2.1 Introduction and main results

In the past years, many authors investigated the complex linear differential equations

$$f^{(k)}+A_{k-1}(z)f^{(k-1)}+\cdots+A_1(z)f'+A_0(z)f=0 \tag{2.1}$$

and

$$f^{(k)}+A_{k-1}(z)f^{(k-1)}+\cdots+A_1(z)f'+A_0(z)f=F(z), \tag{2.2}$$

when $A_j(z)$ $(j=0,1,\cdots,k-1)$, $F(z)$ are entire functions and obtained some valuable results, (see e.g. [2], [11], [17],[18],[19],[20] and [23]).

In 2013, Tu et al. investigated the growth of solutions of equation (2.2) when the dominant coefficient $A_d(z)$ $(0\leq d\leq k-1)$ is of maximal order and being Lacunary series.

Theorem A ([19]) Let $A_j(z)$ $(j=0,1,...,k-1)$, $F(z)$ be entire functions of finite iterated order satisfying

$$\max\{\sigma_p(A_j)\,(j\neq d),\sigma_p(F)\}<\mu_p(A_d)=\sigma_p(A_d)=\sigma<\infty \quad (0\leq d\leq k-1).$$

Suppose that $A_d=\sum_{n=0}^{\infty}c_{\lambda_n}z^{\lambda_n}$ is an entire function such that the sequence of exponents $\{\lambda_n\}$ satisfies the gap condition

$$\frac{\lambda_n}{n}>(\log n)^{2+\eta}\,(\eta>0,\ n\in\mathbb{N}). \tag{2.3}$$

Then every transcendental solution $f(z)$ of (2.2) satisfies $\mu_{p+1}(f) = \sigma_{p+1}(f) = \sigma$. Furthermore, if $F(z) \not\equiv 0$, then every transcendental solution $f(z)$ of (2.2) satisfies

$$\overline{\lambda}_{p+1}(f) = \underline{\lambda}_{p+1}(f) = \overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \sigma.$$

Recently, Huang et al. [11] considered the equation (2.2) with different conditions on the ceofficient $A_d(z)$ and obtained the following result.

Theorem B ([11]) Let $A_j(z)$ $(j = 0, 1, \cdots, k-1)$, $F(z)$ be entire functions. Suppose that there exists some $d \in \{1, \cdots, k-1\}$ such that $\max\{\sigma(A_j), \sigma(F) : j \neq d\} \leq \sigma(A_d) < \infty$, $\max\{\tau(A_j) : \sigma(A_j) = \sigma(A_d), \tau(F)\} < \tau(A_d)$ and $T(r, A_d) \sim \log M(r, A_d)$ as $r \to +\infty$ outside a set of r of finite logarithmic measure. Then we have
(i) Every transcendental solution f of (2.2) satisfies $\sigma_2(f) = \sigma(A_d)$, and (2.2) may have polynomial solution f of degree $< d$.
(ii) If $F(z) \neq 0$, then every nonconstant solution f of (2.2) satisfies $\overline{\lambda}_2(f) = \lambda_2(f) = \sigma_2(f) = \sigma(A_d)$.
(iii) If $d = 1$, then every nonconstant solution f of (2.2) satisfies $\sigma_2(f) = \sigma(A_1)$, Furthermore, if $F(z) \neq 0$, then every nonconstant solution f of (2.2) satisfies $\overline{\lambda}_2(f) = \lambda_2(f) = \sigma_2(f) = \sigma(A_1)$.

As for the linear differential equations

$$A_k(z) f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = 0 \tag{2.4}$$

and

$$A_k(z) f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = F(z), \tag{2.5}$$

where $k \geq 2$, $A_j(z)$ $(j = 0, 1, \cdots, k-1)$, $F(z)$ are entire functions with $A_0 A_k F \not\equiv 0$, many authors investigated the properties of their solutions and obtained some interesting results, (see e.g. [4], [6], [9], [21]). It is well-known that if the coefficient $A_k(z) \equiv 1$, then all solutions of (2.4) and (2.5) are entire functions, but when $A_k(z)$ is a nonconstant entire function, then equation (2.4) or (2.5) can possess meromorphic solutions. For instance, the equation

$$zf''' + 4f'' + \left(-1 - \frac{1}{2}z^2 - z\right) e^{-z} f' + \left(\left(-1 - \frac{1}{2}z^2 + 2z\right) e^{-2z} + ze^{-3z}\right) f = 0$$

has a meromorphic solution $f(z) = \frac{1}{z^2} e^{e^{-z}}$ and the equation

$$z^3 f''' - z^3 f'' - 2z^2 f' - \left(z^3 + 3z^2 - 6\right) f = \left(z^2 - 6\right) \sin z$$

has a meromorphic solution $f(z)=\dfrac{\cos z}{z}$.

In 2015, Wu and Zheng have considered the equations (2.4) and (2.5), and obtained the following result when the coefficient $A_k(z)$ is of maximal order and Fabry gap series.

Theorem C ([21]) Suppose that $k\geq 2$, $A_j(z)$ $(j=0,1,\cdots,k-1)$ are entire functions satisfying $A_k(z)A_0(z)\not\equiv 0$ and $\sigma(A_j)<\sigma(A_k)<\infty$ $(j=0,1,\cdots,k-1)$. Suppose that $A_k(z)=\sum\limits_{n=0}^{\infty}c_{\lambda_n}z^{\lambda_n}$ and the sequence of exponents $\{\lambda_n\}$ satisfies the Fabry gap codition

$$\frac{\lambda_n}{n}\to\infty,\ \ (n\to\infty). \tag{2.6}$$

Then every rational solution $f(z)$ of (2.4) is a polynomial with $\deg f\leq k-1$ and every transcendental meromorphic solution $f(z)$, whose poles are of uniformly bounded multiplicities, of (2.4) such that $\lambda\left(\frac{1}{f}\right)<\mu(f)$, satisfies

$$\overline{\lambda}(f-\varphi)=\lambda(f-\varphi)=\sigma(f)=\infty,\ \overline{\lambda}_2(f-\varphi)=\lambda_2(f-\varphi)=\sigma_2(f)=\sigma(A_k),$$

where $\varphi(z)$ is a finite order meromorphic function and doesn't solve (2.4).

Now, in the light of these theorems, Ferraoun and Belaïdi in [5] asked two questions : First, can we have the same properties as in Theorem B for the solutions of equation (2.2) when the coefficients are of (p,q)-order? and secondly, what about the growth of solutions of the equations (2.4) and (2.5) when we have the arbitrary coefficient $A_s(z)\,(0\leq s\leq k)$ instead of the coefficient $A_k(z)$? In this chapter, they answered these questions and obtained the following results.

Theorem 2.1.1 ([5]) *Let $A_j(z)$ $(j=0,1,\cdots,k)$ with $A_k(z)A_0(z)\not\equiv 0$ be entire functions such that*

$$\max\{\sigma_{(p,q)}(A_j)\ \ j\neq s\}<\sigma_{(p,q)}(A_s)<\infty,\ \ (0\leq s\leq k).$$

Suppose that $A_s(z)=\sum\limits_{n=0}^{\infty}c_{\lambda_n}z^{\lambda_n}$ and the sequence of exponents $\{\lambda_n\}$ satisfies (2.3). Then every rational solution $f(z)$ of (2.4) is a polynomial with $\deg f\leq s-1$ and every transcendental meromorphic solution $f(z)$ of (2.4) such that $\lambda_{(p,q)}\left(\frac{1}{f}\right)<\mu_{(p,q)}(f)$ satisfies

$$\sigma_{(p+1,q)}(f)=\sigma_{(p,q)}(A_s).$$

Theorem 2.1.2 ([5]) *Let* $A_j(z)$ $(j=0,1,\cdots,k)$ *with* $A_k(z)A_0(z)\not\equiv 0$ *be entire functions such that*

$$\max\{\sigma_{(p,q)}(A_j),\quad j\neq s\}<\mu_{(p,q)}(A_s)=\sigma_{(p,q)}(A_s)=\sigma<\infty,\quad (0\leq s\leq k).$$

