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APPLICATIONS OF THE NEVANLINNA THEORY IN THE STUDY OF COMPLEX FUNCTIONAL EQUATIONS

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Abstract

This thesis focuses on Nevanlinna's theory of difference and its applications in the study of complex functional equations. We are interested in some problems related to the growth of solutions of difference equations and also to differential difference equations in general. The substance of these studies is found in the recent counterparts of Nevanlinna's difference theory. The key result is the lemma of the logarithmic derivative in difference form obtained recently by Halburd-Korhonen and Chiang-Feng, independently.

Key words: Linear difference equations, linear differential-difference equations, meromorphic solution, order, type, lower order, lower type, iterated order, iterated type.

Résumé

Cette thèse porte sur la théorie de Nevanlinna aux différences et de ses applications dans l'étude des équations fonctionnelles complexes. On s'intéresse à certains problèmes liés à la croissance des solutions des équations aux différences et également aux différences différentielles en général. Le fond de ces études se trouve dans les contreparties récentes de la théorie différence de Nevanlinna. Le résultat clé est le lemme de la dérivée logarithmique en forme différence obtenu récemment par Halburd-Korhonen et Chiang-Feng, indépendamment.

Mots clés: Équations aux différences linéaires, équations aux différences différentielles linéaires, solution méromorphe, ordre, type, ordre inférieur, type inférieur, ordre itératif, type itératif.

ملخص

تتناول هذه الرسالة نظرية رولف نيفانلينا للاختلافات وتطبيقاتها في دراسة المعادلات الدالية العقدية ، نهتم ببعض المشاكل المتعلقة بنمو حلول المعادلات للاختلافات وكذلك الاختلافات التفاضلية بشكل عام. هذه الدراسة تركز على نظرية رولف نيفانلينا للاختلافات ، والنتيجة الرئيسية هي توطئة المشتق اللوغاريتمي في صيغة الفروق التي حصل عليها مؤخرًا كل من هالبورد-كور هونين وتشياخ-فنج بشكل مستقل.

الكلمات المفتاحية: معادلات الفروق الخطية ، معادلات الفروق التفاضلية الخطية ، حل دالة ميرومورفية، الترتيب ، النوع ، الترتيب الأدنى ، النوع الأدنى ، الترتيب المتكرر، النوع المتكرر.

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Introduction

The appearance of R. Nevanlinna's theory in 1925, which represents the modern theory of the value distribution of meromorphic functions in the complex plane and which is considered to be the first events of mathematics in the twentieth century, which gave very effective tools for the study of the growth and the oscillation of functional equations. For an introduction to the theory of functional equations with the Nevanlinna theory (see, [13, 17, 40]).

There are two questions that have interested several researchers in the topic :

- 1) What are the conditions on the coefficients that ensure that all the solutions to be of infinite order?
- 2) Under what conditions do there exist solutions of finite order in the case when there are coefficients transcendental entire functions?

Several researches have published articles answering these two questions, see for example ([18,19,20,21,25,35]). In [37], Wittich showed that all the solutions of the differential equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f(z) = 0,$$

are of the finite order, if and only if the coefficients are polynomials. Frei [12] extended this result, assuming that if p is the greatest index such that A_p is a transcendental entire coefficient, then there exist at most p linearly independent solutions of finite order. Laine and Bank are considered among the first mathematicians who interested in the oscillation of solutions of linear differential equations of second order with entire coefficients. In 1982, they studied in [22], the distribution of zeros of solutions of the linear differential equation

$$f'' + Af = 0,$$

where A is a polynomial or a transcendental entire function. In 1996, Kwon ([19]) studied the hyper-order of solutions of the differential equation

$$f'' + A(z)f' + B(z)f = 0,$$

where $A(z), B(z) \not\equiv 0$ are entire functions of finite order. Recently, Li and Wang ([25]), Cao, Xu and Chen ([7]) investigated the higher order linear differential equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0,$$

and they proved under some conditions on the coefficients that any solution of these equations is of infinite order.

What concerns the difference equations, recently an active research in this field and several problems were investigated and resolved. The key result here is the difference analogue of the lemma on the logarithmic derivative obtained by Halburd-Korhonen [15, 16] and Chiang-Feng [11], independently. In the recent years, several authors have done a study of the properties of meromorphic solutions of complex linear difference equation

$$A_k(z)f(z + c_k) + A_{k-1}(z)f(z + c_{k-1}) + \cdots + A_1(z)f(z + c_1) + A_0(z)f(z) = 0,$$

where c_k, \dots, c_1 are non-zero distinct complex numbers when one coefficient has maximal order or among coefficients having the maximal order, exactly one has its type strictly greater than others and achieved some important results, see e.g. ([11], [23], [26], [28], [41]).

This thesis consists of an introduction and four chapters. The first chapter, contains reminders and definitions on the theory of Nevanlinna, which we will need in the other chapters.

In the second chapter, we are interested in the study of the growth of meromorphic solutions of non-homogeneous linear difference equation

$$A_n(z)f(z + c_n) + \dots + A_1(z)f(z + c_1) + A_0(z)f(z) = A_{n+1}(z),$$

where $A_{n+1}(z), \dots, A_0(z)$ are entire or meromorphic functions and $c_j (1, \dots, n)$ are non-zero distinct complex numbers. Under some conditions on the (lower) order and the (lower) type of the coefficients, we get some estimates on the lower bound of the order of meromorphic solutions of the above equation. In fact, we extend results of Luo and Zheng, Belaïdi and Benkarouba.

Also, we study the growth of meromorphic solutions of homogeneous and non-homogeneous linear difference equations

$$A_k(z)f(z + c_k) + \dots + A_1(z)f(z + c_1) + A_0(z)f(z) = 0,$$

$$A_k(z)f(z + c_k) + \dots + A_1(z)f(z + c_1) + A_0(z)f(z) = F,$$

where $A_k(z), \dots, A_0(z), F(z)$ are meromorphic functions and $c_j (1, \dots, k)$ are non-zero distinct complex numbers. Under some different conditions on the coefficients, we extend some results due to Zhou and Zheng.

The third chapter is devoted to the study of the growth of meromorphic solutions of the homogeneous and the non-homogeneous linear differential-difference equations

$$\sum_{i=0}^n \sum_{j=0}^m A_{ij} f^{(j)}(z + c_i) = 0,$$

$$\sum_{i=0}^n \sum_{j=0}^m A_{ij} f^{(j)}(z + c_i) = F,$$

where $A_{ij} (i = 0, \dots, n; j = 0, \dots, m), F$ are meromorphic functions and $c_i (0, \dots, n)$ are non-zero distinct complex numbers. Under some conditions on the coefficients, by combining complex differential and difference equations, we extend some results due to Chen and Zheng.

The main goal of the last chapter is to extend and generalize some of our results and those of Zhou and Zheng, for this we will use the concept of iterated p -order, which was introduced by Schönhage and Sato, see [31,32], and its definition was given by Kinnunen in 1998 ([20]). In fact, we obtain lower bound estimates of iterated p -order and p -lower order of meromorphic solutions of some linear differential-difference equations.

Chapter 1

Nevanlinna theory of meromorphic functions

We will quote only the elements necessary for our work in this thesis and for more details, see [1,13,17,40].

1.1 Characteristic function of R. Nevanlinna

Theorem 1.1.1. (*Jensen formula, see [21]*) Let f be a meromorphic function such that $f(0) \neq 0, \infty$ and let a_1, a_2, \dots (resp b_1, b_2, \dots) denote its zeros (resp. poles), each taken into account according to its multiplicity. Then

$$\log |f(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\varphi})| d\varphi + \sum_{|b_i| < r} \log \frac{r}{|b_i|} - \sum_{|a_i| < r} \log \frac{r}{|a_i|}.$$

Definition 1.1.2. (*Unintegrated counting function, see [21]*) Let f be a meromorphic function not being identical equal $a \in \mathbb{C}$. Let $i(z, a, f)$ denote the multiplicity of an a -point of f at z . Then we define

$$n(r, a, f) = n\left(r, \frac{1}{f-a}\right) = n(r, a) := \sum_{|z| \leq r, f(z)=a} i(z, a, f)$$

i.e., $n(r, a, f)$ counts the numbers of the roots of $f(z) = a$ in $|z| \leq r$, each roots according to its multiplicity. For the poles of f , we define similiary

$$n(r, \infty, f) = n(r, f) = n(r, \infty) := \sum_{|z| \leq r, f(z)=\infty} i(z, \infty, f).$$

Example 1.1.3. Let's pose

$$f(z) = \frac{(z+1)^3(z-2)^5}{z(z-1)^2}.$$

For $5 \leq r$, we have

$$n(r, f) = 3, \quad n\left(r, \frac{1}{f}\right) = 8.$$

Definition 1.1.4. (Counting function, see [21]). For a meromorphic function f , we define

$$N\left(r, \frac{1}{f-a}\right) := \int_0^r \frac{n(t, a, f) - n(0, a, f)}{t} dt + n(0, a, f) \log r$$

supposing $f \not\equiv a \in \mathbb{C}$ and

$$N(r, f) := \int_0^r \frac{n(t, \infty, f) - n(0, \infty, f)}{t} dt + n(0, \infty, f) \log r.$$

Definition 1.1.5. (Proximity function, see [21]) For a meromorphic function f , we define

$$m\left(r, \frac{1}{f-a}\right) := \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \frac{1}{f(re^{i\varphi}) - a} \right| d\varphi$$

supposing $f \not\equiv a \in \mathbb{C}$ and

$$m(r, f) := \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\varphi})| d\varphi.$$

Definition 1.1.6. (Characteristic function, see [21]) The characteristic function of a meromorphic function f will be defined as

$$T(r, f) := m(r, f) + N(r, f).$$

Example 1.1.7. For the function $f(z) = e^z$, we have

$$m(r, f) = \frac{r}{\pi}, N(r, f) = 0$$

and

$$T(r, f) = \frac{r}{\pi}.$$

Remark 1.1.8. The characteristic function offers a natural way to define a measure for the growth of a meromorphic function.

Theorem 1.1.9. (First main theorem, see [21]) Let f be a meromorphic function, let $a \in \mathbb{C}$ and let

$$f(z) - a = \sum_{i=m}^{+\infty} c_i z^i, \quad c_m \neq 0, \quad m \in \mathbb{Z},$$

be the Laurent expansion of $f - a$ at the origin. Then

$$T\left(r, \frac{1}{f-a}\right) = T(r, f) - \log |c_m| + \varphi(r, a),$$

where

$$|\varphi(r, a)| \leq \log 2 + \log^+ |a|.$$

Remark 1.1.10. The first main theorem may be expressed as

$$T\left(r, \frac{1}{f-a}\right) = T(r, f) + O(1)$$

for all $a \in \mathbb{C}$. Note the bounded error term $O(1)$ depends on $a \in \mathbb{C}$.

Proposition 1.1.11. (see [13, 21]) Let f, f_1, f_2, \dots, f_n , be meromorphic functions and let $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ such that $\alpha\delta - \beta\gamma \neq 0$. Then

- (a) $T(r, f_1 \cdot f_2 \cdots f_n) \leq \sum_{i=1}^n T(r, f_i)$, for all $r \geq 1$,
- (b) $T(r, f^n) = nT(r, f)$, $n \in \mathbb{N}$,
- (c) $T(r, \sum_{i=1}^n f_i) \leq \sum_{i=1}^n T(r, f_i) + \log n$, $r \geq 1$,
- (d) $T\left(r, \frac{\alpha f + \beta}{\gamma f + \delta}\right) = T(r, f) + O(1)$, with assumption $f \not\equiv -\delta/\gamma$.

1.2 Growth and value distribution of meromorphic function

In the following, we recall some fundamental definitions which are used later.

Definition 1.2.1. ([13], [17], [40]) Let f be a meromorphic function. Then, the order $\rho(f)$ of f is defined by

$$\rho(f) = \limsup_{r \rightarrow +\infty} \frac{\log T(r, f)}{\log r},$$

where $T(r, f)$ is the characteristic function of Nevanlinna (see, [13], [17], [40]). If f is an entire function, then the order of f is defined as

$$\rho(f) = \limsup_{r \rightarrow +\infty} \frac{\log T(r, f)}{\log r} = \limsup_{r \rightarrow +\infty} \frac{\log \log M(r, f)}{\log r},$$

where $M(r, f) = \max_{|z|=r} |f(z)|$.

Definition 1.2.2. ([13], [17], [40]) Let f be a meromorphic function. Then, the lower order $\mu(f)$ of f is defined by

$$\mu(f) = \liminf_{r \rightarrow +\infty} \frac{\log T(r, f)}{\log r}.$$

If f is an entire function, then the lower order of f is defined as

$$\mu(f) = \liminf_{r \rightarrow +\infty} \frac{\log T(r, f)}{\log r} = \liminf_{r \rightarrow +\infty} \frac{\log \log M(r, f)}{\log r}.$$

Definition 1.2.3. ([13], [17]) Let f be a meromorphic function of order $(0 < \rho(f) < \infty)$. Then, the type $\tau(f)$ of f is defined by

$$\tau(f) = \limsup_{r \rightarrow +\infty} \frac{T(r, f)}{r^{\rho(f)}}.$$

If f is an entire function, then the type $\tau_M(f)$ of f with order $(0 < \rho(f) < \infty)$ is defined as

$$\tau_M(f) = \limsup_{r \rightarrow +\infty} \frac{\log M(r, f)}{r^{\rho(f)}}.$$

Definition 1.2.4. ([10], [17]) *The lower type $\tau(f)$ of a meromorphic function f with lower order $(0 < \mu(f) < \infty)$ is defined by*

$$\tau(f) = \liminf_{r \rightarrow +\infty} \frac{T(r, f)}{r^{\mu(f)}}.$$

If f is an entire function, then the lower type $\tau_M(f)$ of f with lower order $(0 < \mu(f) < \infty)$ is defined as

$$\tau_M(f) = \liminf_{r \rightarrow +\infty} \frac{\log M(r, f)}{r^{\mu(f)}}.$$

Definition 1.2.5. ([22]) *Let f be a meromorphic function. Then, the exponent of convergence of poles $\lambda\left(\frac{1}{f}\right)$ of f is defined by*

$$\lambda\left(\frac{1}{f}\right) = \limsup_{r \rightarrow +\infty} \frac{\log N(r, f)}{\log r},$$

where $N(r, f)$ is the integrated counting function of poles of f in $\{z : |z| \leq r\}$.

Example 1.2.6. 1. $\lambda(e^z) = \lambda(P) = 0$, where P is a polynomial.

2. $\lambda(e^z - 12) = \lambda(\sin z) = \lambda(\cos z) = 1$.

1.3 The notion of iterated p -order of a meromorphic function

If the order of entire or meromorphic function is infinite, then we can define the iterated p -order. In the following, we recall some fundamental definitions which are used later.

For all $r \in \mathbb{R}$, we define $\exp_1 r := e^r$ and $\exp_{p+1} r := \exp(\exp_p r)$, $p \in \mathbb{N}$. We also define for all $r > 0$ sufficiently large $\log_1 r := \log r$ and $\log_{p+1} r := \log(\log_p r)$, $p \in \mathbb{N}$. Moreover, we denote by $\exp_0 r := r$, $\log_0 r := r$, $\log_{-1} r := \exp_1 r$ and $\exp_{-1} r := \log_1 r$, see [18].

Definition 1.3.1. ([20]) *Let $p \geq 1$ be an integer. Then, the iterated p -order $\rho_p(f)$ of a meromorphic function f is defined by*

$$\rho_p(f) = \limsup_{r \rightarrow +\infty} \frac{\log_p T(r, f)}{\log r}.$$

For $p = 1$, this quantity is called order and hyper-order when $p = 2$. If f is an entire function, then the iterated p -order of f is defined as

$$\rho_p(f) = \limsup_{r \rightarrow +\infty} \frac{\log_p T(r, f)}{\log r} = \limsup_{r \rightarrow +\infty} \frac{\log_{p+1} M(r, f)}{\log r}.$$

Definition 1.3.2. ([18, 33, 34]) Let $p \geq 1$ be an integer. Then, the iterated lower p -order $\mu_p(f)$ of a meromorphic function f is defined by

$$\mu_p(f) = \liminf_{r \rightarrow +\infty} \frac{\log_p T(r, f)}{\log r}.$$

For $p = 1$, this quantity is called lower order and hyper lower order when $p = 2$. If f is an entire function, then the iterated lower p -order of f is defined as

$$\mu_p(f) = \liminf_{r \rightarrow +\infty} \frac{\log_p T(r, f)}{\log r} = \liminf_{r \rightarrow +\infty} \frac{\log_{p+1} M(r, f)}{\log r}.$$

Definition 1.3.3. ([7, 18]) Let f be a meromorphic function of finite and non-zero iterated p -order $\rho_p(f)$. Then, the iterated p -type $\tau_p(f)$ of f is defined by

$$\tau_p(f) = \limsup_{r \rightarrow +\infty} \frac{\log_{p-1} T(r, f)}{r^{\rho_p(f)}} \quad (p \geq 1 \text{ an integer}).$$

If f is an entire function, then the iterated p -type $\tau_{M,p}(f)$ of f with finite and non-zero iterated p -order $\rho_p(f)$ is defined as

$$\tau_{M,p}(f) = \limsup_{r \rightarrow +\infty} \frac{\log_p M(r, f)}{r^{\rho_p(f)}} \quad (p \geq 1 \text{ an integer}).$$

Definition 1.3.4. ([18]) The iterated lower p -type $\underline{\tau}_p(f)$ of a meromorphic function f of finite and non-zero iterated lower p -order $\mu_p(f)$ is defined by

$$\underline{\tau}_p(f) = \liminf_{r \rightarrow +\infty} \frac{\log_{p-1} T(r, f)}{r^{\mu_p(f)}} \quad (p \geq 1 \text{ an integer}).$$

If f is an entire function, then the iterated lower p -type $\underline{\tau}_{M,p}(f)$ of f with finite and non-zero iterated lower p -order $\mu_p(f)$ is defined as

$$\underline{\tau}_{M,p}(f) = \liminf_{r \rightarrow +\infty} \frac{\log_p M(r, f)}{r^{\mu_p(f)}} \quad (p \geq 1 \text{ an integer}).$$

Definition 1.3.5. ([33, 34]) Let f be a meromorphic function. Then, the iterated p -exponent of convergence of poles $\lambda_p\left(\frac{1}{f}\right)$ of f is defined by

$$\lambda_p\left(\frac{1}{f}\right) = \limsup_{r \rightarrow +\infty} \frac{\log_p N(r, f)}{\log r} \quad (p \geq 1 \text{ an integer}),$$

where $N(r, f)$ is the integrated counting function of poles of f in $\{z : |z| \leq r\}$.

1.4 Linear and logarithmic measure

Definition 1.4.1. The linear measure of a set $E \subset [0, +\infty)$ is defined as

$$m(E) = \int_0^{+\infty} \chi_E(t) dt,$$

and the logarithmic measure of a set $F \subset [1, +\infty)$ is defined by

$$lm(F) = \int_0^{+\infty} \frac{\chi_F(t)}{t} dt,$$

where $\chi_H(t)$ is the characteristic function of a set H .

Example 1.4.2. The linear measure of the set $E = [1, \frac{3}{2}] \cup [\frac{5}{2}, 3] \subset [0, +\infty)$ is

$$m(E) = \int_0^{+\infty} \chi_E(t) dt = \int_1^{\frac{3}{2}} dt + \int_{\frac{5}{2}}^3 dt = 1.$$

The linear measure of the set $E = \mathbb{N}$, is equal zero. Moreover, the linear measure of the countable set is equal zero. The logarithmic measure of a set $F = [1, 2] \subset [1, +\infty)$, is

$$lm(F) = \int_0^{+\infty} \frac{\chi_F(t)}{t} dt = \int_1^2 \frac{dt}{t} = \ln 2.$$

Chapter 2

Growth of meromorphic solutions of linear difference equations

2.1 Introduction and statement of main results

In this chapter, we are interested in the study of the growth of meromorphic solutions of homogeneous and non-homogeneous linear difference equations with entire or meromorphic functions. We impose some conditions on the (lower) order and the (lower) type of the coefficients.

Recently, many articles focused on complex difference equations ([2], [9], [22], [23], [24], [26], [41]). The back-ground for these studies lies in the recent difference counterparts of Nevanlinna theory. The key result here is the difference analogue of the lemma on the logarithmic derivative obtained by Halburd-Korhonen [15, 16] and Chiang-Feng [11], independently. Several authors have investigated the properties of meromorphic solutions of complex linear difference equation

$$A_k(z)f(z + c_k) + A_{k-1}(z)f(z + c_{k-1}) + \cdots + A_1(z)f(z + c_1) + A_0(z)f(z) = 0, \quad (2.1)$$

where c_k, \dots, c_1 are non-zero distinct complex numbers, in the case when one coefficient has maximal order or among coefficients having the maximal order, exactly one has its type strictly greater than others and achieved some important results (see e.g. ([11], [23], [41])).

In [11], Chiang and Feng investigated meromorphic solutions of the linear difference equation

$$A_k(z)f(z + k) + A_{k-1}(z)f(z + k - 1) + \cdots + A_1(z)f(z + 1) + A_0(z)f(z) = 0 \quad (2.2)$$

where $A_k(z), \dots, A_0(z)$ are entire functions and proved the following result.

Theorem 2.1.1. ([11]) *Let $A_0(z), \dots, A_k(z)$ be entire functions. If there exists an integer l ($0 \leq l \leq k$) such that*

$$\rho(A_l) > \max_{0 \leq l \leq k, j \neq l} \{\rho(A_j)\}$$

holds, then every meromorphic solution f ($\neq 0$) of equation (2.2) satisfies $\rho(f) \geq \rho(A_l) + 1$.

For the case when there is more than one of coefficients which have the maximal order, Laine and Yang ([23]) obtained the following result.

Theorem 2.1.2. ([23]) *Let $A_0(z), \dots, A_k(z)$ be entire functions of finite order such that among those having the maximal order*

$$\rho = \max_{0 \leq j \leq k} \{\rho(A_j)\},$$

exactly one has its type strictly greater than the others. Then for every meromorphic solution $f (\neq 0)$ of equation (2.1) we have $\rho(f) \geq \rho + 1$.

By using the concepts of lower order and lower type, Zheng and Tu ([41]) investigated the growth of solutions of equation (2.2) and proved the following result.

Theorem 2.1.3. ([41]) *Let $A_0(z), \dots, A_k(z)$ be entire functions such that there exists an integer l ($0 \leq l \leq n$) satisfying*

$$\max \{\rho(A_j) : j = 0, 1, \dots, k, j \neq l\} \leq \mu(A_l) < \infty$$

and

$$\max \{\tau_M(A_j) : \rho(A_j) = \mu(A_l) > 0 : j = 0, 1, \dots, k, j \neq l\} < \tau_M(A_l) < \infty.$$

Then every meromorphic solution $f (\neq 0)$ of equation (2.2) satisfies $\mu(f) \geq \mu(A_l) + 1$.

When the coefficients $A_0(z), \dots, A_k(z)$ are meromorphic, Latreuch and Belaïdi ([24]) investigated the growth of solutions of equation (2.2) and obtained the following result.

Theorem 2.1.4. ([24]) *Let $A_0(z), \dots, A_k(z)$ be meromorphic functions such that for some integer l ($0 \leq l \leq k$), we have*

$$\lambda\left(\frac{1}{A_l}\right) < \rho(A_l) = \rho \quad (0 < \rho < \infty), \quad \tau(A_l) = \tau \quad (0 < \tau < \infty).$$

Suppose that

$$\max \{\rho(A_j) : 0 \leq j \leq k, j \neq l\} \leq \rho$$

and

$$\sum_{\rho(A_j)=\rho} \tau(A_j) < \tau.$$

If $f (\neq 0)$ is a meromorphic solution of (2.2), then $\rho(f) \geq \rho(A_l) + 1$.

Very recently [28], Luo and Zheng have studied the growth of meromorphic solutions of (2.1) when more than one coefficient has maximal lower order and the lower type strictly greater than the type of other coefficients, and obtained the following Theorems 2.1.5 and 2.1.6.

Theorem 2.1.5. ([28]) *Let $A_j(z)$ ($j = 0, \dots, n$) be entire functions, and let $k, l \in \{0, 1, \dots, n\}$. Suppose that the following three assumptions hold simultaneously:*

- (1) $\max\{\mu(A_k), \rho(A_j), j \neq k, l\} = \rho \leq \mu(A_l) < \infty, \mu(A_l) > 0$;
- (2) $\tau_M(A_l) > \tau_M(A_k)$, when $\mu(A_l) = \mu(A_k)$;
- (3) $\max\{\tau_M(A_j) : \rho(A_j) = \mu(A_l), j \neq k, l\} = \tau_1 < \tau_M(A_l)$, when $\mu(A_l) = \max\{\rho(A_j), j \neq k, l\}$.

Then every meromorphic solution $f \not\equiv 0$ of (2.1) satisfies $\rho(f) \geq \mu(A_l) + 1$.

Theorem 2.1.6. ([28]) *Let $A_j(z)$ ($j = 0, \dots, n$) be meromorphic functions, and let $k, l \in \{0, 1, \dots, n\}$. Suppose that the following four assumptions hold simultaneously:*

- (1) $\max\{\mu(A_k), \rho(A_j), j \neq k, l\} = \rho \leq \mu(A_l) < \infty, \mu(A_l) > 0$;
- (2) $\delta(\infty, A_l) = \delta > 0$;
- (3) $\delta\tau(A_l) > \tau(A_k)$, when $\mu(A_l) = \mu(A_k)$;
- (4) $\max\{\tau(A_j) : \rho(A_j) = \mu(A_l), j \neq k, l\} = \tau_1 < \delta\tau(A_l)$, when $\mu(A_l) = \max\{\rho(A_j), j \neq k, l\}$.

Then every meromorphic solution $f \not\equiv 0$ of (2.1) satisfies $\rho(f) \geq \mu(A_l)$.

Very recently, Zhou and Zheng [41], considered the case of the non-homogeneous equation and got the following result.

Theorem 2.1.7. ([41]) *Let $A_j(z)$ ($j = 0, 1, \dots, k$) and $F(z)$ be meromorphic functions. Suppose there exists an integer l ($0 \leq l \leq k$) such that $A_l(z)$ satisfies*

$$\lambda\left(\frac{1}{A_l}\right) < \rho(A_l) < \infty,$$

$$\max\{\rho(A_j) : j = 0, 1, \dots, k, j \neq l\} \leq \rho(A_l),$$

$$\sum_{\rho(A_j)=\rho(A_l), j \neq l} \tau(A_j) < \tau(A_l) < \infty.$$

(i) *If $\rho(F) < \rho(A_l)$, or $\rho(F) = \rho(A_l)$ and $\sum_{\rho(A_j)=\mu(A_l), j \neq l} \tau(A_j) + \tau(F) < \tau(A_l)$, or $\rho(F) = \rho(A_l)$ and $\sum_{\rho(A_j)=\mu(A_l)>0, j \neq l} \tau(A_j) < \tau(F)$, then every meromorphic solution $f (\not\equiv 0)$ of equation*

$$A_k(z)f(z+c_k) + A_{k-1}(z)f(z+c_{k-1}) + \dots + A_1(z)f(z+c_1) + A_0(z)f(z) = F(z), \quad (2.3)$$

satisfies $\rho(f) \geq \rho(A_l)$. Furthermore, if $F(z) \equiv 0$, then $\rho(f) \geq \rho(A_l) + 1$.

(ii) *If $\rho(F) > \rho(A_l)$, then every meromorphic solution f of (2.3) satisfies $\rho(f) \geq \rho(F)$.*

It is natural to ask an interesting question such as:

Question 2.1.8. *What can be said if we replace the conditions on " $\rho(A_l)$ and $\tau(A_l)$ " in Theorems 2.1.4 and 2.1.7 by the conditions on " $\mu(A_l)$ and $\underline{\tau}(A_l)$ "?*

The aim of this chapter is to extend the results of Theorem 2.1.5 and Theorem 2.1.6, through the study of growth of meromorphic solutions of non-homogeneous linear difference equation

$$A_n f(z + c_n) + A_{n-1} f(z + c_{n-1}) + \cdots + A_1 f(z + c_1) + A_0 f(z) = A_{n+1}(z). \quad (2.4)$$

In the following, we give two results.

Theorem 2.1.9. ([3]) *Let $A_j(z)$ ($j = 0, \dots, n + 1$) be entire functions, and let $k, l \in \{0, 1, \dots, n + 1\}$. Suppose that the following three assumptions hold simultaneously:*

- (1) $\max\{\mu(A_k), \rho(A_j), j \neq k, l\} = \rho \leq \mu(A_l) < \infty, \mu(A_l) > 0$;
- (2) $\underline{\tau}_M(A_l) > \underline{\tau}_M(A_k)$, when $\mu(A_l) = \mu(A_k)$;
- (3) $\max\{\tau_M(A_j) : \rho(A_j) = \mu(A_l), j \neq k, l\} = \tau_1 < \underline{\tau}_M(A_l)$, when $\mu(A_l) = \max\{\rho(A_j), j \neq k, l\}$.

Then every meromorphic solution f of (2.4) satisfies $\rho(f) \geq \mu(A_l)$ if $A_{n+1} \not\equiv 0$. Furthermore, if $A_{n+1} \equiv 0$, then every meromorphic solution $f \not\equiv 0$ of (2.1) satisfies $\rho(f) \geq \mu(A_l) + 1$.

Theorem 2.1.10. ([3]) *Let $A_j(z)$ ($j = 0, \dots, n + 1$) be meromorphic functions, and let $k, l \in \{0, 1, \dots, n + 1\}$. Suppose that the following five assumptions hold simultaneously:*

- (1) $\max\{\mu(A_k), \rho(A_j), j \neq k, l\} = \rho \leq \mu(A_l) < \infty, \mu(A_l) > 0$;
- (2) $\underline{\tau}(A_l) > \underline{\tau}(A_k)$, when $\mu(A_l) = \mu(A_k)$;
- (3)

$$\sum_{\rho(A_j)=\mu(A_l)>0, j \neq k, l} \tau(A_j) < \underline{\tau}(A_l) < +\infty$$

when $\mu(A_l) = \max\{\rho(A_j), j \neq k, l\}$;

- (4)

$$\sum_{\rho(A_j)=\mu(A_l)>0, j \neq k, l} \tau(A_j) + \underline{\tau}(A_k) < \underline{\tau}(A_l) < +\infty$$

when $\mu(A_l) = \mu(A_k) = \max\{\rho(A_j), j \neq k, l\}$;

- (5) $\lambda\left(\frac{1}{A_l}\right) < \mu(A_l) < \infty$.

