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Some results for a four-point boundary value problems for coupled system involving Caputo derivative

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Abstract

In this paper, we prove the existence and uniqueness of solutions for a system for fractional differential equations with four point boundary conditions. The results are obtained using Banach contraction principle and Krasnoselkii's fixed point theorem

$$\begin{cases} D^{\alpha}x(t)+f\left(t,y(t),D^{\delta}y(t)\right)=0, t\in J, \\ D^{\beta}y(t)+g\left(t,x(t),D^{\sigma}x(t)\right)=0, t\in J, \\ x(0)=y(0)=0, x(1)-\lambda_1 x(\eta)=0, y(1)-\lambda_1 y(\eta)=0, \\ x''(0)=y''(0)=0, x''(1)-\lambda_2 x''(\xi)=0, y''(1)-\lambda_2 y''(\xi)=0, \end{cases}$$

where $3<\alpha,\beta\leq 4, \alpha-2<\sigma\leq\alpha-1, \beta-2<\delta\leq\beta-1, 0<\xi,\eta<1$, and $D^{\alpha}, D^{\beta}, D^{\delta}$ and D^{σ}, are the Caputo fractional derivatives, $J=[0,1], \lambda_1, \lambda_2$ are real constants with $\lambda_1\eta\neq 1, \lambda_2\xi\neq 1$ and f, g continuous functions on $[0,1]\times\mathbb{R}^2$.

Keywords: Caputo derivative; Boundary Value Problem; fixed point theorem.

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1 Introduction

Differential equations of fractional order have been shown to be very useful in the study of models of many phenomena in various fields of science and engineering, such as electrochemistry, physics, chemistry, viscoelasticity, control, image and signal processing, biophysics. For more details, we refer the reader to [4, 6, 9, 11, 12, 14, 16, 17] and references therein. There has been a significant progress in the investigation of these equations in recent years, see [5, 7, 8, 14, 15, 26]. More recently, some basic theory for the initial boundary value problems of fractional differential equations has been discussed in [1, 13, 14]. Recently, existence and uniqueness of solutions to boundary value problems for fractional differential equations had attracted the attention of many authors, see for example, [4, 5, 7, 8, 14, 15, 18, 26] and the references therein. The study of coupled system of fractional order is also important as such systems occur in various problems of applied science [3, 10, 19, 20, 23, 25]. In the last decade, many authors have established the existence and uniqueness for solutions of some systems of nonlinear fractional differential equations, one can see [19, 22, 23, 24] and references cited therein. For example in [2, 20, 25] the authors established sufficient conditions for the existence of solutions for a two-point and three-point boundary value problem for a coupled system of fractional differential equations.

In [2, 20, 21, 25], the existence and uniqueness of solutions was investigated for a nonlinear coupled system for fractional differential equations with two-point and three-point boundary conditions by using Schauder's fixed point theorem.

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Motivated by the above mentioned work, this paper deals with the existence of solution for four point boundary value problems for a coupled system of fractional differential equations for the following problem

$$\begin{cases} D^{\alpha}x(t)+f\left(t,y(t),D^{\delta}y(t)\right)=0, t\in J, \\ D^{\beta}y(t)+g(t,x(t),D^{\sigma}x(t))=0, t\in J, \\ x(0)=y(0)=0, x(1)-\lambda_1 x(\eta)=0, y(1)-\lambda_1 y(\eta)=0, \\ x''(0)=y''(0)=0, x''(1)-\lambda_2 x''(\xi)=0, y''(1)-\lambda_2 y''(\xi)=0, \end{cases} \tag{1.1}$$

where $3<\alpha,\beta\leq 4, \alpha-2<\sigma\leq\alpha-1, \beta-2<\delta\leq\beta-1, 0<\xi,\eta<1$, and $D^{\alpha}, D^{\beta}, D^{\delta}$ and D^{σ}, are the Caputo fractional derivatives, $J=[0,1], \lambda_1, \lambda_2$ are real constants with $\lambda_1\eta\neq 1, \lambda_2\xi\neq 1$ and f, g are continuous functions on $[0,1]\times\mathbb{R}^2$.

The rest of this paper is organized as follows. In section 2, we present some preliminaries and lemmas. Section 3 is devoted to existence of solution of problem (1.1). In section 4 examples are treated illustrating our results.

2 Preliminaries

The following notations, definitions, and preliminary facts will be used throughout this paper.

Definition 2.1. *The Riemann-Liouville fractional integral operator of order $\alpha\geq 0$, for a continuous function f on $[0,\infty[$ is defined as:*

$$J^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f(\tau)\,d\tau, \alpha>0, \tag{2.2}$$

$$J^{0}f(t)=f(t),$$

where $\Gamma(\alpha):=\int_0^{\infty}e^{-u}u^{\alpha-1}du$.

Definition 2.2. *The fractional derivative of $f\in C^n([0,\infty[)$ in the Caputo's sense is defined as:*

$$D^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}f^{(n)}(\tau)\,d\tau, n-1<\alpha, n\in N^*. \tag{2.3}$$

For more details about fractional calculus, we refer the reader to [14, 17].

Let us now introduce the spaces

$$X=\{x : x\in C([0,1]), D^{\sigma}x\in C([0,1])\},$$

and

$$Y=\{y : y\in C([0,1]), D^{\delta}y\in C([0,1])\},$$

endowed with the norm

$$\|x\|_X=\|x\|+\|D^{\sigma}x\|,\ \|x\|=\sup_{t\in J}|x(t)|, \|D^{\sigma}x\|=\sup_{t\in J}|D^{\sigma}x(t)|,$$

and

$$\|y\|_Y=\|y\|+\left\|D^{\delta}y\right\|,\ \|y\|=\sup_{t\in J}|y(t)|, \left\|D^{\delta}y\right\|=\sup_{t\in J}\left|D^{\delta}y(t)\right|.$$

Obviously, $(X,\|\,.\,\|_X)$ and $(Y,\|\,.\,\|_Y)$,is a Banach space. The product space $\left(X\times Y,\|(x,y)\|_{X\times Y}\right)$ is also Banach space with norm $\|(x,y)\|_{X\times Y}=\|x\|_X+\|y\|_Y$.

We give the following lemmas [12]:

Lemma 2.1. *For $\alpha>0$, the general solution of the fractional differential equation $D^{\alpha}x(t)=0$ is given by*

$$x(t)=c_0+c_1t+c_2t^2+...+c_{n-1}t^{n-1}, \tag{2.4}$$

where $c_i\in\mathbb{R}, i=0,1,2,..,n-1, n=[\alpha]+1$.

Lemma 2.2. *Let $\alpha > 0$. Then*

$$J^{\alpha}D^{\alpha}x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + ... + c_{n-1}t^{n-1}, \tag{2.5}$$

for some $c_i \in \mathbb{R}, i = 0, 1, 2, ..., n-1, n = [\alpha] + 1$.