Suppose that $A_s(z)=\sum_{n=0}^{\infty}c_{\lambda_n}z^{\lambda_n}$ *and the sequence of exponents* $\{\lambda_n\}$ *satisfies (2.3). Then every rational solution* $f(z)$ *of (2.4) is a polynomial with* $\deg f\leq s-1$ *and every transcendental meromorphic solution* $f(z)$ *of (2.4) such that* $\lambda_{(p,q)}\left(\frac{1}{f}\right)<\mu_{(p,q)}(f)$ *satisfies*

$$\mu_{(p+1,q)}(f)=\sigma_{(p+1,q)}(f)=\sigma_{(p,q)}(A_s)=\sigma.$$

Theorem 2.1.3 ([5]) *Let* $A_j(z)$ $(j=0,1,\cdots,k)$ *with* $A_k(z)A_0(z)\not\equiv 0$ *be entire functions satisfying the hypotheses of Theorem 2.1.1 and* $F(z)\not\equiv 0$ *is an entire function.*
(i) If $\sigma_{(p+1,q)}(F)<\sigma_{(p,q)}(A_s)$, *then every transcendental meromorphic solution* $f(z)$ *of (2.5) such that* $\lambda_{(p,q)}\left(\frac{1}{f}\right)<\mu_{(p,q)}(f)$ *satisfies*

$$\sigma_{(p+1,q)}(f)=\sigma_{(p,q)}(A_s),$$

with at most one exceptional solution f_0 *satisfying* $\sigma_{(p+1,q)}(f_0)<\sigma_{(p,q)}(A_s)$.

(ii) If $\sigma_{(p+1,q)}(F)>\sigma_{(p,q)}(A_s)$, *then every transcendental meromorphic solution* $f(z)$ *of (2.5) such that* $\lambda_{(p,q)}\left(\frac{1}{f}\right)<\mu_{(p,q)}(f)$ *satisfies*

$$\sigma_{(p+1,q)}(f)=\sigma_{(p+1,q)}(F).$$

For equation (2.2), they obtained the following result.

Theorem 2.1.4 ([5]) *Let* $A_j(z)$ $(j=0,1,\cdots,k-1)$, $F(z)$ *be entire funtions. Suppose that there exists some* $s\in\{1,2,\cdots,k-1\}$ *such that*

$$\max\{\sigma_{(p,q)}(A_j)\ \ (j\neq s),\sigma_{(p,q)}(F)\}\leq\sigma_{(p,q)}(A_s)=\sigma<\infty,$$

$$\max\{\tau_{(p,q)}(A_j):\ \sigma_{(p,q)}(A_j)\ \ =\sigma_{(p,q)}(A_s),\tau_{(p,q)}(F)\}<\tau_{(p,q)}(A_s)$$

and that $T(r,A_s)\sim\log M(r,A_s)$ *as* $r\to+\infty$ *outside a set of* r *of finite logarithmic measure. Then*
(i) Every transcendental solution $f(z)$ *of (2.2) satisfes* $\sigma_{(p+1,q)}(f)=\sigma_{(p,q)}(A_s)$, *and every non-transcendental solution* $f(z)$ *of (2.2) is a polynomial of degree* $\deg(f)\leq s-1$.

(ii) If $F(z) \not\equiv 0$, *then every transcendental solution* $f(z)$ *of (2.2) satisfes* $\overline{\lambda}_{(p+1,q)}(f) = \lambda_{(p+1,q)}(f) = \sigma_{(p+1,q)}(f) = \sigma_{(p,q)}(A_s)$.
(iii) If $s = 1$, *then every nonconstant solution* $f(z)$ *of (2.2) satisfes* $\sigma_{(p+1,q)}(f) = \sigma_{(p,q)}(A_1)$ *and if* $F(z) \not\equiv 0$, *then every nonconstant solution* $f(z)$ *of (2.2) satisfes* $\overline{\lambda}_{(p+1,q)}(f) = \lambda_{(p+1,q)}(f) = \sigma_{(p+1,q)}(f) = \sigma_{(p,q)}(A_1)$.

2.2 Preliminary Lemmas

To prove the previous theorems, we need the following lemmas.

Lemma 2.2.1 ([7]) *Let* f *be a transcendental meromorphic function in the plane, and let* $\alpha > 1$ *be a given constant. Then for any given constant and for any given* $\varepsilon > 0$:
(i) There exists a set $E_1 \subset (1, +\infty)$ *that has a finite logarithmic measure, and a constant* $B > 0$ *depending only on* α *such that for all* z *with* $|z| = r \notin [0,1] \cup E_1$, *we have*

$$\left|\frac{f^{(n)}(z)}{f^{(m)}(z)}\right| \leq B\left(\frac{T(\alpha r, f)}{r}(\log^{\alpha} r)\log T(\alpha r, f)\right)^{n-m} (0 \leq m < n).$$

(ii) There exists a set $H_1 \subset [0, 2\pi)$ *that has linear measure zero and a constant* $B > 0$ *depending only on* α, *for any* $\theta \in [0, 2\pi)\backslash H_1$, *there exists a constant* $R_0 = R_0(\theta) > 1$ *such that for all* z *satisfying* $\arg z = \theta$ *and* $|z| = r > R_0$, *we have*

$$\left|\frac{f^{(n)}(z)}{f^{(m)}(z)}\right| \leq B\left(\frac{T(\alpha r, f)}{r}(\log^{\alpha} r)\log T(\alpha r, f)\right)^{n-m} (0 \leq m < n).$$

Lemma 2.2.2 ([5]) *Let* $f(z) = \dfrac{g(z)}{d(z)}$ *be a meromorphic function, where* $g(z)$ *and* $d(z)$ *are entire function satisfying* $\mu_{(p,q)}(g) = \mu_{(p,q)}(f) = \mu \leq \sigma_{(p,q)}(g) = \sigma_{(p,q)}(f) \leq +\infty$ *and* $\lambda_{(p,q)}(d) = \sigma_{(p,q)}(d) = \lambda_{(p,q)}\left(\frac{1}{f}\right) < \mu$.
Then there exists a set $E_2 \subset (1, +\infty)$ *of finite logarithmic measure such that for all* $|z| = r \notin [0,1] \cup E_2$ *and* $|g(z)| = M(r, g)$ *we have*

$$\left|\frac{f(z)}{f^{(k)}(z)}\right| \leq r^{2k}, (k \in \mathbb{N}).$$

Lemma 2.2.3 ([15]) *Let* $f(z) = \sum_{n=0}^{\infty} c_{\lambda_n} z^{\lambda_n}$ *be an entire function and the sequence of exponents* $\{\lambda_n\}$ *satisfies the gap condition (2.3). Then for any given* $\varepsilon > 0$

$$\log L(r, f) > (1 - \varepsilon)\log M(r, f)$$

holds outside a set E_3 *of finite logarithmic measure, where* $M(r, f) = \sup_{|z|=r} |f(z)|$ *and* $L(r, f) = \inf_{|z|=r} |f(z)|$.

Lemma 2.2.4 ([18]) *Let $f(z)$ be an entire function of (p,q)-order satisfies $0 < \sigma_{(p,q)}(f) = \sigma < \infty$. Then for any given $\varepsilon > 0$, there exists a set $E_4 \subset (1,+\infty)$ having infinite logarithmic measure such that for all $r \in E_4$, we have*

$$\sigma = \lim_{r\to+\infty, r\notin E_4} \frac{\log_p T(r,f)}{\log_q r} = \lim_{r\to+\infty, r\notin E_4} \frac{\log_{p+1} M(r,f)}{\log_q r}$$

and

$$M(r,f) > \exp_{p+1}\{(\sigma - \varepsilon)\log_q r\}.$$

Lemma 2.2.5 ([8]) *Let $g:[0,+\infty)\to\mathbb{R}$ and $h:[0,+\infty)\to\mathbb{R}$ be monotone nondecreasing functions such that $g(r) \le h(r)$ for all $r \notin [0,1]\cup E_5$, where $E_5 \subset (1,+\infty)$ is a set of finite logarithmic measure. Then for any $\alpha > 1$, there exists an $r_0 = r_0(\alpha) > 0$ such that $g(r) \le h(\alpha r)$ for all $r > r_0$.*

Lemma 2.2.6 ([5]) *Let $f(z) = \dfrac{g(z)}{d(z)}$ be a meromorphic function, where $g(z)$ and $d(z)$ are entire functions satisfying $\mu_{(p,q)}(g) = \mu_{(p,q)}(f) = \mu \le \sigma_{(p,q)}(g) = \sigma_{(p,q)}(f) \le +\infty$ and $\lambda_{(p,q)}(d) = \sigma_{(p,q)}(d) = \lambda_{(p,q)}\left(\frac{1}{f}\right) < \mu$. Then there exists $E_6 \subset (1,+\infty)$ of finite logarithmic measure such that for all $|z| = r \notin [0,1]\cup E_6$ and $|g(z)| = M(r,g)$ we have*

$$\frac{f^{(n)}(z)}{f(z)} = \left(\frac{v_g(r)}{z}\right)^n (1+o(1)), (n\in\mathbb{N}),$$

where $v_g(r)$ is the central index of $g(z)$.