Then every meromorphic solution f of (2.4) satisfies $\rho(f) \geq \mu(A_l)$ if $A_{n+1} \not\equiv 0$. Furthermore, if $A_{n+1} \equiv 0$, then every meromorphic solution $f \not\equiv 0$ of (2.1) satisfies $\rho(f) \geq \mu(A_l) + 1$.

Further, we give an answer for the above question, by proving the following two results.

Theorem 2.1.11. ([4]) Let $A_j(z)$ ($j = 0, 1, \dots, k$) be meromorphic functions. Suppose there exists an integer l ($0 \leq l \leq k$) such that $A_l(z)$ satisfies

$$\begin{aligned} \lambda\left(\frac{1}{A_l}\right) &< \mu(A_l) < \infty, \\ \max\{\rho(A_j) : j = 0, 1, \dots, k, j \neq l\} &\leq \mu(A_l), \\ \sum_{\rho(A_j)=\mu(A_l), j \neq l} \tau(A_j) &< \underline{\tau}(A_l) < \infty. \end{aligned}$$

Then every meromorphic solution $f(z) \not\equiv 0$ of (2.1) satisfies $\mu(f) \geq \mu(A_l) + 1$.

Theorem 2.1.12. ([4]) Let $A_j(z)$ ($j = 0, 1, \dots, k$) and $F(z) \not\equiv 0$ be meromorphic functions. Suppose there exists an integer l ($0 \leq l \leq k$) such that $A_l(z)$ satisfies

$$\begin{aligned} \lambda\left(\frac{1}{A_l}\right) &< \mu(A_l) < \infty, \\ \max\{\rho(A_j) : j = 0, 1, \dots, k, j \neq l\} &\leq \mu(A_l), \\ \sum_{\rho(A_j)=\mu(A_l), j \neq l} \tau(A_j) &< \underline{\tau}(A_l) < \infty. \end{aligned}$$

(i) If $\rho(F) < \mu(A_l)$, or $\rho(F) = \mu(A_l)$ and $\sum_{\rho(A_j)=\mu(A_l), j \neq l} \tau(A_j) + \tau(F) < \underline{\tau}(A_l)$, or $\mu(F) = \mu(A_l)$ and $\sum_{\rho(A_j)=\mu(A_l), j \neq l} \tau(A_j) + \underline{\tau}(A_l) < \underline{\tau}(F)$, then every meromorphic solution f of (2.3) satisfies $\mu(f) \geq \mu(A_l)$.

(ii) If $\mu(F) > \mu(A_l)$, then every meromorphic solution f of (2.3) satisfies $\mu(f) \geq \mu(F)$.

Remark 2.1.13. Theorem 2.1.11 is an improvement of Theorems 2.1.3 and Theorem 2.1.4, also Theorem 2.1.12 is an improvement of Theorem 2.1.7.

2.2 Some Auxiliary Lemmas

The proofs of our results depend mainly on the following lemmas.

Lemma 2.2.1. ([11]) Let f be a meromorphic function of finite order ρ , and let c_1, c_2 ($c_1 \neq c_2$) be two arbitrary complex numbers. Let $\varepsilon > 0$ be given, then there exists a subset $E_1 \subset (1, +\infty)$ with finite logarithmic measure such that for all $|z| = r \notin [0, 1] \cup E_1$, we have

$$\exp\{-r^{\rho-1+\varepsilon}\} \leq \left| \frac{f(z+c_1)}{f(z+c_2)} \right| \leq \exp\{r^{\rho-1+\varepsilon}\}.$$

Lemma 2.2.2. ([13]) *Let f be a meromorphic function, c be a non-zero complex constant. Then we have that for $r \rightarrow +\infty$*

$$(1 + o(1))T(r - |c|, f(z)) \leq T(r, f(z + c)) \leq (1 + o(1))T(r + |c|, f(z)).$$

Consequently for

$$p \in \mathbb{N}_+ = \{1, 2, \dots\}, \quad \rho_p(f(z + c)) = \rho_p(f), \quad \mu_p(f(z + c)) = \mu_p(f).$$

Lemma 2.2.3. ([8]) *Let f be a meromorphic function of finite order ρ . Then for any given $\varepsilon > 0$, there exists a set $E_2 \subset (1, +\infty)$ having finite linear measure and finite logarithmic measure such that for all z satisfying $|z| = r \notin [0, 1] \cup E_2$ and sufficiently large r , we have*

$$\exp\{-r^{\rho+\varepsilon}\} \leq |f(z)| \leq \exp\{r^{\rho+\varepsilon}\}.$$

Lemma 2.2.4. ([10], [18]) *Let f be an entire function with $\mu(f) < \infty$. Then for any given $\varepsilon (> 0)$, there exists a subset $E_3 \subset (1, +\infty)$ with infinite logarithmic measure such that for all $r \in E_3$, we have*

$$\mu(f) = \lim_{\substack{r \rightarrow +\infty \\ r \in E_3}} \frac{\log \log M(r, f)}{\log r}$$

and

$$M(r, f) < \exp\{r^{\mu(f)+\varepsilon}\}.$$

Lemma 2.2.5. ([10], [39]) *Let f be an entire function with $0 < \mu(f) < \infty$. Then for any given $\varepsilon (> 0)$, there exists a subset $E_4 \subset (1, +\infty)$ with infinite logarithmic measure such that for all $r \in E_4$, we have*

$$\tau_M(f) = \lim_{\substack{r \rightarrow +\infty \\ r \in E_4}} \frac{\log M(r, f)}{r^{\mu(f)}}$$

and

$$M(r, f) < \exp\{(\tau_M(f) + \varepsilon)r^{\mu(f)}\}.$$

Lemma 2.2.6. ([15]) *Let f be a non-constant meromorphic function, $c \in \mathbb{C}$, $\delta < 1$ and $\varepsilon > 0$. Then*

$$m\left(r, \frac{f(z + c)}{f(z)}\right) = o\left(\frac{T(r + |c|, f)^{1+\varepsilon}}{r^\delta}\right),$$

for all r outside of a possible exceptional set $E_1 \subset (1, +\infty)$ with finite logarithmic measure $lm(E_1) = \int_{E_1} \frac{dr}{r} < \infty$.

Lemma 2.2.2 and Lemma 2.2.6 lead to the following lemma.

Lemma 2.2.7. ([15]) *Let f be a non-constant meromorphic function, $c, h \in \mathbb{C}$, $c \neq h$, $\delta < 1$, $\varepsilon > 0$. Then*

$$m\left(r, \frac{f(z + c)}{f(z + h)}\right) = o\left(\frac{T(r + |c - h| + |h|, f)^{1+\varepsilon}}{r^\delta}\right),$$

holds for all r outside of a possible exceptional set $E_2 \subset (1, +\infty)$ with finite logarithmic measure $lm(E_2) = \int_{E_2} \frac{dr}{r} < \infty$.

Lemma 2.2.8. ([2]) *Let η_1, η_2 be two complex numbers such that $\eta_1 \neq \eta_2$ and let f be a finite iterated p -order meromorphic function, $p \geq 1$ is an integer. Let $\rho_p(f) = \rho$ be the iterated p -order of f . Then for each $\varepsilon > 0$, we have*

(i) *If $p = 1$, then (see [11])*

$$m\left(r, \frac{f(z + \eta_1)}{f(z + \eta_2)}\right) = O(r^{\rho-1+\varepsilon}).$$

(ii) *If $p \geq 2$, then*

$$m\left(r, \frac{f(z + \eta_1)}{f(z + \eta_2)}\right) = O(\exp_{p-1}\{r^{\rho+\varepsilon}\}).$$

Lemma 2.2.9. ([4]) *Let f be a meromorphic function with finite iterated lower p -order $\mu_p(f)$. Then, for any given $\varepsilon > 0$, there exists a subset $E_4 \subset (1, +\infty)$ of infinite logarithmic measure such that*

$$T(r, f) < \exp_{p-1}\{r^{\mu_p(f)+\varepsilon}\}.$$

Proof. By definition of iterated lower p -order, there exists a sequence $\{r_n\}_{n=1}^{\infty}$ tending to ∞ satisfying $(1 + \frac{1}{n})r_n < r_{n+1}$ and

$$\lim_{r_n \rightarrow +\infty} \frac{\log_p T(r_n, f)}{\log r_n} = \mu_p(f).$$

Then for any given $\varepsilon > 0$, there exists an integer n_2 such that for all $n \geq n_2$,

$$T(r_n, f) < \exp_{p-1}\left\{r_n^{\mu_p(f)+\frac{\varepsilon}{2}}\right\}.$$

Set $E_4 = \bigcup_{n=n_2}^{+\infty} [\frac{n}{n+1}r_n, r_n]$. Then for $r \in E_4 \subset (1, +\infty)$, we obtain

$$\begin{aligned} T(r, f) &\leq T(r_n, f) < \exp_{p-1}\left\{r_n^{\mu_p(f)+\frac{\varepsilon}{2}}\right\} \\ &\leq \exp_{p-1}\left\{\left(\frac{n+1}{n}r\right)^{\mu_p(f)+\frac{\varepsilon}{2}}\right\} < \exp_{p-1}\{r^{\mu_p(f)+\varepsilon}\} \end{aligned}$$

and $lm(E_4) = \sum_{n=n_2}^{+\infty} \int_{\frac{n}{n+1}r_n}^{r_n} \frac{dt}{t} = \sum_{n=n_2}^{+\infty} \log\left(1 + \frac{1}{n}\right) = \infty$. Thus, Lemma 2.2.9 is proved. \square

Lemma 2.2.10. ([4]) *Let f be a meromorphic function of finite and non-zero iterated lower p -order $\mu_p(f)$. Then, there exists a subset $E_3 \subset (1, +\infty)$ of infinite logarithmic measure such that*

$$\mathcal{I}_p(f) = \lim_{\substack{r \rightarrow +\infty \\ r \in E_3}} \frac{\log_{p-1} T(r, f)}{r^{\mu_p(f)}}.$$

Consequently for any given $\varepsilon > 0$, for all $r \in E_3$

$$T(r, f) < \exp_{p-1}\left\{(\mathcal{I}_p(f) + \varepsilon) r^{\mu_p(f)}\right\}.$$

Proof. By the definition of the iterated lower p -type, there exists a sequence $\{r_n\}_{n=1}^{\infty}$ tending to ∞ satisfying $(1 + \frac{1}{n})r_n < r_{n+1}$ and

$$\mathcal{I}_p(f) = \lim_{r_n \rightarrow +\infty} \frac{\log_{p-1} T(r_n, f)}{r_n^{\mu_p(f)}}.$$

Then for any given $\varepsilon > 0$, there exists an n_1 such that for $n \geq n_1$ and any $r \in [\frac{n}{n+1}r_n, r_n]$, we have

$$\frac{\log_{p-1} T(\frac{n}{n+1}r_n, f)}{r_n^{\mu_p(f)}} \leq \frac{\log_{p-1} T(r, f)}{r^{\mu_p(f)}} \leq \frac{\log_{p-1} T(r_n, f)}{(\frac{n}{n+1}r_n)^{\mu_p(f)}}.$$

It follows that

$$\begin{aligned} \left(\frac{n}{n+1}\right)^{\mu_p(f)} \frac{\log_{p-1} T(\frac{n}{n+1}r_n, f)}{(\frac{n}{n+1}r_n)^{\mu_p(f)}} &\leq \frac{\log_{p-1} T(r, f)}{r^{\mu_p(f)}} \\ &\leq \frac{\log_{p-1} T(r_n, f)}{r_n^{\mu_p(f)}} \left(\frac{n+1}{n}\right)^{\mu_p(f)}. \end{aligned}$$

Set

$$E_3 = \bigcup_{n=n_1}^{+\infty} \left[\frac{n}{n+1}r_n, r_n \right].$$

Then, we have

$$\lim_{\substack{r \rightarrow +\infty \\ r \in E_3}} \frac{\log_{p-1} T(r, f)}{r^{\mu_p(f)}} = \lim_{r_n \rightarrow +\infty} \frac{\log_{p-1} T(r_n, f)}{r_n^{\mu_p(f)}} = \mathcal{I}_p(f)$$

and $lm(E_3) = \int_{E_3} \frac{dr}{r} = \sum_{n=n_1}^{+\infty} \int_{\frac{n}{n+1}r_n}^{r_n} \frac{dt}{t} = \sum_{n=n_1}^{+\infty} \log\left(1 + \frac{1}{n}\right) = +\infty$. \square

Lemma 2.2.11. ([11]) *Let γ, R, R' be real numbers such that $0 < \gamma < 1, R > 0$, and let η be a non-zero complex number. Then there is a positive constant C_γ depending only on γ such that for a given meromorphic function f we have, when $|z| = r$, $\max\{1, r + |\eta|\} < R < R'$, the estimate*

$$\begin{aligned} m\left(r, \frac{f(z+\eta)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+\eta)}\right) &\leq \frac{2|\eta|R}{(R-r-|\eta|)^2} \left(m(R, f) + m\left(R, \frac{1}{f}\right) \right) \\ &+ \frac{2R'}{(R'-R)} \left(\frac{|\eta|}{R-r-|\eta|} + \frac{C_\gamma |\eta|^\gamma}{(1-\gamma)r^\gamma} \right) \left(N(R', f) + N\left(R', \frac{1}{f}\right) \right). \end{aligned}$$

Lemma 2.2.12. ([4]) *Let η_1, η_2 be two complex numbers such that $\eta_1 \neq \eta_2$ and let f be a finite iterated lower p -order meromorphic function. Let $\mu_p(f) = \mu$ be the iterated lower p -order of f . Then, for any given $\varepsilon > 0$, there exists a subset $E_5 \subset (1, +\infty)$ of infinite logarithmic measure such that for $r \in E_5$, we have:*

(i) *If $p = 1$, then*

$$m\left(r, \frac{f(z+\eta_1)}{f(z+\eta_2)}\right) = O(r^{\mu-1+\varepsilon}).$$

(ii) *If $p \geq 2$, then*

$$m\left(r, \frac{f(z+\eta_1)}{f(z+\eta_2)}\right) = O(\exp_{p-1}\{r^{\mu+\varepsilon}\}).$$

Proof. We have

$$\begin{aligned}
m\left(r, \frac{f(z+\eta_1)}{f(z+\eta_2)}\right) &\leq m\left(r, \frac{f(z+\eta_1)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+\eta_2)}\right) \\
&\leq m\left(r, \frac{f(z+\eta_1)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+\eta_1)}\right) \\
&\quad + m\left(r, \frac{f(z)}{f(z+\eta_2)}\right) + m\left(r, \frac{f(z+\eta_2)}{f(z)}\right). \tag{2.5}
\end{aligned}$$

Since f has finite iterated lower p -order $\mu_p(f) = \mu < +\infty$, then by Lemma 2.2.9 for any given ε , $0 < \varepsilon < 2$, there exists a subset $E_5 \subset (1, +\infty)$ of infinite logarithmic measure such that for $r \in E_5$

$$T(r, f) \leq \exp_{p-1}\{r^{\mu+\frac{\varepsilon}{2}}\}. \tag{2.6}$$

By using Lemma 2.2.11, we obtain from inequality (2.5)

$$\begin{aligned}
m\left(r, \frac{f(z+\eta_1)}{f(z+\eta_2)}\right) &\leq \frac{2|\eta_1|R}{(R-r-|\eta_1|)^2} \left(m(R, f) + m\left(R, \frac{1}{f}\right)\right) \\
&\quad + \frac{2R'}{(R'-R)} \left(\frac{|\eta_1|}{R-r-|\eta_1|} + \frac{C_\gamma |\eta_1|^\gamma}{(1-\gamma)r^\gamma}\right) \left(N(R', f) + N\left(R', \frac{1}{f}\right)\right) \\
&\quad \quad + \frac{2|\eta_2|R}{(R-r-|\eta_2|)^2} \left(m(R, f) + m\left(R, \frac{1}{f}\right)\right) \\
&\quad + \frac{2R'}{(R'-R)} \left(\frac{|\eta_2|}{R-r-|\eta_2|} + \frac{C_\gamma |\eta_2|^\gamma}{(1-\gamma)r^\gamma}\right) \left(N(R', f) + N\left(R', \frac{1}{f}\right)\right) \\
&= \left(\frac{2|\eta_1|R}{(R-r-|\eta_1|)^2} + \frac{2|\eta_2|R}{(R-r-|\eta_2|)^2}\right) \left(m(R, f) + m\left(R, \frac{1}{f}\right)\right) \\
&\quad \quad + \frac{2R'}{(R'-R)} \left(\frac{|\eta_1|}{R-r-|\eta_1|} + \frac{C_\gamma |\eta_1|^\gamma}{(1-\gamma)r^\gamma}\right. \\
&\quad \quad \quad \left. + \frac{|\eta_2|}{R-r-|\eta_2|} + \frac{C_\gamma |\eta_2|^\gamma}{(1-\gamma)r^\gamma}\right) \left(N(R', f) + N\left(R', \frac{1}{f}\right)\right). \tag{2.7}
\end{aligned}$$

By choosing $\gamma = 1 - \frac{\varepsilon}{2}$, $R = 2r$, $R' = 3r$ and $r > \max\{|\eta_1|, |\eta_2|, 1/2\}$ in (2.7), we get

$$\begin{aligned}
m\left(r, \frac{f(z+\eta_1)}{f(z+\eta_2)}\right) &\leq \left(\frac{4|\eta_1|r}{(r-|\eta_1|)^2} + \frac{4|\eta_2|r}{(r-|\eta_2|)^2}\right) \left(m(2r, f) + m\left(2r, \frac{1}{f}\right)\right) \\
&\quad + 6 \left(\frac{|\eta_1|}{r-|\eta_1|} + \frac{2C_\gamma |\eta_1|^{1-\frac{\varepsilon}{2}}}{\varepsilon r^{1-\frac{\varepsilon}{2}}} + \frac{|\eta_2|}{r-|\eta_2|} + \frac{2C_\gamma |\eta_2|^{1-\frac{\varepsilon}{2}}}{\varepsilon r^{1-\frac{\varepsilon}{2}}}\right) \left(N(3r, f) + N\left(3r, \frac{1}{f}\right)\right) \\
&\leq 4 \left[\frac{4|\eta_1|r}{(r-|\eta_1|)^2} + \frac{4|\eta_2|r}{(r-|\eta_2|)^2}\right. \\
&\quad \quad \left. + 6 \left(\frac{|\eta_1|}{r-|\eta_1|} + \frac{|\eta_2|}{r-|\eta_2|} + \frac{2C_\gamma (|\eta_1|^{1-\frac{\varepsilon}{2}} + |\eta_2|^{1-\frac{\varepsilon}{2}})}{\varepsilon r^{1-\frac{\varepsilon}{2}}}\right)\right] T(3r, f).
\end{aligned}$$

From this, by using the estimate (2.6), we have for $p = 1$

$$\begin{aligned} m \left(r, \frac{f(z + \eta_1)}{f(z + \eta_2)} \right) &\leq 4K \left[\frac{4|\eta_1|r}{(r - |\eta_1|)^2} + \frac{4|\eta_2|r}{(r - |\eta_2|)^2} \right. \\ &\quad \left. + 6 \left(\frac{|\eta_1|}{r - |\eta_1|} + \frac{|\eta_2|}{r - |\eta_2|} + \frac{2C_\gamma \left(|\eta_1|^{1-\frac{\varepsilon}{2}} + |\eta_2|^{1-\frac{\varepsilon}{2}} \right)}{\varepsilon r^{1-\frac{\varepsilon}{2}}} \right) \right] (3r)^{\mu+\frac{\varepsilon}{2}} \\ &\leq Mr^{\mu-1+\varepsilon}, \end{aligned}$$

where $K > 0$, $M > 0$ are some constants. When $p \geq 2$, we obtain

$$\begin{aligned} m \left(r, \frac{f(z + \eta_1)}{f(z + \eta_2)} \right) &\leq 4K \left[\frac{4|\eta_1|r}{(r - |\eta_1|)^2} + \frac{4|\eta_2|r}{(r - |\eta_2|)^2} \right. \\ &\quad \left. + 6 \left(\frac{|\eta_1|}{r - |\eta_1|} + \frac{|\eta_2|}{r - |\eta_2|} + \frac{2C_\gamma \left(|\eta_1|^{1-\frac{\varepsilon}{2}} + |\eta_2|^{1-\frac{\varepsilon}{2}} \right)}{\varepsilon r^{1-\frac{\varepsilon}{2}}} \right) \right] \exp_{p-1} \left\{ (3r)^{\mu+\frac{\varepsilon}{2}} \right\} \\ &\leq M_1 \exp_{p-1} \left\{ r^{\mu+\varepsilon} \right\}, \end{aligned}$$

where $M_1 > 0$ is some constant. This completes the proof of Lemma 2.2.12. \square

Remark 2.2.13. We note that the Lemmas 2.2.2, 2.2.6, 2.2.7, 2.2.8 and 2.2.11 hold without any finite order conditions on meromorphic functions.

2.3 Proof of the Theorems

Proof of Theorem 2.1.9 If f has infinite order, then the result holds. Now we suppose that $\rho(f) < \infty$. We divide (2.4) by $f(z + c_l)$ to get

$$-A_l(z) = \sum_{j=1, j \neq k, l}^n A_j(z) \frac{f(z + c_j)}{f(z + c_l)} + A_k(z) \frac{f(z + c_k)}{f(z + c_l)} + A_0(z) \frac{f(z)}{f(z + c_l)} - \frac{A_{n+1}(z)}{f(z + c_l)}. \quad (2.8)$$

Therefore

$$\begin{aligned} |A_l(z)| &\leq \sum_{j=1, j \neq k, l}^n |A_j(z)| \left| \frac{f(z + c_j)}{f(z + c_l)} \right| + |A_k(z)| \left| \frac{f(z + c_k)}{f(z + c_l)} \right| \\ &\quad + |A_0(z)| \left| \frac{f(z)}{f(z + c_l)} \right| + \left| \frac{A_{n+1}(z)}{f(z + c_l)} \right|. \end{aligned} \quad (2.9)$$

It follows by Lemma 2.2.1 that for any $\varepsilon (> 0)$, there exists a subset $E_1 \subset (1, +\infty)$ with finite logarithmic measure such that for all $|z| = r \notin [0, 1] \cup E_1$, we have

$$\left| \frac{f(z)}{f(z + c_l)} \right| \leq \exp\{r^{\rho-1+\varepsilon}\}, \quad \left| \frac{f(z + c_j)}{f(z + c_l)} \right| \leq \exp\{r^{\rho-1+\varepsilon}\}, \quad j \neq l. \quad (2.10)$$

By Lemma 2.2.2, we have

$$\rho(f(z + c_l)) = \rho \left(\frac{1}{f(z + c_l)} \right) = \rho(f).$$

Then by Lemma 2.2.3, for the above ε , there exists a subset $E_2 \subset (1, +\infty)$, having finite logarithmic measure such that for all z satisfying $|z| = r \notin [0, 1] \cup E_2$ and sufficiently large r , we have

$$\left| \frac{1}{f(z + c_l)} \right| \leq \exp\{r^{\rho(f)+\varepsilon}\}. \quad (2.11)$$

In the following, we divide the proof into four cases:

(i) We suppose that $\rho < \mu(A_l)$

By the definition of $\rho(A_j)$ for the above ε and sufficiently large r , we have

$$|A_j(z)| \leq \exp\{r^{\rho(A_j)+\varepsilon}\} \leq \exp\{r^{\rho+\varepsilon}\}, \quad j \neq k, l. \quad (2.12)$$

By the definition of $\mu(A_l)$, for sufficiently small $\varepsilon > 0$ and sufficiently large r , we have

$$|A_l(z)| \geq \exp\{r^{\mu(A_l)-\varepsilon}\}. \quad (2.13)$$

By the definition of $\mu(A_k)$ and Lemma 2.2.4, for any given $\varepsilon (> 0)$, there exists a subset $E_3 \subset (1, +\infty)$ with infinite logarithmic measure such that for all $r \in E_3$, we have

$$|A_k(z)| \leq \exp\{r^{\mu(A_k)+\varepsilon}\}. \quad (2.14)$$

By substituting (2.10) – (2.14) into (2.9), for all z satisfying $|z| = r \in E_3 \setminus ([0, 1] \cup E_1 \cup E_2)$, we obtain

$$\begin{aligned} \exp\{r^{\mu(A_l)-\varepsilon}\} &\leq (n-1) \exp\{r^{\rho+\varepsilon}\} \exp\{r^{\rho(f)-1+\varepsilon}\} + \exp\{r^{\mu(A_k)+\varepsilon}\} \exp\{r^{\rho(f)-1+\varepsilon}\} \\ &\quad + \exp\{r^{\rho+\varepsilon}\} \exp\{r^{\rho(f)+\varepsilon}\}. \end{aligned} \quad (2.15)$$

Now, we may choose sufficiently small ε satisfying $0 < 3\varepsilon < \mu(A_l) - \rho$, we deduce from (2.15) that for $|z| = r \in E_3 \setminus ([0, 1] \cup E_1 \cup E_2)$, $r \rightarrow +\infty$

$$\exp\{r^{\mu(A_l)-2\varepsilon}\} \leq \exp\{r^{\rho(f)+\varepsilon}\},$$

that is, $\mu(A_l) \leq \rho(f) + 3\varepsilon$, since $\varepsilon > 0$ is arbitrary, then $\rho(f) \geq \mu(A_l)$.

(ii) $\max\{\rho(A_j), j \neq k, l\} = \alpha < \mu(A_k) = \mu(A_l)$, $\underline{\tau}_M(A_l) > \underline{\tau}_M(A_k)$.

By the definition of $\rho(A_j)$ for any given $\varepsilon (> 0)$ and sufficiently large r , we have

$$|A_j(z)| \leq \exp\{r^{\rho(A_j)+\varepsilon}\} \leq \exp\{r^{\alpha+\varepsilon}\}, \quad j \neq k, l. \quad (2.16)$$

By the definition of $\underline{\tau}_M(A_l)$, for sufficiently small $\varepsilon > 0$ and sufficiently large r , we have

$$|A_l(z)| \geq \exp\{(\underline{\tau}_M(A_l) - \varepsilon) r^{\mu(A_l)}\}. \quad (2.17)$$

By the definition of $\underline{\tau}_M(A_k)$ and Lemma 2.2.5, for any given $\varepsilon (> 0)$, there exists a subset $E_4 \subset (1, +\infty)$ with infinite logarithmic measure such that for all $r \in E_4$, we have

$$|A_k(z)| \leq \exp\{(\underline{\tau}_M(A_k) + \varepsilon) r^{\mu(A_k)}\} = \exp\{(\underline{\tau}_M(A_k) + \varepsilon) r^{\mu(A_l)}\}. \quad (2.18)$$

By substituting (2.10), (2.11) and (2.16) – (2.18) into (2.9), for $r \in |z| = r \in E_4 \setminus ([0, 1] \cup E_1 \cup E_2)$, we have

$$\exp\{(\underline{\tau}_M(A_l) - \varepsilon) r^{\mu(A_l)}\} \leq (n-1) \exp\{r^{\alpha+\varepsilon}\} \exp\{r^{\rho(f)-1+\varepsilon}\}$$

$$+ \exp\{(\underline{\tau}_M(A_k) + \varepsilon) r^{\mu(A_l)}\} \exp\{r^{\rho(f)-1+\varepsilon}\} + \exp\{r^{\alpha+\varepsilon}\} \exp\{r^{\rho(f)+\varepsilon}\}. \quad (2.19)$$

Now, we may choose sufficiently small ε , $0 < 2\varepsilon < \min\{\mu(A_l) - \alpha, \underline{\tau}_M(A_l) - \underline{\tau}_M(A_k)\}$, then from (2.19) for $r \in E_4 \setminus ([0, 1] \cup E_1 \cup E_2)$ sufficiently large r , we obtain

$$\exp\{(\underline{\tau}_M(A_l) - \underline{\tau}_M(A_k) - 2\varepsilon) r^{\mu(A_l)-\varepsilon}\} \leq \exp\{r^{\rho(f)+\varepsilon}\},$$

that is, $\mu(A_l) \leq \rho(f) + 2\varepsilon$, since $\varepsilon > 0$ is arbitrary, then $\rho(f) \geq \mu(A_l)$.

(iii) We suppose $\mu(A_k) < \max\{\rho(A_j), j \neq k, l\} = \mu(A_l)$ and $\max\{\tau_M(A_j) : \rho(A_j) = \mu(A_l), j \neq k, l\} = \tau_1 < \underline{\tau}_M(A_l)$. By the definitions of $\rho(A_j)$ and $\tau(A_j)$, for any given $\varepsilon > 0$ and sufficiently large r , we have

$$|A_j(z)| \leq \begin{cases} \exp\{r^{\rho(A_j)+\varepsilon}\} \leq \exp\{r^{\mu(A_l)-\varepsilon}\}, & \text{if } \rho(A_j) < \mu(A_l), j \neq k, l, \\ \exp\{(\tau_1 + \varepsilon) r^{\mu(A_l)}\}, & \text{if } \rho(A_j) = \mu(A_l), j \neq k, l. \end{cases} \quad (2.20)$$

Then, by substituting (2.10), (2.11), (2.14), (2.17) and (2.20) into (2.9), for all z satisfying $|z| = r \in E_3 \setminus ([0, 1] \cup E_1 \cup E_2)$ sufficiently large r , we have

$$\begin{aligned} \exp\{(\underline{\tau}_M(A_l) - \varepsilon) r^{\mu(A_l)}\} &\leq O\left(\exp\{(\tau_1 + \varepsilon) r^{\mu(A_l)}\} \exp\{r^{\rho(f)-1+\varepsilon}\}\right) \\ &\quad + O\left(\exp\{r^{\mu(A_l)-\varepsilon}\} \exp\{r^{\rho(f)-1+\varepsilon}\}\right) \\ &\quad + \exp\{r^{\mu(A_k)+\varepsilon}\} \exp\{r^{\rho(f)-1+\varepsilon}\} + \exp\{(\tau_1 + \varepsilon) r^{\mu(A_l)}\} \exp\{r^{\rho(f)+\varepsilon}\}. \end{aligned} \quad (2.21)$$

Now, we may choose sufficiently small ε satisfying

$$0 < 2\varepsilon < \min\{\mu(A_l) - \mu(A_k), \underline{\tau}_M(A_l) - \tau_1\},$$

then from (2.21) for sufficiently large $r \in E_3 \setminus ([0, 1] \cup E_1 \cup E_2)$, we get

$$\exp\{(\underline{\tau}_M(A_l) - \tau_1 - 2\varepsilon) r^{\mu(A_l)-\varepsilon}\} \leq \exp\{r^{\rho(f)+\varepsilon}\},$$

that is, $\mu(A_l) \leq \rho(f) + 2\varepsilon$, since $\varepsilon > 0$ is arbitrary, then $\rho(f) \geq \mu(A_l)$.