We need also the following auxiliary result:

Lemma 2.3. *Let $g \in C([0,1])$, the solution of the equation*

$$D^{\alpha}x(t) + g(t) = 0, t \in J,\ 3 < \alpha \leq 4, \tag{2.6}$$

subject to the boundary condition

$$\begin{aligned} x(0) = 0,\ x(1) - \lambda_1 x(\eta) = 0, \\ x''(0) = 0,\ x''(1) - \lambda_2 x''(\xi) = 0, \end{aligned} \tag{2.7}$$

is given by:

$$\begin{aligned} x(t) = & -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)\, ds \\ & + \frac{\lambda_1 t}{(\lambda_1\eta - 1)\Gamma(\alpha)} \int_0^{\eta} (\eta - s)^{\alpha-1} g(s)\, ds \\ & - \frac{t}{(\lambda_1\eta - 1)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} g(s)\, ds \\ & + \frac{(\lambda_2 - \lambda_2\lambda_1\eta^3)\, t + (\lambda_2\lambda_1\eta - \lambda_2)\, t^3}{6(\lambda_1\eta - 1)(\lambda_2\xi - 1)\Gamma(\alpha-2)} \int_0^{\xi} (\xi - s)^{\alpha-3} g(s)\, ds \\ & - \frac{(1 - \lambda_1\eta^3)\, t + (\lambda_1\eta - 1)\, t^3}{6(\lambda_1\eta - 1)(\lambda_2\xi - 1)\Gamma(\alpha-2)} \int_0^1 (1-s)^{\alpha-3} g(s)\, ds. \end{aligned} \tag{2.8}$$

Proof. For $c_i \in \mathbb{R}, i = 0, 1, 2, 3$, and by Lemmas ((2.1), (2.2)), the general solution of (2.6) is given by

$$x(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)\, ds - c_0 - c_1 t - c_2 t^2 - c_3 t^3 \tag{2.9}$$

Using the boundary condition (2.7), we have $c_0 = c_2 = 0$, and

$$\begin{aligned} c_1 = & -\frac{\lambda_1}{(\lambda_1\eta - 1)\Gamma(\alpha)} \int_0^{\eta} (\eta - s)^{\alpha-1} g(s)\, ds \\ & + \frac{1}{(\lambda_1\eta - 1)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} g(s)\, ds \\ & - \frac{\lambda_2(1 - \lambda_1\eta^3)}{6(\lambda_1\eta - 1)(\lambda_2\xi - 1)\Gamma(\alpha-2)} \int_0^{\xi} (\xi - s)^{\alpha-3} g(s)\, ds \\ & + \frac{(1 - \lambda_1\eta)}{6(\lambda_1\eta - 1)(\lambda_2\xi - 1)\Gamma(\alpha-2)} \int_0^1 (1-s)^{\alpha-3} g(s)\, ds \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} c_3 = & -\frac{\lambda_2}{6(\lambda_2\xi - 1)\Gamma(\alpha-2)} \int_0^{\xi} (\xi - s)^{\alpha-3} g(s)\, ds \\ & + \frac{1}{6(\lambda_2\xi - 1)\Gamma(\alpha-2)} \int_0^1 (1-s)^{\alpha-3} g(s)\, ds \end{aligned} \tag{2.11}$$

Substituting the value of c_1 and c_3 in (2.9), we obtain the desired quantity in Lemma. □

3 Main Results

Let us the take of convenience, we set:

$$\begin{aligned}
M_1 &= \tfrac{|\lambda_1\eta-1|+|\lambda_1|\eta^{\alpha}+1}{|\lambda_1\eta-1|\Gamma(\alpha+1)} \\
&+ \tfrac{\left(\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|+|\lambda_2\lambda_1\eta-\lambda_2|\right)\xi^{\alpha-2}+\left|1-\lambda_1\eta^3\right|+|\lambda_1\eta-1|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}, \\
M_2 &= \tfrac{1}{\Gamma(\alpha-\sigma+1)} + \tfrac{|\lambda_1|\eta^{\alpha}+1}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\sigma)} \\
&+ \tfrac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}+\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\sigma)} + \tfrac{|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}+|\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\sigma)}, \\
M_3 &= \tfrac{|\lambda_1\eta-1|+|\lambda_1|\eta^{\beta}+1}{|\lambda_1\eta-1|\Gamma(\beta+1)} + \tfrac{\left(\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|+|\lambda_2\lambda_1\eta-\lambda_2|\right)\xi^{\beta-2}+\left|1-\lambda_1\eta^3\right|+|\lambda_1\eta-1|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)}, \\
M_4 &= \tfrac{1}{\Gamma(\beta-\delta+1)} + \tfrac{|\lambda_1|\eta^{\beta}+1}{|\lambda_1\eta-1|\Gamma(\beta+1)\Gamma(2-\delta)} \\
&+ \tfrac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\beta-2}+\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)\Gamma(2-\delta)} + \tfrac{|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\beta-2}+|\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)\Gamma(4-\delta)}, \\
L_1 &= \tfrac{\left(\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|+|\lambda_2\lambda_1\eta-\lambda_2|\right)\xi^{\alpha-2}+\left|1-\lambda_1\eta^3\right|+|\lambda_1\eta-1|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}, \\
L_2 &= \tfrac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}+\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\sigma)} + \tfrac{|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}+|\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\sigma)}, \\
L_3 &= \tfrac{\left(\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|+|\lambda_2\lambda_1\eta-\lambda_2|\right)\xi^{\beta-2}+\left|1-\lambda_1\eta^3\right|+|\lambda_1\eta-1|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)}, \\
L_4 &= \tfrac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\beta-2}+\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)\Gamma(2-\delta)} + \tfrac{|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\beta-2}+|\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)\Gamma(4-\delta)}.
\end{aligned} \tag{3.12}$$

Now list the following hypotheses for convenience:
$(H1)$: There exist two constants k_1 and k_2 such that for all $t \in [0,1]$ and $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$, we have

$$\begin{aligned}
|f(t, x_1, y_1) - f(t, x_2, y_2)| &\leq k_1 (|x_1 - x_2| + |y_1 - y_2|), \\
|g(t, x_1, y_1) - g(t, x_2, y_2)| &\leq k_2 (|x_1 - x_2| + |y_1 - y_2|).
\end{aligned} \tag{3.13}$$

$(H2))$: The functions $f, g : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous
$(H3)$: There exists positive constants N_1 and N_2 such that

$$|f(t,x,y)| \leq N_1,\ |g(t,x,y)| \leq N_2 \text{ for each } t \in J \text{ and all } x, y \in \mathbb{R}.$$

Our first result is based on Banach contraction principle:

Theorem 3.1. *Assume that the hypothesis $(H1)$holds.*
If

$$k_1 (M_1 + M_2) + k_2 (M_3 + M_4) < 1, \tag{3.14}$$

then the boundary value problem (1.1) has a unique solution.

Proof. Consider the operator $\phi : X \times Y \to X \times Y$ defined by:

$$\phi(x,y)(t) := (\phi_1 y(t), \phi_2 x(t)), \tag{3.15}$$

where

$$\begin{aligned}
\phi_1 y(t) &:= -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(s, y(s), D^{\delta} y(s)\right) ds \\
&+ \frac{\lambda_1 t}{(\lambda_1\eta - 1)\Gamma(\alpha)} \int_0^{\eta} (\eta - s)^{\alpha-1} f\left(s, y(s), D^{\delta} y(s)\right) ds \\
&- \frac{t}{(\lambda_1\eta - 1)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f\left(s, y(s), D^{\delta} y(s)\right) ds \\
&+ \frac{\left(\lambda_2 - \lambda_2\lambda_1\eta^3\right) t + (\lambda_2\lambda_1\eta - \lambda_2) t^3}{6(\lambda_1\eta - 1)(\lambda_2\xi - 1)\Gamma(\alpha - 2)} \int_0^{\xi} (\xi - s)^{\alpha-3} f\left(s, y(s), D^{\delta} y(s)\right) ds \\
&- \frac{\left(1 - \lambda_1\eta^3\right) t + (\lambda_1\eta - 1) t^3}{6(\lambda_1\eta - 1)(\lambda_2\xi - 1)\Gamma(\alpha - 2)} \int_0^1 (1-s)^{\alpha-3} f\left(s, y(s), D^{\delta} y(s)\right) ds,
\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}\phi_2 x(t) &:= -\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} g(s,x(s),D^{\sigma}x(s))\,ds \\ &+ \frac{\lambda_1 t}{(\lambda_1\eta-1)\Gamma(\beta)}\int_0^{\eta}(\eta-s)^{\alpha-1} g(s,x(s),D^{\sigma}x(s))\,ds \\ &- \frac{t}{(\lambda_1\eta-1)\Gamma(\beta)}\int_0^1 (1-s)^{\alpha-1} g(s,x(s),D^{\sigma}x(s))\,ds \\ &+ \frac{(\lambda_2-\lambda_2\lambda_1\eta^3)t+(\lambda_2\lambda_1\eta-\lambda_2)t^3}{6(\lambda_1\eta-1)(\lambda_2\xi-1)\Gamma(\beta-2)}\int_0^{\xi}(\xi-s)^{\alpha-3} g(s,x(s),D^{\sigma}x(s))\,ds \\ &- \frac{(1-\lambda_1\eta^3)t+(\lambda_1\eta-1)t^3}{6(\lambda_1\eta-1)(\lambda_2\xi-1)\Gamma(\beta-2)}\int_0^1 (1-s)^{\alpha-3} g(s,x(s),D^{\sigma}x(s))\,ds.\end{aligned} \tag{3.17}$$