Lemma 2.2.7 ([5]) *Let $f(z)$ be an entire function such that $\sigma_{(p,q)}(f) = \sigma < +\infty$. Then there exist entire functions $\beta(z)$ and $D(z)$ such that*

$$f(z) = \beta(z)\, e^{D(z)},$$

$$\sigma_{(p,q)}(f) = \max\{\sigma_{(p,q)}(\beta), \sigma_{(p,q)}\left(e^{D(z)}\right)\}$$

and

$$\sigma_{(p,q)}(\beta) = \overline{\lim_{r\to+\infty}} \frac{\log_p N\left(r, \frac{1}{f}\right)}{\log_q r}.$$

Moreover, for any given $\varepsilon > 0$, we have

$$|\beta(z)| \ge \exp\{-\exp_p\{(\sigma_{(p,q)}(\beta)+\varepsilon)\log_q r\}\} \ (r \notin E_7),$$

where $E_7 \subset (1,+\infty)$ is a set of r of finite linear measure.

Lemma 2.2.8 ([5]) *Let $f(z)$ be an entire function such that $\sigma_{(p,q)}(f) = \sigma < +\infty$. Then, there exists a set $E_8 \subset (1, +\infty)$ of r of finite linear measure such that for any given $\varepsilon > 0$, we have*

$$\exp\{-\exp_p\{(\sigma+\varepsilon)\log_q r\}\} \leq |f(z)| \leq \exp_{p+1}\{(\sigma+\varepsilon)\log_q r\} \ (r \notin E_8).$$

Lemma 2.2.9 ([18]) *Let $f(z)$ be an entire function of (p,q)-order, and let $v_f(r)$ be a central index of $f(z)$. Then*

$$\sigma_{(p,q)}(f) = \overline{\lim_{r\to+\infty}} \frac{\log_p v_f(r)}{\log_q r}.$$

Lemma 2.2.10 ([5]) *Let $f(z)$ be an entire function with $\sigma_{(p,q)}(f) = \sigma, 0 < \sigma < \infty$. Then for any given $\beta < \sigma$, there exists a set E_9 having infinite logarithmic measure such that for all $|z| = r \in E_9$, we have*

$$\log_{p+1} M(r,f) > \beta \log_q r,$$

where $M(r,f) = \sup_{|z|=r} |f(z)|$.

Lemma 2.2.11 ([5]) *Let $f(z) = \sum_{n=0}^{\infty} c_{\lambda_n} z^{\lambda_n}$ be an entire function with $\sigma_{(p,q)}(f) = \sigma, 0 < \sigma < \infty$. If the sequence of exponent $\{\lambda_n\}$ satisfies (2.3), then for any given $\beta < \sigma$, there exists a set E_{10} having infinite logarithmic measure such that for all $|z| = r \in E_{10}$, we have*

$$|f(z)| > \exp_{p+1}\{\beta \log_q r\}.$$

Lemma 2.2.12 ([5]) *Let $f(z)$ be an entire function with $\mu_{(p,q)}(f) = \mu < \infty$. Then for any given $\varepsilon > 0$, there exists a set $E_{11} \subset (1, +\infty)$ having infinite logarithmic measure such that for all $|z| = r \in E_{11}$, we have*

$$\mu_{(p,q)}(f) = \lim_{r\to+\infty, r\in E_{11}} \frac{\log_{p+1} M(r,f)}{\log_q r}$$

and

$$M(r,f) < \exp_{p+1}\{(\mu+\varepsilon)\log_q r\}.$$

Lemma 2.2.13 ([17]) *If $f(z)$ is a meromorphic function, then $\sigma_{(p,q)}(f') = \sigma_{(p,q)}(f)$.*

Lemma 2.2.14 ([19]) *Let $f(z)$ be a transcendental entire function, and let $z_r = re^{i\theta_r}$ be a point satisfying $|f(z_r)| = M(r,f)$. Then, there exists a constant $\delta_r > 0$ such that for all z satisfying $|z| = r \notin E_{12}$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$, we have*

$$\left|\frac{f(z)}{f^{(j)}(z)}\right| \leq 2r^j, \quad (j \in \mathbb{N}).$$

Lemma 2.2.15 ([18]) *Let $f(z)$ be an entire function of (p,q)-order satisfying $0<\sigma_{(p,q)}(f)=\sigma<\infty$ and $0<\tau_{(p,q)}(f)=\tau<\infty$. Then for any given $\beta<\tau$, there exists a set $E_{13}\subset[1,+\infty)$ that has an infinite logarithmic measure such that for all $|z|=r\in E_{13}$, we have*

$$\log_p M(r,f)>\beta\left(\log_{q-1} r\right)^{\sigma}\quad (r\in E_{13}).$$

Lemma 2.2.16 ([5]) *Let $f(z)$ be a transcendental entire function satisfying $0<\sigma_{(p,q)}(f)=\sigma<\infty$, $0<\tau_{(p,q)}(f)=\tau<\infty$ and $T(r,f)\sim\log M(r,f)$ as $r\to+\infty$ outside a set of r of finite logarithmic measure. Then for any given $\beta<\tau$, there exists a set $E_{14}\subset(0,+\infty)$ having infinite logarithmic measure and a set $H_2\subset[0,2\pi)$ that has linear measure zero such that for all z satisfying $|z|=r\in E_{14}$ and $\arg z=\theta\in[0,2\pi)\diagdown H_2$, we have*

$$\left|f\left(re^{i\theta}\right)\right|>\exp_p\{\beta\left(\log_{q-1} r\right)^{\sigma}\}.$$

Lemma 2.2.17 ([17]) *Let $A_0(z),A_1(z),...,A_{k-1}(z)$ and $F(z)\not\equiv 0$ be meromorphic functions. If $f(z)$ is a meromorphic solution to (2.2) satisfying*

$$\max\{\sigma_{(p+1,q)}(F),\ \sigma_{(p+1,q)}(A_j):j=0,1,\cdots,k-1\}<\sigma_{(p+1,q)}(f),$$

then we have

$$\overline{\lambda}_{(p+1,q)}(f)=\lambda_{(p+1,q)}(f)=\sigma_{(p+1,q)}(f).$$

2.3 Proof of Theorem 2.1.1

Assume that $f(z)$ is a rational solution of (2.4). If either $f(z)$ is a rational function, which has a pole at z_0 of degree $m\geq 1$, or $f(z)$ is a polynomial with $\deg f\geq s$, then $f^{(s)}(z)\not\equiv 0$. Since $\max\{\sigma_{(p,q)}(A_j),\ j\neq s\}<\sigma_{(p,q)}(A_s)<\infty$, $(0\leq s\leq k)$, then

$$\sigma_{(p,q)}(0)=\sigma_{(p,q)}(A_k(z)\ f^{(k)}+A_{k-1}(z)f^{(k-1)}+\cdots+A_1(z)f'+A_0(z)f) \tag{2.3.1}$$

We have $\max\{\sigma_{(p,q)}(A_j),\ j\neq s\}<\sigma_{(p,q)}(A_s)<\infty$, $(0\leq s\leq k)$, and $\sigma_{(p,q)}(0)=0$ and $\sigma_{(p,q)}\left(A_k(z)f^{(k)}+A_{k-1}(z)f^{(k-1)}+\cdots+A_1(z)f'+A_0(z)f\ \right)\leq\max\{\sigma_{(p,q)}(A_j),\ j\neq s\}$. So from (2.3.1) we have this result

$$0\leq\max\{\sigma_{(p,q)}(A_j),\ j\neq s\}<\sigma_{_{(p,q)}}(A_s)>0,$$

which is a contradiction. Therefore, $f(z)$ must be a polynomial with $\deg f\leq s-1$.