(iv) We suppose that $\max\{\rho(A_j), j \neq k, l\} = \mu(A_k) = \mu(A_l)$ and

$$\max\{\underline{\tau}_M(A_k), \tau_M(A_j) : \rho(A_j) = \mu(A_l), j \neq k, l\} = \tau_2 < \underline{\tau}_M(A_l).$$

Then, by substituting (2.10), (2.11), (2.17), (2.18) and (2.20) into (2.9), for all z satisfying $|z| = r \in E_4 \setminus ([0, 1] \cup E_1 \cup E_2)$ sufficiently large, we obtain

$$\begin{aligned} \exp\{(\underline{\tau}_M(A_l) - \varepsilon) r^{\mu(A_l)}\} &\leq O\left(\exp\{(\tau_2 + \varepsilon) r^{\mu(A_l)}\} \exp\{r^{\rho(f)-1+\varepsilon}\}\right) \\ &\quad + O\left(\exp\{r^{\mu(A_l)-\varepsilon}\} \exp\{r^{\rho(f)-1+\varepsilon}\}\right) \\ &\quad + \exp\{(\underline{\tau}_M(A_k) + \varepsilon) r^{\mu(A_l)}\} \exp\{r^{\rho(f)-1+\varepsilon}\} + \exp\{(\tau_2 + \varepsilon) r^{\mu(A_l)}\} \exp\{r^{\rho(f)+\varepsilon}\}. \end{aligned} \quad (2.22)$$

Now, we may choose sufficiently small ε satisfying

$$0 < 2\varepsilon < \underline{\tau}_M(A_l) - \tau_2,$$

then from (2.22) for sufficiently large $|z| = r \in E_4 \setminus ([0, 1] \cup E_1 \cup E_2)$, we get

$$\exp\{(\underline{\tau}_M(A_l) - \tau_2 - 2\varepsilon) r^{\mu(A_l)-\varepsilon}\} \leq \exp\{r^{\rho(f)+\varepsilon}\},$$

that is, $\mu(A_l) \leq \rho(f) + 2\varepsilon$, since $\varepsilon > 0$ is arbitrary, then $\rho(f) \geq \mu(A_l)$.

Further, if $A_{n+1} \equiv 0$, then by Theorem 2.1.1, every meromorphic solution $f \not\equiv 0$ of (1.2) satisfies $\rho(f) \geq \mu(A_l) + 1$.

Proof of Theorem 2.1.10 If f has infinite order, then the result holds. Now, we suppose that $\rho(f) < \infty$. It follows by (2.8) and Lemma 2.2.2 that

$$\begin{aligned}
T(r, A_l(z)) &= m(r, A_l(z)) + N(r, A_l(z)) \leq \sum_{j=0, j \neq k, l}^{n+1} m(r, A_j(z)) + m(r, A_k(z)) \\
&+ \sum_{j=1, j \neq l}^n m\left(r, \frac{f(z+c_j)}{f(z+c_l)}\right) + m\left(r, \frac{f(z)}{f(z+c_l)}\right) + m\left(r, \frac{1}{f(z+c_l)}\right) \\
&+ N(r, A_l(z)) + O(1) \leq \sum_{j=0, j \neq k, l}^{n+1} T(r, A_j(z)) + T(r, A_k(z)) + N(r, A_l(z)) \\
&+ T\left(r, \frac{1}{f(z+c_l)}\right) + \sum_{j=1, j \neq l}^n m\left(r, \frac{f(z+c_j)}{f(z+c_l)}\right) + m\left(r, \frac{f(z)}{f(z+c_l)}\right) + O(1) \\
&\leq \sum_{j=0, j \neq k, l}^{n+1} T(r, A_j(z)) + T(r, A_k(z)) + N(r, A_l(z)) \\
&+ 2T(r + |c_l|, f(z)) + \sum_{j=1, j \neq l}^n m\left(r, \frac{f(z+c_j)}{f(z+c_l)}\right) + m\left(r, \frac{f(z)}{f(z+c_l)}\right) + O(1) \\
&\leq \sum_{j=0, j \neq k, l}^{n+1} T(r, A_j(z)) + T(r, A_k(z)) + N(r, A_l(z)) \\
&+ 2T(2r, f) + \sum_{j=1, j \neq l}^n m\left(r, \frac{f(z+c_j)}{f(z+c_l)}\right) + m\left(r, \frac{f(z)}{f(z+c_l)}\right) + O(1). \tag{2.23}
\end{aligned}$$

By Lemma 2.2.8, for any given $\varepsilon (> 0)$, we have

$$m\left(r, \frac{f(z)}{f(z+c_l)}\right) = O(r^{\rho(f)-1+\varepsilon}), \quad m\left(r, \frac{f(z+c_j)}{f(z+c_l)}\right) = O(r^{\rho(f)-1+\varepsilon}), \quad j \neq l. \tag{2.24}$$

By the definition of $\lambda\left(\frac{1}{A_l}\right)$, for the above ε and sufficiently large r , we have

$$N(r, A_l) \leq r^{\lambda\left(\frac{1}{A_l}\right)+\varepsilon}. \tag{2.25}$$

(i) We suppose that $\rho < \mu(A_l)$.

By the definition of $\rho(A_j)$ and $\rho(f)$ for the above ε and sufficiently large r , we have

$$T(r, A_j) \leq r^{\rho(A_j)+\varepsilon} \leq r^{\rho+\varepsilon}, \quad j \neq k, l, \tag{2.26}$$

$$T(r, f) \leq r^{\rho(f)+\varepsilon}. \tag{2.27}$$

By the definition of $\mu(A_l)$, for sufficiently small $\varepsilon > 0$ and sufficiently large r , we have

$$T(r, A_l) \geq r^{\mu(A_l)-\varepsilon}. \quad (2.28)$$

By the definition of $\mu(A_k)$ and Lemma 2.2.9, for any given $\varepsilon(> 0)$, there exists a subset $E_5 \subset (1, +\infty)$ with infinite logarithmic measure such that for all $r \in E_5$, we have

$$T(r, A_k) \leq r^{\mu(A_k)+\varepsilon}. \quad (2.29)$$

By substituting (2.24) – (2.29) into (2.23) for sufficiently large $r \in E_5$, we obtain

$$r^{\mu(A_l)-\varepsilon} \leq nr^{\rho+\varepsilon} + r^{\mu(A_k)+\varepsilon} + r^{\lambda\left(\frac{1}{A_l}\right)+\varepsilon} + 2(2r)^{\rho(f)+\varepsilon} + O(r^{\rho(f)-1+\varepsilon}). \quad (2.30)$$

Now, we may choose sufficiently small ε satisfying

$$0 < 3\varepsilon < \min \left\{ \mu(A_l) - \rho, \mu(A_l) - \lambda \left(\frac{1}{A_l} \right) \right\},$$

we deduce from (2.30) that for $|z| = r \in E_5$, $r \rightarrow +\infty$

$$r^{\mu(A_l)-2\varepsilon} \leq r^{\rho(f)+\varepsilon},$$

that is, $\mu(A_l) \leq \rho(f) + 3\varepsilon$, since $\varepsilon > 0$ is arbitrary, then $\rho(f) \geq \mu(A_l)$.

Further, if $A_{n+1} \equiv 0$, then by substituting (2.24) – (2.26), (2.28) and (2.29) into (2.23) for sufficiently large $r \in E_5$, we obtain

$$r^{\mu(A_l)-\varepsilon} \leq (n-1)r^{\rho+\varepsilon} + r^{\mu(A_k)+\varepsilon} + r^{\lambda\left(\frac{1}{A_l}\right)+\varepsilon} + O(r^{\rho(f)-1+\varepsilon}). \quad (2.31)$$

Now, we may choose sufficiently small ε satisfying

$$0 < 3\varepsilon < \min \left\{ \mu(A_l) - \rho, \mu(A_l) - \lambda \left(\frac{1}{A_l} \right) \right\},$$

we deduce from (2.31) that for $|z| = r \in E_5$, $r \rightarrow +\infty$

$$r^{\mu(A_l)-2\varepsilon} \leq r^{\rho(f)-1+\varepsilon},$$

that is, $\mu(A_l) \leq \rho(f) - 1 + 3\varepsilon$, since $\varepsilon > 0$ is arbitrary, then $\rho(f) \geq \mu(A_l) + 1$.

(ii) $\max\{\rho(A_j), j \neq k, l\} = \alpha < \mu(A_k) = \mu(A_l)$, $\underline{\tau}(A_l) > \underline{\tau}(A_k)$.

By the definition of $\rho(A_j)$, for any given $\varepsilon(> 0)$ and sufficiently large r , we have

$$T(r, A_j) \leq r^{\rho(A_j)+\varepsilon} \leq r^{\alpha+\varepsilon}, \quad j \neq k, l, \quad (2.32)$$

By the definition of $\underline{\tau}(A_l)$, for sufficiently small $\varepsilon > 0$ and sufficiently large r , we have

$$T(r, A_l) \geq (\underline{\tau}(A_l) - \varepsilon) r^{\mu(A_l)}. \quad (2.33)$$

By the definition of $\underline{\tau}(A_k)$ and Lemma 2.2.10, for any given $\varepsilon(> 0)$, there exists a subset $E_6 \subset (1, +\infty)$ with infinite logarithmic measure such that for all $r \in E_6$, we have

$$T(r, A_k) \leq (\underline{\tau}(A_k) + \varepsilon) r^{\mu(A_k)} = (\underline{\tau}(A_k) + \varepsilon) r^{\mu(A_l)}. \quad (2.34)$$

By substituting (2.24), (2.25), (2.27), (2.32) – (2.34) into (2.23), for sufficiently large $r \in E_6$, we obtain

$$\begin{aligned} (\underline{\tau}(A_l) - \varepsilon) r^{\mu(A_l)} &\leq nr^{\alpha+\varepsilon} + (\underline{\tau}(A_k) + \varepsilon) r^{\mu(A_l)} \\ &+ r^{\lambda\left(\frac{1}{A_l}\right)+\varepsilon} + 2(2r)^{\rho(f)+\varepsilon} + O(r^{\rho(f)-1+\varepsilon}). \end{aligned} \quad (2.35)$$

Now, we may choose sufficiently small ε satisfying

$$0 < 2\varepsilon < \min \left\{ \mu(A_l) - \alpha, \underline{\tau}(A_l) - \underline{\tau}(A_k), \mu(A_l) - \lambda \left(\frac{1}{A_l} \right) \right\},$$

then from (2.35) for $r \in E_6$ sufficiently large, we get

$$(\underline{\tau}(A_l) - \underline{\tau}(A_k) - 2\varepsilon) r^{\mu(A_l)-\varepsilon} \leq r^{\rho(f)+\varepsilon},$$

that is, $\mu(A_l) \leq \rho(f) + 2\varepsilon$, since $\varepsilon > 0$ is arbitrary, then $\rho(f) \geq \mu(A_l)$.

Further, if $A_{n+1} \equiv 0$, then by substituting (2.24), (2.25), (2.32) – (2.34) into (2.23) for sufficiently large $r \in E_6$, we obtain

$$\begin{aligned} (\underline{\tau}(A_l) - \varepsilon) r^{\mu(A_l)} &\leq (n-1) r^{\alpha+\varepsilon} + (\underline{\tau}(A_k) + \varepsilon) r^{\mu(A_l)} \\ &+ r^{\lambda\left(\frac{1}{A_l}\right)+\varepsilon} + O(r^{\rho(f)-1+\varepsilon}). \end{aligned} \quad (2.36)$$

Now, we may choose sufficiently small ε satisfying

$$0 < 2\varepsilon < \min \left\{ \mu(A_l) - \alpha, \underline{\tau}(A_l) - \underline{\tau}(A_k), \mu(A_l) - \lambda \left(\frac{1}{A_l} \right) \right\},$$

we deduce from (2.36) that for $|z| = r \in E_6$, $r \rightarrow +\infty$

$$(\underline{\tau}(A_l) - \underline{\tau}(A_k) - 2\varepsilon) r^{\mu(A_l)-\varepsilon} \leq r^{\rho(f)-1+\varepsilon},$$

that is, $\mu(A_l) \leq \rho(f) - 1 + 2\varepsilon$, since $\varepsilon > 0$ is arbitrary, then $\rho(f) \geq \mu(A_l) + 1$.

(iii) We suppose $\mu(A_k) < \max\{\rho(A_j), j \neq k, l\} = \mu(A_l)$ and

$$\sum_{\rho(A_j)=\mu(A_l)>0, j \neq k, l} \tau(A_j) < \underline{\tau}(A_l) < +\infty.$$

Then, there exists a set $J \subseteq \{0, 1, \dots, n+1\} \setminus \{k, l\}$ such that for $j \in J$, we have $\rho(A_j) = \mu(A_l)$ with $\sum_{j \in J} \tau(A_j) = \tau_1 < \underline{\tau}(A_l)$ and for $j \in \{0, 1, \dots, n+1\} \setminus (J \cup \{k, l\})$, we have $\rho(A_j) < \mu(A_l)$. Hence, for any given $\varepsilon > 0$ and sufficiently large r , we have

$$T(r, A_j) \leq (\tau(A_j) + \varepsilon) r^{\mu(A_l)}, \quad j \in J \quad (2.37)$$

and

$$T(r, A_j) \leq r^{\mu(A_l)-\varepsilon}, \quad j \in \{0, 1, \dots, n+1\} \setminus (J \cup \{k, l\}). \quad (2.38)$$

Then, by substituting (2.24), (2.25), (2.27), (2.29), (2.33), (2.37) and (2.38) into (2.23), for all z satisfying $|z| = r \in E_5$ sufficiently large r , we obtain

$$(\underline{\tau}(A_l) - \varepsilon) r^{\mu(A_l)} \leq \sum_{j \in J} (\tau(A_j) + \varepsilon) r^{\mu(A_l)} + \sum_{j \in \{0, 1, \dots, n+1\} \setminus (J \cup \{k, l\})} r^{\mu(A_l)-\varepsilon} + r^{\mu(A_k)+\varepsilon}$$

$$\begin{aligned}
+r^\lambda \left(\frac{1}{A_l}\right)^{+\varepsilon} + 2(2r)^{\rho(f)+\varepsilon} + O(r^{\rho(f)-1+\varepsilon}) &\leq (\tau_1 + n\varepsilon) r^{\mu(A_l)} + O(r^{\mu(A_l)-\varepsilon}) + r^{\mu(A_k)+\varepsilon} \\
+r^\lambda \left(\frac{1}{A_l}\right)^{+\varepsilon} + 2(2r)^{\rho(f)+\varepsilon} + O(r^{\rho(f)-1+\varepsilon}). &
\end{aligned} \tag{2.39}$$

Now, we may choose sufficiently small ε satisfying

$$0 < \varepsilon < \min \left\{ \frac{\mu(A_l) - \mu(A_k)}{2}, \frac{\underline{\tau}(A_l) - \tau_1}{n+1}, \frac{\mu(A_l) - \lambda \left(\frac{1}{A_l}\right)}{2} \right\},$$

then from (2.39) for sufficiently large $|z| = r \in E_5$, we get

$$(\underline{\tau}(A_l) - \tau_1 - (n+1)\varepsilon) r^{\mu(A_l)-\varepsilon} \leq r^{\rho(f)+\varepsilon},$$

that is, $\mu(A_l) \leq \rho(f) + 2\varepsilon$, since $\varepsilon > 0$ is arbitrary, then $\rho(f) \geq \mu(A_l)$.

Further, if $A_{n+1} \equiv 0$, then by substituting (2.24), (2.25), (2.29), (2.33), (2.37) and (2.38) into (2.23) for sufficiently large $r \in E_5$, we obtain

$$\begin{aligned}
(\underline{\tau}(A_l) - \varepsilon) r^{\mu(A_l)} &\leq (\tau_1 + (n-1)\varepsilon) r^{\mu(A_l)} + O(r^{\mu(A_l)-\varepsilon}) + r^{\mu(A_k)+\varepsilon} \\
+r^\lambda \left(\frac{1}{A_l}\right)^{+\varepsilon} + O(r^{\rho(f)-1+\varepsilon}). &
\end{aligned} \tag{2.40}$$

Now, we may choose sufficiently small ε satisfying

$$0 < \varepsilon < \min \left\{ \frac{\mu(A_l) - \mu(A_k)}{2}, \frac{\underline{\tau}(A_l) - \tau_1}{n+1}, \frac{\mu(A_l) - \lambda \left(\frac{1}{A_l}\right)}{2} \right\},$$

we deduce from (2.40) that for $|z| = r \in E_5$, $r \rightarrow +\infty$

$$(\underline{\tau}(A_l) - \tau_1 - n\varepsilon) r^{\mu(A_l)-\varepsilon} \leq r^{\rho(f)-1+\varepsilon},$$

that is, $\mu(A_l) \leq \rho(f) - 1 + 2\varepsilon$, since $\varepsilon > 0$ is arbitrary, then $\rho(f) \geq \mu(A_l) + 1$.

(iv) We suppose that $\max\{\rho(A_j), j \neq k, l\} = \mu(A_k) = \mu(A_l)$ and

$$\sum_{\rho(A_j)=\mu(A_l)>0, j \neq k, l} \tau(A_j) + \underline{\tau}(A_k) < \underline{\tau}(A_l) < +\infty.$$

Then, by substituting (2.24), (2.25), (2.27), (2.33), (2.34), (2.37) and (2.38) into (2.26), for all z satisfying $|z| = r \in E_6$ sufficiently large, we have

$$\begin{aligned}
(\underline{\tau}(A_l) - \varepsilon) r^{\mu(A_l)} &\leq \sum_{j \in J} (\tau(A_j) + \varepsilon) r^{\mu(A_l)} + \sum_{j \in \{0,1,\dots,n+1\} \setminus (J \cup \{k,l\})} r^{\mu(A_l)-\varepsilon} \\
&+ (\underline{\tau}(A_k) + \varepsilon) r^{\mu(A_l)} + r^\lambda \left(\frac{1}{A_l}\right)^{+\varepsilon} + 2(2r)^{\rho(f)+\varepsilon} + O(r^{\rho(f)-1+\varepsilon}) \\
&\leq (\tau_1 + \underline{\tau}(A_k) + (n+1)\varepsilon) r^{\mu(A_l)} + O(r^{\mu(A_l)-\varepsilon}) \\
&+ r^\lambda \left(\frac{1}{A_l}\right)^{+\varepsilon} + 2(2r)^{\rho(f)+\varepsilon} + O(r^{\rho(f)-1+\varepsilon}).
\end{aligned} \tag{2.41}$$

Now, we may choose sufficiently small ε satisfying

$$0 < \varepsilon < \min \left\{ \frac{\underline{\tau}(A_l) - \tau_1 - \underline{\tau}(A_k)}{n+2}, \frac{\mu(A_l) - \lambda\left(\frac{1}{A_l}\right)}{2} \right\},$$

then from (2.41) for sufficiently large $|z| = r \in E_6$, we get

$$(\underline{\tau}(A_l) - \tau_1 - \underline{\tau}(A_k) - (n+2)\varepsilon) r^{\mu(A_l)-\varepsilon} \leq r^{\rho(f)+\varepsilon},$$

that is, $\mu(A_l) \leq \rho(f) + 2\varepsilon$, since $\varepsilon > 0$ is arbitrary, then $\rho(f) \geq \mu(A_l)$.

Further, if $A_{n+1} \equiv 0$, then by substituting (2.24), (2.25), (2.33), (2.34), (2.37) and (2.38) into (2.23), for sufficiently large $r \in E_6$, we obtain

$$\begin{aligned} (\underline{\tau}(A_l) - \varepsilon) r^{\mu(A_l)} &\leq (\tau_1 + \underline{\tau}(A_k) + n\varepsilon) r^{\mu(A_l)} + O(r^{\mu(A_l)-\varepsilon}) \\ &\quad + r^{\lambda\left(\frac{1}{A_l}\right)+\varepsilon} + O(r^{\rho(f)-1+\varepsilon}). \end{aligned} \quad (2.42)$$

Now, we may choose sufficiently small ε satisfying

$$0 < \varepsilon < \min \left\{ \frac{\underline{\tau}(A_l) - \tau_1 - \underline{\tau}(A_k)}{n+1}, \frac{\mu(A_l) - \lambda\left(\frac{1}{A_l}\right)}{2} \right\},$$

we deduce from (2.42) that for $|z| = r \in E_6$, $r \rightarrow +\infty$

$$(\underline{\tau}(A_l) - \tau_1 - \underline{\tau}(A_k) - (n+1)\varepsilon) r^{\mu(A_l)-\varepsilon} \leq r^{\rho(f)-1+\varepsilon},$$

that is, $\mu(A_l) \leq \rho(f) - 1 + 2\varepsilon$, since $\varepsilon > 0$ is arbitrary, then $\rho(f) \geq \mu(A_l) + 1$.

Proof of Theorem 2.1.11. Let $f \not\equiv 0$ be a meromorphic solution of (2.1). If f has infinite lower order, then the result holds. Now, we suppose that $\mu(f) < \infty$. We divide equation (2.1) by $f(z + c_l)$ to get

$$-A_l(z) = \sum_{j=1, j \neq l}^k A_j(z) \frac{f(z + c_j)}{f(z + c_l)} + A_0(z) \frac{f(z)}{f(z + c_l)}. \quad (2.43)$$

From Lemma 2.2.12, it follows that for any given $\varepsilon > 0$, there exists a subset $E_5 \subset (1, +\infty)$ of infinite logarithmic measure such that for $r \in E_5$, we have

$$\begin{aligned} T(r, A_l(z)) &= m(r, A_l(z)) + N(r, A_l(z)) \leq \sum_{j=0, j \neq l}^k m(r, A_j(z)) \\ &\quad + \sum_{j=1, j \neq l}^k m\left(r, \frac{f(z + c_j)}{f(z + c_l)}\right) + m\left(r, \frac{f(z)}{f(z + c_l)}\right) + N(r, A_l(z)) + O(1) \\ &\leq \sum_{j=0, j \neq l}^k T(r, A_j(z)) + \sum_{j=1, j \neq l}^k O(r^{\mu(f)-1+\varepsilon}) + O(r^{\mu(f)-1+\varepsilon}) + N(r, A_l(z)) + O(1) \end{aligned}$$

$$\leq \sum_{j=0, j \neq l}^k T(r, A_j(z)) + N(r, A_l(z)) + O(r^{\mu(f)-1+\varepsilon}). \quad (2.44)$$

First, we suppose that $b = \max\{\rho(A_j) : j = 0, 1, \dots, k, j \neq l\} < \mu(A_l) = \mu$. Then, for any given ε ($0 < 2\varepsilon < \mu - b$) and sufficiently large r , we have

$$T(r, A_l) \geq r^{\mu(A_l)-\varepsilon} \quad (2.45)$$

and

$$T(r, A_j) \leq r^{b+\varepsilon} \quad (j = 0, 1, \dots, k, j \neq l). \quad (2.46)$$

By the definition of $\lambda\left(\frac{1}{A_l}\right)$, for any given ε ($0 < 2\varepsilon < \mu - \lambda\left(\frac{1}{A_l}\right)$) and sufficiently large r , we have

$$N(r, A_l) \leq r^{\lambda\left(\frac{1}{A_l}\right)+\varepsilon}. \quad (2.47)$$

By substituting the assumptions (2.45), (2.46) and (2.47) into (2.44), for any given ε ($0 < 2\varepsilon < \min\left\{\mu - b, \mu - \lambda\left(\frac{1}{A_l}\right)\right\}$) and sufficiently large $r \in E_5$, we obtain

$$r^{\mu-\varepsilon} \leq kr^{b+\varepsilon} + r^{\lambda\left(\frac{1}{A_l}\right)+\varepsilon} + O(r^{\mu(f)-1+\varepsilon}).$$

So

$$(1 - o(1))r^{\mu-\varepsilon} \leq O(r^{\mu(f)-1+\varepsilon}),$$

that is, $\mu(f) \geq \mu + 1 - 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we get $\mu(f) \geq \mu(A_l) + 1$.

Assume

$$\max\{\rho(A_j) : j = 0, 1, \dots, k, j \neq l\} = \mu(A_l) = \mu$$

and $\tau_1 = \sum_{\rho(A_j)=\mu(A_l), j \neq l} \tau(A_j) < \underline{\tau}(A_l) = \tau$. Then, there exists a set $J \subseteq \{j = 0, 1, \dots, k\} \setminus \{l\}$ such that for $j \in J$, we have $\rho(A_j) = \mu(A_l) = \mu$ with $\tau_1 = \sum_{j \in J} \tau(A_j) < \underline{\tau}(A_l) = \tau$ and for $j \in \{0, 1, \dots, l-1, l+1, \dots, k\} \setminus J$, we have $\rho(A_j) < \mu(A_l) = \mu$. Hence, for any given ε ($0 < \varepsilon < \frac{\tau-\tau_1}{k+1}$) and sufficiently large r , we have

$$T(r, A_j) \leq (\tau(A_j) + \varepsilon)r^{\mu(A_l)}, \quad j \in J \quad (2.48)$$

and

$$T(r, A_j) \leq r^{\mu_0}, \quad j \in \{0, 1, \dots, l-1, l+1, \dots, k\} \setminus J, \quad (2.49)$$

where $0 < \mu_0 < \mu$. By the definition of lower type, for the above ε and sufficiently large r , we have

$$T(r, A_l) \geq (\tau - \varepsilon)r^{\mu(A_l)}. \quad (2.50)$$

Now, we may choose sufficiently small ε satisfying $0 < \varepsilon < \min\left\{\frac{\mu-\lambda\left(\frac{1}{A_l}\right)}{2}, \frac{\tau-\tau_1}{k+1}\right\}$, by substituting the assumptions (2.47), (2.48), (2.49) and (2.50) into (2.44), for sufficiently large $r \in E_5$, we obtain

$$\begin{aligned} (\tau - \varepsilon)r^\mu &\leq \sum_{j \in J} (\tau(A_j) + \varepsilon)r^\mu + \sum_{j \in \{0, 1, \dots, l-1, l+1, \dots, k\} \setminus J} r^{\mu_0} + r^{\lambda\left(\frac{1}{A_l}\right)+\varepsilon} \\ &+ O(r^{\mu(f)-1+\varepsilon}) \leq (\tau_1 + k\varepsilon)r^\mu + kr^{\mu_0} + r^{\lambda\left(\frac{1}{A_l}\right)+\varepsilon} + O(r^{\mu(f)-1+\varepsilon}). \end{aligned}$$

It follows that

$$(1 - o(1))(\tau - \tau_1 - (k+1)\varepsilon)r^\mu \leq O(r^{\mu(f)-1+\varepsilon}),$$

that is, $\mu(f) \geq \mu + 1 - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we get $\mu(f) \geq \mu(A_l) + 1$.

Proof of Theorem 2.1.12.

Let f be a meromorphic solution of (2.3). If f has infinite lower order, then the result holds. Now, we suppose that $\mu(f) < \infty$. We divide equation (2.3) by $f(z + c_l)$ to get

$$-A_l(z) = \sum_{j=1, j \neq l}^k A_j(z) \frac{f(z + c_j)}{f(z + c_l)} + A_0(z) \frac{f(z)}{f(z + c_l)} - \frac{F(z)}{f(z + c_l)}. \quad (2.51)$$

From Lemma 2.2.2 and Lemma 2.2.12, for any given $\varepsilon (> 0)$, there exists a subset $E_5 \subset (1, +\infty)$ of infinite logarithmic measure such that for all $r \in E_5$, we have

$$\begin{aligned} T(r, A_l(z)) &= m(r, A_l(z)) + N(r, A_l(z)) \\ &\leq \sum_{j=0, j \neq l}^k m(r, A_j(z)) + \sum_{j=1, j \neq l}^k m\left(r, \frac{f(z + c_j)}{f(z + c_l)}\right) + m\left(r, \frac{f(z)}{f(z + c_l)}\right) \\ &\quad + m\left(r, \frac{1}{f(z + c_l)}\right) + m(r, F(z)) + N(r, A_l(z)) + O(1) \\ &\leq \sum_{j=0, j \neq l}^k T(r, A_j(z)) + \sum_{j=1, j \neq l}^k O(r^{\mu-1+\varepsilon}) \\ &\quad + O(r^{\mu-1+\varepsilon}) + T(r, f(z + c_l)) + T(r, F(z)) + N(r, A_l(z)) + O(1) \\ &\leq \sum_{j=0, j \neq l}^k T(r, A_j(z)) + O(r^{\mu-1+\varepsilon}) + (1 + o(1))T(r + |c_l|, f(z)) \\ &\quad + T(r, F(z)) + N(r, A_l(z)) \leq \sum_{j=0, j \neq l}^k T(r, A_j(z)) + O(r^{\mu-1+\varepsilon}) \\ &\quad + T(r, F(z)) + O(r^{\mu+\varepsilon}) + N(r, A_l(z)) \\ &\leq \sum_{j=0, j \neq l}^k T(r, A_j(z)) + T(r, F(z)) + N(r, A_l(z)) + O(r^{\mu-1+\varepsilon}) + O(r^{\mu+\varepsilon}). \end{aligned} \quad (2.52)$$

(i) If $\rho(F) < \mu(A_l) = \mu$, then for any given ε ($0 < 2\varepsilon < \mu - \rho(F)$) and sufficiently large r , we have

$$T(r, F) \leq r^{\rho(F)+\varepsilon}. \quad (2.53)$$

First, we suppose that $b = \max\{\rho(A_j) : j = 0, 1, \dots, k, j \neq l\} < \mu(A_l) = \mu$. By substituting the assumptions (2.45), (2.46), (2.47) and (2.53) into (4.1), for any given ε ($0 < \varepsilon < \min\left\{\frac{\mu-b}{2}, \frac{\mu-\lambda\left(\frac{1}{A_l}\right)}{2}, \frac{\mu-\rho(F)}{2}\right\}$) and sufficiently large $r \in E_5$, we obtain

$$r^{\mu-\varepsilon} \leq kr^{b+\varepsilon} + r^{\rho(F)+\varepsilon} + r^{\lambda\left(\frac{1}{A_l}\right)+\varepsilon} + O(r^{\mu-1+\varepsilon}) + O(r^{\mu+\varepsilon}).$$

So

$$(1 - o(1)) r^{\mu - \varepsilon} \leq O(r^{\mu + \varepsilon}),$$

that is, $\mu(f) \geq \mu - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we get $\mu(f) \geq \mu(A_l)$.