We shall prove that ϕ is a contraction mapping :

For $(x,y),(x_1,y_1)\in X\times Y$ and for each $t\in J$, we have

$$\begin{aligned}|\phi_1 y(t)-\phi_1 y_1(t)| &\le \tfrac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|f\left(s,y(s),D^{\delta}y(s)\right)-f\left(s,y_1(s),D^{\delta}y_1(s)\right)\right|ds \\ &+ \tfrac{|\lambda_1|t}{|\lambda_1\eta-1|\Gamma(\alpha)}\int_0^{\eta}(\eta-s)^{\alpha-1}\left|f\left(s,y(s),D^{\delta}y(s)\right)-f\left(s,y_1(s),D^{\delta}y_1(s)\right)\right|ds \\ &+ \tfrac{t}{|\lambda_1\eta-1|\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\left|f\left(s,y(s),D^{\delta}y(s)\right)-f\left(s,y_1(s),D^{\delta}y_1(s)\right)\right|ds \\ &+ \tfrac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|t+|\lambda_2\lambda_1\eta-\lambda_2|t^3}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)}\int_0^{\xi}(\xi-s)^{\alpha-3}\left|f\left(s,y(s),D^{\delta}y(s)\right)-f\left(s,y_1(s),D^{\delta}y_1(s)\right)\right|ds \\ &+ \tfrac{\left|1-\lambda_1\eta^3\right|t+|\lambda_1\eta-1|t^3}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)}\int_0^1 (1-s)^{\alpha-3}\left|f\left(s,y(s),D^{\delta}y(s)\right)-f\left(s,y_1(s),D^{\delta}y_1(s)\right)\right|ds.\end{aligned}$$

Using the $(H1)$,we obtain

$$\begin{aligned}|\phi_1 y(t)-\phi_1 y_1(t)| &\le \tfrac{k_1(|\lambda_1\eta-1|+|\lambda_1|\eta^{\alpha}+1)\left(\|y-y_1\|+\left\|D^{\delta}y-D^{\delta}y_1\right\|\right)}{|\lambda_1\eta-1|\Gamma(\alpha+1)} \\ &+ \tfrac{k_1\left[\left(\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|+|\lambda_2\lambda_1\eta-\lambda_2|\right)\xi^{\alpha-2}+\left|1-\lambda_1\eta^3\right|+|\lambda_1\eta-1|\right]\left(\|y-y_1\|+\left\|D^{\delta}y-D^{\delta}y_1\right\|\right)}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}.\end{aligned}$$

Consequently,we have

$$|\phi_1 x(t)-\phi_1 y(t)| \le k_1 M_1\left(\|y-y_1\|+\left\|D^{\delta}y-D^{\delta}y_1\right\|\right). \tag{3.18}$$

Which implies that

$$\|\phi_1(x)-\phi_1(y)\| \le k_1 M_1\left(\|y-y_1\|+\left\|D^{\delta}y-D^{\delta}y_1\right\|\right), \tag{3.19}$$

and

$$\begin{aligned}|D^{\sigma}\phi_1 y(t)-D^{\sigma}\phi_1 y_1(t)| &\le \tfrac{1}{\Gamma(\alpha-\sigma)}\int_0^t (t-s)^{\alpha-\sigma-1}\left|f\left(s,y(s),D^{\delta}y(s)\right)-f\left(s,y_1(s),D^{\delta}y_1(s)\right)\right|ds \\ &+ \tfrac{|\lambda_1|t^{1-\sigma}}{|\lambda_1\eta-1|\Gamma(\alpha)\Gamma(2-\sigma)}\int_0^{\eta}(\eta-s)^{\alpha-1}\left|f\left(s,y(s),D^{\delta}y(s)\right)-f\left(s,y_1(s),D^{\delta}y_1(s)\right)\right|ds \\ &+ \tfrac{t^{1-\sigma}}{|(\lambda_1\eta-1)|\Gamma(\alpha)\Gamma(2-\sigma)}\int_0^1 (1-s)^{\alpha-1}\left|f\left(s,y(s),D^{\delta}y(s)\right)-f\left(s,y_1(s),D^{\delta}y_1(s)\right)\right|ds \\ &+ \left(\begin{array}{c}\tfrac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|t^{1-\sigma}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(2-\sigma)} \\ +\tfrac{|\lambda_2\lambda_1\eta-\lambda_2|t^{3-\sigma}}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(4-\sigma)}\end{array}\right)\int_0^{\xi}(\xi-s)^{\alpha-3}\left|f\left(s,y(s),D^{\delta}y(s)\right)-f\left(s,y_1(s),D^{\delta}y_1(s)\right)\right|ds\end{aligned}$$

$$+\left(\begin{array}{c}\frac{\left|1-\lambda_1\eta^3\right|t^{1-\sigma}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(2-\sigma)}\\+\frac{|\lambda_1\eta-1|t^{3-\sigma}}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(4-\sigma)}\end{array}\right)\int_0^1(1-s)^{\alpha-3}\left|f\left(s,y(s),D^{\delta}y(s)\right)-f\left(s,y_1(s),D^{\delta}y_1(s)\right)\right|ds.$$

By $(H1)$, yields

$$\begin{aligned}|D^{\sigma}\phi_1 y(t)-D^{\sigma}\phi_1 y_1(t)|\leq{}&\frac{k_1\left(\|y-y_1\|+\left\|D^{\delta}y-D^{\delta}y_1\right\|\right)}{\Gamma(\alpha-\sigma+1)}+\frac{k_1[|\lambda_1|\eta^{\alpha}+1]\left(\|y-y_1\|+\left\|D^{\delta}y-D^{\delta}y_1\right\|\right)}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\sigma)}\\&+\frac{k_1\left[\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}+\left|1-\lambda_1\eta^3\right|\right]\left(\|y-y_1\|+\left\|D^{\delta}y-D^{\delta}y_1\right\|\right)}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\sigma)}\\&+\frac{k_1\left[|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}+|\lambda_1\eta-1|\right]\left(\|y-y_1\|+\left\|D^{\delta}y-D^{\delta}y_1\right\|\right)}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\sigma)}.\end{aligned}$$

Hence,we have

$$\begin{aligned}|D^{\sigma}\phi_1 y(t)-D^{\sigma}\phi_1 y_1(t)|\leq{}&k_1\left[\frac{1}{\Gamma(\alpha-\sigma+1)}+\frac{|\lambda_1|\eta^{\alpha}+1}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\sigma)}\right]\left(\|y-y_1\|+\left\|D^{\delta}y-D^{\delta}y_1\right\|\right)\\&+k_1\left[\begin{array}{c}\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}+\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\sigma)}\\+\frac{|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}+|\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\sigma)}\end{array}\right]\left(\|y-y_1\|+\left\|D^{\delta}y-D^{\delta}y_1\right\|\right).\end{aligned}$$