Now, we assume that $f(z)$ is a transcendental meromorphic solution of (2.4) such that $\lambda_{(p,q)}\left(\frac{1}{f}\right) < \mu_{(p,q)}(f)$.

By Lemma 2.2.1, there exists a constant $B > 0$ and a set $E_1 \subset (1, +\infty)$ of finite logarithmic measure such that for all z satisfying $|z| = r \notin [0,1] \cup E_1$, we have

$$\begin{aligned}\left|\frac{f^{(j)}(z)}{f(z)}\right| &\leq B\left(\frac{T(2r,f)}{r}\left(\log^2 r\right)\log T(2r,f)\right)^j, \ (1 \leq j \leq k)\\ &\leq BT(2r,f)^j\, T(2r,f)\\ &\leq B\left(T(2r,f)\right)^{j+1}\\ &\leq B\left(T(2r,f)\right)^{k+1}, \ (1 \leq j \leq k) \qquad (2.3.2)\end{aligned}$$

Since $\lambda_{(p,q)}\left(\frac{1}{f}\right) < \mu_{(p,q)}(f)$, then by Hadamard's factorization theorem, we can write f as $f(z) = \frac{g(z)}{d(z)}$, where $g(z)$ and $d(z)$ are entire functions satisfying

$$\mu_{(p,q)}(g) = \mu_{(p,q)}(f) = \mu \leq \sigma_{(p,q)}(g) = \sigma_{(p,q)}(f),$$

$$\lambda_{(p,q)}(d) = \sigma_{(p,q)}(d) = \lambda_{(p,q)}\left(\frac{1}{f}\right) < \mu.$$

Then by Lemma 2.2.2, there exists a set $E_2 \subset (1, +\infty)$ of finite logarithmic measure such that for all $|z| = r \notin [0,1] \cup E_2$ and $|g(z)| = M(r,g)$ and for r sufficiently large, we have

$$\left|\frac{f(z)}{f^{(s)}(z)}\right| \leq r^{2s}, \ (s \in \mathbb{N}). \qquad (2.3.3)$$

Set $\alpha = \max\{\sigma_{(p,q)}(A_j)\ j \neq s\} < \sigma_{(p,q)}(A_s) = \sigma < \infty$. Then, for any given $\varepsilon\ (0 < 2\varepsilon < \sigma - \alpha)$, we have

$$\begin{aligned}&\frac{\log_{p+1} M(r,A_j)}{\log_q r} \leq \alpha + \varepsilon\\ &\Rightarrow \log_{p+1} M(r,A_j) \leq (\alpha+\varepsilon)\log_q r\\ &\Rightarrow M(r,A_j) \leq \exp_{p+1}\{(\alpha+\varepsilon)\log_q r\},\end{aligned}$$

that is,

$$|A_j(z)| \leq \exp_{p+1}\{(\alpha+\varepsilon)\log_q r\}, j \neq s. \qquad (2.3.4)$$

By Lemma 2.2.3 and Lemma 2.2.4, there exists a set $E_{15} \subset (1, +\infty)$ having infinite logarithmic measure such that for all $r \in E_{15}$, we have

$$|A_s(z)| \geq L(r,A_s) > (M(r,A_s))^{1-\varepsilon} \geq \left(\exp_{p+1}\left\{\left(\sigma - \frac{\varepsilon}{2}\right)\log_q r\right\}\right)^{1-\varepsilon} \geq \exp_{p+1}\{(\sigma-\varepsilon)\log_q r\}. \qquad (2.3.5)$$

By (2.4), we have

$$A_k(z)\ f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_{s+1}(z) f^{(s+1)}$$

$$+A_s(z)\ f^{(s)} + A_{s-1}(z) f^{(s-1)} + \cdots + A_1(z) f' + A_0(z) f = 0$$

$$\Rightarrow -A_s(z)\ f^{(s)} = A_k(z)\ f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots$$

$$+A_{s+1}(z) f^{(s+1)} + A_{s-1}(z) f^{(s-1)} + \cdots + A_1(z) f' + A_0(z) f$$

$$\Longrightarrow -A_s(z)\ f^{(s)} = f\left[A_k(z)\frac{f^{(k)}}{f} + A_{k-1}(z)\frac{f^{(k-1)}}{f} + \cdots\right.$$

$$\left.+A_{s+1}(z)\frac{f^{(s+1)}}{f} + A_{s-1}(z)\frac{f^{(s-1)}}{f} + \cdots + A_1(z)\frac{f'}{f} + A_0(z)\right]$$

$$\Longrightarrow -A_s(z) = \frac{f}{f^{(s)}}\left[A_k(z)\frac{f^{(k)}}{f} + A_{k-1}(z)\frac{f^{(k-1)}}{f} + \cdots + A_{s+1}(z)\frac{f^{(s+1)}}{f}\right.$$

$$\left.+A_{s-1}(z)\frac{f^{(s-1)}}{f} + \cdots + A_1(z)\frac{f'}{f} + A_0(z)\right]$$

$$\Longrightarrow |A_s(z)| \le \left|\frac{f}{f^{(s)}}\right|\left[|A_k(z)|\left|\frac{f^{(k)}}{f}\right| + |A_{k-1}(z)||\frac{f^{(k-1)}}{f}| + \cdots + |A_{s+1}(z)|\left|\frac{f^{(s+1)}}{f}\right|\right.$$

$$\left.+|A_{s-1}(z)|\left|\frac{f^{(s-1)}}{f}\right| + \cdots + |A_1(z)|\left|\frac{f'}{f}\right| + |A_0(z)|\right]. \tag{2.3.6}$$

Hence, by substituting (2.3.2) - (2.3.5) into (2.3.6), we obtain for all z satisfying $r \in E_{15}\setminus(E_1 \cup E_2 \cup [0,1])$

$$\exp_{p+1}\{(\sigma-\varepsilon)\log_q r\} \le r^{2s}\exp_{p+1}\{(\alpha+\varepsilon)\log_q r\}kB(T(2r,f))^{k+1}.$$

Since $\alpha < \sigma$, then, for any given ε $(0 < 2\varepsilon < \sigma - \alpha)$, we have

$$\exp\{(1-o(1))\exp_p\{(\sigma-\varepsilon)\log_q r\}\} \le r^{2s}kB(T(2r,f))^{k+1}. \tag{2.3.7}$$

By (2.3.7) and Lemma 2.2.5, we have

$$\sigma_{(p+1,q)}(f) \ge \sigma_{(p,q)}(A_s).$$

Now, we prove that $\sigma_{(p+1,q)}(f) \le \sigma_{(p,q)}(A_s)$. We can rewrite (2.4) as

$$A_k(z) f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_{s+1}(z) f^{(s+1)} + A_s(z) f^{(s)}$$

$$+A_{s-1}(z) f^{(s-1)} + \cdots + A_1(z) f' + A_0(z) f = 0$$

$$\Rightarrow -A_k(z)\frac{f^{(k)}}{f} = A_{k-1}(z)\frac{f^{(k-1)}}{f} + \cdots + A_{s+1}(z)\frac{f^{(s+1)}}{f} + A_s(z)\frac{f^{(s)}}{f}$$

$$+A_{s-1}(z)\frac{f^{(s-1)}}{f} + \cdots + A_1(z)\frac{f'}{f} + A_0(z)\ . \tag{2.3.8}$$

By Lemma 2.2.6, there exists a set $E_6 \subset (1,+\infty)$ of finite logarithmic measure such that for all $|z| = r \notin [0,1] \cup E_6$ and $|g(z)| = M(r,g)$, we have

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{v_g(r)}{z}\right)^j (1+o(1)), \ \ (j = 0,\cdots,k). \tag{2.3.9}$$

Since $\max\{\sigma_{(p,q)}(A_j),\ j \neq s\} < \sigma_{(p,q)}(A_s) < \infty$, then for sufficiently large r, we have

$$|A_j(z)| \leq \exp_p\{\left(\sigma_{(p,q)}(A_s)+\varepsilon\right)\log_q r\}, (j = 0,\cdots,k). \tag{2.3.10}$$