Assume that

$$\max\{\rho(A_j) : j = 0, 1, \dots, k, j \neq l\} = \mu(A_l) = \mu$$

and $\tau_1 = \sum_{\rho(A_j)=\mu(A_l), j \neq l} \tau(A_j) < \tau(A_l) = \tau$. By substituting the assumptions (2.47), (2.48), (2.49), (2.50) and (2.53) into (4.1), for any given

$$\varepsilon \left(0 < \varepsilon < \min \left\{ \frac{\tau - \tau_1}{k+1}, \frac{\mu - \lambda\left(\frac{1}{A_l}\right)}{2}, \frac{\mu - \rho(F)}{2} \right\} \right)$$

and sufficiently large $r \in E_5$, we obtain

$$\begin{aligned} (\tau - \varepsilon) r^\mu &\leq \sum_{j \in J} (\tau(A_j) + \varepsilon) r^\mu + \sum_{j \in \{0, 1, \dots, l-1, l+1, \dots, k\} \setminus J} r^{\mu_0} \\ &\quad + r^{\rho(F) + \varepsilon} + r^{\lambda\left(\frac{1}{A_l}\right) + \varepsilon} + O(r^{\mu-1+\varepsilon}) + O(r^{\mu+\varepsilon}) \\ &\leq (\tau_1 + k\varepsilon) r^\mu + k r^{\mu_0} + r^{\rho(F) + \varepsilon} + r^{\lambda\left(\frac{1}{A_l}\right) + \varepsilon} + O(r^{\mu-1+\varepsilon}) + O(r^{\mu+\varepsilon}). \end{aligned}$$

So

$$(1 - o(1)) (\tau - \tau_1 - (k+1)\varepsilon) r^\mu \leq O(r^{\mu+\varepsilon}),$$

which implies $\mu(f) \geq \mu(A_l)$.

If $\rho(F) = \mu(A_l) = \mu$ and $\sum_{\rho(A_j)=\mu(A_l), j \neq l} \tau(A_j) + \tau(F) = \tau_1 + \tau(F) < \tau(A_l) = \tau$, then for any given $\varepsilon > 0$ and sufficiently large r , we have

$$T(r, F) \leq (\tau(F) + \varepsilon) r^{\mu(A_l)}. \quad (2.54)$$

Now, we may choose sufficiently small ε satisfying

$$0 < \varepsilon < \min \left\{ \frac{\mu - \lambda\left(\frac{1}{A_l}\right)}{2}, \frac{\tau - \tau_1 - \tau(F)}{k+2} \right\},$$

by substituting the assumptions (2.47), (2.48), (2.49), (2.50) and (2.54) into (4.1), for sufficiently large $r \in E_5$, we obtain

$$\begin{aligned} (\tau - \varepsilon) r^\mu &\leq \sum_{j \in J} (\tau(A_j) + \varepsilon) r^\mu + \sum_{j \in \{0, 1, \dots, l-1, l+1, \dots, k\} \setminus J} r^{\mu_0} \\ &\quad + (\tau(F) + \varepsilon) r^\mu + r^{\lambda\left(\frac{1}{A_l}\right) + \varepsilon} + O(r^{\mu-1+\varepsilon}) + O(r^{\mu+\varepsilon}) \\ &\leq (\tau_1 + k\varepsilon) r^\mu + k r^{\mu_0} + (\tau(F) + \varepsilon) r^\mu + r^{\lambda\left(\frac{1}{A_l}\right) + \varepsilon} + O(r^{\mu-1+\varepsilon}) + O(r^{\mu+\varepsilon}). \end{aligned}$$

It follows that

$$(1 - o(1)) (\tau - \tau_1 - \tau(F) - (k+2)\varepsilon) r^\mu \leq O(r^{\mu+\varepsilon}),$$

which implies $\mu(f) \geq \mu(A_l)$.

If $\mu(F) = \mu(A_l) = \mu$ and $\tau_1 + \tau(A_l) < \tau(F)$, then for any given $\varepsilon > 0$ and sufficiently large r , we have

$$T(r, F) > (\tau(F) - \varepsilon)r^{\mu(A_l)}. \quad (2.55)$$

By Lemma 2.2.10, for the above ε , there exists a subset E_3 with infinite logarithmic measure such that for all $r \in E_3$, we have

$$T(r, A_l) \leq (\tau(A_l) + \varepsilon)r^{\mu(A_l)}. \quad (2.56)$$

By (2.45) and Lemma 2.2.2, we obtain

$$\begin{aligned} T(r, F(z)) &\leq \sum_{j=0}^k T(r, A_j(z)) + \sum_{j=1}^k T(r, f(z + c_j)) + T(r, f(z)) + O(1) \\ &\leq \sum_{j=0}^k T(r, A_j(z)) + (1 + o(1))kT(r + |c_s|, f(z)) + T(r, f(z)) + O(1) \\ &\leq \sum_{j=0, j \neq l}^k T(r, A_j(z)) + T(r, A_l(z)) + (2k + 1)T(2r, f(z)), \quad |c_s| = \max_{1 \leq j \leq k} \{|c_j|\}. \end{aligned} \quad (2.57)$$

Now, we may choose sufficiently small ε satisfying $0 < \varepsilon < \frac{\tau(F) - \tau_1 - \tau}{k+2}$, by substituting (2.48), (2.49), (2.55) and (2.56) into (2.57) that for $r \in E_3$ sufficiently large, we get

$$\begin{aligned} (\tau(F) - \varepsilon)r^\mu &\leq \sum_{j \in J} (\tau(A_j) + \varepsilon)r^\mu + \sum_{j \in \{0, 1, \dots, l-1, l+1, \dots, k\} \setminus J} r^{\mu_0} \\ &\quad + (\tau + \varepsilon)r^\mu + (2k + 1)T(2r, f(z)) \\ &\leq (\tau_1 + k\varepsilon)r^\mu + kr^{\mu_0} + (\tau + \varepsilon)r^\mu + (2k + 1)T(2r, f(z)). \end{aligned}$$

It follows that

$$(1 - o(1))(\tau(F) - \tau_1 - \tau - (k + 2)\varepsilon)r^\mu \leq (2k + 1)T(2r, f(z)),$$

which implies $\mu(f) \geq \mu(A_l)$.

(ii) Next, we consider the case $\mu(F) > \mu(A_l) = \mu$. Let f be a meromorphic solution of (2.45). Then, for any given ε ($0 < 2\varepsilon < \mu(F) - \mu$) and sufficiently large r , we have

$$T(r, F) \geq r^{\mu(F) - \varepsilon}. \quad (2.58)$$

By Lemma 2.2.9, for the above ε , there exists a subset E_4 with infinite logarithmic measure such that for all $r \in E_4$, we have

$$T(r, A_l) \leq r^{\mu(A_l) + \varepsilon}. \quad (2.59)$$

If $b = \max\{\rho(A_j) : j = 0, 1, \dots, k, j \neq l\} < \mu(A_l) = \mu$, then by substituting the assumptions (2.46), (2.58) and (2.59) into (2.57), for any given ε ($0 < 2\varepsilon < \mu(F) - \mu$) and sufficiently large $r \in E_4$, we obtain

$$r^{\mu(F) - \varepsilon} \leq kr^{b + \varepsilon} + r^{\mu + \varepsilon} + (2k + 1)T(2r, f(z)).$$

Thus

$$(1 - o(1)) r^{\mu(F) - \varepsilon} \leq (2k + 1) T(2r, f(z)),$$

that is, $\mu(f) \geq \mu(F) - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we get $\mu(f) \geq \mu(F)$.

If

$$\max\{\rho(A_j) : j = 0, 1, \dots, k, j \neq l\} = \mu(A_l) = \mu$$

and $\tau_1 = \sum_{\rho(A_j) = \mu(A_l), j \neq l} \tau(A_j) < \tau(A_l) = \tau$, then by substituting the assumptions (2.48), (2.49), (2.56) and (2.58) into (2.57), for any given ε ($0 < 2\varepsilon < \mu(F) - \mu$) and sufficiently large $r \in E_3$, we obtain

$$\begin{aligned} r^{\mu(F) - \varepsilon} &\leq \sum_{j \in J} (\tau(A_j) + \varepsilon) r^\mu + \sum_{j \in \{0, 1, \dots, l-1, l+1, \dots, k\} \setminus J} r^{\mu_0} + T(r, A_l(z)) \\ &+ (2k + 1) T(2r, f(z)) \leq (\tau_1 + k\varepsilon) r^\mu + kr^{\mu_0} + (\tau + \varepsilon) r^\mu + (2k + 1) T(2r, f(z)). \end{aligned}$$

Thus

$$(1 - o(1)) r^{\mu(F) - \varepsilon} \leq (2k + 1) T(2r, f(z)),$$

that is, $\mu(f) \geq \mu(F) - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we get $\mu(f) \geq \mu(F)$.

2.4 Examples

Example 2.4.1. Consider the non-homogeneous difference equation with entire coefficients

$$A_3 f(z + \sqrt{\pi}) + A_2 f(z - \sqrt{\pi}) + A_1 f(z + \frac{\sqrt{\pi}}{2}) + A_0 f(z) = A_4. \quad (2.60)$$

Case 1. $\max\{\rho(A_j), \mu(A_k) : j \neq l, k\} < \mu(A_l)$. In (4.1), for

$$\begin{aligned} A_0(z) &= 1, \quad A_1(z) = e^{-4\sqrt{\pi}z + 3\pi}, \\ A_2(z) &= -2e^{8\sqrt{\pi}z - 12\pi}, \quad A_3(z) = e^{-4z^2 + z + 8\pi}, \quad A_4(z) = e^z, \end{aligned}$$

we have

$$\max\{\rho(A_j) \ (j = 0, 1, 4), \mu(A_2)\} = 1 < \mu(A_3) = 2.$$

As we see, the conditions of Theorem 2.1.9 are verified. The function

$$f(z) = e^{4(z^2 - \pi) - 8\sqrt{\pi}z}$$

is a solution of equation (2.60) which satisfies $\rho(f) = 2 \geq \mu(A_3) = 2$.

Case 2. $\max\{\rho(A_j) : j \neq k, l\} < \mu(A_k) = \mu(A_l)$, $\tau_M(A_l) > \tau_M(A_k)$. In (2.60), for

$$\begin{aligned} A_0(z) &= e^z, \quad A_1(z) = e^{z(1 - 4\sqrt{\pi}) + 3\pi}, \quad A_2(z) = -2e^{z(1 + 8\sqrt{\pi}) - 12\pi} + e^{z^2 + 16\sqrt{\pi}z}, \\ A_3(z) &= e^{-4z^2 + z + 8\pi} - e^{z^2 + 16\pi}, \quad A_4(z) = e^z, \end{aligned}$$

we get

$$\max\{\rho(A_j) \ (j = 0, 1, 4)\} = 1 < \mu(A_2) = \mu(A_3) = 2, \quad \tau_M(A_2) = 1 < 4 = \tau_M(A_3).$$

Obviously, the conditions of Theorem 2.1.9 are satisfied. The function

$$f(z) = e^{4(z^2-\pi)-8\sqrt{\pi}z}$$

is a solution of equation (2.60) and satisfies $\rho(f) = 2 \geq \mu(A_3) = 2$.

Case 3. $\mu(A_k) < \max\{\rho(A_j), j \neq k, l\} = \mu(A_l)$. In (2.60), for

$$\begin{aligned} A_0(z) &= 1, & A_1(z) &= e^{-4\sqrt{\pi}z+3\pi}, \\ A_2(z) &= -2e^{8\sqrt{\pi}z-12\pi}, & A_3(z) &= e^{-3z^2+8\pi}, & A_4(z) &= e^{z^2}, \end{aligned}$$

we have

$$\begin{aligned} \mu(A_0) &= 0 < \max\{\rho(A_j) \ (j = 1, 2, 4)\} = 2 = \mu(A_3), \\ \tau_M(A_4) &= 1 < 3 = \tau_M(A_3). \end{aligned}$$

Hence, the conditions of Theorem 2.1.9 are verified. The function

$$f(z) = e^{4(z^2-\pi)-8\sqrt{\pi}z}$$

is a solution of equation (2.60) satisfying $\rho(f) = 2 \geq \mu(A_3) = 2$.

Case 4. $\max\{\rho(A_j) : j \neq k, l\} = \mu(A_k) = \mu(A_l)$ and $\max\{\tau_M(A_k), \tau_M(A_j) : \rho(A_j) = \mu(A_l), j \neq k, l\} < \tau_M(A_l)$. In (2.60), for

$$\begin{aligned} A_0(z) &= e^{z^2}, & A_1(z) &= e^{z^2-4\sqrt{\pi}z+3\pi}, \\ A_2(z) &= -2e^{z^2+8\sqrt{\pi}z-12\pi}, & A_3(z) &= e^{-6z^2}, & A_4(z) &= e^{-2z^2-8\pi}, \end{aligned}$$

we get

$$\begin{aligned} \max\{\rho(A_j) \ (j = 0, 1, 4)\} &= \mu(A_2) = \mu(A_3) = 2, \\ \max\{\tau_M(A_2), \tau_M(A_j), \ (j = 0, 1, 4)\} &= 2 < 6 = \tau_M(A_3). \end{aligned}$$

We see that the conditions of Theorem 2.1.9 are satisfied. The function

$$f(z) = e^{4(z^2-\pi)-8\sqrt{\pi}z}$$

is a solution of equation (2.60) which satisfies $\rho(f) = 2 \geq \mu(A_3) = 2$.

Example 2.4.2. Consider the non-homogeneous difference equation with meromorphic coefficients

$$A_2 f(z + i\pi^2) + A_1 f(z + i\frac{\pi^2}{2}) + A_0 f(z) = A_3(z). \quad (2.61)$$

Case 1. $\max\{\rho(A_j), \mu(A_k) : j \neq l, k\} < \mu(A_l)$. In (2.61), for

$$\begin{aligned} A_0(z) &= e^{24\pi iz^2}, & A_1(z) &= \frac{e^{-\frac{8}{\pi}z^3-12\pi iz^2}}{\cos(\frac{iz}{\pi})}, \\ A_2(z) &= e^{24\pi^3 z+8\pi^5 i}, & A_3(z) &= -e^{-6\pi^3 z-i\pi^5} \sec^2\left(\frac{iz}{\pi}\right), \end{aligned}$$

we have

$$\lambda\left(\frac{1}{A_1}\right) = 1 < \mu(A_1) = 3,$$

$$\max\{\rho(A_j), \mu(A_0) : j = 2, 3\} = 2 < \mu(A_1) = 3.$$

It's clear that the conditions of Theorem 2.1.10 are satisfied. The function

$$f(z) = \frac{e^{\frac{8}{\pi}z^3}}{\sin\left(\frac{iz}{\pi}\right)}$$

is a solution of equation (2.61) satisfying $\rho(f) = 3 \geq \mu(A_3) = 3$.

Case 2. $\max\{\rho(A_j) : j \neq k, l\} = \alpha < \mu(A_k) = \mu(A_l)$, $\tau_M(A_l) > \tau_M(A_k)$. In (2.61), for

$$A_0(z) = -e^{z^3+12\pi iz^2-6\pi^3z} + e^{24\pi iz^2}, \quad A_1(z) = e^{-\frac{8}{\pi}z^3+6\pi^3z} - \left(\cot\left(\frac{iz}{\pi}\right)\right) e^{z^3+i\pi^5},$$

$$A_2(z) = e^{24\pi^3z+8\pi^5i}, \quad A_3(z) = -e^{12\pi iz^2-i\pi^5} \sec\left(\frac{iz}{\pi}\right),$$

we have

$$\lambda\left(\frac{1}{A_1}\right) = 1 < \mu(A_1) = 3, \quad \max\{\rho(A_j) : j = 2, 3\} = 2 < \mu(A_0) = \mu(A_1) = 3,$$

$$\tau(A_1) = \frac{8}{\pi^2} > \frac{1}{\pi} = \tau(A_0).$$

Obviously, the conditions of Theorem 2.1.10 are satisfied. The function

$$f(z) = \frac{e^{\frac{8}{\pi}z^3}}{\sin\left(\frac{iz}{\pi}\right)}$$

is a solution of equation (2.61) and f satisfies $\rho(f) = 3 \geq \mu(A_1) = 3$.

Case 3. We suppose $\mu(A_k) < \max\{\rho(A_j), j \neq k, l\} = \mu(A_l)$ and

$$\sum_{\rho(A_j)=\mu(A_l)>0, j \neq k, l} \tau(A_j) < \tau(A_l) < +\infty.$$

In (2.61), for

$$A_0(z) = e^{24\pi iz^2}, \quad A_1(z) = \frac{e^{-\frac{12}{\pi}z^3-12\pi iz^2+6\pi^3z}}{\cos\left(\frac{iz}{\pi}\right)},$$

$$A_2(z) = e^{24\pi^3z+8\pi^5i}, \quad A_3(z) = -e^{-\frac{4}{\pi}z^3-\pi^5i} \sec^2\left(\frac{iz}{\pi}\right),$$

we get

$$\lambda\left(\frac{1}{A_1}\right) = 1 < \mu(A_1) = 3, \quad \mu(A_2) = 1 < \max\{\rho(A_j) : j = 0, 3\} = \mu(A_1) = 3,$$

$$\sum_{\rho(A_j)=\mu(A_l)>0, j \neq k, l} \tau(A_j) = \tau(A_3) = \frac{4}{\pi^2} < \tau(A_1) = \frac{12}{\pi^2}.$$

Obviously, the conditions of Theorem 2.1.10 are satisfied. The function

$$f(z) = \frac{e^{\frac{8}{\pi}z^3}}{\sin\left(\frac{iz}{\pi}\right)}$$

is a solution of equation (2.61) which satisfies $\rho(f) = 3 \geq \mu(A_1) = 3$.

Case 4. $\max\{\rho(A_j), j \neq k, l\} = \mu(A_k) = \mu(A_l)$ and

$$\sum_{\rho(A_j)=\mu(A_l)>0, j \neq k, l} \tau(A_j) + \underline{\tau}(A_k) < \underline{\tau}(A_l) < +\infty.$$

In (2.61), for

$$A_0(z) = e^{\frac{iz^3}{\pi} + 24\pi iz^2}, \quad A_1(z) = \frac{e^{-\frac{12}{\pi}z^3 - 12\pi iz^2 + 6\pi^3 z}}{\cos\left(\frac{iz}{\pi}\right)},$$

$$A_2(z) = e^{\frac{iz^3}{\pi} + 24\pi^3 z + 8\pi^5 i}, \quad A_3(z) = -e^{-\frac{4}{\pi}z^3 - \pi^5 i} \sec^2\left(\frac{iz}{\pi}\right),$$

we have

$$\lambda\left(\frac{1}{A_1}\right) = 1 < \mu(A_1) = 3, \quad \max\{\rho(A_j) : j = 2, 3\} = \mu(A_0) = \mu(A_1) = 3,$$

$$\begin{aligned} \sum_{\rho(A_j)=\mu(A_l)>0, j \neq k, l} \tau(A_j) + \underline{\tau}(A_k) &= \tau(A_2) + \tau(A_3) + \underline{\tau}(A_0) \\ &= \frac{1}{\pi^2} + \frac{4}{\pi^2} + \frac{1}{\pi^2} = \frac{6}{\pi^2} < \underline{\tau}(A_1) = \frac{12}{\pi^2}. \end{aligned}$$

Obviously, the conditions of Theorem 2.1.10 are satisfied. The function

$$f(z) = \frac{e^{\frac{8}{\pi}z^3}}{\sin\left(\frac{iz}{\pi}\right)}$$

is a solution of equation (2.61) and f satisfies $\rho(f) = 3 \geq \mu(A_1) = 3$.

Example 2.4.3. Consider the homogeneous difference equation with meromorphic coefficients

$$A_2(z) f(z+2) + A_1(z) f(z+1) + A_0(z) f(z) = 0, \quad (2.62)$$

where

$$A_0(z) = -e^{-z^2 \log \pi} \sec^2(z), \quad A_1(z) = e^{(4z^2 + 3z + 1) \log \frac{1}{\pi}},$$

$$A_2(z) = e^{(7z^2 + 12z + 8) \log \frac{1}{\pi}} \tan^2(z).$$

Clearly, $A_j(z)$, $j = 0, 1, 2$ satisfy

$$\lambda\left(\frac{1}{A_2}\right) = 1 < \mu(A_2) = 2,$$

$$\max\{\rho(A_j) : j = 0, 1\} = \mu(A_2) = 2,$$

$$\tau(A_0) + \tau(A_1) = \frac{\log \pi}{\pi} + \frac{4 \log \pi}{\pi} = \frac{5 \log \pi}{\pi} < \frac{7 \log \pi}{\pi} = \underline{\tau}(A_2).$$

As we see, the conditions of Theorem 2.1.11 are verified, where $l = 2$. The function

$$f(z) = e^{z^3 \log \pi}$$

is a solution of equation (2.62) and f satisfies $\mu(f) = 3 \geq \mu(A_2) + 1 = 2 + 1$.

Example 2.4.4. (i) Consider the non-homogeneous difference equation with meromorphic coefficients

$$A_2(z) f\left(z + \frac{\pi}{2}\right) + A_1(z) f\left(z - \frac{\pi}{2}\right) + A_0(z) f(z) = F(z), \quad (2.63)$$

where

$$\begin{aligned} A_0(z) &= e^{z^2 + 6\pi z}, & A_1(z) &= e^{-6z^2 - \frac{3}{2}\pi^2} \tan^2(4z), \\ A_2(z) &= -e^{z^2 - \frac{3}{2}\pi^2}, & F(z) &= \frac{e^{-6\pi z}}{\cos^2(4z)}. \end{aligned}$$

Case 1. $\rho(F) < \mu(A_l)$. We have

$$\lambda\left(\frac{1}{A_1}\right) = 1 < \mu(A_1) = 2, \quad \max\{\rho(A_j) : j = 0, 2\} = \mu(A_1) = 2,$$

$$\sum_{\rho(A_j)=\mu(A_1), j \neq 1} \tau(A_j) = \tau(A_0) + \tau(A_2) = \frac{2}{\pi} < \frac{6}{\pi} = \underline{\tau}(A_1),$$

$$\rho(F) = 1 < \mu(A_1) = 2.$$

As we see, the conditions of Theorem 2.1.12 (i) are verified. The meromorphic function

$$f(z) = \frac{e^{6z^2}}{\sin^2(4z)}$$

is a solution of equation (2.63) and f satisfies

$$\mu(f) = 2 \geq \mu(A_1) = 2.$$

Case 2. $\rho(F) = \mu(A_l)$ and $\sum \tau(A_j) + \tau(F) < \underline{\tau}(A_l)$. In (2.63), for

$$A_0(z) = e^{z^2 + \frac{3}{2}\pi^2}, \quad A_1(z) = e^{-8z^2 + 6\pi z} \tan^2(4z),$$

$$A_2(z) = -e^{z^2 - 6\pi z}, \quad F(z) = \frac{e^{-2z^2 + \frac{3}{2}\pi^2}}{\cos^2(4z)},$$

we have

$$\lambda\left(\frac{1}{A_1}\right) = 1 < \mu(A_1) = 2,$$

$$\max\{\rho(A_j) : j = 0, 2\} = \mu(A_1) = \rho(F) = 2,$$

$$\sum_{\rho(A_j)=\mu(A_1), j \neq 1} \tau(A_j) + \tau(F) = \tau(A_0) + \tau(A_2) + \tau(F) = \frac{1}{\pi} + \frac{1}{\pi} + \frac{2}{\pi} = \frac{4}{\pi} < \frac{8}{\pi} = \underline{\tau}(A_1).$$

Hence, the conditions of Theorem 2.1.12 (i) are verified. We see that the meromorphic function $f(z) = \frac{e^{6z^2}}{\sin^2(4z)}$ is a solution of (2.63) that satisfies $\mu(f) = 2 \geq \mu(A_1) = 2$.

Case 3. $\mu(F) = \mu(A_l)$ and $\sum \tau(A_j) + \underline{\tau}(A_l) < \underline{\tau}(F)$. In (2.63), for

$$A_0(z) = e^{z^2 + \frac{3}{2}\pi^2}, \quad A_1(z) = e^{3z^2 + 6\pi z} \tan^2(4z),$$

$$A_2(z) = -e^{z^2 - 6\pi z}, \quad F(z) = \frac{e^{9z^2 + \frac{3}{2}\pi^2}}{\cos^2(4z)},$$

we have

$$\lambda\left(\frac{1}{A_1}\right) = 1 < \mu(A_1) = 2,$$

$$\max\{\rho(A_j) : j = 0, 2\} = \mu(A_1) = \mu(F) = 2,$$

$$\sum_{\rho(A_j)=\mu(A_1), j \neq 1} \tau(A_j) + \underline{\tau}(A_1) = \tau(A_0) + \tau(A_2) + \underline{\tau}(A_1) = \frac{1}{\pi} + \frac{1}{\pi} + \frac{3}{\pi} = \frac{5}{\pi} < \frac{9}{\pi} = \underline{\tau}(F).$$

Thus, the conditions of Theorem 2.1.12 (i) are satisfied. We see that the meromorphic function $f(z) = \frac{e^{6z^2}}{\sin^2(4z)}$ is a solution of (2.63) that satisfies $\mu(f) = 2 \geq \mu(A_1) = 2$. (ii) $\mu(F) > \mu(A_l)$. In (2.63), for

$$A_0(z) = e^{3\pi z}, \quad A_1(z) = e^{8\pi z - \frac{3}{2}\pi^2},$$

$$A_2(z) = -e^{-3\pi z - \frac{3}{2}\pi^2}, \quad F(z) = \frac{e^{6z^2 + 2\pi z}}{\sin^2(4z)},$$

we have

$$\lambda\left(\frac{1}{A_1}\right) = 0 < \mu(A_1) = 1,$$

$$\max\{\rho(A_j) : j = 0, 2\} = \mu(A_1) = 1,$$

$$\sum_{\rho(A_j)=\mu(A_1), j \neq 1} \tau(A_j) = \tau(A_0) + \tau(A_2) = 6 < 8 = \underline{\tau}(A_1),$$

$$\mu(F) = 2 > \mu(A_1) = 1.$$

Obviously, the conditions of Theorem 2.1.12 (ii) are verified. The meromorphic function

$$f(z) = \frac{e^{6z^2}}{\sin^2(4z)}$$

is solution of equation (2.63) and f satisfies

$$\mu(f) = 2 \geq \mu(F) = 2.$$

Chapter 3

Growth of meromorphic solutions of linear differential-difference equations

3.1 Introduction and statement of main results

In this chapter, we devoted to the study of the growth of meromorphic solutions of the homogeneous and the non-homogeneous linear differential-difference equations

$$\sum_{i=0}^n \sum_{j=0}^m A_{ij} f^{(j)}(z + c_i) = 0, \quad (3.1)$$

$$\sum_{i=0}^n \sum_{j=0}^m A_{ij} f^{(j)}(z + c_i) = F, \quad (3.2)$$

where A_{ij} ($i = 0, \dots, n; j = 0, \dots, m$), F are meromorphic functions and c_i ($0, \dots, n$) are non-zero distinct complex numbers. Under the sufficient condition that there exists one coefficient having the maximal lower order or having the maximal lower type strictly greater than the order or the type of other coefficients, we obtain estimates of the lower bound of the order of meromorphic solutions of the above equation.

Historically, the study of complex differential-difference equations can be traced back to Naftalevich's research. Using operator theory and iteration method, Naftalevich [29] ever considered the meromorphic solutions on complex differential-difference equations. However, there are few investigations on complex differential-difference field using Nevanlinna theory.

Very recently, Zhou and Zheng ([42]) have investigated the growth of solutions of equations (3.1) and (3.2) and have obtained the following results.

Theorem 3.1.1. ([42]) *Let $A_{ij}(z)$ ($i = 0, \dots, n; j = 0, \dots, m$) and $F(z)$ be meromorphic functions. Suppose there exists an integer l ($0 \leq l \leq n$) such that $A_{l0}(z)$ satisfies*

$$\lambda \left(\frac{1}{A_{l0}} \right) < \rho(A_{l0}) < \infty$$

and

$$\max \{ \rho(A_{ij}) : i = 0, 1, \dots, n; j = 0, 1, \dots, m, (i, j) \neq (l, 0) \} \leq \rho(A_{l0}) < \infty,$$

$$\tau_1 = \sum_{\rho(A_{ij})=\rho(A_{l_0}), (i,j)\neq(l,0)} \tau(A_{ij}) < \tau(A_{l_0}) = \tau < \infty.$$

- (i) If $\rho(F) < \rho(A_{l_0})$, or $\rho(F) = \rho(A_{l_0})$ and $\sum_{\rho(A_{ij})=\rho(A_{l_0}), (i,j)\neq(l,0)} \tau(A_{ij}) + \tau(F) < \tau(A_{l_0}) = \tau$, or $\rho(F) = \rho(A_{l_0})$ and $\sum_{\rho(A_{ij})=\rho(A_{l_0}), (i,j)\neq(l,0)} \tau(A_{ij}) < \tau(F)$, then every meromorphic solution f of (3.2) satisfies $\rho(f) \geq \rho(A_{l_0})$ when $F(z) \not\equiv 0$. Further, if $F(z) \equiv 0$, then every meromorphic solution $f(z) \not\equiv 0$ of (3.1) satisfies $\rho(f) \geq \rho(A_{l_0}) + 1$.
- (ii) If $\rho(F) > \rho(A_{l_0})$. then every meromorphic solution f of (3.2) satisfies $\rho(f) \geq \rho(F)$.