Therefore,

$$|D^{\sigma}\phi_1 y(t)-D^{\sigma}\phi_1 y_1(t)|\leq k_1M_2\left(\|y-y_1\|+\left\|D^{\delta}y-D^{\delta}y_1\right\|\right).\tag{3.20}$$

And consequently,

$$\|D^{\sigma}\phi_1(y)-D^{\sigma}\phi_1(y_1)\|\leq k_1M_2\left(\|y-y_1\|+\left\|D^{\delta}y-D^{\delta}y_1\right\|\right).\tag{3.21}$$

By (3.19) and (3.21),we can write

$$\|\phi_1(y)-\phi_1(y_1)\|_X\leq k_1(M_1+M_2)\left(\|y-y_1\|+\left\|D^{\delta}y-D^{\delta}y_1\right\|\right).\tag{3.22}$$

With the same arguments as before, we have

$$\|\phi_2(x)-\phi_2(x_1)\|_Y\leq k_2(M_3+M_4)\left(\|x-x_1\|+\|D^{\sigma}x-D^{\sigma}x_1\|\right).\tag{3.23}$$

And by,(3.22) and (3.23) we obtain

$$\|\phi(x,y)-\phi(x_1,y_1)\|_{X\times Y}\leq\left[\begin{array}{c}k_1(M_1+M_2)\\+k_2(M_3+M_4)\end{array}\right]\|(x-x_1,y-y_1)\|_{X\times Y}.\tag{3.24}$$

Consequently by (3.14) , we conclude that ϕ is contraction. As a consequence of Banach fixed point theorem, we deduce that ϕ has a fixed point which is a solution of the boundary value problem (1.1) . □

Now, we use Krasnselskii's fixed point theorem to prove the following result:

Theorem 3.2. *Assume that the hypotheses $(H1)-(H2)$ and $(H3)$ are satisfied,such that*

$$k_1\theta_1+k_2\theta_2<1,\tag{3.25}$$

where

$$\begin{aligned}\theta_1&=\frac{|\lambda_1\eta-1|+|\lambda_1|\eta^{\alpha}+1}{|\lambda_1\eta-1|\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha-\sigma+1)}+\frac{|\lambda_1|\eta^{\alpha}+1}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\sigma)},\\\theta_2&=\frac{|\lambda_1\eta-1|+|\lambda_1|\eta^{\beta}+1}{|\lambda_1\eta-1|\Gamma(\beta+1)}+\frac{1}{\Gamma(\beta-\delta+1)}+\frac{|\lambda_1|\eta^{\beta}+1}{|\lambda_1\eta-1|\Gamma(\beta+1)\Gamma(2-\delta)},\end{aligned}$$

if there exist $\mu\in\mathbb{R}$ such that

$$N_1(M_1+M_2)+N_2(M_3+M_4)\leq\mu,\tag{3.26}$$

then, the problem (1.1) has at least a solution.

Proof. We shall use Krasnselskii's fixed point theorem to prove that ϕ has at least a fixed point on $X \times Y$.
Suppose that (3.26) holds and let us take

$$\phi(x,y)(t) := T(x,y)(t) + R(x,y)(t), \tag{3.27}$$

where

$$T(x,y)(t) := (T_1 y(t), T_2 x(t)), \tag{3.28}$$

$$\begin{aligned} T_1 y(t) = &-\tfrac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(s, y(s), D^\delta y(s)\right) ds \\ &+ \tfrac{\lambda_1 t}{(\lambda_1\eta-1)\Gamma(\alpha)} \int_0^\eta (\eta-s)^{\alpha-1} f\left(s, y(s), D^\delta y(s)\right) ds \\ &- \tfrac{t}{(\lambda_1\eta-1)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f\left(s, y(s), D^\delta y(s)\right) ds, \end{aligned} \tag{3.29}$$

$$\begin{aligned} T_2 x(t) = &-\tfrac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g\left(s, x(s), D^\sigma x(s)\right) ds \\ &+ \tfrac{\lambda_1 t}{(\lambda_1\eta-1)\Gamma(\beta)} \int_0^\eta (\eta-s)^{\alpha-1} g\left(s, x(s), D^\sigma x(s)\right) ds \\ &- \tfrac{t}{(\lambda_1\eta-1)\Gamma(\beta)} \int_0^1 (1-s)^{\alpha-1} g\left(s, x(s), D^\sigma x(s)\right) ds, \end{aligned} \tag{3.30}$$

and

$$R(x,y)(t) := (R_1 y(t), R_2 x(t)), \tag{3.31}$$

where

$$\begin{aligned} R_1 y(t) = &\tfrac{(\lambda_2-\lambda_2\lambda_1\eta^3)t+(\lambda_2\lambda_1\eta-\lambda_2)t^3}{6(\lambda_1\eta-1)(\lambda_2\xi-1)\Gamma(\alpha-2)} \int_0^\xi (\xi-s)^{\alpha-3} f\left(s, y(s), D^\delta y(s)\right) ds \\ &- \tfrac{(1-\lambda_1\eta^3)t+(\lambda_1\eta-1)t^3}{6(\lambda_1\eta-1)(\lambda_2\xi-1)\Gamma(\alpha-2)} \int_0^1 (1-s)^{\alpha-3} f\left(s, y(s), D^\delta y(s)\right) ds, \end{aligned} \tag{3.32}$$

$$\begin{aligned} R_2 x(t) = &\tfrac{(\lambda_2-\lambda_2\lambda_1\eta^3)t+(\lambda_2\lambda_1\eta-\lambda_2)t^3}{6(\lambda_1\eta-1)(\lambda_2\xi-1)\Gamma(\beta-2)} \int_0^\xi (\xi-s)^{\alpha-3} g\left(s, x(s), D^\sigma x(s)\right) ds \\ &- \tfrac{(1-\lambda_1\eta^3)t+(\lambda_1\eta-1)t^3}{6(\lambda_1\eta-1)(\lambda_2\xi-1)\Gamma(\beta-2)} \int_0^1 (1-s)^{\alpha-3} g\left(s, x(s), D^\sigma x(s)\right) ds. \end{aligned} \tag{3.33}$$

The proof will be given in several steps.

Step1: We shall prove that for any $(x,y), (x_1,y_1) \in B_\mu$, then $T(x,y) + R(x_1,y_1) \in B_\mu$, Such that $B_\mu = \{(x,y) \in X \times Y; \|(x,y)\|_{X\times Y} \le \mu\}$.

For any $(x,y), (x_1,y_1) \in B_\mu$ and for each $t \in J$ we have:

$$\begin{aligned} |T_1 y(t) + R_1 y_1(t)| = &| -\tfrac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(s, y(s), D^\delta y(s)\right) ds \\ &+ \tfrac{\lambda_1 t}{(\lambda_1\eta-1)\Gamma(\alpha)} \int_0^\eta (\eta-s)^{\alpha-1} f\left(s, y(s), D^\delta y(s)\right) ds \\ &- \tfrac{t}{(\lambda_1\eta-1)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f\left(s, y(s), D^\delta y(s)\right) ds \\ &+ \tfrac{(\lambda_2-\lambda_2\lambda_1\eta^3)t+(\lambda_2\lambda_1\eta-\lambda_2)t^3}{6(\lambda_1\eta-1)(\lambda_2\xi-1)\Gamma(\alpha-2)} \int_0^\xi (\xi-s)^{\alpha-3} f\left(s, y(s), D^\delta y(s)\right) ds \\ &- \tfrac{(1-\lambda_1\eta^3)t+(\lambda_1\eta-1)t^3}{6(\lambda_1\eta-1)(\lambda_2\xi-1)\Gamma(\beta-2)} \int_0^1 (1-s)^{\alpha-3} f\left(s, y(s), D^\delta y(s)\right) ds \,| \end{aligned}$$