By Lemma 2.2.8, there exists a set $E_8 \subset (1,+\infty)$ of finite linear measure (and so of finite logarithmic measure) such that for all $|z| = r \notin E_8$, we have

$$|A_k(z)| \geq \exp\{-\exp_p\{\left(\sigma_{(p,q)}(A_k)+\varepsilon\right)\log_q r\}\}$$

$$\geq \exp\{-\exp_p\{\left(\sigma_{(p,q)}(A_s)+\varepsilon\right)\log_q r\}\}. \tag{2.3.11}$$

From (2.3.8) and (2.3.9), for z all satisfying $|z| = r \notin [0,1] \cup E_6$ and $|g(z)| = M(r,g)$, we have

$$-A_k(z)\ \left(\frac{v_g(r)}{z}\right)^k (1+o(1)) = A_{k-1}(z)\left(\frac{v_g(r)}{z}\right)^{k-1}(1+o(1))$$

$$+\cdots + A_{s+1}(z)\left(\frac{v_g(r)}{z}\right)^{s+1}(1+o(1)) + A_s(z)\left(\frac{v_g(r)}{z}\right)^{s}(1+o(1))$$

$$+A_{s-1}(z)\left(\frac{v_g(r)}{z}\right)^{s-1}(1+o(1)) + \cdots + A_1(z)\left(\frac{v_g(r)}{z}\right)^{1}(1+o(1)) + A_0(z)$$

It follows that

$$|A_k(z)|\left|\left(\frac{v_g(r)}{z}\right)^k\right||(1+o(1))| \leq |A_{k-1}(z)|\left|\left(\frac{v_g(r)}{z}\right)^{k-1}\right||(1+o(1))|$$

$$+\cdots + |A_{s+1}(z)|\left|\left(\frac{v_g(r)}{z}\right)^{s+1}\right||(1+o(1))| + |A_s(z)|\left|\left(\frac{v_g(r)}{z}\right)^{s}\right||(1+o(1))|$$

$$+|A_{s-1}(z)|\left|\left(\frac{v_g(r)}{z}\right)^{s-1}\right||(1+o(1))| + \cdots + |A_1(z)|\left|\left(\frac{v_g(r)}{z}\right)^{1}\right||(1+o(1))| + |A_0(z)| \tag{2.3.12}$$

and by (2.3.10) - (2.3.12) for z all satisfying $|z| = r \notin ([0,1] \cup E_6 \cup E_8)$ and $|g(z)| = M(r,g)$, we have

$$\exp\{-\exp_p\{\left(\sigma_{(p,q)}(A_s)+\varepsilon\right)\log_q r\}\}\left(\frac{v_g(r)}{r}\right)|(1+o(1))|$$

$$\leq k\exp_{p+1}\{\left(\sigma_{(p,q)}(A_s)+\varepsilon\right)\log_q r\}|(1+o(1))|$$

$$\Rightarrow -\exp_p\{\left(\sigma_{(p,q)}(A_s)+\varepsilon\right)\log_q r\}+\log\left(\frac{v_g(r)}{r}\right)$$

$$\leq \log k+\exp_p\{\left(\sigma_{(p,q)}(A_s)+\varepsilon\right)\log_q r\}$$

$$\Longrightarrow \log v_g(r)-\log r\leq \log k+2\exp_p\{\left(\sigma_{(p,q)}(A_s)+\varepsilon\right)\log_q r\}$$

$$\Longrightarrow \log_{p+1} v_g(r)-\log_{p+1} r\leq \log_{p+1} k+\log_p 2+\left(\sigma_{(p,q)}(A_s)+\varepsilon\right)\log_q r$$

$$\Longrightarrow \frac{\log_{p+1} v_g(r)}{\log_q r}-\frac{\log_{p+1} r}{\log_q r}\leq \frac{\log_{p+1} k}{\log_q r}+\frac{\log_p 2}{\log_q r}+\sigma_{(p,q)}(A_s)+\varepsilon$$

$$\Longrightarrow \overline{\lim_{r\to+\infty}}\frac{\log_{p+1} v_g(r)}{\log_q r}\leq \sigma_{(p,q)}(A_s)+\varepsilon \tag{2.3.13}$$

Since $\varepsilon > 0$ is arbitrary, then by (2.3.13), Lemma 2.2.5 and Lemma 2.2.9, we have $\sigma_{(p+1,q)}(g) \leq \sigma_{(p,q)}(A_s)$, that is $\sigma_{(p+1,q)}(f) \leq \sigma_{(p,q)}(A_s)$. Therefore, we get $\sigma_{(p+1,q)}(f) = \sigma_{(p,q)}(A_s)$.

2.4 Proof of Theorem 2.1.2

Assume that $f(z)$ is a rational solution of (2.4). By the same reasoning as in the proof of Theorem 2.1.1, it is clear that $f(z)$ is a polynomial with $\deg f \leq s-1$. Now, we assume that $f(z)$ is a transcendental meromorphic solution of (2.4) such that $\lambda_{(p,q)}\left(\frac{1}{f}\right) < \mu_{(p,q)}(f)$. By Theorem 2.1.1, we have $\sigma_{(p+1,q)}(f) = \sigma_{(p,q)}(A_s) = \sigma$. Then, we only need to prove that $\mu_{(p+1,q)}(f) = \mu_{(p,q)}(A_s) = \sigma$. Since $\max\{\sigma_{(p,q)}(A_j),\ (j \neq s)\} < \sigma$, then there exist constants α_1, β_1 satisfying $\max\{\sigma_{(p,q)}(A_j), (\ j \neq s)\} < \alpha_1 < \beta_1 < \sigma$. Then

$$\frac{\log_{p+1} M(r,A_j)}{\log_q r}\leq \alpha_1$$

$$\Rightarrow \log_{p+1} M(r,A_j)\leq \alpha_1\log_q r$$

$$\Rightarrow M(r,A_j)\leq \exp_{p+1}\{\alpha_1\log_q r\}, j\neq s,$$

$$\Rightarrow |A_j(z)|\leq \exp_{p+1}\{\alpha_1\log_q r\}, j\neq s. \tag{2.4.1}$$

Also, we have $A_s(z) = \sum_{n=0}^{\infty} c_{\lambda_n} z^{\lambda_n}$ such that the sequence of exponents $\{\lambda_n\}$ satisfies (2.3) and $\mu_{(p,q)}(A_s) = \sigma_{(p,q)}(A_s) = \sigma$. Then, by Lemma 2.2.11, there exists a set E_{10} having infinite logarithmic measure such that for all z satisfying $|z| = r \in E_{10}$, we have

$$\frac{\log_{p+1} M(r, A_s)}{\log_q r} > \beta_1$$

$$\Rightarrow \log_{p+1} M(r, A_s) > \beta_1 \log_q r$$

$$\Rightarrow M(r, A_s) > \exp_{p+1}\{\beta_1 \log_q r\}$$

$$\Rightarrow |A_s(z)| > \exp_{p+1}\{\beta_1 \log_q r\}. \tag{2.4.2}$$

Hence, by substituting (2.4.1), (2.4.2), (2.3.2), (2.3.3) into (2.3.6), for all z satisfying $|z| = r \in E_{10} \setminus ([0,1] \cup E_1 \cup E_2)$, we have

$$\exp_{p+1}\{\beta_1 \log_q r\} \leq B \exp_{p+1}\{\alpha_1 \log_q r\} r^{2s} k [T(2r, f)]^{k+1}. \tag{2.4.3}$$

Since β_1 is aribitrarily close to σ, then by (2.4.3) and Lemma 2.2.5, we obtain

$$\mu_{(p+1,q)}(f) \geq \sigma = \mu_{(p,q)}(A_s).$$

On the other hand, by (2.4), we have

$$\begin{aligned} |A_k(z)| \left|\frac{f^{(k)}}{f}\right| \leq\ & |A_{k-1}(z)| \left|\frac{f^{(k-1)}}{f}\right| + \cdots + |A_{s+1}(z)| \left|\frac{f^{(s+1)}}{f}\right| + |A_s(z)| \left|\frac{f^{(s)}}{f}\right| \\ & + |A_{s-1}(z)| \left|\frac{f^{(s-1)}}{f}\right| + \cdots + |A_1(z)| \left|\frac{f'}{f}\right| + |A_0(z)|. \end{aligned} \tag{2.4.4}$$