Very recently, after that in [10], Chen and Zheng have investigated the growth of solutions of equation (3.1), and obtained the following results.

Theorem 3.1.2. ([10]) Let $A_{ij}(z)$ ($i = 0, \dots, n; j = 0, \dots, m$) be entire functions, and $a, l \in \{0, 1, \dots, n\}, b \in \{0, 1, \dots, m\}$ such that $(a, b) \neq (l, 0)$. Suppose that the following three assumptions hold simultaneously :

- (1) $\max\{\mu(A_{ab}), \rho(A_{ij}), (i, j) \neq (a, b), (l, 0)\} = \rho \leq \mu(A_{l_0}) < \infty, \mu(A_{l_0}) > 0$;
- (2) $\underline{\tau}_M(A_{l_0}) > \underline{\tau}_M(A_{ab})$, when $\mu(A_{l_0}) = \mu(A_{ab})$;
- (3) $\underline{\tau}_M(A_{l_0}) > \max\{\tau_M(A_{ij}) : \rho(A_{ij}) = \mu(A_{l_0}), (i, j) \neq (a, b), (l, 0)\}$, when $\mu(A_{l_0}) = \max\{\rho(A_{ij}) : (i, j) \neq (a, b), (l, 0)\}$.

Then any non zero meromorphic solution f of (3.1) satisfies $\rho(f) \geq \mu(A_{l_0}) + 1$.

Theorem 3.1.3. ([10]) Let $A_{ij}(z)$ ($i = 0, \dots, n; j = 0, \dots, m$) be meromorphic functions, and $a, l \in \{0, 1, \dots, n\}, b \in \{0, 1, \dots, m\}$ such that $(a, b) \neq (l, 0)$. Suppose that the following four assumptions hold simultaneously:

- (1) $\delta(\infty, A_{l_0}) = \liminf_{r \rightarrow +\infty} \frac{m(r, A_{l_0})}{T(r, A_{l_0})} = \delta > 0$;
- (2) $\max\{\mu(A_{ab}), \rho(A_{ij}), (i, j) \neq (a, b), (l, 0)\} = \rho \leq \mu(A_{l_0}) < \infty, \mu(A_{l_0}) > 0$;
- (3) $\delta \underline{\tau}(A_{l_0}) > \underline{\tau}(A_{ab})$, when $\mu(A_{l_0}) = \mu(A_{ab})$;
- (4) $\delta \underline{\tau}(A_{l_0}) > \max\{\tau(A_{ij}) : \rho(A_{ij}) = \mu(A_{l_0}), (i, j) \neq (a, b), (l, 0)\}$ when $\mu(A_{l_0}) = \max\{\rho(A_{ij}) : (i, j) \neq (a, b), (l, 0)\}$.

Then any non zero meromorphic solution f of (3.1) satisfies $\rho(f) \geq \mu(A_{l_0}) + 1$.

In the following two theorems, by combining complex differential and difference equations, we extend the results of Theorems 3.1.2 and 3.1.3 for equation (3.2). Let us define

$$S := \{F, A_{ij} : (i, j) \neq (l, 0), (k, p)\}, \quad \rho(S) := \max\{\rho(g) : g \in S\}.$$

Theorem 3.1.4. ([5]) Consider a delay-differential equation of type (3.2) with entire coefficients. Suppose that one of the coefficients A_{l_0} with $\mu(A_{l_0}) > 0$ is dominating in the sense that:

- (1) $\rho := \max\{\mu(A_{kp}), \rho(S)\} \leq \mu(A_{l_0}) < \infty$;

(2) $\tau_M(A_{l0}) > \tau_M(A_{kp})$, whenever $\mu(A_{l0}) = \mu(A_{kp})$;

(3) $\tau_1 := \max\{\tau_M(g) : \rho(g) = \mu(A_{l0}), g \in S\} < \tau_M(A_{l0})$, whenever $\mu(A_{l0}) = \rho(S)$.

Then every meromorphic solution f of (3.2) satisfies $\rho(f) \geq \mu(A_{l0})$ if $F(z) \not\equiv 0$. Further, if $F(z) \equiv 0$, then every meromorphic solution $f \not\equiv 0$ of (3.1) satisfies $\rho(f) \geq \mu(A_{l0}) + 1$.

Theorem 3.1.5. ([5]) Consider a delay-differential equation of type (3.2) with meromorphic coefficients. Suppose that one of the coefficients A_{l0} is dominating in the sense that:

(1) $\rho := \max\{\mu(A_{kp}), \rho(S)\} \leq \mu(A_{l0}) < \infty$;

(2) $\tau(A_{l0}) > \tau(A_{kp})$, whenever $\mu(A_{l0}) = \mu(A_{kp})$;

(3)
$$\tau_1 = \sum_{\rho(A_{ij})=\mu(A_{l0}), (i,j) \neq (l,0), (k,p)} \tau(A_{ij}) + \tau(F) < \tau(A_{l0}) < +\infty$$

whenever $\mu(A_{l0}) = \rho(S)$;

(4)
$$\tau_1 + \tau(A_{kp}) < \tau(A_{l0}) < +\infty$$

whenever $\mu(A_{l0}) = \mu(A_{kp}) = \rho(S)$;

(5) $\lambda\left(\frac{1}{A_{l0}}\right) < \mu(A_{l0}) < \infty$.

Then every meromorphic solution f of (3.2) satisfies $\rho(f) \geq \mu(A_{l0})$ if $F(z) \not\equiv 0$. Further, if $F(z) \equiv 0$, then every meromorphic solution $f \not\equiv 0$ of (3.1) satisfies $\rho(f) \geq \mu(A_{l0}) + 1$.

Related to the Theorem 3.1.1, a natural question arises here as follows : What happens if we replace the conditions on " $\rho(A_{l0})$ and $\tau(A_{l0})$ " in Theorem 3.1.1 by the conditions on " $\mu(A_{l0})$ and $\tau(A_{l0})$ ". Concerning this question, we obtain the following results.

Theorem 3.1.6. ([6]) Let $A_{ij}(z)$ ($i \in I = \{0, 1, \dots, n\}$; $j \in J = \{0, 1, \dots, m\}$) be meromorphic functions. Suppose there exists an integer l ($0 \leq l \leq n$) such that $A_{l0}(z)$ satisfies

$$\lambda\left(\frac{1}{A_{l0}}\right) < \mu(A_{l0}) < \infty$$

and

$$\max\{\rho(A_{ij}) : i = 0, 1, \dots, n; j = 0, 1, \dots, m, (i, j) \neq (l, 0)\} \leq \mu(A_{l0}) < \infty,$$

$$\tau_1 = \sum_{\rho(A_{ij})=\mu(A_{l0}), (i,j) \neq (l,0)} \tau(A_{ij}) < \tau(A_{l0}) = \tau < \infty.$$

Then every meromorphic solution $f(z) \not\equiv 0$ of (3.1) satisfies $\rho(f) \geq \mu(A_{l0}) + 1$.

Theorem 3.1.7. ([6]) Let $A_{ij}(z)$ ($i \in I = \{0, 1, \dots, n\}$; $j \in J = \{0, 1, \dots, m\}$) and $F(z) \not\equiv 0$ be meromorphic functions. Suppose there exists an integer l ($0 \leq l \leq n$) such that $A_{l0}(z)$ satisfies

$$\lambda\left(\frac{1}{A_{l0}}\right) < \mu(A_{l0}) < \infty$$

and

$$\max\{\rho(A_{ij}) : i = 0, 1, \dots, n; j = 0, 1, \dots, m, (i, j) \neq (l, 0)\} \leq \mu(A_{l0}) < \infty,$$

$$\tau_1 = \sum_{\rho(A_{ij}) = \mu(A_{l0}), (i, j) \neq (l, 0)} \tau(A_{ij}) < \tau(A_{l0}) = \tau < \infty.$$

- (i) If $\rho(F) < \mu(A_{l0})$, or $\rho(F) = \mu(A_{l0})$ and $\tau_1 + \tau(F) < \tau$, or $\mu(F) = \mu(A_{l0})$ and $\tau_1 + \tau < \tau(F)$, then every meromorphic solution f of (3.2) satisfies $\rho(f) \geq \mu(A_{l0})$.
(ii) If $\mu(F) > \mu(A_{l0})$, then every meromorphic solution f of (3.2) satisfies $\rho(f) \geq \mu(F)$.

Remark 3.1.8. Theorems 3.1.4 and 3.1.5 are generalization of our theorems obtained very recently in ([3]), for the growth of meromorphic solutions to non-homogeneous linear differential-difference equations, see Theorems 2.1.9 and 2.1.10 in Chapter 2.

3.2 Some Preliminary Lemmas

For the proofs of our results, we will use the Lemmas 2.2.1 – 2.2.11, see Chapter 2, and the following lemmas.

Lemma 3.2.1. ([14]) Let f be a transcendental meromorphic function of finite order $\rho(f)$, and let k and j be integers satisfying $k > j \geq 0$. Then for any given $\varepsilon (> 0)$, there exists a subset $E_1 \subset (1, +\infty)$ which has finite logarithmic measure such that for all z satisfying $|z| = r \notin [0, 1] \cup E_1$, we have

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq |z|^{(k-j)(\rho(f)-1+\varepsilon)}.$$

Lemma 3.2.2. ([17]) (Logarithmic Derivative Lemma) Let f be a meromorphic function and $k \geq 1$ be an integer. Then, we have

$$m\left(r, \frac{f^{(k)}}{f}\right) = O(\log T(r, f) + \log r)$$

possibly outside a set $E_6 \subset [0, +\infty)$ of a finite linear measure. If $\rho(f) < \infty$, then

$$m\left(r, \frac{f^{(k)}}{f}\right) = O(\log r).$$

Lemma 3.2.3. ([13, 17, 40]) Let f be a meromorphic function and $k \geq 1$ be an integer. Then, we have

$$T(r, f^{(k)}) \leq (k+1)T(r, f) + S(r, f),$$

where $S(r, f)$ denotes any quantity that satisfies the condition $S(r, f) = o(T(r, f))$ as $r \rightarrow +\infty$ possibly outside an exceptional set $E_6 \subset (0, +\infty)$ of r of finite linear measure. If f is of finite order, then $S(r, f) = o(T(r, f))$ as $r \rightarrow +\infty$.

3.3 Proof of the Theorems

Proof of Theorem 3.1.4 If f has infinite order, then the result holds. Now, we suppose that $\rho(f) < \infty$. We divide (3.2) by $f(z + c_l)$ to get

$$\begin{aligned} -A_{l0}(z) &= \sum_{i=0, i \neq l, k}^n \sum_{j=0}^m A_{ij} \frac{f^{(j)}(z + c_i)}{f(z + c_i)} \frac{f(z + c_i)}{f(z + c_l)} + \sum_{j=0, j \neq p}^m A_{kj} \frac{f^{(j)}(z + c_k)}{f(z + c_k)} \frac{f(z + c_k)}{f(z + c_l)} \\ &+ A_{kp} \frac{f^{(p)}(z + c_k)}{f(z + c_k)} \frac{f(z + c_k)}{f(z + c_l)} + \sum_{j=1}^m A_{lj} \frac{f^{(j)}(z + c_l)}{f(z + c_l)} - \frac{F(z)}{f(z + c_l)}. \end{aligned} \quad (3.3)$$

Therefore

$$\begin{aligned} |A_{l0}(z)| &\leq \sum_{i=0, i \neq l, k}^n \sum_{j=0}^m |A_{ij}| \left| \frac{f^{(j)}(z + c_i)}{f(z + c_i)} \right| \left| \frac{f(z + c_i)}{f(z + c_l)} \right| \\ &+ \sum_{j=0, j \neq p}^m |A_{kj}| \left| \frac{f^{(j)}(z + c_k)}{f(z + c_k)} \right| \left| \frac{f(z + c_k)}{f(z + c_l)} \right| \\ &+ |A_{kp}| \left| \frac{f^{(p)}(z + c_k)}{f(z + c_k)} \right| \left| \frac{f(z + c_k)}{f(z + c_l)} \right| + \sum_{j=1}^m |A_{lj}| \left| \frac{f^{(j)}(z + c_l)}{f(z + c_l)} \right| + \left| \frac{F(z)}{f(z + c_l)} \right|. \end{aligned} \quad (3.4)$$

From Lemma 3.2.1 and Lemma 2.2.2, for any given $\varepsilon (> 0)$, there exists a subset $E_1 \subset (1, +\infty)$ which has finite logarithmic measure such that for all z satisfying $|z| = r \notin [0, 1] \cup E_1$, we have

$$\left| \frac{f^{(j)}(z + c_i)}{f(z + c_i)} \right| \leq |z|^{j(\rho(f+c_i)-1+\varepsilon)} = |z|^{j(\rho(f)-1+\varepsilon)}, \quad (i, j) \neq (l, 0). \quad (3.5)$$

It follows by Lemma 2.2.1 that for any $\varepsilon (> 0)$, there exists a subset $E_2 \subset (1, +\infty)$ with finite logarithmic measure such that for all z satisfying $|z| = r \notin [0, 1] \cup E_2$, we have

$$\left| \frac{f(z + c_i)}{f(z + c_l)} \right| \leq \exp\{r^{\rho(f)-1+\varepsilon}\}, \quad i \neq l. \quad (3.6)$$

From Lemma 2.2.2, we get

$$\rho(f(z + c_l)) = \rho\left(\frac{1}{f(z + c_l)}\right) = \rho(f).$$

So, by Lemma 2.2.3, for any given $\varepsilon > 0$, there exists a subset $E_3 \subset (1, +\infty)$ having finite linear measure and finite logarithmic measure such that for all z satisfying $|z| = r \notin [0, 1] \cup E_3$ sufficiently large, we have

$$\left| \frac{1}{f(z + c_l)} \right| \leq \exp\{r^{\rho(f)+\varepsilon}\}. \quad (3.7)$$

In the following, we divide the proof into four cases:

Case (i). We suppose that $\rho < \mu(A_{l0})$.

For $g \in S$, by the definition of $\rho(S)$, for any given $\varepsilon > 0$ and sufficiently large r , we have

$$|g(z)| \leq \exp\{r^{\rho(S)+\varepsilon}\} \leq \exp\{r^{\rho+\varepsilon}\}. \quad (3.8)$$

It follows by the definition of $\mu(A_{l_0})$, for sufficiently small $\varepsilon > 0$ and sufficiently large r , we have

$$|A_{l_0}(z)| \geq \exp\{r^{\mu(A_{l_0})-\varepsilon}\}. \quad (3.9)$$

It also follows by the definition of $\mu(A_{k_p})$ and Lemma 2.2.4, for any given $\varepsilon(> 0)$, there exists a subset $E_4 \subset (1, +\infty)$ with infinite logarithmic measure such that for all $r \in E_4$, we have

$$|A_{k_p}(z)| \leq \exp\{r^{\mu(A_{k_p})+\varepsilon}\}. \quad (3.10)$$

By substituting (3.5) – (3.10) into (3.4), for all z satisfying $|z| = r \in E_4 \setminus ([0, 1] \cup E_1 \cup E_2 \cup E_3)$, we obtain

$$\begin{aligned} \exp\{r^{\mu(A_{l_0})-\varepsilon}\} &\leq \sum_{i=0, i \neq l, k}^n \sum_{j=0}^m \exp\{r^{\rho+\varepsilon}\} |z|^{j(\rho(f)-1+\varepsilon)} \exp\{r^{\rho(f)-1+\varepsilon}\} \\ &\quad + \sum_{j=0, j \neq p}^m \exp\{r^{\rho+\varepsilon}\} |z|^{j(\rho(f)-1+\varepsilon)} \exp\{r^{\rho(f)-1+\varepsilon}\} \\ &\quad + |z|^{p(\rho(f)-1+\varepsilon)} \exp\{r^{\mu(A_{k_p})+\varepsilon}\} \exp\{r^{\rho(f)-1+\varepsilon}\} \\ &\quad + \sum_{j=1}^m \exp\{r^{\rho+\varepsilon}\} |z|^{j(\rho(f)-1+\varepsilon)} + \exp\{r^{\rho+\varepsilon}\} \exp\{r^{\rho(f)+\varepsilon}\}. \\ &\leq ((n-1)(m+1) + 2m) r^{m(\rho(f)-1+\varepsilon)} \exp\{r^{\rho+\varepsilon}\} \exp\{r^{\rho(f)-1+\varepsilon}\} \\ &\quad + r^{p(\rho(f)-1+\varepsilon)} \exp\{r^{\mu(A_{k_p})+\varepsilon}\} \exp\{r^{\rho(f)-1+\varepsilon}\} + \exp\{r^{\rho+\varepsilon}\} \exp\{r^{\rho(f)+\varepsilon}\}. \end{aligned} \quad (3.11)$$

Now, we may choose sufficiently small ε satisfying $0 < 3\varepsilon < \mu(A_{l_0}) - \rho$, we deduce from (3.11) that for $|z| = r \in E_4 \setminus ([0, 1] \cup E_1 \cup E_2 \cup E_3)$, $r \rightarrow +\infty$

$$\exp\{r^{\mu(A_{l_0})-2\varepsilon}\} \leq \exp\{r^{\rho(f)+\varepsilon}\}.$$

Therefore, $\mu(A_{l_0}) \leq \rho(f) + 3\varepsilon$, since $\varepsilon > 0$ is arbitrary, then $\rho(f) \geq \mu(A_{l_0})$.

Further, if $F \equiv 0$, then by substituting (3.5), (3.6) and (3.8) – (3.10) into (3.4), for all z satisfying $|z| = r \in E_4 \setminus ([0, 1] \cup E_1 \cup E_2)$, we obtain

$$\begin{aligned} \exp\{r^{\mu(A_{l_0})-\varepsilon}\} &\leq (nm + n + m - 1) r^{m(\rho(f)-1+\varepsilon)} \exp\{r^{\rho+\varepsilon}\} \exp\{r^{\rho(f)-1+\varepsilon}\} \\ &\quad + r^{p(\rho(f)-1+\varepsilon)} \exp\{r^{\mu(A_{k_p})+\varepsilon}\} \exp\{r^{\rho(f)-1+\varepsilon}\}. \end{aligned} \quad (3.12)$$

By choosing sufficiently small ε satisfying $0 < 3\varepsilon < \mu(A_{l_0}) - \rho$, we deduce from (3.12) that for $|z| = r \in E_4 \setminus ([0, 1] \cup E_1 \cup E_2)$, $r \rightarrow +\infty$

$$\exp\{r^{\mu(A_{l_0})-2\varepsilon}\} \leq \exp\{r^{\rho(f)-1+\varepsilon}\},$$

that is, $\mu(A_{l_0}) \leq \rho(f) - 1 + 3\varepsilon$, since $\varepsilon > 0$ is arbitrary, then $\rho(f) \geq \mu(A_{l_0}) + 1$.

Case (ii) We suppose that $\beta = \rho(S) < \mu(A_{l_0}) = \mu(A_{k_p})$, $\underline{\tau}_M(A_{l_0}) > \underline{\tau}_M(A_{k_p})$.

For $g \in S$, by the definition of $\rho(S)$, for any given $\varepsilon(> 0)$, and sufficiently large r , we have

$$|g(z)| \leq \exp\{r^{\rho(S)+\varepsilon}\} \leq \exp\{r^{\beta+\varepsilon}\}. \quad (3.13)$$

From the definition of $\mathcal{I}_M(A_{l_0})$, for sufficiently small $\varepsilon > 0$ and sufficiently large r , we have

$$|A_{l_0}(z)| \geq \exp\{(\mathcal{I}_M(A_{l_0}) - \varepsilon) r^{\mu(A_{l_0})}\}. \quad (3.14)$$

Also, from the definition of $\mathcal{I}_M(A_{kp})$ and Lemma 2.2.5, for any given $\varepsilon (> 0)$, there exists a subset $E_5 \subset (1, +\infty)$ with infinite logarithmic measure such that for all $r \in E_5$, we have

$$|A_{kp}(z)| \leq \exp\{(\mathcal{I}_M(A_{kp}) + \varepsilon) r^{\mu(A_{kp})}\} = \exp\{(\mathcal{I}_M(A_{kp}) + \varepsilon) r^{\mu(A_{l_0})}\}. \quad (3.15)$$

By substituting (3.5) – (3.7) and (3.13) – (3.15) into (3.4), for all z satisfying $|z| = r \in E_5 \setminus ([0, 1] \cup E_1 \cup E_2 \cup E_3)$, we get

$$\begin{aligned} & \exp\{(\mathcal{I}_M(A_{l_0}) - \varepsilon) r^{\mu(A_{l_0})}\} \\ & \leq (nm + n + m - 1) r^{m(\rho(f)-1+\varepsilon)} \exp\{r^{\beta+\varepsilon}\} \exp\{r^{\rho(f)-1+\varepsilon}\} \\ & + r^{p(\rho(f)-1+\varepsilon)} \exp\{(\mathcal{I}_M(A_{kp}) + \varepsilon) r^{\mu(A_{l_0})}\} \exp\{r^{\rho(f)-1+\varepsilon}\} + \exp\{r^{\beta+\varepsilon}\} \exp\{r^{\rho(f)+\varepsilon}\}. \end{aligned} \quad (3.16)$$

Therefore, we may choose sufficiently small ε , $0 < 2\varepsilon < \min\{\mu(A_{l_0}) - \beta, \mathcal{I}_M(A_{l_0}) - \mathcal{I}_M(A_{kp})\}$, then from (3.16) for $r \in E_5 \setminus ([0, 1] \cup E_1 \cup E_2 \cup E_3)$ sufficiently large, we obtain

$$\exp\{(\mathcal{I}_M(A_{l_0}) - \mathcal{I}_M(A_{kp}) - 2\varepsilon) r^{\mu(A_{l_0})-\varepsilon}\} \leq \exp\{r^{\rho(f)+\varepsilon}\}.$$

Then, $\mu(A_{l_0}) \leq \rho(f) + 2\varepsilon$, since $\varepsilon > 0$ is arbitrary, so $\rho(f) \geq \mu(A_{l_0})$.

Further, if $F \equiv 0$, then by substituting (3.5), (3.6) and (3.13) – (3.15) into (3.4), for all z satisfying $|z| = r \in E_5 \setminus ([0, 1] \cup E_1 \cup E_2)$, we have

$$\begin{aligned} & \exp\{(\mathcal{I}_M(A_{l_0}) - \varepsilon) r^{\mu(A_{l_0})}\} \\ & \leq (nm + n + m - 1) r^{m(\rho(f)-1+\varepsilon)} \exp\{r^{\beta+\varepsilon}\} \exp\{r^{\rho(f)-1+\varepsilon}\} \\ & + r^{p(\rho(f)-1+\varepsilon)} \exp\{(\mathcal{I}_M(A_{kp}) + \varepsilon) r^{\mu(A_{l_0})}\} \exp\{r^{\rho(f)-1+\varepsilon}\}. \end{aligned} \quad (3.17)$$

Now, we may choose sufficiently small ε , $0 < 2\varepsilon < \min\{\mu(A_{l_0}) - \beta, \mathcal{I}_M(A_{l_0}) - \mathcal{I}_M(A_{kp})\}$, then from (3.17) for $r \in E_5 \setminus ([0, 1] \cup E_1 \cup E_2)$ sufficiently large, we get

$$\exp\{(\mathcal{I}_M(A_{l_0}) - \mathcal{I}_M(A_{kp}) - 2\varepsilon) r^{\mu(A_{l_0})-\varepsilon}\} \leq \exp\{r^{\rho(f)-1+\varepsilon}\},$$

that is, $\mu(A_{l_0}) \leq \rho(f) - 1 + 2\varepsilon$, since $\varepsilon > 0$ is arbitrary, then $\rho(f) \geq \mu(A_{l_0}) + 1$.

Case (iii) When $\mu(A_{l_0}) = \rho(S) > \mu(A_{kp})$, and

$$\max\{\tau_M(g) : \rho(g) = \mu(A_{l_0}), g \in S\} = \tau_1 < \mathcal{I}_M(A_{l_0}).$$

For $g \in S$, by the definitions of $\rho(S)$ and $\tau_M(g)$, for any given $\varepsilon > 0$ and sufficiently large r , we have

$$|g(z)| \leq \begin{cases} \exp\{r^{\rho(S)+\varepsilon}\} \leq \exp\{r^{\mu(A_{l_0})-\varepsilon}\}, & \text{if } \rho(S) < \mu(A_{l_0}), \\ \exp\{(\tau_1 + \varepsilon) r^{\mu(A_{l_0})}\}, & \text{if } \rho(S) = \mu(A_{l_0}). \end{cases} \quad (3.18)$$

Then, by substituting (3.5) – (3.7), (3.10), (3.14) and (3.18) into (3.4), for all z satisfying $|z| = r \in E_4 \setminus ([0, 1] \cup E_1 \cup E_2 \cup E_3)$ sufficiently large, we obtain

$$\exp\{(\mathcal{I}_M(A_{l_0}) - \varepsilon) r^{\mu(A_{l_0})}\}$$

$$\begin{aligned}
&\leq O\left(r^{m(\rho(f)-1+\varepsilon)} \exp\{(\tau_1 + \varepsilon) r^{\mu(A_{l_0})}\} \exp\{r^{\rho(f)-1+\varepsilon}\}\right) \\
&\quad + O\left(r^{m(\rho(f)-1+\varepsilon)} \exp\{r^{\mu(A_{l_0})-\varepsilon}\} \exp\{r^{\rho(f)-1+\varepsilon}\}\right) \\
&\quad + r^{p(\rho(f)-1+\varepsilon)} \exp\{r^{\mu(A_{k_p})+\varepsilon}\} \exp\{r^{\rho(f)-1+\varepsilon}\} \\
&\quad + O\left(r^{m(\rho(f)-1+\varepsilon)} \exp\{r^{\mu(A_{l_0})-\varepsilon}\}\right) \\
&+ O\left(r^{m(\rho(f)-1+\varepsilon)} \exp\{(\tau_1 + \varepsilon) r^{\mu(A_{l_0})}\}\right) + \exp\{(\tau_1 + \varepsilon) r^{\mu(A_{l_0})}\} \exp\{r^{\rho(f)+\varepsilon}\}. \quad (3.19)
\end{aligned}$$

Now, we may choose sufficiently small ε satisfying

$$0 < 2\varepsilon < \min\{\mu(A_{l_0}) - \mu(A_{k_p}), \underline{\tau}_M(A_{l_0}) - \tau_1\},$$

then from (3.19) for sufficiently large $r \in E_4 \setminus ([0, 1] \cup E_1 \cup E_2 \cup E_3)$, we get

$$\exp\{(\underline{\tau}_M(A_{l_0}) - \tau_1 - 2\varepsilon) r^{\mu(A_{l_0})-\varepsilon}\} \leq \exp\{r^{\rho(f)+\varepsilon}\}.$$

That means, $\mu(A_{l_0}) \leq \rho(f) + 2\varepsilon$, since $\varepsilon > 0$ is arbitrary, then $\rho(f) \geq \mu(A_{l_0})$. Further, if $F \equiv 0$, then by substituting (3.5), (3.6), (3.10), (3.14) and (3.18) into (3.4), for all z satisfying $|z| = r \in E_4 \setminus ([0, 1] \cup E_1 \cup E_2)$ sufficiently large, we have

$$\begin{aligned}
&\exp\{(\underline{\tau}_M(A_{l_0}) - \varepsilon) r^{\mu(A_{l_0})}\} \\
&\leq O\left(r^{m(\rho(f)-1+\varepsilon)} \exp\{(\tau_1 + \varepsilon) r^{\mu(A_{l_0})}\} \exp\{r^{\rho(f)-1+\varepsilon}\}\right) \\
&\quad + O\left(r^{m(\rho(f)-1+\varepsilon)} \exp\{r^{\mu(A_{l_0})-\varepsilon}\} \exp\{r^{\rho(f)-1+\varepsilon}\}\right) \\
&\quad + r^{p(\rho(f)-1+\varepsilon)} \exp\{r^{\mu(A_{k_p})+\varepsilon}\} \exp\{r^{\rho(f)-1+\varepsilon}\} \\
&+ O\left(r^{m(\rho(f)-1+\varepsilon)} \exp\{r^{\mu(A_{l_0})-\varepsilon}\}\right) + O\left(r^{m(\rho(f)-1+\varepsilon)} \exp\{(\tau_1 + \varepsilon) r^{\mu(A_{l_0})}\}\right). \quad (3.20)
\end{aligned}$$

Now, we may choose sufficiently small ε satisfying

$$0 < 2\varepsilon < \min\{\mu(A_{l_0}) - \mu(A_{k_p}), \underline{\tau}_M(A_{l_0}) - \tau_1\},$$

then from (3.20) for sufficiently large $r \in E_4 \setminus ([0, 1] \cup E_1 \cup E_2)$, we get

$$\exp\{(\underline{\tau}_M(A_{l_0}) - \tau_1 - 2\varepsilon) r^{\mu(A_{l_0})-\varepsilon}\} \leq \exp\{r^{\rho(f)-1+\varepsilon}\}.$$

That means, $\mu(A_{l_0}) \leq \rho(f) - 1 + 2\varepsilon$, since $\varepsilon > 0$ is arbitrary, then $\rho(f) \geq \mu(A_{l_0}) + 1$.