then,

$$
\begin{aligned}
|T_1 y(t) + R_1 y_1(t)| \leq & \tfrac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left| f\left(s, y(s), D^\delta y(s)\right)\right| ds \\
& + \tfrac{|\lambda_1|}{|\lambda_1 \eta - 1|\Gamma(\alpha)} \int_0^\eta (\eta - s)^{\alpha-1} \left| f\left(s, y(s), D^\delta y(s)\right)\right| ds \\
& + \tfrac{1}{|\lambda_1 \eta - 1|\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \left| f\left(s, y(s), D^\delta y(s)\right)\right| ds \\
& + \tfrac{\left|\lambda_2 - \lambda_2 \lambda_1 \eta^3\right| + |\lambda_2 \lambda_1 \eta - \lambda_2|}{6|\lambda_1 \eta - 1||\lambda_2 \xi - 1|\Gamma(\alpha-2)} \int_0^\xi (\xi - s)^{\alpha-3} \left| f\left(s, y_1(s), D^\delta y_1(s)\right)\right| ds \\
& + \tfrac{\left|1 - \lambda_1 \eta^3\right| + |\lambda_1 \eta - 1|}{6|\lambda_1 \eta - 1||\lambda_2 \xi - 1|\Gamma(\alpha-2)} \int_0^1 (1-s)^{\alpha-3} \left| f\left(s, y_1(s), D^\delta y_1(s)\right)\right| ds.
\end{aligned}
$$

Using the $(H3)$, we obtain

$$
\begin{aligned}
|T_1 y(t) + R_1 y_1(t)| \leq & N_1 \left[\tfrac{|\lambda_1 \eta - 1| + |\lambda_1| \eta^\alpha + 1}{|\lambda_1 \eta - 1|\Gamma(\alpha+1)} \right] \\
& + N_1 \left[\tfrac{\left[\left(\left|\lambda_2 - \lambda_2 \lambda_1 \eta^3\right| + |\lambda_2 \lambda_1 \eta - \lambda_2|\right)\xi^{\alpha-2} + \left|1 - \lambda_1 \eta^3\right| + |\lambda_1 \eta - 1|\right]}{6|\lambda_1 \eta - 1||\lambda_2 \xi - 1|\Gamma(\alpha-1)} \right].
\end{aligned}
$$

Consequently,

$$|T_1 y(t) + R_1 y_1(t)| \leq N_1 M_1.$$

Thus,

$$\|T_1(y) + R_1(y_1)\| \leq N_1 M_1, \tag{3.34}$$

and

$$
\begin{aligned}
|D^\sigma T_1 y(t) + D^\sigma R_1 y_1(t)| \leq & \tfrac{1}{\Gamma(\alpha-\sigma)} \int_0^t (t-s)^{\alpha-\sigma-1} \left| f\left(s, y(s), D^\delta y(s)\right)\right| ds \\
& + \tfrac{|\lambda_1|}{|\lambda_1 \eta - 1|\Gamma(\alpha)\Gamma(2-\sigma)} \int_0^\eta (\eta - s)^{\alpha-1} \left| f\left(s, y(s), D^\delta y(s)\right)\right| ds \\
& + \tfrac{1}{|\lambda_1 \eta - 1|\Gamma(\alpha)\Gamma(2-\sigma)} \int_0^1 (1-s)^{\alpha-1} \left| f\left(s, y(s), D^\delta y(s)\right)\right| ds \\
& + \left[\begin{array}{c} \tfrac{\left|\lambda_2 - \lambda_2 \lambda_1 \eta^3\right|}{6|\lambda_1 \eta - 1||\lambda_2 \xi - 1|\Gamma(\alpha-2)\Gamma(2-\sigma)} \\ + \tfrac{|\lambda_2 \lambda_1 \eta - \lambda_2|}{|\lambda_1 \eta - 1||\lambda_2 \xi - 1|\Gamma(\alpha-2)\Gamma(4-\sigma)} \end{array} \right] \int_0^\xi (\xi - s)^{\alpha-3} \left| f\left(s, y_1(s), D^\delta y_1(s)\right)\right| ds \\
& + \left[\begin{array}{c} \tfrac{\left|1 - \lambda_1 \eta^3\right|}{6|\lambda_1 \eta - 1||\lambda_2 \xi - 1|\Gamma(\alpha-2)\Gamma(2-\sigma)} \\ + \tfrac{|\lambda_1 \eta - 1|}{|\lambda_1 \eta - 1||\lambda_2 \xi - 1|\Gamma(\alpha-2)\Gamma(4-\sigma)} \end{array} \right] \int_0^1 (1-s)^{\alpha-3} \left| f\left(s, y_1(s), D^\delta y_1(s)\right)\right| ds.
\end{aligned}
$$

By $(H3)$,we have

$$
\begin{aligned}
|D^\sigma T_1 y(t) + D^\sigma R_1 y_1(t)| \leq & N_1 \left[\tfrac{1}{\Gamma(\alpha-\sigma+1)} + \tfrac{|\lambda_1| \eta^\alpha + 1}{|\lambda_1 \eta - 1|\Gamma(\alpha+1)\Gamma(2-\sigma)} \right] \\
& + N_1 \left[\tfrac{\left|\lambda_2 - \lambda_2 \lambda_1 \eta^3\right| \xi^{\alpha-2} + \left|1 - \lambda_1 \eta^3\right|}{6|\lambda_1 \eta - 1||\lambda_2 \xi - 1|\Gamma(\alpha-1)\Gamma(2-\sigma)} + \tfrac{|\lambda_2 \lambda_1 \eta - \lambda_2| \xi^{\alpha-2} + |\lambda_1 \eta - 1|}{|\lambda_1 \eta - 1||\lambda_2 \xi - 1|\Gamma(\alpha-1)\Gamma(4-\sigma)} \right].
\end{aligned}
$$

Consequently we obtain

$$|D^\sigma T_1 y(t) + D^\sigma R_1 y_1(t)| \leq N_1 M_2.$$

Hence,

$$\|D^\sigma T_1(y) + D^\sigma R_1(y_1)\| \leq N_1 M_2. \tag{3.35}$$

Combining (3.34) and (3.35) yields

$$\|T_1(y) + R_1(y_1)\|_X \leq N_1 (M_1 + M_2). \tag{3.36}$$

Analogously, we have

$$\|T_2(x) + R_2(x_1)\|_Y \leq N_2 (M_3 + M_4). \tag{3.37}$$

Hence, it follows from (3.36) and (3.37) that

$$\|T(x,y)+R(x_1,y_1)\|_{X\times Y}\leq N_1(M_1+M_2)+N_2(M_3+M_4)<\mu. \tag{3.38}$$

Step2: We shall prove that R is continuous and compact.

$[1^*]$: The continuity of f and g implies that the operator R is continuous.

$[2^*]$: Now, we prove that R maps bounded sets into bounded sets of $X\times Y$.