By Lemma 2.2.12, for any given $\varepsilon > 0$, there exists a set $E_{11} \subset (1, +\infty)$ having infinite logarithmic measure such that for all z satisfying $|z| = r \in E_{11}$, one has

$$|A_j(z)| < \exp_{p+1}\{\left(\mu_{(p,q)}(A_s) + \varepsilon\right) \log_q r\}, j = 0, \cdots, k. \tag{2.4.5}$$

By Lemma 2.2.8, there exists a set $E_8 \subset (1, +\infty)$ of finite logarithmic measure such that for all $|z| = r \notin E_8$, we have

$$\begin{aligned} |A_k(z)| &\geq \exp\{-\exp_p\{\left(\sigma_{(p,q)}(A_k) + \varepsilon\right) \log_q r\}\} \\ &\geq \exp\{-\exp_p\{\left(\sigma_{(p,q)}(A_s) + \varepsilon\right) \log_q r\}\} \\ &= \exp\{-\exp_p\{\left(\mu_{(p,q)}(A_s) + \varepsilon\right) \log_q r\}\}. \end{aligned} \tag{2.4.6}$$

From (2.3.9), (2.4.4) - (2.4.6), for all z satisfying $|z| = r \in E_{11} \setminus (E_6 \cup E_8)$ and $|g(z)| = M(r, g)$, we have

$$\exp\{-\exp_p\{(\mu_{(p,q)}(A_s) + \varepsilon)\log_q r\}\}\left(\frac{v_g(r)}{r}\right)^k |1 + o(1)|$$

$$\leq \exp_{p+1}\{(\mu_{(p,q)}(A_s) + \varepsilon)\log_q r\}\left(\frac{v_g(r)}{r}\right)^{k-1} |1 + o(1)|$$

$$+ \cdots + \exp_{p+1}\{(\mu_{(p,q)}(A_s) + \varepsilon)\log_q r\}\left(\frac{v_g(r)}{r}\right)^{s+1} |1 + o(1)|$$

$$+ \exp_{p+1}\{(\mu_{(p,q)}(A_s) + \varepsilon)\log_q r\}\left(\frac{v_g(r)}{r}\right)^{s} |1 + o(1)|$$

$$+ \exp_{p+1}\{(\mu_{(p,q)}(A_s) + \varepsilon)\log_q r\}\left(\frac{v_g(r)}{r}\right)^{s-1} |1 + o(1)|$$

$$+ \cdots + \exp_{p+1}\{(\mu_{(p,q)}(A_s) + \varepsilon)\log_q r\}\left(\frac{v_g(r)}{r}\right)|1 + o(1)| + \exp_{p+1}\{(\mu_{(p,q)}(A_s) + \varepsilon)\log_q r\},$$

that is, for all z satisfying $|z| = r \in E_{11} \setminus (E_6 \cup E_8)$ and $|g(z)| = M(r, g)$, we obtain

$$\exp\{-\exp_p\{(\mu_{(p,q)}(A_s) + \varepsilon)\log_q r\}\}\ \left(\frac{v_g(r)}{r}\right)|1 + o(1)|$$

$$\leq k\,|1 + o(1)|\exp_{p+1}\{(\mu_{(p,q)}(A_s) + \varepsilon)\log_q r\,. \tag{2.4.7}$$

It follows that

$$\lim_{r\to+\infty}\frac{\log_{p+1} v_g(r)}{\log_q r} \leq \mu_{(p,q)}(A_s) + \varepsilon. \tag{2.4.8}$$

Since $\varepsilon > 0$ is arbitrary, then by (2.4.8), Lemma 2.2.5 and Lemma 2.2.9, we have

$$\mu_{(p+1,q)}(g) \leq \mu_{(p,q)}(A_s),$$

that is,

$$\mu_{(p+1,q)}(f) \leq \mu_{(p,q)}(A_s).$$

Therefore, we get

$$\mu_{(p+1,q)}(f) = \mu_{(p,q)}(A_s) = \sigma.$$

2.5 Proof of Theorem 2.1.3

(i) We assume that $f(z)$ is a transcendental meromorphic solution of (2.5) such that $\lambda_{(p,q)}\left(\frac{1}{f}\right) < \mu_{(p,q)}(f)$, and $\{f_1, f_2, \cdots, f_k\}$ is a meromorphic solution base of the corresponding homogeneous equation (2.4) of (2.5). By Theorem 2.1.1, we get that

$$\sigma_{(p+1,q)}(f_j) = \sigma_{(p,q)}(A_s), (j = 1, 2, \cdots, k).$$

By the elementary theory of differential equations, all solutions of (2.5) can be represented in the form

$$f(z) = f_0(z) + B_1 f_1(z) + B_2 f_2(z) + \cdots + B_k f_k(z), \tag{2.5.1}$$

where $B_1, \cdots, B_k \in \mathbb{C}$ and the function f_0 has the form

$$f_0(z) = C_1(z) f_1(z) + C_2(z) f_2(z) + \cdots + C_k(z) f_k(z), \tag{2.5.2}$$

where $C_1(z), \cdots, C_k(z)$ are suitable meromorphic functions satisfying

$$C_j' = F.G_j(f_1, \cdots, f_k).[W(f_1, \cdots, f_k)]^{-1}, j = 1, 2, \cdots, k, \tag{2.5.3}$$

where $G_j(f_1, \cdots, f_k)$ are differential polynomials in $f_1, \cdots, f_k$ and their derivatives with constant coefficients, and $W(f_1, \cdots, f_k)$ is the Wronskian of $f_1, \cdots, f_k$. Since the Wronskian $W(f_1, \cdots, f_k)$ is a differential polynomial in $f_1, \cdots, f_k$, it is easy to obtain

$$\sigma_{(p+1,q)}(W) \leq \max\{\sigma_{(p+1,q)}(f_j) : j = 1, 2, \cdots, k\} = \sigma_{(p,q)}(A_s). \tag{2.5.4}$$

Also, we have that $G_j(f_1, \cdots, f_k)$ are differential polynomials in $f_1, \cdots, f_k$ and their derivatives with constant coefficients. Then, we have

$$\sigma_{(p+1,q)}(G_j) \leq \max\{\sigma_{(p+1,q)}(f_j) : j = 1, 2, \cdots, k\} = \sigma_{(p,q)}(A_s), (j = 1, 2, \cdots, k). \tag{2.5.5}$$

By Lemma 2.2.13 and (2.5.5), for $j = 1, \cdots, k$, we have

$$\sigma_{(p+1,q)}(C_j) = \sigma_{(p+1,q)}(C_j') \leq \max\{\sigma_{(p+1,q)}(F), \sigma_{(p,q)}(A_s)\} = \sigma_{(p,q)}(A_s). \tag{2.5.6}$$

Hence, from (2.5.1), (2.5.2) and (2.5.6), we obtain

$$\sigma_{(p+1,q)}(f) \leq \max\{\sigma_{(p+1,q)}(C_j), \sigma_{(p+1,q)}(f_j) : j = 1, 2, \cdots, k\} = \sigma_{(p,q)}(A_s).$$

Now we assert that all meromorphic solutions f of equation (2.5) such that $\lambda_{(p,q)}\left(\frac{1}{f}\right) < \mu_{(p,q)}(f)$ satisfy $\sigma_{(p+1,q)}(f) = \sigma_{(p,q)}(A_s)$, with at most one exceptional solution f_0 with

$$\sigma_{(p+1,q)}(f_0) < \sigma_{(p,q)}(A_s).$$

In fact, if there exists another meromorphic solution f_1 of (2.5) satisfying $\sigma_{(p+1,q)}(f_1) < \sigma_{(p,q)}(A_s)$, then $f_0 - f_1$ is a nonzero meromorphic solution of (2.4) and satisfies $\sigma_{(p+1,q)}(f_0 - f_1) < \sigma_{(p,q)}(A_s)$. But by Theorem 2.1.1 we have any meromorphic solution of (2.4) such that $\lambda_{(p,q)}\left(\frac{1}{f}\right) < \mu_{(p,q)}(f)$ staisfies $\sigma_{(p+1,q)}(f) = \sigma_{(p,q)}(A_s)$. This is a contradiction. Therefore, we have that all meromorphic solutions f of equation (2.5) such that $\lambda_{(p,q)}\left(\frac{1}{f}\right) < \mu_{(p,q)}(f)$, satisfy $\sigma_{(p+1,q)}(f) = \sigma_{(p,q)}(A_s)$, with at most one exceptional solution f_0 with $\sigma_{(p+1,q)}(f_0) < \sigma_{(p,q)}(A_s)$.