Case (iv) We suppose that $\rho(S) = \mu(A_{k_p}) = \mu(A_{l_0})$ and

$$\max\{\underline{\tau}_M(A_{k_p}), \tau_M(g) : \rho(g) = \mu(A_{l_0}), g \in S\} = \tau_2 < \underline{\tau}_M(A_{l_0}).$$

It follows by substituting (3.5) – (3.7), (3.14), (3.15) and (3.18) into (3.4), for all z satisfying $|z| = r \in E_5 \setminus ([0, 1] \cup E_1 \cup E_2 \cup E_3)$ sufficiently large, we have

$$\begin{aligned}
&\exp\{(\underline{\tau}_M(A_{l_0}) - \varepsilon) r^{\mu(A_{l_0})}\} \\
&\leq O\left(r^{m(\rho(f)-1+\varepsilon)} \exp\{(\tau_2 + \varepsilon) r^{\mu(A_{l_0})}\} \exp\{r^{\rho(f)-1+\varepsilon}\}\right) \\
&\quad + O\left(r^{m(\rho(f)-1+\varepsilon)} \exp\{r^{\mu(A_{l_0})-\varepsilon}\} \exp\{r^{\rho(f)-1+\varepsilon}\}\right) \\
&+ r^{p(\rho(f)-1+\varepsilon)} \exp\{(\underline{\tau}_M(A_{k_p}) + \varepsilon) r^{\mu(A_{l_0})}\} \exp\{r^{\rho(f)-1+\varepsilon}\}
\end{aligned}$$

$$\begin{aligned}
& +O\left(r^{m(\rho(f)-1+\varepsilon)} \exp\{r^{\mu(A_{l_0})-\varepsilon}\}\right) + O\left(r^{m(\rho(f)-1+\varepsilon)} \exp\{(\tau_2 + \varepsilon) r^{\mu(A_{l_0})}\}\right) \\
& \quad + \exp\{(\tau_2 + \varepsilon) r^{\mu(A_{l_0})}\} \exp\{r^{\rho(f)+\varepsilon}\}. \tag{3.21}
\end{aligned}$$

Now, we may choose sufficiently small ε satisfying

$$0 < 2\varepsilon < \underline{\tau}_M(A_{l_0}) - \tau_2,$$

from (3.21) for sufficiently large $r \in E_5 \setminus ([0, 1] \cup E_1 \cup E_2 \cup E_3)$, we get

$$\exp\{(\underline{\tau}_M(A_{l_0}) - \tau_2 - 2\varepsilon) r^{\mu(A_{l_0})-\varepsilon}\} \leq \exp\{r^{\rho(f)+\varepsilon}\},$$

that means, $\mu(A_{l_0}) \leq \rho(f) + 2\varepsilon$, since $\varepsilon > 0$ is arbitrary, then $\rho(f) \geq \mu(A_{l_0})$. Further, if $F \equiv 0$, by substituting (3.6), (3.7), (3.14), (3.15) and (3.18) into (3.4), for all z satisfying $|z| = r \in E_5 \setminus ([0, 1] \cup E_1 \cup E_2)$ sufficiently large, we have

$$\begin{aligned}
& \exp\{(\underline{\tau}_M(A_{l_0}) - \varepsilon) r^{\mu(A_{l_0})}\} \leq O\left(r^{m(\rho(f)-1+\varepsilon)} \exp\{(\tau_2 + \varepsilon) r^{\mu(A_{l_0})}\} \exp\{r^{\rho(f)-1+\varepsilon}\}\right) \\
& \quad + O\left(r^{m(\rho(f)-1+\varepsilon)} \exp\{r^{\mu(A_{l_0})-\varepsilon}\} \exp\{r^{\rho(f)-1+\varepsilon}\}\right) \\
& \quad + r^{p(\rho(f)-1+\varepsilon)} \exp\{(\underline{\tau}_M(A_{kp}) + \varepsilon) r^{\mu(A_{l_0})}\} \exp\{r^{\rho(f)-1+\varepsilon}\} \\
& \quad + O\left(r^{m(\rho(f)-1+\varepsilon)} \exp\{r^{\mu(A_{l_0})-\varepsilon}\}\right) + O\left(r^{m(\rho(f)-1+\varepsilon)} \exp\{(\tau_2 + \varepsilon) r^{\mu(A_{l_0})}\}\right). \tag{3.22}
\end{aligned}$$

Now, we may choose sufficiently small ε satisfying

$$0 < 2\varepsilon < \underline{\tau}_M(A_{l_0}) - \tau_2,$$

from (3.22) for sufficiently large $r \in E_5 \setminus ([0, 1] \cup E_1 \cup E_2)$, we get

$$\exp\{(\underline{\tau}_M(A_{l_0}) - \tau_2 - 2\varepsilon) r^{\mu(A_{l_0})-\varepsilon}\} \leq \exp\{r^{\rho(f)-1+\varepsilon}\}.$$

That means, $\mu(A_{l_0}) \leq \rho(f) - 1 + 2\varepsilon$, since $\varepsilon > 0$ is arbitrary, then $\rho(f) \geq \mu(A_{l_0}) + 1$. The proof of Theorem 3.1.4 is complete.

Proof of Theorem 3.1.5 If f has infinite order, then the result holds. Now, we suppose that $\rho(f) < \infty$. By (3.3), we have

$$\begin{aligned}
T(r, A_{l_0}(z)) &= m(r, A_{l_0}(z)) + N(r, A_{l_0}(z)) \\
&\leq \sum_{i=0, i \neq l, k}^n \sum_{j=0}^m m(r, A_{ij}(z)) + m(r, A_{kp}(z)) \\
&\quad + \sum_{j=0, j \neq p}^m m(r, A_{kj}(z)) + \sum_{j=1}^m m(r, A_{lj}(z)) + \sum_{i=0, i \neq l, k}^n \sum_{j=0}^m m\left(r, \frac{f^{(j)}(z + c_i)}{f(z + c_i)}\right) \\
&\quad + \sum_{i=0, i \neq l, k}^n m\left(r, \frac{f(z + c_i)}{f(z + c_l)}\right) + \sum_{j=1}^m m\left(r, \frac{f^{(j)}(z + c_k)}{f(z + c_k)}\right) + 2m\left(r, \frac{f(z + c_k)}{f(z + c_l)}\right) \\
&\quad + \sum_{j=1}^m m\left(r, \frac{f^{(j)}(z + c_l)}{f(z + c_l)}\right) + m(r, F(z)) + m\left(r, \frac{1}{f(z + c_l)}\right)
\end{aligned}$$

$$\begin{aligned}
& +N(r, A_{l_0}(z)) + O(1) \leq \sum_{i=0, i \neq l, k}^n \sum_{j=0}^m T(r, A_{ij}(z)) + T(r, A_{kp}(z)) \\
& + \sum_{j=0, j \neq p}^m T(r, A_{kj}(z)) + \sum_{j=1}^m T(r, A_{lj}(z)) + \sum_{i=0, i \neq l, k}^n \sum_{j=1}^m m \left(r, \frac{f^{(j)}(z + c_i)}{f(z + c_i)} \right) \\
& + \sum_{i=0, i \neq l, k}^n m \left(r, \frac{f(z + c_i)}{f(z + c_l)} \right) + \sum_{j=1}^m m \left(r, \frac{f^{(j)}(z + c_k)}{f(z + c_k)} \right) + 2m \left(r, \frac{f(z + c_k)}{f(z + c_l)} \right) \\
& + \sum_{j=1}^m m \left(r, \frac{f^{(j)}(z + c_l)}{f(z + c_l)} \right) + T(r, F(z)) + T \left(r, \frac{1}{f(z + c_l)} \right) + N(r, A_{l_0}(z)) + O(1).
\end{aligned}$$

By Lemma 2.2.2 and the first main theorem of Nevanlinna, when r sufficiently large, we have

$$T \left(r, \frac{1}{f(z + c_l)} \right) = T(r, f(z + c_l)) + O(1) \leq (1 + o(1))T(r + |c_l|, f) \leq 2T(2r, f).$$

So, for r sufficiently large we obtain

$$\begin{aligned}
T(r, A_{l_0}(z)) & \leq \sum_{i=0, i \neq l, k}^n \sum_{j=0}^m T(r, A_{ij}(z)) + T(r, A_{kp}(z)) \\
& + \sum_{j=0, j \neq p}^m T(r, A_{kj}(z)) + \sum_{j=1}^m T(r, A_{lj}(z)) \\
& + \sum_{i=0, i \neq l, k}^n \sum_{j=1}^m m \left(r, \frac{f^{(j)}(z + c_i)}{f(z + c_i)} \right) + \sum_{i=0, i \neq l, k}^n m \left(r, \frac{f(z + c_i)}{f(z + c_l)} \right) + T(r, F(z)) \\
& + 2T(2r, f) + \sum_{j=1}^m m \left(r, \frac{f^{(j)}(z + c_l)}{f(z + c_l)} \right) \\
& + \sum_{j=1}^m m \left(r, \frac{f^{(j)}(z + c_k)}{f(z + c_k)} \right) + 2m \left(r, \frac{f(z + c_k)}{f(z + c_l)} \right) + N(r, A_{l_0}(z)) + O(1). \quad (3.23)
\end{aligned}$$

By Lemma 2.2.8, for any given $\varepsilon (> 0)$, we have

$$m \left(r, \frac{f(z)}{f(z + c_l)} \right) = O(r^{\rho(f)-1+\varepsilon}), \quad m \left(r, \frac{f(z + c_j)}{f(z + c_l)} \right) = O(r^{\rho(f)-1+\varepsilon}), \quad j \neq l. \quad (3.24)$$

It follows by Lemma 3.2.2, there exists a subset $E_6 \subset [0, +\infty[$ of a finite linear measure such that for all $r \notin E_6$ sufficiently large, we have

$$m \left(r, \frac{f^{(j)}(z + c_i)}{f(z + c_i)} \right) = O(\log r) \quad (i = 0, \dots, n; j = 1, \dots, m). \quad (3.25)$$

From the definition of $\lambda \left(\frac{1}{A_{l_0}} \right)$, for any given $\varepsilon > 0$ and sufficiently large r , we have

$$N(r, A_{l_0}) \leq r^{\lambda \left(\frac{1}{A_{l_0}} \right) + \varepsilon}. \quad (3.26)$$

In the following, we divide the proof into four cases:

Case (i) We suppose that $\rho < \mu(A_{l_0})$.

For $g \in S$, from the definition of $\rho(S)$ and $\rho(f)$ for any given $\varepsilon > 0$ and sufficiently large r , we have

$$T(r, g) \leq r^{\rho(S)+\varepsilon} \leq r^{\rho+\varepsilon}, \quad (3.27)$$

$$T(r, f) \leq r^{\rho(f)+\varepsilon}. \quad (3.28)$$

It follows from the definition of $\mu(A_{l_0})$, for sufficiently small $\varepsilon > 0$ and sufficiently large r , we have

$$T(r, A_{l_0}) \geq r^{\mu(A_{l_0})-\varepsilon}. \quad (3.29)$$

It follows from the definition of $\mu(A_{kp})$ and Lemma 2.2.9, for any given $\varepsilon (> 0)$, there exists a subset $E_7 \subset (1, +\infty)$ with infinite logarithmic measure such that for all $r \in E_7$, we have

$$T(r, A_{kp}) \leq r^{\mu(A_{kp})+\varepsilon}. \quad (3.30)$$

By substituting (3.24) – (3.30) into (3.23) for sufficiently large $r \in E_7 \setminus E_6$, we obtain

$$\begin{aligned} r^{\mu(A_{l_0})-\varepsilon} &\leq ((n-1)(m+1) + 2m) r^{\rho+\varepsilon} + r^{\mu(A_{kp})+\varepsilon} + O(r^{\rho(f)-1+\varepsilon}) \\ &\quad + 2(2r)^{\rho(f)+\varepsilon} + r^{\rho+\varepsilon} + r^{\lambda\left(\frac{1}{A_{l_0}}\right)+\varepsilon} + O(\log r). \end{aligned} \quad (3.31)$$

We may choose sufficiently small ε satisfying

$$0 < 3\varepsilon < \min \left\{ \mu(A_{l_0}) - \rho, \mu(A_{l_0}) - \lambda \left(\frac{1}{A_{l_0}} \right) \right\},$$

it follows from (3.31) that for $r \in E_7 \setminus E_6$, $r \rightarrow +\infty$

$$r^{\mu(A_{l_0})-2\varepsilon} \leq r^{\rho(f)+\varepsilon},$$

that means, $\mu(A_{l_0}) \leq \rho(f) + 3\varepsilon$, since $\varepsilon > 0$ is arbitrary, then $\rho(f) \geq \mu(A_{l_0})$.

Further, if $F \equiv 0$, then by substituting (3.24) – (3.27), (3.29) and (3.30) into (3.23) for sufficiently large $r \in E_7 \setminus E_6$, we obtain

$$\begin{aligned} r^{\mu(A_{l_0})-\varepsilon} &\leq ((n-1)(m+1) + 2m) r^{\rho+\varepsilon} + r^{\mu(A_{kp})+\varepsilon} + O(r^{\rho(f)-1+\varepsilon}) \\ &\quad + r^{\lambda\left(\frac{1}{A_{l_0}}\right)+\varepsilon} + O(\log r). \end{aligned} \quad (3.32)$$

We may choose sufficiently small ε satisfying

$$0 < 3\varepsilon < \min \left\{ \mu(A_{l_0}) - \rho, \mu(A_{l_0}) - \lambda \left(\frac{1}{A_{l_0}} \right) \right\},$$

from (3.32) that for $r \in E_7 \setminus E_6$, $r \rightarrow +\infty$

$$r^{\mu(A_{l_0})-2\varepsilon} \leq r^{\rho(f)-1+\varepsilon},$$

that means, $\mu(A_{l_0}) \leq \rho(f) - 1 + 3\varepsilon$, since $\varepsilon > 0$ is arbitrary, then $\rho(f) \geq \mu(A_{l_0}) + 1$.

Case (ii) We suppose that

$$\rho(S) = \beta < \mu(A_{l_0}) = \mu(A_{kp}), \underline{\tau}(A_{l_0}) > \underline{\tau}(A_{kp}).$$

For $g \in S$, by the definition of $\rho(S)$, for any given $\varepsilon(> 0)$ and sufficiently large r , we obtain

$$T(r, g) \leq r^{\rho(S)+\varepsilon} \leq r^{\beta+\varepsilon}. \quad (3.33)$$

From the definition of $\underline{\tau}(A_{l_0})$, for sufficiently small $\varepsilon > 0$ and sufficiently large r , we have

$$T(r, A_{l_0}) \geq (\underline{\tau}(A_{l_0}) - \varepsilon) r^{\mu(A_{l_0})}. \quad (3.34)$$

It follows from the definition of $\underline{\tau}(A_{kp})$ and Lemma 2.2.10, for any given $\varepsilon(> 0)$, there exists a subset $E_8 \subset (1, +\infty)$ with infinite logarithmic measure such that for all $r \in E_8$, we have

$$T(r, A_{kp}) \leq (\underline{\tau}(A_{kp}) + \varepsilon) r^{\mu(A_{kp})} \leq (\underline{\tau}(A_{kp}) + \varepsilon) r^{\mu(A_{l_0})}. \quad (3.35)$$

By substituting (3.24) – (3.26), (3.28) and (3.33) – (3.35) into (3.23), for sufficiently large $r \in E_8 \setminus E_6$, we obtain

$$\begin{aligned} (\underline{\tau}(A_{l_0}) - \varepsilon) r^{\mu(A_{l_0})} &\leq ((n-1)(m+1) + 2m) r^{\beta+\varepsilon} + (\underline{\tau}(A_{kp}) + \varepsilon) r^{\mu(A_{l_0})} \\ &+ O(r^{\rho(f)-1+\varepsilon}) + 2(2r)^{\rho(f)+\varepsilon} + r^{\beta+\varepsilon} + r^{\lambda\left(\frac{1}{A_{l_0}}\right)+\varepsilon} + O(\log r). \end{aligned} \quad (3.36)$$

Now, we may choose sufficiently small ε satisfying

$$0 < 2\varepsilon < \min \left\{ \mu(A_{l_0}) - \beta, \underline{\tau}(A_{l_0}) - \underline{\tau}(A_{kp}), \mu(A_{l_0}) - \lambda \left(\frac{1}{A_{l_0}} \right) \right\},$$

so from (3.36) for sufficiently large $r \in E_8 \setminus E_6$, we have

$$(\underline{\tau}(A_{l_0}) - \underline{\tau}(A_{kp}) - 2\varepsilon) r^{\mu(A_{l_0})-\varepsilon} \leq r^{\rho(f)+\varepsilon},$$

that means, $\mu(A_{l_0}) \leq \rho(f) + 2\varepsilon$, since $\varepsilon > 0$ is arbitrary, then $\rho(f) \geq \mu(A_{l_0})$.

Further, if $F \equiv 0$, then by substituting (3.24) – (3.26) and (3.33) – (3.35) into (3.23), for sufficiently large $r \in E_8 \setminus E_6$, we obtain

$$\begin{aligned} (\underline{\tau}(A_{l_0}) - \varepsilon) r^{\mu(A_{l_0})} &\leq ((n-1)(m+1) + 2m) r^{\beta+\varepsilon} + (\underline{\tau}(A_{kp}) + \varepsilon) r^{\mu(A_{l_0})} \\ &+ O(r^{\rho(f)-1+\varepsilon}) + r^{\lambda\left(\frac{1}{A_{l_0}}\right)+\varepsilon} + O(\log r). \end{aligned} \quad (3.37)$$

Now, we may choose sufficiently small ε satisfying

$$0 < 2\varepsilon < \min \left\{ \mu(A_{l_0}) - \beta, \underline{\tau}(A_{l_0}) - \underline{\tau}(A_{kp}), \mu(A_{l_0}) - \lambda \left(\frac{1}{A_{l_0}} \right) \right\}.$$

From (3.37) for sufficiently large $r \in E_8 \setminus E_6$, we get

$$(\underline{\tau}(A_{l_0}) - \underline{\tau}(A_{kp}) - 2\varepsilon) r^{\mu(A_{l_0})-\varepsilon} \leq r^{\rho(f)-1+\varepsilon},$$

that means, $\mu(A_{l_0}) \leq \rho(f) - 1 + 2\varepsilon$, since $\varepsilon > 0$ is arbitrary, then $\rho(f) \geq \mu(A_{l_0}) + 1$.

Case (iii) We suppose that $\mu(A_{l_0}) = \rho(S) > \mu(A_{kp})$ and

$$\tau_1 = \sum_{\rho(A_{ij})=\mu(A_{l_0}), (i,j) \neq (l,0), (k,p)} \tau(A_{ij}) + \tau(F) < \underline{\tau}(A_{l_0}).$$

Then, there exists a subset $J \subseteq \{0, 1, \dots, n\} \times \{0, 1, \dots, m\} \setminus \{(l, 0), (k, p)\}$ such that for all $(i, j) \in J$, when $\rho(A_{ij}) = \mu(A_{l_0})$, we have $\sum_{(i,j) \in J} \tau(A_{ij}) < \underline{\tau}(A_{l_0}) - \tau(F)$, and for $(i, j) \in \Pi = \{0, 1, \dots, n\} \times \{0, 1, \dots, m\} \setminus (J \cup \{(l, 0), (k, p)\})$ we have $\rho(A_{ij}) < \mu(A_{l_0})$. Hence, for any given $\varepsilon > 0$ and sufficiently large r , we get

$$T(r, A_{ij}) \leq \begin{cases} (\tau(A_{ij}) + \varepsilon) r^{\mu(A_{l_0})}, & \text{if } (i, j) \in J, \\ r^{\rho(A_{ij}) + \varepsilon} \leq r^{\mu(A_{l_0}) - \varepsilon}, & \text{if } (i, j) \in \Pi \end{cases} \quad (3.38)$$

and

$$T(r, F) \leq \begin{cases} (\tau(F) + \varepsilon) r^{\mu(A_{l_0})}, & \text{if } \rho(F) = \mu(A_{l_0}), \\ r^{\rho(F) + \varepsilon} \leq r^{\mu(A_{l_0}) - \varepsilon}, & \text{if } \rho(F) < \mu(A_{l_0}). \end{cases} \quad (3.39)$$

Then, by substituting (3.24) – (3.26), (3.28), (3.30), (3.34), (3.38) and (3.39) into (3.23), for all z satisfying $|z| = r \in E_7 \setminus E_6$ sufficiently large r , we obtain

$$\begin{aligned} (\underline{\tau}(A_{l_0}) - \varepsilon) r^{\mu(A_{l_0})} &\leq \sum_{(i,j) \in J} (\tau(A_{ij}) + \varepsilon) r^{\mu(A_{l_0})} + \sum_{(i,j) \in \Pi} r^{\mu(A_{l_0}) - \varepsilon} + r^{\mu(A_{kp}) + \varepsilon} \\ &+ (\tau(F) + \varepsilon) r^{\mu(A_{l_0})} + r^{\lambda\left(\frac{1}{A_{l_0}}\right) + \varepsilon} + 2(2r)^{\rho(f) + \varepsilon} + O(r^{\rho(f) - 1 + \varepsilon}) + O(\ln r) \\ &\leq (\tau_1 + (nm + n + m)\varepsilon) r^{\mu(A_{l_0})} + O(r^{\mu(A_{l_0}) - \varepsilon}) + r^{\mu(A_{kp}) + \varepsilon} \\ &+ r^{\lambda\left(\frac{1}{A_{l_0}}\right) + \varepsilon} + 2(2r)^{\rho(f) + \varepsilon} + O(r^{\rho(f) - 1 + \varepsilon}) + O(\log r). \end{aligned} \quad (3.40)$$

Now, we choose sufficiently small ε satisfying

$$0 < \varepsilon < \min \left\{ \frac{\mu(A_{l_0}) - \mu(A_{kp})}{2}, \frac{\underline{\tau}(A_{l_0}) - \tau_1}{nm + n + m + 1}, \frac{\mu(A_{l_0}) - \lambda\left(\frac{1}{A_{l_0}}\right)}{2} \right\},$$

then from (3.40) for sufficiently large $r \in E_7 \setminus E_6$, we get

$$(\underline{\tau}(A_{l_0}) - \tau_1 - (nm + n + m + 1)\varepsilon) r^{\mu(A_{l_0}) - \varepsilon} \leq r^{\rho(f) + \varepsilon},$$

that means, $\mu(A_{l_0}) \leq \rho(f) + 2\varepsilon$, since $\varepsilon > 0$ is arbitrary, then $\rho(f) \geq \mu(A_{l_0})$.

Further, if $F \equiv 0$, then by substituting (3.24) – (3.26), (3.30), (3.34) and (3.38) into (3.23), for all z satisfying $|z| = r \in E_7 \setminus E_6$ sufficiently large r , we obtain

$$\begin{aligned} (\underline{\tau}(A_{l_0}) - \varepsilon) r^{\mu(A_{l_0})} &\leq (\tau_1 + (nm + n + m - 1)\varepsilon) r^{\mu(A_{l_0})} + O(r^{\mu(A_{l_0}) - \varepsilon}) \\ &+ r^{\mu(A_{kp}) + \varepsilon} + r^{\lambda\left(\frac{1}{A_{l_0}}\right) + \varepsilon} + O(r^{\rho(f) - 1 + \varepsilon}) + O(\log r). \end{aligned} \quad (3.41)$$

Now, we choose sufficiently small ε satisfying

$$0 < \varepsilon < \min \left\{ \frac{\mu(A_{l_0}) - \mu(A_{kp})}{2}, \frac{\underline{\tau}(A_{l_0}) - \tau_1}{nm + n + m}, \frac{\mu(A_{l_0}) - \lambda\left(\frac{1}{A_{l_0}}\right)}{2} \right\},$$

then from (3.41) for sufficiently large $r \in E_7 \setminus E_6$, we get

$$(\underline{\tau}(A_{l_0}) - \tau_1 - (nm + n + m)\varepsilon) r^{\mu(A_{l_0}) - \varepsilon} \leq r^{\rho(f) - 1 + \varepsilon},$$

that means, $\mu(A_{l_0}) \leq \rho(f) - 1 + 2\varepsilon$, since $\varepsilon > 0$ is arbitrary, then $\rho(f) \geq \mu(A_{l_0}) + 1$.

Case (iv) We suppose that $\rho(S) = \mu(A_{l_0}) = \mu(A_{kp})$ with $\tau_1 + \underline{\tau}(A_{kp}) < \underline{\tau}(A_{l_0})$.

It follows by substituting (3.24) – (3.26), (3.28), (3.34), (3.35), (3.38) and (3.39) into (3.23), for all sufficiently large $r \in E_8 \setminus E_6$, we have

$$\begin{aligned} (\underline{\tau}(A_{l_0}) - \varepsilon) r^{\mu(A_{l_0})} &\leq (\tau_1 + (nm + n + m)\varepsilon) r^{\mu(A_{l_0})} + O(r^{\mu(A_{l_0}) - \varepsilon}) \\ &+ (\underline{\tau}(A_{kp}) + \varepsilon) r^{\mu(A_{l_0})} + r^{\lambda\left(\frac{1}{A_{l_0}}\right) + \varepsilon} + 2(2r)^{\rho(f) + \varepsilon} + O(r^{\rho(f) - 1 + \varepsilon}) + O(\log r). \end{aligned} \quad (3.42)$$

Now, we choose sufficiently small ε satisfying

$$0 < \varepsilon < \min \left\{ \frac{\underline{\tau}(A_{l_0}) - \tau_1 - \underline{\tau}(A_{kp})}{nm + n + m + 2}, \frac{\mu(A_{l_0}) - \lambda\left(\frac{1}{A_{l_0}}\right)}{2} \right\},$$

then from (3.42) for sufficiently large $r \in E_8 \setminus E_6$, we get

$$(\underline{\tau}(A_{l_0}) - \tau_1 - \underline{\tau}(A_{kp}) - (nm + n + m + 2)\varepsilon) r^{\mu(A_{l_0}) - \varepsilon} \leq r^{\rho(f) + \varepsilon},$$

that means, $\mu(A_{l_0}) \leq \rho(f) + 2\varepsilon$, since $\varepsilon > 0$ is arbitrary, then $\rho(f) \geq \mu(A_{l_0})$.

Further, if $F \equiv 0$, then by It follows by substituting (3.24) – (3.26), (3.34), (3.35) and (3.38), for all sufficiently large $r \in E_8 \setminus E_6$, we have

$$\begin{aligned} (\underline{\tau}(A_{l_0}) - \varepsilon) r^{\mu(A_{l_0})} &\leq (\tau_1 + (nm + n + m - 1)\varepsilon) r^{\mu(A_{l_0})} + O(r^{\mu(A_{l_0}) - \varepsilon}) \\ &+ (\underline{\tau}(A_{kp}) + \varepsilon) r^{\mu(A_{l_0})} + r^{\lambda\left(\frac{1}{A_{l_0}}\right) + \varepsilon} + O(r^{\rho(f) - 1 + \varepsilon}) + O(\log r). \end{aligned} \quad (3.43)$$

Now, we choose sufficiently small ε satisfying

$$0 < \varepsilon < \min \left\{ \frac{\underline{\tau}(A_{l_0}) - \tau_1 - \underline{\tau}(A_{kp})}{nm + n + m + 1}, \frac{\mu(A_{l_0}) - \lambda\left(\frac{1}{A_{l_0}}\right)}{2} \right\},$$

then from (3.43) for sufficiently large $r \in E_8 \setminus E_6$, we get

$$(\underline{\tau}(A_{l_0}) - \tau_1 - \underline{\tau}(A_{kp}) - (nm + n + m + 1)\varepsilon) r^{\mu(A_{l_0}) - \varepsilon} \leq r^{\rho(f) - 1 + \varepsilon},$$

so that means, $\mu(A_{l_0}) \leq \rho(f) - 1 + 2\varepsilon$, since $\varepsilon > 0$ is arbitrary, then $\rho(f) \geq \mu(A_{l_0}) + 1$.

The proof of Theorem 3.1.5 is complete.

Proof of Theorem 3.1.6 Let $f \not\equiv 0$ be a meromorphic solution of (3.1). If f has infinite order, then the result holds. Now, we suppose that $\rho(f) < \infty$. Firstly we divide equation (3.1) by $f(z + c_l)$ to get

$$-A_{l_0}(z) = \sum_{i=0, i \neq l}^n \sum_{j=0}^m A_{ij} \frac{f^{(j)}(z + c_i) f(z + c_i)}{f(z + c_i) f(z + c_l)} + \sum_{j=1}^m A_{lj} \frac{f^{(j)}(z + c_l)}{f(z + c_l)}. \quad (3.44)$$

By virtue of Lemma 2.2.8 and Lemma 3.2.2, for any given $\varepsilon (> 0)$, there exists a subset $E_1 \subset (0, +\infty)$ of finite linear measure, such that for all $r \notin E_1$, we have

$$T(r, A_{l_0}(z)) = m(r, A_{l_0}(z)) + N(r, A_{l_0}(z)) \leq \sum_{i=0, i \neq l}^n \sum_{j=0}^m m(r, A_{ij}(z)) + \sum_{j=1}^m m(r, A_{lj}(z))$$

$$\begin{aligned}
& + \sum_{i=0}^n \sum_{j=1}^m m \left(r, \frac{f^{(j)}(z+c_i)}{f(z+c_i)} \right) + \sum_{i=0, i \neq l}^n m \left(r, \frac{f(z+c_i)}{f(z+c_i)} \right) + N(r, A_{l0}(z)) + O(1) \\
& \leq \sum_{i=0, i \neq l}^n \sum_{j=0}^m T(r, A_{ij}(z)) + \sum_{j=1}^m T(r, A_{lj}(z)) + O(\log r) + O(r^{\rho(f)-1+\varepsilon}) + N(r, A_{l0}(z)) + O(1).
\end{aligned} \tag{3.45}$$

We suppose that $\alpha = \max \{ \rho(A_{ij}) : (i, j) \neq (l, 0) \} < \mu(A_{l0}) = \mu$. Then, by the definition of $\mu(A_{l0})$ and $\rho(A_{ij})$ for any given ε ($0 < 2\varepsilon < \mu - \alpha$) and sufficiently large r , we have

$$T(r, A_{l0}) \geq r^{\mu-\varepsilon} \tag{3.46}$$

and

$$T(r, A_{ij}) \leq r^{\rho(A_{ij})+\varepsilon} \leq r^{\alpha+\varepsilon}, \quad (i, j) \neq (l, 0). \tag{3.47}$$

It follows from the definition of $\lambda\left(\frac{1}{A_{l0}}\right)$, for any given ε ($0 < 2\varepsilon < \mu - \lambda\left(\frac{1}{A_{l0}}\right)$) and sufficiently large r , we have

$$N(r, A_{l0}) \leq r^{\lambda\left(\frac{1}{A_{l0}}\right)+\varepsilon}. \tag{3.48}$$

By substituting (3.46), (3.47) and (3.48) into (3.45), for any given

$$\varepsilon \left(0 < 2\varepsilon < \min \left\{ \mu - \alpha, \mu - \lambda\left(\frac{1}{A_{l0}}\right) \right\} \right)$$

and sufficiently large $r \notin E_1$, we obtain

$$r^{\mu-\varepsilon} \leq n(m+1)r^{\alpha+\varepsilon} + mr^{\alpha+\varepsilon} + r^{\lambda\left(\frac{1}{A_{l0}}\right)+\varepsilon} + O(r^{\rho(f)-1+\varepsilon}) + O(\log r).$$

So

$$(1 - o(1)) r^{\mu-\varepsilon} \leq O(r^{\rho(f)-1+\varepsilon}),$$

that is, $\rho(f) \geq \mu + 1 - 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, then we get $\rho(f) \geq \mu(A_{l0}) + 1$. Now, assume

$$\max \{ \rho(A_{ij}) : (i, j) \neq (l, 0) \} = \mu(A_{l0}) = \mu$$

and

$$\tau_1 = \sum_{\rho(A_{ij})=\mu(A_{l0}), (i,j) \neq (l,0)} \tau(A_{ij}) < \underline{\tau}(A_{l0}) = \tau.$$

Then, there exists a set $\Gamma \subseteq I \times J \setminus \{(l, 0)\}$ such that for $(i, j) \in \Gamma$, we have $\rho(A_{ij}) = \mu$ with $\tau_1 = \sum_{(i,j) \in \Gamma} \tau(A_{ij}) < \tau$, and for $(i, j) \in I \times J \setminus (\Gamma \cup \{(l, 0)\})$ we have $\rho(A_{ij}) < \mu$.