For $(x,y)\in B_\mu$ and for each $t\in J$,we have:

$$\begin{aligned}|R_1y(t)|&\leq \tfrac{|\lambda_2-\lambda_2\lambda_1\eta^3|t+|\lambda_2\lambda_1\eta-\lambda_2|t^3}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)}\int_0^\xi(\xi-s)^{\alpha-3}\left|f\left(s,y(s),D^\delta y(s)\right)\right|ds\\&+\tfrac{|1-\lambda_1\eta^3|t+|\lambda_1\eta-1|t^3}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)}\int_0^1(1-s)^{\alpha-3}\left|f\left(s,y(s),D^\delta y(s)\right)\right|ds.\end{aligned}$$

Using the $(H3)$, we obtain

$$\begin{aligned}|R_1y(t)|&\leq \tfrac{N_1\left[\left(|\lambda_2-\lambda_2\lambda_1\eta^3|+|\lambda_2\lambda_1\eta-\lambda_2|\right)\xi^{\alpha-2}+|1-\lambda_1\eta^3|+|\lambda_1\eta-1|\right]}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}\\&\leq N_1\left(\tfrac{\left(|\lambda_2-\lambda_2\lambda_1\eta^3|+|\lambda_2\lambda_1\eta-\lambda_2|\right)\xi^{\alpha-2}+|1-\lambda_1\eta^3|+|\lambda_1\eta-1|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}\right).\end{aligned}$$

Thus,

$$|R_1y(t)|\leq N_1L_1, t\in J,$$

Therefore,

$$\|R_1(y)\|\leq N_1L_1. \tag{3.39}$$

On the other hand,

$$\begin{aligned}|D^\sigma R_1y(t)|&\leq \tfrac{1}{\Gamma(\alpha-2)}\left(\begin{array}{c}\frac{|\lambda_2-\lambda_2\lambda_1\eta^3|t^{1-\sigma}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(2-\sigma)}+\\ \frac{|\lambda_2\lambda_1\eta-\lambda_2|t^{3-\sigma}}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(4-\sigma)}\end{array}\right)\int_0^\xi(\xi-s)^{\alpha-3}\left|f\left(s,y(s),D^\delta y(s)\right)\right|ds\\&+\tfrac{1}{\Gamma(\alpha-2)}\left(\begin{array}{c}\frac{|1-\lambda_1\eta^3|t^{1-\sigma}}{6\Gamma(2-\sigma)|\lambda_1\eta-1||\lambda_2\xi-1|}+\\ \frac{|\lambda_1\eta-1|t^{3-\sigma}}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(4-\sigma)}\end{array}\right)\int_0^1(1-s)^{\alpha-3}\left|f\left(s,y(s),D^\delta y(s)\right)\right|ds.\end{aligned}$$

By $(H3)$, we have,

$$\begin{aligned}|D^\sigma\phi_1y(t)|&\leq N_1\left[\tfrac{|\lambda_2-\lambda_2\lambda_1\eta^3|\xi^{\alpha-2}+|1-\lambda_1\eta^3|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\sigma)}+\tfrac{|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}+|\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\sigma)}\right]\\&\leq N_1\left[\tfrac{|\lambda_2-\lambda_2\lambda_1\eta^3|\xi^{\alpha-2}+|1-\lambda_1\eta^3|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\sigma)}+\tfrac{|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}+|\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\sigma)}\right].\end{aligned}$$

Consequently we obtain,

$$|D^\sigma R_1y(t)|\leq N_1L_2, t\in J.$$

Therefore,

$$\|D^\sigma R_1(y)\|\leq N_1L_2. \tag{3.40}$$

Hence, from (3.39) and (3.40), we have

$$\|R_1(y)\|_X\leq N_1(L_1+L_2). \tag{3.41}$$

Similarly, it can be shown that,

$$\|R_2(x)\|_Y\leq N_2(L_3+L_4). \tag{3.42}$$

It follows from (3.41) and (3.42) that

$$\|R(x,y)\|_{X\times Y}\leq N_1(L_1+L_2)+N_2(L_3+L_4). \tag{3.43}$$

Consequently

$$\|R(x,y)\|_{X\times Y}<\infty.$$

$[3^*]$: In the end we show that R is equicontinuous on J:
Let $t_1,t_2\in J$, such that $t_1<t_2$ and $(x,y)\in B_\mu$, .Then, we have:

$$\begin{aligned}|R_1y(t_2)-R_1y(t_1)|\leq{}&\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|(t_2-t_1)+|\lambda_2\lambda_1\eta-\lambda_2|\left(t_2^3-t_1^3\right)}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)}\int_0^\xi(\xi-s)^{\alpha-3}\left|f\left(s,y(s),D^\delta y(s)\right)\right|ds\\&+\frac{\left|1-\lambda_1\eta^3\right|(t_1-t_2)+|\lambda_1\eta-1|\left(t_1^3-t_2^3\right)}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)}\int_0^1(1-s)^{\alpha-3}\left|f\left(s,y(s),D^\delta y(s)\right)\right|ds.\end{aligned}$$

Using the $(H3)$,we obtain

$$\begin{aligned}|R_1y(t_2)-R_1y(t_1)|\leq{}&\frac{N_1\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}(t_2-t_1)\\&+\frac{N_1\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}(t_1-t_2)\\&+\frac{N_1|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}\left(t_2^3-t_1^3\right)+\frac{N_1|\lambda_1\eta-1|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}\left(t_1^3-t_2^3\right),\end{aligned}\tag{3.44}$$

and

$$\begin{aligned}|D^\sigma R_1y(t_2)-D^\sigma R_1y(t_1)|\leq{}&\left[\begin{array}{c}\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\left(t_2^{1-\sigma}-t_1^{1-\sigma}\right)}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(2-\sigma)}\\+\frac{|\lambda_2\lambda_1\eta-\lambda_2|\left(t_2^{3-\sigma}-t_1^{3-\sigma}\right)}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(4-\sigma)}\end{array}\right]\int_0^\xi(\xi-s)^{\alpha-3}\left|f\left(s,y(s),D^\delta y(s)\right)\right|ds\\&+\left[\begin{array}{c}\frac{\left|1-\lambda_1\eta^3\right|\left(t_1^{1-\sigma}-t_2^{1-\sigma}\right)}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(2-\sigma)}\\+\frac{|\lambda_1\eta-1|\left(t_1^{3-\sigma}-t_2^{3-\sigma}\right)}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(4-\sigma)}\end{array}\right]\int_0^1(1-s)^{\alpha-3}\left|f\left(s,y(s),D^\delta y(s)\right)\right|ds,\end{aligned}$$

by $(H3)$, we have:

$$\begin{aligned}|D^\sigma R_1y(t_2)-D^\sigma R_1y(t_1)|\leq{}&\frac{N_1\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\sigma)}\left(t_2^{1-\sigma}-t_1^{1-\sigma}\right)\\&+\frac{N_1\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\sigma)}\left(t_1^{1-\sigma}-t_2^{1-\sigma}\right)\\&+\frac{N_1|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\sigma)}\left(t_2^{3-\sigma}-t_1^{3-\sigma}\right)\\&+\frac{N_1|\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\sigma)}\left(t_1^{3-\sigma}-t_2^{3-\sigma}\right).\end{aligned}\tag{3.45}$$

Hence,by (3.44) and (3.45), we obtain

$$\begin{aligned}\|R_1y(t_2)-R_1y(t_1)\|_X\leq{}&\frac{N_1\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}(t_2-t_1)+\frac{N_1\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}(t_1-t_2)\\&+\frac{N_1|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}\left(t_2^3-t_1^3\right)+\frac{N_1|\lambda_1\eta-1|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}\left(t_1^3-t_2^3\right)\\&+\frac{N_1\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\sigma)}\left(t_2^{1-\sigma}-t_1^{1-\sigma}\right)\\&+\frac{N_1\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\sigma)}\left(t_1^{1-\sigma}-t_2^{1-\sigma}\right)\\&+\frac{N_1|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\sigma)}\left(t_2^{3-\sigma}-t_1^{3-\sigma}\right)\\&+\frac{N_1|\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\sigma)}\left(t_1^{3-\sigma}-t_2^{3-\sigma}\right).\end{aligned}\tag{3.46}$$

Analogously, we can obtain

$$\begin{aligned}\left\|R_2x(t_2)-R_2x(t_1)\right\|_Y \leq{}& \tfrac{N_2\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\beta-2}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)}(t_2-t_1)+\tfrac{N_2\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)}(t_1-t_2)\\&+\tfrac{N_2|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\beta-2}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)}\left(t_2^3-t_1^3\right)+\tfrac{N_2|\lambda_1\eta-1|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)}\left(t_1^3-t_2^3\right)\\&+\tfrac{N_2\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\beta-2}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)\Gamma(2-\delta)}\left(t_2^{1-\delta}-t_1^{1-\delta}\right)\\&+\tfrac{N_2\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)\Gamma(2-\delta)}\left(t_1^{1-\delta}-t_2^{1-\delta}\right)\\&+\tfrac{N_2|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\beta-2}}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)\Gamma(4-\delta)}\left(t_2^{3-\delta}-t_1^{3-\delta}\right)\\&+\tfrac{N_2|\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)\Gamma(4-\delta)}\left(t_1^{3-\delta}-t_2^{3-\delta}\right).\end{aligned}\tag{3.47}$$

Thanks to (3.46) and (3.47), can state that $\|\phi(x,y)(t_2)-\phi(x,y)(t_1)\|\to 0$ as $t_1\to t_2$. Then, as a consequence of steps $([1^*],[2^*],[3^*])$; we can conclude that R is continuous and compact.