(ii) From (2.5), by a simple consideration of order, we get $\sigma_{(p+1,q)}(f) \geq \sigma_{(p+1,q)}(F)$. By Lemma 2.2.13 and (2.5.3) - (2.5.5), for $j = 1, 2, \cdots, k$, we have

$$\sigma_{(p+1,q)}(C_j) = \sigma_{(p+1,q)}\left(C_j'\right) \leq \max\{\sigma_{(p+1,q)}(F), \sigma_{(p,q)}(A_s)\} \leq \sigma_{(p+1,q)}(F). \tag{2.5.7}$$

By (2.5.1), (2.5.2) and (2.5.7), we have

$$\begin{aligned}\sigma_{(p+1,q)}(f) &\leq \max\{\sigma_{(p+1,q)}(C_j), \sigma_{(p+1,q)}(f_j) : j = 1, 2, \cdots, k\}\\ &\leq \max\{\sigma_{(p+1,q)}(F), \sigma_{(p,q)}(A_s)\}\\ &\leq \sigma_{(p+1,q)}(F).\end{aligned}$$

Therefore, we get $\sigma_{(p+1,q)}(f) = \sigma_{(p+1,q)}(F)$.

2.6 Proof of Theorem 2.1.4

(i) Suppose that $f(z)$ is a transcendental solution of (2.2). On one hand, by (2.2), we get

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = F(z)$$

$$\frac{f^{(k)}}{f} + A_{k-1}(z)\frac{f^{(k-1)}}{f} + \cdots + A_1(z)\frac{f'}{f} + A_0(z) = \frac{F(z)}{f}$$

$$\left|\frac{f^{(k)}}{f}\right| \leq |A_{k-1}(z)|\left|\frac{f^{(k-1)}}{f}\right| + \cdots + |A_1(z)|\left|\frac{f'}{f}\right| + |A_0(z)| + \left|\frac{F}{f}\right|. \tag{2.6.1}$$

By Wiman-Valiron's theory [12, p.187-199], there exists a set $E_{16} \subset (1, +\infty)$ of finite logarithmic measure such that for all $|z| = r \notin [0,1] \cup E_{16}$ and $|f(z)| = M(r, f) > 1$, we have

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{v_f(r)}{z}\right)^j (1 + o(1)), (j = 0, \cdots, k). \tag{2.6.2}$$

By the definition of the (p,q)-order, for any given $\varepsilon > 0$ and for sufficiently large r, we have

$$|A_j(z)| \leq \exp_{p+1}\{(\sigma+\varepsilon)\log_q r\}, j \neq s \tag{2.6.3}$$

and

$$|F(z)| \leq \exp_{p+1}\{(\sigma+\varepsilon)\log_q r\}. \tag{2.6.4}$$

Since $|f(z)| = M(r,f) > 1$, then by (2.6.4), for sufficiently large r, we have

$$\left|\frac{F(z)}{f(z)}\right| = \frac{|F(z)|}{M(r,f)} \leq \exp_{p+1}\{(\sigma+\varepsilon)\log_q r\}. \tag{2.6.5}$$

By substituting (2.6.2) , (2.6.3) and (2.6.5) into (2.6.1), for sufficiently large $r \notin [0,1] \cup E_{16}$, we obtain

$$\left(\frac{v_f(r)}{r}\right)^k |1+o(1)| \leq \exp_{p+1}\{(\sigma+\varepsilon)\log_q r\}\left(\frac{v_f(r)}{r}\right)^{k-1}|1+o(1)|+$$

$$\cdots+\exp_{p+1}\{(\sigma+\varepsilon)\log_q r\}\left(\frac{v_f(r)}{r}\right)^{1}|1+o(1)|+\exp_{p+1}\{(\sigma+\varepsilon)\log_q r\}\left(\frac{v_f(r)}{r}\right)|1+o(1)|$$

$$\Longrightarrow \left(\frac{v_f(r)}{r}\right)^k |1+o(1)| \leq (k+1)\exp_{p+1}\{(\sigma+\varepsilon)\log_q r\}\left(\frac{v_f(r)}{r}\right)^{k-1}|1+o(1)|$$

$$\Longrightarrow \left(\frac{v_f(r)}{r}\right)|1+o(1)| \leq (k+1)\exp_{p+1}\{(\sigma+\varepsilon)\log_q r\}. \tag{2.6.6}$$

By (2.6.6), Lemma 2.2.5 and Lemma 2.2.9, we obtain $\sigma_{(p+1,q)}(f) \leq \sigma_{(p,q)}(A_s)+\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we get $\sigma_{(p+1,q)}(f) \leq \sigma_{(p,q)}(A_s)$.

On the other hand, by (2.2), we obtain

$$|A_s(z)| \leq \left|\frac{f}{f^{(s)}}\right|\left[\left|\frac{f^{(k)}}{f}\right|+|A_{k-1}(z)|\left|\frac{f^{(k-1)}}{f}\right|+\cdots+|A_{s+1}(z)|\left|\frac{f^{(s+1)}}{f}\right|\right.$$

$$\left.+|A_{s-1}(z)|\left|\frac{f^{(s-1)}}{f}\right|+\cdots+|A_1(z)|\left|\frac{f'}{f}\right|+|A_0(z)|+\left|\frac{F}{f}\right|\right]. \tag{2.6.7}$$

For each sufficiently large circle $|z| = r$, we take $z_r = re^{i\theta_r}$ satisfying $|f(z_r)| = M(r,f) > 1$. Then, by Lemma 2.2.14, there exists a constant $\delta_r > 0$ and a set E_{12} such that for all z satisfying $|z| = r \notin E_{12}$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$, we have

$$\left|\frac{f(z)}{f^{(s)}(z)}\right| \leq 2r^s. \tag{2.6.8}$$

By Lemma 2.2.1, there exist a set $H_1 \subset [0, 2\pi)$ that has linear measure zero and a constant $B > 0$ such that for all z satisfying $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$ and for sufficiently large r, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq B\left(T(2r, f)\right)^{k+1}, 1 \leq j \leq k. \tag{2.6.9}$$

We choose α_2, β_2 satisfying $\max\{\tau_{(p,q)}(A_j) : \sigma_{(p,q)}(A_j) = \sigma_{(p,q)}(A_s), \tau_{(p,q)}(F)\} < \alpha_2 < \beta_2 < \tau_{(p,q)}(A_s)$. Since $|f(z) - f(z_r)| < \varepsilon$ and $|f(z_r)| \to \infty$ as $r \to +\infty$, for all suficiently large $|z| = r \notin E_{12}$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$, we have

$$|A_j(z)| \leq \exp_p\left\{\alpha_2\left(\log_{q-1} r\right)^{\sigma_{(p,q)}(A_s)}\right\}, \ j \neq s \tag{2.6.10}$$

and

$$\left|\frac{F(z)}{f(z)}\right| \leq |F(z)| \leq \exp_p\left\{\alpha_2\left(\log_{q-1} r\right)^{\sigma_{(p,q)}(A_s)}\right\}. \tag{2.6.11}$$

Since $T(r, A_s) \sim \log M(r, A_s)$ as $r \to +\infty$ $(r \notin E_{12})$, by Lemma 2.2.16, for any $\beta_2 < \tau_{(p,q)}(A_s)$, there exists a set $E_{14} \subset (0, +\infty)$ having infinite logarithmic measure and a set $H_2 \subset [0, 2\pi)$ that has linear measure zero such that for all z satisfying $|z| = r \in E_{14}$ and $\arg z = \theta \in [0, 2\pi) \setminus H_2$, we have

$$|A_s(z)| > \exp_p\left\{\beta_2\left(\log_{q-1} r\right)^{\sigma_{(p,q)}(A_s)}\right\}, \ j \neq s. \tag{2.6.12}$$