Hence, for any given ε ($0 < \varepsilon < \frac{\tau-\tau_1}{nm+n+m+1}$) and sufficiently large r , we have

$$T(r, A_{ij}) \leq (\tau(A_{ij}) + \varepsilon) r^\mu, \quad (i, j) \in \Gamma \tag{3.49}$$

and

$$T(r, A_{ij}) \leq r^{\mu-\varepsilon}, \quad (i, j) \in I \times J \setminus (\Gamma \cup \{(l, 0)\}). \tag{3.50}$$

By the definition of $\underline{\tau}(A_{l0})$, for the above ε and sufficiently large r , we have

$$T(r, A_{l0}) \geq (\tau - \varepsilon) r^\mu. \tag{3.51}$$

Now, we may choose sufficiently small ε satisfying $0 < \varepsilon < \min \left\{ \frac{\mu - \lambda \left(\frac{1}{A_{l_0}} \right)}{2}, \frac{\tau - \tau_1}{nm + n + m + 1} \right\}$, by substituting the assumptions (3.48)–(3.51) into (3.45), for sufficiently large $r \notin E_1$, we obtain

$$(\tau - \varepsilon) r^\mu \leq \sum_{(i,j) \in \Gamma} (\tau(A_{ij}) + \varepsilon) r^\mu + \sum_{(i,j) \in I \times J \setminus (\Gamma \cup \{(l,0)\})} r^{\mu - \varepsilon} + r^{\lambda \left(\frac{1}{A_{l_0}} \right) + \varepsilon} + O(r^{\rho(f) - 1 + \varepsilon})$$

$$+ O(\log r) \leq (\tau_1 + (nm + n + m)\varepsilon) r^\mu + O(r^{\mu - \varepsilon}) + r^{\lambda \left(\frac{1}{A_l} \right) + \varepsilon} + O(r^{\rho(f) - 1 + \varepsilon}) + O(\log r).$$

So

$$(1 - o(1))(\tau - \tau_1 - (nm + n + m + 1)\varepsilon) r^\mu \leq O(r^{\rho(f) - 1 + \varepsilon}),$$

which implies $\rho(f) \geq \mu + 1 - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, then we get $\rho(f) \geq \mu(A_{l_0}) + 1$.

Proof of Theorem 3.1.7 Let f be a meromorphic solution of (3.2). If f has infinite order, then the result holds. Now, we suppose that $\rho(f) < \infty$. We divide equation (3.2) by $f(z + c_l)$ to get

$$-A_{l_0}(z) = \sum_{i=0, i \neq l}^n \sum_{j=0}^m A_{ij} \frac{f^{(j)}(z + c_i)}{f(z + c_i)} \frac{f(z + c_i)}{f(z + c_l)} + \sum_{j=1}^m A_{lj} \frac{f^{(j)}(z + c_l)}{f(z + c_l)} - \frac{F(z)}{f(z + c_l)}. \quad (3.52)$$

By virtue of Lemma 3.2.2 and Lemma 2.2.8, for any given $\varepsilon (> 0)$, there exists a subset $E_1 \subset (0, +\infty)$ of finite linear measure such that for all $r \notin E_1$, we have

$$T(r, A_{l_0}(z)) = m(r, A_{l_0}(z)) + N(r, A_{l_0}(z)) \leq \sum_{i=0, i \neq l}^n \sum_{j=0}^m m(r, A_{ij}(z)) + \sum_{j=1}^m m(r, A_{lj}(z))$$

$$+ \sum_{i=0}^n \sum_{j=1}^m m\left(r, \frac{f^{(j)}(z + c_i)}{f(z + c_i)}\right) + \sum_{i=0, i \neq l}^n m\left(r, \frac{f(z + c_i)}{f(z + c_l)}\right) + m(r, F(z)) + m\left(r, \frac{1}{f(z + c_l)}\right)$$

$$+ N(r, A_{l_0}(z)) + O(1) \leq \sum_{i=0, i \neq l}^n \sum_{j=0}^m T(r, A_{ij}(z)) + \sum_{j=1}^m T(r, A_{lj}(z)) + O(\log r)$$

$$+ O(r^{\rho(f) - 1 + \varepsilon}) + T(r, F(z)) + T\left(r, \frac{1}{f(z + c_l)}\right) + N(r, A_{l_0}(z)) + O(1). \quad (3.53)$$

By Lemma 2.2.2 and the first main theorem of Nevanlinna, when r sufficiently large, we have

$$T\left(r, \frac{1}{f(z + c_l)}\right) = T(r, f(z + c_l)) + O(1) \leq (1 + o(1))T(r + |c_l|, f) \leq 2T(2r, f). \quad (3.54)$$

So, for $r \notin E_1$ sufficiently large we obtain

$$T(r, A_{l_0}(z)) \leq \sum_{i=0, i \neq l}^n \sum_{j=0}^m T(r, A_{ij}(z)) + \sum_{j=1}^m T(r, A_{lj}(z)) + O(\log r)$$

$$+ O(r^{\rho(f) - 1 + \varepsilon}) + T(r, F(z)) + 2T(2r, f) + N(r, A_{l_0}(z)) + O(1).$$

$$\begin{aligned} &\leq \sum_{i=0, i \neq l}^n \sum_{j=0}^m T(r, A_{ij}(z)) + \sum_{j=1}^m T(r, A_{lj}(z)) + O(\log r) \\ &\quad + O(r^{\rho(f)-1+\varepsilon}) + T(r, F(z)) + O(r^{\rho(f)+\varepsilon}) + N(r, A_{l0}(z)) + O(1). \end{aligned} \quad (3.55)$$

(i) If $\rho(F) < \mu$, then for any given ε ($0 < 2\varepsilon < \mu - \rho(F)$) and sufficiently large r , we have

$$T(r, F) \leq r^{\rho(F)+\varepsilon}. \quad (3.56)$$

Assume $\alpha = \max\{\rho(A_{ij}) : (i, j) \neq (l, 0)\} < \mu$. By substituting (3.46) – (3.48) and (3.56) into (3.55), for any given ε ($0 < \varepsilon < \min\left\{\frac{\mu-\alpha}{2}, \frac{\mu-\lambda\left(\frac{1}{A_{l0}}\right)}{2}, \frac{\mu-\rho(F)}{2}\right\}$) and sufficiently large $r \notin E_1$, we obtain

$$\begin{aligned} r^{\mu-\varepsilon} &\leq \sum_{i=0, i \neq l}^n \sum_{j=0}^m r^{\alpha+\varepsilon} + \sum_{j=1}^m r^{\alpha+\varepsilon} + O(r^{\rho(f)-1+\varepsilon}) + r^{\rho(F)+\varepsilon} \\ &\quad + O(r^{\rho(f)+\varepsilon}) + r^{\lambda\left(\frac{1}{A_{l0}}\right)+\varepsilon} + O(\log r) \\ &\leq n(m+1)r^{\alpha+\varepsilon} + mr^{\alpha+\varepsilon} + O(r^{\rho(f)-1+\varepsilon}) + r^{\rho(F)+\varepsilon} + O(r^{\rho(f)+\varepsilon}) + r^{\lambda\left(\frac{1}{A_{l0}}\right)+\varepsilon} + O(\log r). \end{aligned}$$

Then

$$(1 - o(1)) r^{\mu-\varepsilon} \leq O(r^{\rho(f)+\varepsilon}),$$

which implies $\rho(f) \geq \mu - \varepsilon$, since $\varepsilon > 0$ arbitrary number, then $\rho(f) \geq \mu$.

Assume

$$\max\{\rho(A_{ij}) : (i, j) \neq (l, 0)\} = \mu(A_{l0}) = \mu \quad \text{and} \quad \tau_1 < \tau.$$

By substituting the assumptions (3.48) – (3.51) and (3.56) into (3.55), for any given ε satisfying

$$0 < \varepsilon < \min\left\{\frac{\tau - \tau_1}{mn + n + m + 1}, \frac{\mu - \lambda\left(\frac{1}{A_{l0}}\right)}{2}, \frac{\mu - \rho(F)}{2}\right\}$$

and sufficiently large $r \notin E_1$, we obtain

$$\begin{aligned} (\tau - \varepsilon) r^\mu &\leq \sum_{(i,j) \in \Gamma} (\tau(A_{ij}) + \varepsilon) r^\mu + \sum_{(i,j) \in I \times J \setminus (\Gamma \cup \{(l,0)\})} r^{\mu-\varepsilon} + r^{\rho(F)+\varepsilon} + O(r^{\rho(f)-1+\varepsilon}) \\ &\quad + r^{\rho(F)+\varepsilon} + O(r^{\rho(f)+\varepsilon}) + r^{\lambda\left(\frac{1}{A_{l0}}\right)+\varepsilon} + O(\log r) \leq (\tau_1 + (mn + n + m)\varepsilon) r^\mu + O(r^{\mu-\varepsilon}) \\ &\quad + r^{\rho(F)+\varepsilon} + O(r^{\rho(f)-1+\varepsilon}) + r^{\rho(F)+\varepsilon} + O(r^{\rho(f)+\varepsilon}) + r^{\lambda\left(\frac{1}{A_l}\right)+\varepsilon} + O(\log r). \end{aligned}$$

Then

$$(1 - o(1)) (\tau - \tau_1 - (mn + n + m + 1)\varepsilon) r^\mu \leq O(r^{\rho(f)+\varepsilon}),$$

which implies $\rho(f) \geq \mu = \mu(A_{l0})$.

If $\rho(F) = \mu(A_{l0}) = \mu$ and $\tau_1 + \tau(F) < \tau$. Then, for any given $\varepsilon > 0$ and sufficiently large r , we have

$$T(r, F) \leq (\tau(F) + \varepsilon) r^\mu. \quad (3.57)$$

Now, we may choose sufficiently small ε satisfying

$$0 < \varepsilon < \min \left\{ \frac{\mu - \lambda \left(\frac{1}{A_{l_0}} \right)}{2}, \frac{\tau - \tau_1 - \tau(F)}{nm + n + m + 2} \right\},$$

by substituting the assumptions (3.48) – (3.51) and (3.57) into (3.55), for sufficiently large $r \notin E_1$, we obtain

$$\begin{aligned} (\tau - \varepsilon) r^\mu &\leq \sum_{(i,j) \in \Gamma} (\tau(A_{ij}) + \varepsilon) r^\mu + \sum_{(i,j) \in I \times J \setminus (\Gamma \cup \{(l,0)\})} r^{\mu - \varepsilon} + O(r^{\rho(f) - 1 + \varepsilon}) + (\tau(F) + \varepsilon) r^\mu \\ &\quad + O(r^{\rho(f) + \varepsilon}) + r^{\lambda \left(\frac{1}{A_{l_0}} \right) + \varepsilon} + O(\log r) \\ &\leq (\tau_1 + (nm + n + m)\varepsilon) r^\mu + O(r^{\mu - \varepsilon}) + O(r^{\rho(f) - 1 + \varepsilon}) + (\tau(F) + \varepsilon) r^\mu \\ &\quad + O(r^{\rho(f) + \varepsilon}) + r^{\lambda \left(\frac{1}{A_{l_0}} \right) + \varepsilon} + O(\log r). \end{aligned}$$

It follows that

$$(1 - o(1)) (\tau - \tau_1 - \tau(F) - (nm + n + m + 2)\varepsilon) r^\mu \leq +O(r^{\rho(f) + \varepsilon}),$$

which gives $\rho(f) \geq \mu = \mu(A_{l_0})$.

If $\mu(F) = \mu(A_{l_0}) = \mu$ and $\tau_1 + \tau < \underline{\tau}(F)$. Then for any given $\varepsilon > 0$ and sufficiently large r , we have

$$T(r, F) > (\underline{\tau}(F) - \varepsilon) r^\mu. \quad (3.58)$$

By Lemma 2.2.10, for the above $\varepsilon > 0$, there exists a subset E_4 with infinite logarithmic measure such that for all $r \in E_4$, we have

$$T(r, A_{l_0}) \leq (\tau + \varepsilon) r^\mu. \quad (3.59)$$

From (3.2), by using Lemma 2.2.2 and Lemma 3.2.3, we get

$$\begin{aligned} T(r, F(z)) &\leq \sum_{i=0, i \neq l}^n \sum_{j=0}^m T(r, A_{ij}(z)) + T(r, A_{l_0}(z)) + \sum_{j=1}^m T(r, A_{lj}(z)) \\ &\quad + \sum_{i=0}^n \sum_{j=0}^m T(r, f^{(j)}(z + c_i)) + O(1) \leq \sum_{i=0, i \neq l}^n \sum_{j=0}^m T(r, A_{ij}(z)) + T(r, A_{l_0}(z)) \\ &\quad + \sum_{j=1}^m T(r, A_{lj}(z)) + \sum_{i=0}^n \sum_{j=0}^m ((j+1) T(r, f(z + c_i)) + S(r, f)) \\ &\leq \sum_{i=0, i \neq l}^n \sum_{j=0}^m T(r, A_{ij}(z)) + T(r, A_{l_0}(z)) + \sum_{j=1}^m T(r, A_{lj}(z)) + O(T(2r, f(z))) + S(r, f). \end{aligned} \quad (3.60)$$

Now, we may choose sufficiently small ε satisfying $0 < \varepsilon < \frac{\underline{\tau}(F) - \tau_1 - \underline{\tau}(A_{l_0})}{nm + n + m + 2}$, by substituting (3.49), (3.50), (3.58) and (3.59) into (3.60) for all $r \in E_4$ sufficiently large, we get

$$(\underline{\tau}(F) - \varepsilon) r^\mu \leq \sum_{(i,j) \in \Gamma} (\tau(A_{ij}) + \varepsilon) r^\mu + (\tau + \varepsilon) r^\mu + \sum_{(i,j) \in I \times J \setminus (\Gamma \cup \{(l,0)\})} r^{\mu - \varepsilon} + O(T(2r, f(z)))$$

$$+o(T(r, f)) \leq (\tau_1 + \tau + (nm + n + m + 1)\varepsilon) r^\mu + O(r^{\mu-\varepsilon}) + O(T(2r, f(z))) + o(T(r, f)).$$

It follows that

$$(1 - o(1)) (\underline{\tau}(F) - \tau_1 - \underline{\tau}(A_{l_0}) - (nm + n + m + 2)\varepsilon) r^\mu \leq O(T(2r, f(z))) + o(T(r, f)),$$

which implies $\rho(f) \geq \mu = \mu(A_{l_0})$.

(ii) We consider the case $\mu(F) > \mu(A_{l_0}) = \mu$. Then, for any given ε ($0 < 2\varepsilon < \mu(F) - \mu$) and sufficiently large r , we have

$$T(r, F) \geq r^{\mu(F)-\varepsilon}. \quad (3.61)$$

By Lemma 2.2.9, for any given $\varepsilon > 0$, there exists a subset E_5 with infinite logarithmic measure such that for all $r \in E_5$, we have

$$T(r, A_{l_0}) \leq r^{\mu+\varepsilon}. \quad (3.62)$$

If $\alpha = \max\{\rho(A_{ij}) : (i, j) \neq (l, 0)\} < \mu(A_{l_0}) = \mu$, then by substituting (3.47), (3.61) and (3.62) into (3.60), for any given ε ($0 < 2\varepsilon < \mu(F) - \mu$) and sufficiently large $r \in E_5$, we obtain

$$r^{\mu(F)-\varepsilon} \leq n(m+1)r^{\alpha+\varepsilon} + r^{\mu+\varepsilon} + mr^{\alpha+\varepsilon} + O(T(2r, f(z))) + o(T(r, f)).$$

Thus

$$(1 - o(1)) r^{\mu(F)-\varepsilon} \leq O(T(2r, f(z))) + o(T(r, f)),$$

that is $\rho(f) \geq \mu(F) - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, then we get $\rho(f) \geq \mu(F)$.

If

$$\max\{\rho(A_{ij}) : (i, j) \neq (l, 0)\} = \mu(A_{l_0}) = \mu \quad \text{and} \quad \tau_1 < \tau.$$

Then, by substituting the assumptions (3.49), (3.50), (3.59) and (3.61) into (3.60), for any given ε ($0 < 2\varepsilon < \mu(F) - \mu$) and sufficiently large $r \in E_4$, we obtain

$$\begin{aligned} r^{\mu(F)-\varepsilon} &\leq \sum_{(i,j) \in \Gamma} (\tau(A_{ij}) + \varepsilon) r^\mu + \sum_{(i,j) \in I \times J \setminus (\Gamma \cup \{(l,0)\})} r^{\mu-\varepsilon} \\ &\quad + (\tau + \varepsilon) r^\mu + O(T(2r, f(z))) + o(T(r, f)) \\ &\leq (\tau_1 + n(m+1)\varepsilon) r^\mu + O(r^{\mu-\varepsilon}) + (\tau + \varepsilon) r^\mu + O(T(2r, f(z))) + o(T(r, f)). \end{aligned}$$

Thus

$$(1 - o(1)) r^{\mu(F)-\varepsilon} \leq O(T(2r, f(z))) + o(T(r, f)),$$

which gives $\rho(f) \geq \mu(F) - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, then we get $\rho(f) \geq \mu(F)$. The proof of Theorem 3.1.7 is complete.

3.4 Examples

Example 3.4.1. Consider the non-homogeneous differential-difference equation with entire coefficients

$$A_{02}(z)f''(z) + A_{11}(z)f'(z+1) + A_{01}(z)f'(z) + A_{10}(z)f(z+1) + A_{00}(z)f(z) = F(z). \quad (3.63)$$

Case 1. $\rho(S) < \mu(A_{10})$. In (3.63) for

$$\begin{aligned} A_{00}(z) &= \pi^2 + 2\pi^4 z^2, & A_{10}(z) &= e^{-\pi^2 z^2 - \pi^2}, & A_{01}(z) &= 2\pi^2(z+1)e^{2\pi^2 z + \pi^2}, \\ A_{11}(z) &= -2\pi^2 z, & A_{02}(z) &= -\frac{1}{2}, & F(z) &= e^{2\pi^2 z}, \end{aligned}$$

we have

$$\max\{\rho(F), \mu(A_{11}), \rho(A_{ij}) : (i, j) \neq (1, 0), (1, 1)\} = 1 < \mu(A_{10}) = 2.$$

We see that the conditions of Theorem 3.1.4 are verified. The function

$$f(z) = e^{\pi^2 z^2}$$

is a solution of equation (3.63) which satisfies $\rho(f) = 2 \geq \mu(A_{10}) = 2$.

Case 2. $\rho(S) < \mu(A_{10}) = \mu(A_{kp})$ with $\underline{\tau}_M(A_{10}) > \underline{\tau}_M(A_{kp})$. In (3.63) for

$$\begin{aligned} A_{00}(z) &= 2\pi^2, & A_{10}(z) &= 2\pi^2(z+1)e^{z^2} + e^{-\pi^2 z^2 - \pi^2}, & A_{01}(z) &= 2\pi^2 z, \\ A_{11}(z) &= -e^{z^2}, & A_{02}(z) &= -1, & F(z) &= e^{2\pi^2 z}, \end{aligned}$$

we get $\max\{\rho(F), \rho(A_{ij}) : (i, j) \neq (1, 0), (1, 1)\} = 1 < \mu(A_{10}) = \mu(A_{11}) = 2$ and $\underline{\tau}_M(A_{10}) = \pi^2 > \underline{\tau}_M(A_{11}) = 1$. Hence, the conditions of Theorem 3.1.4 are satisfied. The function

$$f(z) = e^{\pi^2 z^2}$$

is a solution of equation (3.63) and f satisfies $\rho(f) = 2 \geq \mu(A_{10}) = 2$.

Case 3. $\mu(A_{10}) = \rho(S) > \mu(A_{kp})$ with

$$\underline{\tau}_M(A_{10}) > \tau_1 = \max\{\tau_M(g) : \rho(g) = \mu(A_{10}), g \in S\}.$$

In (3.63) for

$$\begin{aligned} A_{00}(z) &= \pi^2 + 2\pi^4 z^2, & A_{10}(z) &= e^{-\frac{4}{5}\pi^2 z^2 - \pi^2}, & A_{01}(z) &= 2\pi^2(z+1)e^{2\pi^2 z + \pi^2}, \\ A_{11}(z) &= -2\pi^2 z, & A_{02}(z) &= -\frac{1}{2}, & F(z) &= e^{\frac{1}{5}\pi^2 z^2 + 2\pi^2 z}, \end{aligned}$$

we have $\mu(A_{10}) = \max\{\rho(F), \rho(A_{ij}) : (i, j) \neq (1, 0), (0, 1)\} = 2 > \mu(A_{01}) = 1$ and $\underline{\tau}_M(A_{10}) = \frac{4\pi^2}{5} > \tau_1 = \tau_M(F) = \frac{\pi^2}{5}$. Obviously, the conditions of Theorem 3.1.4 are verified. The function

$$f(z) = e^{\pi^2 z^2}$$

is a solution of equation (3.63) which satisfies $\rho(f) = 2 \geq \mu(A_{10}) = 2$.

Case 4. $\mu(A_{10}) = \mu(A_{kp}) = \rho(S)$, and $\underline{\tau}_M(A_{10}) > \max\{\tau_1, \underline{\tau}_M(A_{kp})\}$. In (3.63) for

$$\begin{aligned} A_{00}(z) &= 2\pi^2, & A_{10}(z) &= 2\pi^2(z+1)e^{z^2} + e^{-\frac{4}{5}\pi^2 z^2 - \pi^2}, & A_{01}(z) &= 2\pi^2 z, \\ A_{11}(z) &= -e^{z^2}, & A_{02}(z) &= -1, & F(z) &= e^{\frac{1}{5}\pi^2 z^2 + 2\pi^2 z}, \end{aligned}$$

we get $\mu(A_{10}) = \mu(A_{11}) = \max\{\rho(F), \rho(A_{ij}) : (i, j) \neq (1, 0), (1, 1)\} = 2$ and

$$\underline{\tau}_M(A_{10}) = \frac{4\pi^2}{5} > \max\{\tau_1, \underline{\tau}_M(A_{11})\} = \max\{\tau_M(F), \underline{\tau}_M(A_{11})\} = \frac{\pi^2}{5}.$$

We see that the conditions of Theorem 3.1.4 are satisfied. The function

$$f(z) = e^{\pi^2 z^2}$$

is a solution of equation (3.63) which satisfies $\rho(f) = 2 \geq \mu(A_{10}) = 2$.

Example 3.4.2. Consider the homogeneous differential-difference equation with entire coefficients

$$A_{11}(z)g'(z-1) + A_{20}(z)g(z+3) + A_{00}(z)g(z) = 0. \quad (3.64)$$

Case 1. $\max\{\mu(A_{kp}), \rho(A_{ij}) : (i, j) \neq (l, 0), (k, p)\} < \mu(A_{l0})$. In (3.64), for

$$A_{00}(z) = 1, \quad A_{20}(z) = (4\pi i(1-z) - e^{4\pi iz}) e^{-16\pi iz}, \quad A_{11}(z) = 1,$$

we have

$$\max\{\mu(A_{11}), \rho(A_{ij}) : (i, j) \neq (2, 0), (1, 1)\} = 0 < \mu(A_{20}) = 1.$$

So, the conditions of Theorem 3.1.4 are satisfied. The meromorphic function

$$g(z) = e^{2\pi iz^2}$$

is a solution of equation (3.64) and g satisfies $\rho(g) = 2 \geq \mu(A_{20}) + 1 = 2$.

Case 2. $\max\{\rho(A_{ij}) : (i, j) \neq (l, 0), (k, p)\} < \mu(A_{l0}) = \mu(A_{kp})$, with $\tau_M(A_{l0}) > \tau_M(A_{kp})$. In (3.64), for

$$A_{00}(z) = 1, \quad A_{20}(z) = (4\pi i(1-z) - e^{2\pi iz}) e^{-14\pi iz}, \quad A_{11}(z) = e^{2\pi iz},$$

we get $\mu(A_{20}) = \mu(A_{11}) = 1 > \max\{\rho(A_{ij}) : (i, j) \neq (2, 0), (1, 1)\} = 0$ and $\tau_M(A_{20}) = 14\pi > \tau_M(A_{11}) = 2\pi$. Obviously, the conditions of Theorem 3.1.4 are verified. The meromorphic function

$$g(z) = e^{2\pi iz^2}$$

is a solution of equation (3.64) and g satisfies $\rho(g) = 2 \geq \mu(A_{20}) + 1 = 2$.

Case 3. $\mu(A_{l0}) = \mu(A_{kp}) = \max\{\rho(A_{ij}) : (i, j) \neq (l, 0), (k, p)\}$ with $\tau_M(A_{l0}) > \tau_1 = \max\{\tau_M(A_{ij}) : \rho(A_{ij}) = \mu(A_{l0}), (i, j) \neq (l, 0), (k, p)\}$. In (3.64), for

$$A_{00}(z) = e^{-2\pi iz}, \quad A_{20}(z) = (4\pi i(1-z) - 1) e^{-14\pi iz}, \quad A_{11}(z) = e^{2\pi iz},$$

we have $\mu(A_{20}) = \mu(A_{11}) = \max\{\rho(A_{ij}) : (i, j) \neq (2, 0), (1, 1)\} = 1$ and $\tau_M(A_{20}) = 14\pi > \max\{\tau_M(A_{00}), \tau_M(A_{11})\} = 2\pi$. It is clear that the conditions of Theorem 3.1.4 are satisfied. The meromorphic function

$$g(z) = e^{2\pi iz^2}$$

is a solution of equation (3.64) and g satisfies $\rho(g) = 2 \geq \mu(A_{20}) + 1 = 2$.

Example 3.4.3. Consider the non-homogeneous differential-difference equation with meromorphic coefficients

$$A_{11}(z)f'(z-1) + A_{01}(z)f'(z) + A_{20}(z)f(z+1) + A_{10}(z)f(z-1) = F(z). \quad (3.65)$$

Case 1. $\max\{\mu(A_{kp}), \rho(S)\} < \mu(A_{l0})$. In (3.65) for

$$A_{10}(z) = e^{-\pi^3 z^3 + 3\pi^3 z^2 - 3\pi^3 z + \pi^3}, \quad A_{20}(z) = 3\pi^3 (2z-1) e^{-3\pi^3 z^2 - 3\pi^3 z - \pi^3},$$

$$A_{01}(z) = -1, \quad A_{11}(z) = e^{3\pi^3 z^2 - 3\pi^3 z + \pi^3}, \quad F(z) = \tan(\pi z),$$

we have

$$\max\{\rho(F), \mu(A_{11}), \rho(A_{ij}) : (i, j) \neq (1, 0), (1, 1)\} = 2 < \mu(A_{10}) = 3,$$

$$\lambda\left(\frac{1}{A_{10}}\right) = 0 < \mu(A_{10}) = 3.$$

It is easy to see that the conditions of Theorem 3.1.5 are verified. The function

$$f(z) = e^{\pi^3 z^3} \tan(\pi z)$$

is a solution of equation (3.65) which satisfies $\rho(f) = 3 \geq \mu(A_{10}) = 3$.

Case 2. $\rho(S) < \mu(A_{10}) = \mu(A_{kp})$ with $\underline{\tau}(A_{10}) > \underline{\tau}(A_{kp})$. In (3.65) for

$$\begin{aligned} A_{10}(z) &= e^{-\pi^3 z^3 + 3\pi^3 z^2 - 3\pi^3 z + \pi^3} \\ &+ \left(3\pi^3 (2z - 1) - 3z^2 \pi^3 - \pi \tan(\pi z) + \frac{\pi}{\tan(\pi z)} \right) e^{-z^3}, \\ A_{20}(z) &= 3\pi^3 z^2 + \pi \tan(\pi z) + \frac{\pi}{\tan(\pi z)}, \quad A_{01}(z) = -e^{3\pi^3 z^2 + 3\pi^3 z + \pi^3}, \\ A_{11}(z) &= e^{-z^3}, \quad F(z) = \tan(\pi z), \end{aligned}$$

we get

$$\max\{\rho(F), \rho(A_{ij}) : (i, j) \neq (1, 0), (1, 1)\} = 2 < \mu(A_{10}) = \mu(A_{11}) = 3,$$

$$\lambda\left(\frac{1}{A_{10}}\right) = 1 < \mu(A_{10}) = 3$$

and

$$\underline{\tau}(A_{10}) = \pi^2 > \underline{\tau}(A_{11}) = \frac{1}{\pi}.$$

Hence, the conditions of Theorem 3.1.5 are satisfied. The function

$$f(z) = e^{\pi^3 z^3}$$

is a solution of equation (3.65) and f satisfies $\rho(f) = 3 \geq \mu(A_{10}) = 3$.