Step3: Now, we prove that T is contraction mapping.

Let $(x,y),(x_1,y_1)\in X\times Y$. Then, for each $t\in J$, we have

$$\begin{aligned}\left|T_1y(t)-T_1y_1(t)\right|\leq{}& \tfrac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\left|\begin{array}{c}f\left(s,y(s),D^\delta y(s)\right)\\-f\left(s,y_1(s),D^\delta y_1(s)\right)\end{array}\right|ds\\&+\tfrac{\lambda_1 t}{(\lambda_1\eta-1)\Gamma(\alpha)}\int_0^\eta(\eta-s)^{\alpha-1}\left|\begin{array}{c}f\left(s,y(s),D^\delta y(s)\right)\\-f\left(s,y_1(s),D^\delta y_1(s)\right)\end{array}\right|ds\\&+\tfrac{t}{(\lambda_1\eta-1)\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}\left|\begin{array}{c}f\left(s,y(s),D^\delta y(s)\right)\\-f\left(s,y_1(s),D^\delta y_1(s)\right)\end{array}\right|ds.\end{aligned}$$

Thanks to $(H1)$, we can write

$$\begin{aligned}\left|T_1y(t)-T_1y_1(t)\right|\leq{}&\tfrac{k_1}{\Gamma(\alpha+1)}\left(\|y-y_1\|+\left\|D^\delta y-D^\delta y_1\right\|\right)\\&+\tfrac{k_1(|\lambda_1|\eta^\alpha+1)}{|\lambda_1\eta-1|\Gamma(\alpha+1)}\left(\|y-y_1\|+\left\|D^\delta y-D^\delta y_1\right\|\right).\end{aligned}$$

Consequently,

$$\left\|T_1(y)-T_1(y_1)\right\|\leq\tfrac{k_1[|\lambda_1\eta-1|+|\lambda_1|\eta^\alpha+1]}{|\lambda_1\eta-1|\Gamma(\alpha+1)}\left(\|y-y_1\|+\left\|D^\delta y-D^\delta y_1\right\|\right),\tag{3.48}$$

and

$$\begin{aligned}\left|D^\sigma T_1y(t)-D^\sigma T_1y_1(t)\right|\leq{}&\tfrac{1}{\Gamma(\alpha-\sigma)}\int_0^t(t-s)^{\alpha-\sigma-1}\left|\begin{array}{c}f\left(s,y(s),D^\delta y(s)\right)\\-f\left(s,y_1(s),D^\delta y_1(s)\right)\end{array}\right|ds\\&+\tfrac{|\lambda_1|t^{1-\sigma}}{|\lambda_1\eta-1|\Gamma(\alpha)\Gamma(2-\sigma)}\int_0^\eta(\eta-s)^{\alpha-1}\left|\begin{array}{c}f\left(s,y(s),D^\delta y(s)\right)\\-f\left(s,y_1(s),D^\delta y_1(s)\right)\end{array}\right|ds\\&+\tfrac{t^{1-\sigma}}{|\lambda_1\eta-1|\Gamma(\alpha)\Gamma(2-\sigma)}\int_0^1(1-s)^{\alpha-1}\left|\begin{array}{c}f\left(s,y(s),D^\delta y(s)\right)\\-f\left(s,y_1(s),D^\delta y_1(s)\right)\end{array}\right|ds.\end{aligned}$$

By $(H1)$,yields

$$\begin{aligned}\left|D^\sigma T_1y(t)-D^\sigma T_1y_1(t)\right|\leq{}&\tfrac{k_1}{\Gamma(\alpha-\sigma+1)}\left(\|y-y_1\|+\left\|D^\delta y-D^\delta y_1\right\|\right)\\&+\tfrac{k_1|\lambda_1|\eta^\alpha}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\sigma)}\left(\|y-y_1\|+\left\|D^\delta y-D^\delta y_1\right\|\right)\\&+\tfrac{k_1}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\sigma)}\left(\|y-y_1\|+\left\|D^\delta y-D^\delta y_1\right\|\right).\end{aligned}$$

Hence,

$$\left\| D^{\sigma}T_1(y) - D^{\sigma}T_1(y_1) \right\| \le k_1 \left[\begin{array}{c} \frac{1}{\Gamma(\alpha-\sigma+1)} \\ +\frac{|\lambda_1|\eta^{\alpha}+1}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\sigma)} \end{array} \right] \left(\|y-y_1\| + \left\| D^{\delta}y - D^{\delta}y_1 \right\| \right). \quad (3.49)$$

By (3.48) and (3.49) we can write

$$\|T_1(y) - T_1(y_1)\|_X \le k_1 \left[\begin{array}{c} \frac{|\lambda_1\eta-1|+|\lambda_1|\eta^{\alpha}+1}{|\lambda_1\eta-1|\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha-\sigma+1)} \\ +\frac{|\lambda_1|\eta^{\alpha}+1}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\sigma)} \end{array} \right] \left(\|y-y_1\| + \left\| D^{\delta}y - D^{\delta}y_1 \right\| \right).$$

Thus,

$$\|T_1(y) - T_1(y_1)\|_X \le k_1\theta_1 \left(\|y-y_1\| + \left\| D^{\delta}y - D^{\delta}y_1 \right\| \right). \quad (3.50)$$

Analogously, we can get

$$\|T_2(x) - T_2(x_1)\|_Y \le k_2\theta_2 \left(\|x-x_1\| + \|D^{\sigma}x - D^{\sigma}x_1\| \right). \quad (3.51)$$

It follows from (3.50) and (3.51) that

$$\|T(x,y) - T(x_1,y_1)\|_{X\times Y} \le [k_1\theta_1 + k_2\theta_2] \left(\|(x-x_1, y-y_1)\|_{X\times Y} \right).$$

Using the condition (3.25) we conclude that T is a contraction mapping.

As a consequence of Krasnoselskii's fixed point theorem we deduce that ϕ has a fixed point which is a solution of (1.1) . □

4 Examples

In this section we give an example to illustrate the usefulness of our main results.