Substituting (2.6.8) - (2.6.12) into (2.6.7), for all z satisfying $|z| = r \in E_{14} \setminus E_{12}$ and $\arg z = \theta \in [0, 2\pi) \setminus (H_1 \cup H_2)$, we get

$$\exp_p\{\beta_2\left(\log_{q-1} r\right)^{\sigma_{(p,q)}(A_s)}\} \leq 2r^s\left(\exp_p\{\alpha_2\left(\log_{q-1} r\right)^{\sigma_{(p,q)}(A_s)}\} B\left(T(2r, f)\right)^k\right.$$

$$\left. + \exp_p\{\alpha_2\left(\log_{q-1} r\right)^{\sigma_{(p,q)}(A_s)}\} B\left(T(2r, f)\right)^{k-1} + \cdots + \exp_p\{\alpha_2\left(\log_{q-1} r\right)^{\sigma_{(p,q)}(A_s)}\} B\left(T(2r, f)\right)\right)$$

$$\Longrightarrow \exp_p\{\beta_2\left(\log_{q-1} r\right)^{\sigma_{(p,q)}(A_s)}\} \leq 2r^s \exp_p\{\alpha_2\left(\log_{q-1} r\right)^{\sigma_{(p,q)}(A_s)}\} (k+1) B\left(T(2r, f)\right)^{k+1} \tag{2.6.13}$$

Since $\alpha_2 < \beta_2$, then

$$\exp\{(1 - o(1)) \exp_{p-1}\{\beta_2\left(\log_{q-1} r\right)^{\sigma_{(p,q)}(A_s)}\} \leq 2r^s (k+1) B\left(T(2r, f)\right)^{k+1}. \tag{2.6.14}$$

By (2.6.14) and Lemme 2.2.5, we obtain

$$\sigma_{(p+1,q)}(f) \geq \sigma_{(p,q)}(A_s).$$

Thus, we have

$$\sigma_{(p+1,q)}(f) = \sigma_{(p,q)}(A_s).$$

Now, if $f(z)$ is a polynomial solution of (2.2) with $\deg(f) \geq s$, then $f^{(s)}(z) \not\equiv 0$. If

$$\max\{\sigma_{(p,q)}(A_j), \sigma_{(p,q)}(F),\ j \neq s\} < \sigma_{(p,q)}(A_s) < \infty,$$

and

$$-A_s(z) f^{(s)} = f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_{s+1}(z) f^{(s+1)}$$
$$+A_{s-1}(z) f^{(s-1)} + \cdots + A_1(z) f' + A_0(z) f - F(z)$$

then

$$\sigma_{(p,q)}(A_s) = \sigma_{(p,q)}(-A_s f^{(s)}) = \sigma_{(p,q)}(f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_{s+1}(z) f^{(s+1)}$$
$$+A_s(z)\ f^{(s)} + A_{s-1}(z) f^{(s-1)} + \cdots + A_1(z) f' + A_0(z) f - F(z))$$
$$\leq \max\{\sigma_{(p,q)}(A_j), \sigma_{(p,q)}(F),\ j \neq s\} < \sigma_{(p,q)}(A_s),$$

which is a contradiction.

If $\max\{\sigma_{(p,q)}(A_j), \sigma_{(p,q)}(F), j \neq s\} = \sigma_{(p,q)}(A_s) = \sigma$ and $\max\{\tau_{(p,q)}(A_j) : \sigma_{(p,q)}(A_j), \tau_{(p,q)}(F)\} < \tau_{(p,q)}(A_s)$, then we choose α_2, β_2 satisfying $\max\{\tau_{(p,q)}(A_j) : \sigma_{(p,q)}(A_j) = \sigma_{(p,q)}(A_s), \tau_{(p,q)}(F)\} < \alpha_2 < \beta_2 < \tau_{(p,q)}(A_s)$.

By Lemma 2.2.15, there exists a set E_{13} having infinite logarithmic measure such that for all z satisfying $|z| = r \in E_{13}$, we have

$$|A_s(z)| > \exp_p\left\{\beta_2 \left(\log_{q-1} r\right)^{\sigma}\right\} \tag{2.6.15}$$

and for sufficiently large r

$$|F(z)| \leq \exp_p\left\{\alpha_2 \left(\log_{q-1} r\right)^{\sigma}\right\}, \quad |A_j(z)| \leq \exp_p\left\{\alpha_2 \left(\log_{q-1} r\right)^{\sigma}\right\}, j \neq s. \tag{2.6.16}$$

Hence, from (2.6.7), (2.6.15) and (2.6.16), for all z satisfying $|z| = r \in E_{13}$, we have

$$\exp_p\left\{\beta_2 \left(\log_{q-1} r\right)^{\sigma}\right\} \leq (k+1) r^M \exp_p\left\{\alpha_2 \left(\log_{q-1} r\right)^{\sigma}\right\},$$

where M is a constant.

It follows that

$$\beta_2 \left(\log_{q-1} r\right)^{\sigma} \leq \log_p(k+1) + \log_p r^M + \alpha_2 \left(\log_{q-1} r\right)^{\sigma}$$

$$\Rightarrow \beta_2 \leq \frac{\log_p (k+1)}{\left(\log_{q-1} r\right)^{\sigma}} + \frac{\log_p r^M}{\left(\log_{q-1} r\right)^{\sigma}} + \alpha_2,$$

which gives $\beta_2 \leq \alpha_2$, as $r \to +\infty$. This is a contradiction. Therefore, $f(z)$ must be a polynomial with $\deg f \leq s-1$.

(ii) If $F(z) \not\equiv 0$, then we find from Lemma 2.2.17 that every transcendental solution $f(z)$ of (2.2) satisfies $\overline{\lambda}_{(p+1,q)}(f) = \lambda_{(p+1,q)}(f) = \sigma_{(p+1,q)}(f) = \sigma_{(p,q)}(A_s)$.

(iii) If $s = 1$ and $f(z)$ is a polynomial solution of (2.2), then by (ii), we get that $\deg f \leq s-1$. Thus $f(z)$ must be a constant.

By (i) and (ii), every nonconstant solution $f(z)$ of (2.2) satisfies $\sigma_{(p+1,q)}(f) = \sigma_{(p,q)}(A_1)$ and $\overline{\lambda}_{(p+1,q)}(f) = \lambda_{(p+1,q)}(f) = \sigma_{(p+1,q)}(f) = \sigma_{(p,q)}(A_1)$ if $F(z) \not\equiv 0$.

Conclusion

Throughout this work, by using the (p,q)-order and other data, we have discussed some results about the growth of (p,q)-order of the solutions to linear differential equations of the forms

$$f^{(k)}+A_{k-1}(z)f^{(k-1)}+\cdots+A_1(z)f'+A_0(z)f=0,$$

$$f^{(k)}+A_{k-1}(z)f^{(k-1)}+\cdots+A_1(z)f'+A_0(z)f=F(z),$$

$$A_k(z)\ f^{(k)}+A_{k-1}(z)f^{(k-1)}+\cdots+A_1(z)f'+A_0(z)f=0$$

and

$$A_k(z)\ f^{(k)}+A_{k-1}(z)f^{(k-1)}+\cdots+A_1(z)f'+A_0(z)f=F(z)$$

where $A_j(z)$ and $F(z)$ are entire functions and some arbitrary coefficient $A_s(z)\,(0\leq s\leq k)$ being Lacunary series.

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وزارة التعليم العالي والبحث العلمي

جامعة عبد الحميد بن باديس-مستغانم-
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والمكلف بإنجاز مذكرة ماستر بعنوان:

On the (p,q)-order of solutions of some complex linear differential equations

أصرح بشرفي أني ألتزم بمراعاة المعايير العلمية والمنهجية ومعايير الأخلاقيات العلمية والنزاهة الأكاديمية المطلوبة في إنجاز البحث، وأتحمل المسؤولية الشخصية عن كل المحتوى المتضمن في البحث المذكور أعلاه.

التاريخ : 2025/06/11

إمضاء المعني

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