Example 4.1. *Let us consider the following system of fractional boundary value problem:*

$$\begin{cases} D^{\frac{7}{2}}x(t) + \frac{\sqrt{\pi}e^{-\pi t^2}\cos(\pi t)\left(y(t)+D^{\frac{5}{2}}y(t)\right)}{\left(5\sqrt{\pi}+7e^t\right)\left(1+y(t)+D^{\frac{5}{2}}y(t)\right)} + \ln\left(1+t^2\right) = 0, t \in J, \\ D^{\frac{11}{3}}y(t) + \frac{\sqrt{\pi}e^{-\pi t^2}\cos(\pi t)\left(x(t)+D^{\frac{9}{4}}x(t)\right)}{\left(5\sqrt{\pi}+7e^t\right)\left(1+x(t)+D^{\frac{9}{4}}x(t)\right)} + \ln\left(1+t^2\right) = 0, t \in J, \\ x(0) = 0, x(1) - \frac{3}{4}x\left(\frac{1}{3}\right) = 0, y(0) = 0, y(1) - \frac{3}{4}y\left(\frac{1}{3}\right) = 0, \\ x''(0) = 0, x''(1) - \frac{4}{5}x''\left(\frac{2}{3}\right) = 0, y''(0) = 0, y''(1) - \frac{4}{5}y''\left(\frac{2}{3}\right) = 0, \end{cases}$$

Set

$$f(t,x,y) = g(t,x,y) = \frac{\sqrt{\pi}e^{-\pi t}|\cos(\pi t)|(|x|+|y|)}{\left(5\sqrt{\pi}+7e^t\right)^2(1+|x|+|y|)} + \ln\left(1+t^2\right), t \in [0,1], x,y \in [0,\infty),$$

For $t \in J = [0,1]$ and $x_1, y_1, x_2, y_2 \in [0,\infty)$,we have:

$$\begin{aligned} |f(t,x,y) - f(t,x_1,y_1)| &= \frac{\sqrt{\pi}e^{-\pi t}|\cos(\pi t)|}{\left(5\sqrt{\pi}+7e^t\right)^2}\left|\frac{x+y}{(1+|x|+|y|)} - \frac{x_1+y_1}{(1+|x_1|+|y_1|)}\right| \\ &\le \frac{\sqrt{\pi}e^{-\pi t}|\cos(\pi t)|(|x-x_1|+|y-y_1|)}{\left(5\sqrt{\pi}+7e^t\right)^2(1+|x|+|y|)(1+|x_1|+|y_1|)} \\ &\le \frac{\sqrt{\pi}e^{-\pi t}|\cos(\pi t)|(|x-x_1|+|y-y_1|)}{\left(5\sqrt{\pi}+7e^t\right)^2} \\ &\le \frac{\sqrt{\pi}}{\left(5\sqrt{\pi}+7\right)^2}(|x-x_1|+|y-y_1|). \end{aligned}$$

Hence the condition $(H1)$ holds with $k_1 = k_2 = \frac{\sqrt{\pi}}{(5\sqrt{\pi}+7)^2}$.

For $\alpha = \frac{7}{2}, \beta = \frac{11}{3}, \sigma = \frac{9}{4}, \delta = \frac{5}{2}$ *and* $\lambda_1 = \frac{3}{4}, \lambda_2 = \frac{4}{5} = \eta = \frac{1}{3}, \xi = \frac{2}{3}$, *we have:*

$$M_1 = 1,089, M_2 = 3,503, M_3 = 0,909, M_4 = 3,089,$$

and,

$$k_1 (M_1 + M_2) + k_2 (M_3 + M_4) = 0,0605075.$$

Therefore,

$$k_1 (M_1 + M_2) + k_2 (M_3 + M_4) < 1.$$

Hence, the condition (3.14) *of Theorem* (3.1) *is satisfied. Therefore the boundary value problem* (1.1) *has a unique solution. So, a simple computation shows that*

$$\theta_1 = 1,283, \theta_2 = 1,058,$$

and, we have

$$k_1\theta_1 + k_2\theta_2 = 0,0164904.$$

Using the condition (3.25), *we get,*

$$k_1\theta_1 + k_2\theta_2 < 1.$$

Therefore it follow from Theorem (3.2) *that the boundary value problem* (1.1) *has a solution.*

Example 4.2. *Consider the following system of fractional Bounded value problem:*

$$\begin{cases} D^{\frac{7}{2}} x(t) + \frac{\left|D^{\frac{7}{3}} y(t)\right|}{5\pi\left(\sqrt{\pi}+2e^t\right)} + \frac{e^{-t^2}|y(t)|}{5\pi\left(\sqrt{\pi}e^t+2\right)^2(1+|y(t)|)} = 0, t \in J, \\ D^{\frac{11}{3}} y(t) + \frac{|x(t)|}{14\sqrt{\pi}(1+|x(t)|)} + \frac{|\cos(\pi t)|\left|D^{\frac{5}{2}} x(t)\right|}{7\sqrt{\pi}(t+1)^2} = 0, t \in J, \\ x(0) = 0, x(1) - \frac{2}{3}x\left(\frac{1}{5}\right) = 0, y(0) = 0, y(1) - \frac{2}{3}y\left(\frac{1}{5}\right) \\ x''(0) = 0, x''(1) - \frac{1}{2}x''\left(\frac{1}{4}\right) = 0, y''(0) = 0, y''(1) - \frac{1}{2}y''\left(\frac{1}{4}\right) = 0. \end{cases}$$

For this example, we have

$$f(t,x,y) = \frac{|x|}{5\pi\left(\sqrt{\pi}+2e^t\right)} + \frac{e^{-t^2}|y|}{5\pi\left(\sqrt{\pi}e^t+2\right)^2(1+|y|)}, t \in [0,1], x, y \in [0,\infty),$$
$$g(t,x,y) = \frac{|x|}{14\sqrt{\pi}(1+|x|)} + \frac{|\cos(\pi t)||y|}{7\sqrt{\pi}(t+1)^2}, t \in [0,1], x, y \in [0,\infty).$$

For $t \in J = [0,1]$ *and* $x, y, x_1, y_1 \in [0,\infty)$. *Then we have:*

$$\begin{aligned} |f(t,x,y) - f(t,x_1,y_1)| &= \frac{e^{-t^2}|x-x_1|}{5\left(\sqrt{\pi}e^t+2\right)^2(1+|x|)(1+|x_1|)} + \frac{|y-y_1|}{5\pi\left(\sqrt{\pi}+2e^t\right)} \\ &\leq \frac{e^{-t^2}}{5\pi\left(\sqrt{\pi}e^t+2\right)^2}|x-x_1| + \frac{1}{5\pi\left(\sqrt{\pi}+2e^t\right)}|y-y_1| \\ &\leq \frac{1}{5\pi\left(\sqrt{\pi}+2\right)^2}(|x-x_1| + |y-y_1|), \end{aligned}$$

and

$$\begin{aligned} |g(t,x,y) - g(t,x_1,y_1)| &= \frac{|x-x_1|}{14\sqrt{\pi}(1+|x|)(1+|x_1|)} + \frac{|\cos(\pi t)||y-y_1|}{7\sqrt{\pi}(t+1)^2} \\ &\leq \frac{1}{14\sqrt{\pi}}|x-x_1| + \frac{|\cos(\pi t)|}{7\sqrt{\pi}(t+1)^2}|y-y_1| \\ &\leq \frac{1}{14\sqrt{\pi}}(|x-x_1| + |y-y_1|). \end{aligned}$$

So,we can take: $k_1 = \frac{1}{5\pi(\sqrt{\pi}+2)^2}$ *and* $k_2 = \frac{1}{14\sqrt{\pi}}$.
It, follows then that

$$M_1 = 0,51254, M_2 = 1,9531, M_3 = 0,43113, M_4 = 1,32719,$$

and

$$k_1 (M_1 + M_2) + k_2 (M_3 + M_4) = \frac{2,46564}{5\pi \left(\sqrt{\pi} + 2\right)^2} + \frac{1,75832}{14\sqrt{\pi}} = 0,081914 < 1.$$

Hence by Theorem (3.1) *the boundary value problem* (1.1) *has a unique solution.*
Now, using the condition (3.25), *we get:*

$$k_1\theta_1 + k_2\theta_2 = \frac{1,15722}{5\pi \left(\sqrt{\pi} + 2\right)^2} + \frac{0,96759}{14\sqrt{\pi}} = 0.044183.$$

Therefore,

$$k_1\theta_1 + k_2\theta_2 < 1.$$

By Theorem (3.2) *the boundary value problem* (1.1) *has a solution.*

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