Case 3. $\mu(A_{10}) = \rho(S) > \mu(A_{kp})$ with

$$\underline{\tau}(A_{10}) > \tau_1 = \sum_{\rho(A_{ij}) = \mu(A_{10}), (i,j) \neq (l,0), (k,p)} \tau(A_{ij}) + \tau(F).$$

In (3.65) for

$$\begin{aligned} A_{10}(z) &= e^{-2\pi^3 z^3 + 3\pi^3 z^2 - 3\pi^3 z + \pi^3}, \quad A_{20}(z) = 3\pi^3 (2z - 1) e^{-3\pi^3 z^2 - 3\pi^3 z - \pi^3}, \\ A_{01}(z) &= -1, \quad A_{11}(z) = e^{3\pi^3 z^2 - 3\pi^3 z + \pi^3}, \quad F(z) = \frac{\tan(\pi z)}{e^{\pi^3 z^3}}, \end{aligned}$$

we have

$$\mu(A_{10}) = \max\{\rho(F), \rho(A_{ij}) : (i, j) \neq (1, 0), (1, 1)\} = 3 > \mu(A_{11}) = 2,$$

$$\lambda\left(\frac{1}{A_{10}}\right) = 0 < \mu(A_{10}) = 3$$

and $\underline{\tau}(A_{10}) = 2\pi^2 > \tau_1 = \tau(F) = \pi^2$. We can see that the conditions of Theorem 3.1.5 are verified. The function

$$f(z) = e^{\pi^3 z^3} \tan(\pi z)$$

is a solution of equation (3.65) which satisfies $\rho(f) = 3 \geq \mu(A_{10}) = 3$.

Case 4. $\mu(A_{10}) = \mu(A_{k0}) = \rho(S)$, and $\underline{\tau}(A_{10}) > \tau_1 + \underline{\tau}(A_{kp})$. In (3.65) for

$$A_{10}(z) = e^{-2\pi^3 z^3 + 3\pi^3 z^2 - 3\pi^3 z + \pi^3}, \quad A_{20}(z) = 3\pi^3 (2z - 1) e^{(\frac{\pi}{4}z)^3 - 3\pi^3 z^2 - 3\pi^3 z - \pi^3},$$

$$A_{01}(z) = -e^{(\frac{\pi}{4}z)^3}, \quad A_{11}(z) = e^{(\frac{\pi}{4}z)^3 + 3\pi^3 z^2 - 3\pi^3 z + \pi^3}, \quad F(z) = \frac{\tan(\pi z)}{e^{\pi^3 z^3}},$$

we get

$$\mu(A_{10}) = \mu(A_{11}) = \max\{\rho(F), \rho(A_{ij}) : (i, j) \neq (1, 0), (1, 1)\} = 3,$$

$$\lambda\left(\frac{1}{A_{10}}\right) = 0 < \mu(A_{10}) = 3$$

and $\tau_1 + \underline{\tau}(A_{11}) = \tau(A_{01}) + \tau(A_{20}) + \tau(F) + \underline{\tau}(A_{11}) = \left(\frac{2}{4^3} + 1\right)\pi^2 + \frac{\pi^2}{4^3} = \frac{67}{64}\pi^2 < \underline{\tau}(A_{10}) = 2\pi^2$. Obviously, the conditions of Theorem 3.1.5 are satisfied. The function

$$f(z) = e^{\pi^3 z^3} \tan(\pi z)$$

is a solution of equation (3.65) which satisfies $\rho(f) = 3 \geq \mu(A_{10}) = 3$.

Example 3.4.4. Consider the homogeneous (differential)-difference equation with meromorphic coefficients

$$A_{11}(z)h'(z + i\pi) + A_{20}(z)h(z + 2i\pi) + A_{00}(z)h(z) = 0. \quad (3.66)$$

Case 1. $\max\{\mu(A_{kp}), \rho(A_{ij}) : (i, j) \neq (l, 0), (k, p)\} < \mu(A_{l0})$. In (3.66), for

$$A_{00}(z) = -1, \quad A_{20}(z) = e^{12\pi z^2 + 24\pi^2 iz - 16\pi^3} - e^{6\pi z^2 + 18\pi^2 iz - 14\pi^3},$$

$$A_{11}(z) = \frac{\cos(2iz)}{6i(z + i\pi)^2 \cos(2iz) + 2i \sin(2iz)},$$

we have $\max\{\mu(A_{11}), \rho(A_{ij}) : (i, j) \neq (2, 0), (1, 1)\} = 1 < \mu(A_{20}) = 2$ and

$$\lambda\left(\frac{1}{A_{20}}\right) = 0 < \mu(A_{20}) = 2.$$

Obviously, the conditions of Theorem 3.1.5 are verified. The meromorphic function

$$h(z) = \frac{e^{2iz^3}}{\cos(2iz)}$$

is a solution of equation (3.66) and h satisfies $\rho(h) = 3 \geq \mu(A_{20}) + 1 = 3$.

Case 2. $\max\{\rho(A_{ij}) : (i, j) \neq (l, 0), (k, p)\} < \mu(A_{l0}) = \mu(A_{kp})$ with $\underline{\tau}(A_{l0}) > \underline{\tau}(A_{kp})$. In (3.66), for

$$A_{00}(z) = 1, \quad A_{20}(z) = -2e^{12\pi z^2 + 24\pi^2 iz - 16\pi^3},$$

$$A_{11}(z) = \frac{e^{6\pi z^2 + 6\pi^2 iz - 2\pi^3} \cos(2iz)}{6i(z + i\pi)^2 \cos(2iz) + 2i \sin(2iz)},$$

we get

$$\mu(A_{20}) = \mu(A_{11}) = 2 > \max\{\rho(A_{ij}) : (i, j) \neq (2, 0), (1, 1)\} = \rho(A_{00}) = 0,$$

$$\lambda\left(\frac{1}{A_{20}}\right) = 0 < \mu(A_{20}) = 2$$

and

$$\tau(A_{20}) = 12 > \tau(A_{11}) = 6.$$

It is clear that the conditions of Theorem 3.1.5 are satisfied. The meromorphic function

$$h(z) = \frac{e^{2iz^3}}{\cos(2iz)}$$

is a solution of equation (3.66) and h satisfies $\rho(h) = 3 \geq \mu(A_{20}) + 1 = 3$.

Case 3. $\mu(A_{l0}) = \mu(A_{kp}) = \max\{\rho(A_{ij}) : (i, j) \neq (l, 0), (k, p)\}$, with

$$\tau(A_{l0}) > \sum_{\rho(A_{ij})=\mu(A_{l0}), (i,j)\neq(l,0),(k,p)} \tau(A_{ij}) + \tau(A_{kp}).$$

In (3.66), for

$$\begin{aligned} A_{00}(z) &= e^{\pi z^2}, & A_{20}(z) &= -2e^{13\pi z^2 + 24\pi^2 iz - 16\pi^3}, \\ A_{11}(z) &= \frac{e^{7\pi z^2 + 6\pi^2 iz - 2\pi^3} \cos(2iz)}{6i(z + i\pi)^2 \cos(2iz) + 2i \sin(2iz)}, \end{aligned}$$

we have

$$\mu(A_{20}) = \mu(A_{11}) = \max\{\rho(A_{ij}) : (i, j) \neq (2, 0), (1, 1)\} = \rho(A_{00}) = 2,$$

$$\lambda\left(\frac{1}{A_{20}}\right) = 0 < \mu(A_{20}) = 2$$

and

$$\tau(A_{20}) = 13 > \tau(A_{00}) + \tau(A_{11}) = 1 + 7 = 8.$$

It is easy to see that the conditions of Theorem 3.1.5 are verified. The meromorphic function

$$h(z) = \frac{e^{2iz^3}}{\cos(2iz)}$$

is a solution of equation (3.66) and h satisfies $\rho(h) = 3 \geq \mu(A_{20}) + 1 = 3$.

Example 3.4.5. Consider the homogeneous differential-difference equation with meromorphic coefficients

$$A_{20}f(z + i\pi^2) + A_{10}f\left(z + \frac{i\pi^2}{2}\right) + A_{01}f'(z) + A_{00}f(z) = 0. \quad (3.67)$$

Case 1. $\max \{\rho(A_{ij}) : (i, j) \neq (l, 0)\} < \mu(A_{l0})$. In (3.67), for

$$A_{00}(z) = e^{-(6\pi^3 z + i\pi^5)} - 1, \quad A_{01}(z) = \frac{\pi \sin\left(\frac{iz}{\pi}\right)}{24z^2 \sin\left(\frac{iz}{\pi}\right) - i \cos\left(\frac{iz}{\pi}\right)},$$

$$A_{10}(z) = \left(e^{-12\pi iz^2} - e^{12\pi iz^2}\right) \cot\left(\frac{iz}{\pi}\right), \quad A_{20}(z) = e^{18\pi^3 z + 7i\pi^5},$$

we have

$$\lambda\left(\frac{1}{A_{10}}\right) = 1 < \mu(A_{10}) = 2, \quad \max \{\rho(A_{ij}) : (i, j) \neq (1, 0)\} = 1 < \mu(A_{10}) = 2.$$

As we see, the conditions of Theorem 3.1.6 are verified. The meromorphic function

$$f(z) = \frac{e^{\frac{8z^3}{\pi}}}{\sin\left(\frac{iz}{\pi}\right)}$$

is a solution of equation (3.67) and f satisfies

$$\rho(f) = 3 \geq \mu(A_{10}) + 1.$$

Case 2. $\max \{\rho(A_{ij}) : (i, j) \neq (l, 0)\} = \mu(A_{l0}) = \mu$ and

$$\tau_1 = \sum_{\rho(A_{ij})=\mu(A_{l0}), (i,j) \neq (l,0)} \tau(A_{ij}) < \tau(A_{l0}) = \tau.$$

In (4.1), for

$$A_{00}(z) = e^{-(6\pi^3 z + i\pi^5)} - 1, \quad A_{01}(z) = \frac{\pi \sin\left(\frac{iz}{\pi}\right)}{24z^2 \sin\left(\frac{iz}{\pi}\right) - i \cos\left(\frac{iz}{\pi}\right)},$$

$$A_{10}(z) = \left(e^{-12\pi iz^2} - e^{24\pi iz^2}\right) \cot\left(\frac{iz}{\pi}\right), \quad A_{20}(z) = e^{12\pi iz^2 + 18\pi^3 z + 7i\pi^5},$$

we have

$$\lambda\left(\frac{1}{A_{10}}\right) = 1 < \mu(A_{10}) = 2$$

and

$$\max \{\rho(A_{ij}) : (i, j) \neq (1, 0)\} = \rho(A_{20}) = \mu(A_{10}) = 2,$$

$$\tau_1 = \sum_{\rho(A_{ij})=\mu(A_{10}), (i,j) \neq (1,0)} \tau(A_{ij}) = \tau(A_{20}) = 12 < \tau(A_{10}) = 24.$$

It is clear that the conditions of Theorem 3.1.6 are satisfied. The meromorphic function

$$f(z) = \frac{e^{\frac{8z^3}{\pi}}}{\sin\left(\frac{iz}{\pi}\right)}$$

is a solution of equation (3.67) and f satisfies

$$\rho(f) = 3 \geq \mu(A_{10}) + 1.$$

Example 3.4.6. (i) Consider the non-homogeneous differential-difference equation with meromorphic coefficients

$$A_{11}f'(z + i\pi) + A_{10}f(z + i\pi) + A_{01}f'(z) + A_{00}f(z) = F(z). \quad (3.68)$$

Case 1. $\rho(F) < \mu(A_{10})$. In (3.68), for

$$A_{00}(z) = \frac{1}{\pi} \left(2iz + \cot\left(\frac{z}{\pi}\right) \right) e^{\pi z}, \quad A_{01}(z) = e^{\pi z},$$

$$A_{10}(z) = \left(\frac{2i}{\pi}z - 2 + \frac{1}{\pi} \cot\left(\frac{z}{\pi} + i\right) \right) e^{-2z} + e^{\frac{iz^2}{\pi}},$$

$$A_{11}(z) = e^{-2z}, \quad F(z) = -\frac{e^{2z}}{\sin\left(\frac{z}{\pi} + i\right)},$$

we have

$$\lambda\left(\frac{1}{A_{10}}\right) = 1 < \mu(A_{10}) = 2, \quad \max\{\rho(A_{ij}) : (i, j) \neq (1, 0)\} = 1 < \mu(A_{10}) = 2,$$

$$\rho(F) = 1 < \mu(A_{10}) = 2.$$

Obviously, the conditions of Theorem 3.1.7 (i) are verified. The meromorphic function

$$f(z) = \frac{e^{-\frac{iz^2}{\pi}}}{\sin\left(\frac{z}{\pi}\right)}$$

is a solution of equation (3.68) and f satisfies

$$\rho(f) = 2 \geq \mu(A_{10}) = 2.$$

Case 2. $\rho(F) = \mu(A_{10})$ and $\sum_{\rho(A_{ij})=\mu(A_{10}), (i,j) \neq (1,0)} \tau(A_{ij}) + \tau(F) < \underline{\tau}(A_{10})$. In (3.68), for

$$A_{00}(z) = \frac{1}{\pi} \left(2iz + \cot\left(\frac{z}{\pi}\right) \right) e^{\frac{z^2}{4\pi}}, \quad A_{01}(z) = e^{\frac{z^2}{4\pi}},$$

$$A_{10}(z) = \left(\frac{2i}{\pi}z - 2 + \frac{1}{\pi} \cot\left(\frac{z}{\pi} + i\right) \right) e^{-\frac{iz^2}{3\pi}} + e^{\frac{2iz^2}{\pi}},$$

$$A_{11}(z) = e^{-\frac{iz^2}{3\pi}}, \quad F(z) = -\frac{e^{\frac{iz^2}{\pi} + 2z}}{\sin\left(\frac{z}{\pi} + i\right)},$$

we have

$$\lambda\left(\frac{1}{A_{10}}\right) = 1 < \mu(A_{10}) = 2,$$

$$\max\{\rho(A_{ij}) : (i, j) \neq (1, 0)\} = \mu(A_{10}) = \mu(F) = 2,$$

$$\sum_{\rho(A_{ij})=\mu(A_{10}), (i,j) \neq (1,0)} \tau(A_{ij}) + \tau(F) = \frac{1}{4\pi^2} + \frac{1}{4\pi^2} + \frac{1}{3\pi^2} + \frac{1}{\pi^2} = \frac{11}{6\pi^2} < \underline{\tau}(A_{10}) = \frac{2}{\pi^2}.$$

Thus, the conditions of Theorem 3.1.7 (i) are satisfied. We see that the meromorphic function $f(z) = \frac{e^{-\frac{iz^2}{\pi}}}{\sin\left(\frac{z}{\pi}\right)}$ is a solution of (3.68) that satisfies $\rho(f) = 2 \geq \mu(A_{10}) = 2$.

Case 3. $\mu(F) = \mu(A_{10})$ and $\sum_{\rho(A_{ij})=\mu(A_{10}), (i,j)\neq(1,0)} \tau(A_{ij}) + \tau(A_{10}) < \tau(F)$. In (3.68), for

$$\begin{aligned} A_{00}(z) &= \frac{1}{\pi} \left(2iz + \cot\left(\frac{z}{\pi}\right) \right) e^{\frac{z^2}{4\pi}}, \quad A_{01}(z) = e^{\frac{z^2}{4\pi}}, \\ A_{10}(z) &= \left(\frac{2i}{\pi}z - 2 + \frac{1}{\pi} \cot\left(\frac{z}{\pi} + i\right) \right) e^{-\frac{iz^2}{3\pi}} + e^{-\frac{2iz^2}{\pi}}, \\ A_{11}(z) &= e^{-\frac{iz^2}{3\pi}}, \quad F(z) = -\frac{e^{-\frac{3iz^2}{\pi}+2z}}{\sin\left(\frac{z}{\pi} + i\right)}, \end{aligned}$$

we have

$$\begin{aligned} \lambda\left(\frac{1}{A_{10}}\right) &= 1 < \mu(A_{10}) = 2, \\ \max\{\rho(A_{ij}) : (i,j) \neq (1,0)\} &= \mu(A_{10}) = \mu(F) = 2, \\ \sum_{\rho(A_{ij})=\mu(A_{10}), (i,j)\neq(1,0)} \tau(A_{ij}) + \tau(A_{10}) &= \tau(A_{00}) + \tau(A_{01}) + \tau(A_{11}) + \tau(A_{10}) \\ &= \frac{1}{4\pi^2} + \frac{1}{4\pi^2} + \frac{1}{3\pi^2} + \frac{2}{\pi^2} = \frac{17}{6\pi^2} < \tau(F) = \frac{3}{\pi^2}. \end{aligned}$$

Hence, the conditions of Theorem 3.1.7 (i) are verified. We see that the meromorphic function $f(z) = \frac{e^{-\frac{iz^2}{\pi}}}{\sin\left(\frac{z}{\pi}\right)}$ is a solution of (3.68) that satisfies $\rho(f) = 2 \geq \mu(A_{10}) = 2$.

(ii) $\mu(F) > \mu(A_{10})$. In (3.68), for

$$\begin{aligned} A_{00}(z) &= 2iz + \cot\left(\frac{z}{\pi}\right), \quad A_{01}(z) = \pi, \\ A_{10}(z) &= 2iz - 2\pi + \cos\left(\frac{z}{\pi} + i\right) + e^{-2z}, \\ A_{11}(z) &= \pi, \quad F(z) = -\frac{e^{-\frac{iz^2}{\pi}}}{\sin\left(\frac{z}{\pi} + i\right)}, \end{aligned}$$

we have

$$\begin{aligned} \lambda\left(\frac{1}{A_{10}}\right) &= 0 < \mu(A_{10}) = 1, \\ \max\{\rho(A_{ij}) : (i,j) \neq (1,0)\} &= \mu(A_{10}) = 1, \\ \sum_{\rho(A_{ij})=\mu(A_{10}), (i,j)\neq(1,0)} \tau(A_{ij}) &= \tau(A_{00}) = \frac{1}{\pi^2} < \tau(A_{10}) = \frac{2}{\pi}. \end{aligned}$$

Obviously, the conditions of Theorem 3.1.7 (ii) are verified. The meromorphic function

$$f(z) = \frac{e^{-\frac{iz^2}{\pi}}}{\sin\left(\frac{z}{\pi}\right)}$$

is solution of equation (3.68) and f satisfies

$$\rho(f) = 2 \geq \mu(F) = 2.$$

Chapter 4

Lower bound estimates of the iterated p -order and p -lower order of meromorphic solutions of linear differential-difference equations

4.1 Introduction and statement of main results

In this chapter, we will continue the study of the lower bound of the order of solutions of difference equation and differential-difference equation. By using the concept of p -iterated, we make an extension of some recent results obtained by Zhou and Zheng.

Very recently, Zhou and Zheng [42] have obtained the following results.

Theorem 4.1.1. ([42]) *Let $p \geq 2$, $A_{ij}(z)$ ($i = 0, \dots, n; j = 0, \dots, m$) and $F(z)$ be meromorphic functions. Suppose there exists an integer l ($0 \leq l \leq n$) such that $A_{l0}(z)$ satisfies*

$$\lambda_p\left(\frac{1}{A_{l0}}\right) < \rho_p(A_{l0}) < \infty$$

and

$$\max\{\rho_p(A_{ij}) : i = 0, \dots, n; j = 0, \dots, m, (i, j) \neq (l, 0)\} \leq \rho_p(A_{l0}) < \infty,$$

$$\max\{\tau_p(A_{ij}) : \rho_p(A_{ij}) = \rho_p(A_{l0}), (i, j) \neq (l, 0)\} < \tau_p(A_{l0}) < \infty.$$

- (i) *If $\rho_p(F) < \rho_p(A_{l0})$, or $\rho_p(F) = \rho_p(A_{l0})$ and $\tau_p(F) \neq \tau_p(A_{l0})$, then every meromorphic solution $f(z) \not\equiv 0$ of (3.2) satisfies $\rho_p(f) \geq \rho_p(A_{l0})$.*
- (ii) *If $\rho_p(F) > \rho_p(A_{l0})$, then every meromorphic solution f of (3.2) satisfies $\rho_p(f) \geq \rho_p(F)$.*

Related to the theorems above, a natural question arises:

Question 4.1.2. *What can be said about the growth of meromorphic solutions in Theorem 2.1.1 and Theorem 2.1.7 of Chapter 2, when we use the concepts of iterated lower p -order and iterated lower p -type?*

The aim of this chapter is to give an answer for the above question, and we obtain the following results.

Theorem 4.1.3. ([4]) *Let $p \geq 2$ be an integer and $A_j(z)$ ($j = 0, 1, \dots, k$), $F(z) \not\equiv 0$ be meromorphic functions. Suppose there exists $l \in \{0, 1, \dots, k\}$ such that*

$$\lambda_p \left(\frac{1}{A_l} \right) < \mu_p(A_l) < \infty,$$

$$\max \{ \rho_p(A_j) : j = 0, 1, \dots, k, j \neq l \} \leq \mu_p(A_l),$$

$$\max \{ \tau_p(A_j) : \rho_p(A_j) = \mu_p(A_l), (j \neq l) \} < \tau_p(A_l) < \infty.$$

(i) *If $\rho_p(F) < \mu_p(A_l)$, or $\rho_p(F) = \mu_p(A_l)$ and $\tau_p(F) < \tau_p(A_l) = \tau$, or $\mu_p(F) = \mu_p(A_l)$ and $\tau_p(F) > \tau_p(A_l)$, then every meromorphic solution f of (2.3) satisfies $\mu_p(f) \geq \mu_p(A_l)$.*

(ii) *If $\mu_p(F) > \mu_p(A_l)$, then every meromorphic solution f of (2.3) satisfies $\mu_p(f) \geq \mu_p(F)$.*

Theorem 4.1.4. ([6]) *Let $p \geq 2$ be an integer and $A_{ij}(z)$ ($i \in I = \{0, 1, \dots, n\}$; $j \in J = \{0, 1, \dots, m\}$), $F(z) \not\equiv 0$ be meromorphic functions. Suppose there exists an integer l ($0 \leq l \leq n$) such that $A_{l0}(z)$ satisfies*

$$\lambda_p \left(\frac{1}{A_{l0}} \right) < \mu_p(A_{l0}) < \infty$$

and

$$\max \{ \rho_p(A_{ij}) : i = 0, 1, \dots, n; j = 0, 1, \dots, m, (i, j) \neq (l, 0) \} \leq \mu_p(A_{l0}) < \infty,$$

$$\tau_1 = \max \{ \tau_p(A_{ij}) : \rho_p(A_{ij}) = \mu_p(A_{l0}), (i, j) \neq (l, 0) \} < \tau_p(A_{l0}) = \tau < \infty.$$

(i) *If $\rho_p(F) < \mu_p(A_{l0})$, or $\rho_p(F) = \mu_p(A_{l0})$ and $\tau_p(F) < \tau$, or $\mu_p(F) = \mu_p(A_{l0})$ and $\tau_p(F) > \tau$, then every meromorphic solution f of (3.2) satisfies $\rho_p(f) \geq \mu_p(A_{l0})$.*

(ii) *If $\mu_p(F) > \mu_p(A_{l0})$, then every meromorphic solution f of (3.2) satisfies $\rho_p(f) \geq \mu_p(F)$.*

Remark 4.1.5. *Theorems 4.1.3 is an improvement of Theorem 1.4 [2] and Theorem 1.5, and generalization of our result obtained, very recently in [4]. [42].*

Remark 4.1.6. *Furthermore, Theorems 4.1.4 is a generalization of our result obtained, very recently in [6].*

4.2 Proof of the Theorems

For the proofs of our results, we will use the Lemmas 2.2.9-2.2.10, see [Chapter 2] and the Lemma 3.2.2, see [Chapter 3].

Proof of Theorem 4.1.3 Let f be a meromorphic solution of (2.3). By according to (2.51) and from Lemma 2.2.2 and Lemma 2.2.7, it follows that for any given $\varepsilon (> 0)$, we have

$$\begin{aligned}
& T(r, A_l(z)) = m(r, A_l(z)) + N(r, A_l(z)) \\
& \leq \sum_{j=0, j \neq l}^k m(r, A_j(z)) + \sum_{j=1, j \neq l}^k m\left(r, \frac{f(z+c_j)}{f(z+c_l)}\right) + m\left(r, \frac{f(z)}{f(z+c_l)}\right) \\
& \quad + m\left(r, \frac{1}{f(z+c_l)}\right) + m(r, F(z)) + N(r, A_l(z)) + O(1) \\
& \leq \sum_{j=0, j \neq l}^k T(r, A_j(z)) + \sum_{j=1, j \neq l}^k o\left(\frac{(T(r+|c_j-c_l|+|c_l|, f))^{1+\varepsilon}}{r^\delta}\right) \\
& \quad + o\left(\frac{(T(r+2|c_l|, f))^{1+\varepsilon}}{r^\delta}\right) + T(r, f(z+c_l)) + T(r, F(z)) + N(r, A_l(z)) + O(1) \\
& \leq \sum_{j=0, j \neq l}^k T(r, A_j(z)) + o\left(\frac{(T(2r, f))^{1+\varepsilon}}{r^\delta}\right) + 2T(2r, f(z)) + T(r, F(z)) + N(r, A_l(z)) \\
& \leq \sum_{j=0, j \neq l}^k T(r, A_j(z)) + T(r, F(z)) + N(r, A_l(z)) + 3(T(2r, f(z)))^2 \quad (4.1)
\end{aligned}$$

for all r outside of a possible exceptional set $E_2 \subset (1, +\infty)$ of finite logarithmic measure.

(i) If $\rho_p(F) < \mu_p(A_l) = \mu$, then for any given ε ($0 < 2\varepsilon < \mu - \rho_p(F)$) and sufficiently large r , we have

$$T(r, F) \leq \exp_{p-1} \{r^{\rho_p(F)+\varepsilon}\}. \quad (4.2)$$

First, we suppose that $b = \max \{\rho_p(A_j) : j = 0, 1, \dots, k, j \neq l\} < \mu_p(A_l) = \mu$. Then, for any given ε ($0 < 2\varepsilon < \mu - b$) and sufficiently large r , we have

$$T(r, A_l) \geq \exp_{p-1} \{r^{\mu_p(A_l)-\varepsilon}\} \quad (4.3)$$

and

$$T(r, A_j) \leq \exp_{p-1} \{r^{b+\varepsilon}\} \quad (j = 0, 1, \dots, k, j \neq l). \quad (4.4)$$

By the definition of $\lambda_p\left(\frac{1}{A_l}\right)$, for any given ε ($0 < 2\varepsilon < \mu - \lambda_p\left(\frac{1}{A_l}\right)$) and sufficiently large r , we obtain

$$N(r, A_l) \leq \exp_{p-1} \left\{r^{\lambda_p\left(\frac{1}{A_l}\right)+\varepsilon}\right\}. \quad (4.5)$$

By substituting the assumptions (4.2), (4.3), (4.4) and (4.5) into (4.1), for any given ε ($0 < \varepsilon < \min \left\{\frac{\mu-b}{2}, \frac{\mu-\lambda_p\left(\frac{1}{A_l}\right)}{2}, \frac{\mu-\rho_p(F)}{2}\right\}$) and sufficiently large $r \notin E_2$, we have

$$\exp_{p-1} \{r^{\mu-\varepsilon}\} \leq k \exp_{p-1} \{r^{b+\varepsilon}\} + \exp_{p-1} \{r^{\rho_p(F)+\varepsilon}\}$$

$$+ \exp_{p-1} \left\{ r^{\lambda_p \left(\frac{1}{A_l} \right) + \varepsilon} \right\} + 3 (T(2r, f(z)))^2.$$

So

$$(1 - o(1)) \exp_{p-1} \{ r^{\mu - \varepsilon} \} \leq 3 (T(2r, f(z)))^2,$$

that is, $\mu_p(f) \geq \mu - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we get $\mu_p(f) \geq \mu_p(A_l)$. Assume

$$\max \{ \rho_p(A_j) : j = 0, 1, \dots, k, j \neq l \} = \mu_p(A_l) = \mu,$$

$$\tau_1 = \max \{ \tau_p(A_j) : \rho_p(A_j) = \mu_p(A_l), (j \neq l) \} < \tau_p(A_l) = \tau.$$

Then, there exists a set $J \subseteq \{j = 0, 1, \dots, k\} \setminus \{l\}$ such that for $j \in J$, we have $\rho_p(A_j) = \mu_p(A_l) = \mu$ with $\tau_1 = \max \{ \tau_p(A_j) : \rho_p(A_j) = \mu_p(A_l), (j \neq l) \} < \tau_p(A_l) = \tau$ and for $j \in \{0, 1, \dots, l-1, l+1, \dots, k\} \setminus J$, we have $\rho_p(A_j) < \mu_p(A_l) = \mu$. Hence, for any given ε ($0 < \varepsilon < \frac{\tau - \tau_1}{2}$) and sufficiently large r , we have

$$T(r, A_j) \leq \exp_{p-1} \{ (\tau_1 + \varepsilon) r^{\mu_p(A_l)} \}, \quad j \in J \quad (4.6)$$

and

$$T(r, A_j) \leq \exp_{p-1} \{ r^{\mu_0} \}, \quad j \in \{0, 1, \dots, l-1, l+1, \dots, k\} \setminus J, \quad (4.7)$$

where $0 < \mu_0 < \mu$. By the definition of lower p -type, for the above ε and sufficiently large r , we have

$$T(r, A_l) \geq \exp_{p-1} \{ (\tau - \varepsilon) r^{\mu_p(A_l)} \}. \quad (4.8)$$

By substituting the assumptions (4.5), (4.6), (4.7), (4.8) into (4.1), for any given ε ($0 < 2\varepsilon < \min \left\{ \mu - \lambda_p \left(\frac{1}{A_l} \right), \mu - \rho_p(F), \tau - \tau_1 \right\}$) and sufficiently large $r \notin E_2$, we obtain

$$\begin{aligned} \exp_{p-1} \{ (\tau - \varepsilon) r^\mu \} &\leq O \left(\exp_{p-1} \{ (\tau_1 + \varepsilon) r^\mu \} \right) + O \left(\exp_{p-1} \{ r^{\mu_0} \} \right) \\ &+ \exp_{p-1} \{ r^{\rho_p(F) + \varepsilon} \} + \exp_{p-1} \left\{ r^{\lambda_p \left(\frac{1}{A_l} \right) + \varepsilon} \right\} + 3 (T(2r, f(z)))^2. \end{aligned}$$

So

$$(1 - o(1)) \exp_{p-1} \{ (\tau - \varepsilon) r^\mu \} \leq 3 (T(2r, f(z)))^2,$$

which implies $\mu_p(f) \geq \mu_p(A_l)$.

If $\rho_p(F) = \mu_p(A_l) = \mu$ and $\tau_p(F) < \tau_p(A_l) = \tau$, then for any given ε ($0 < \varepsilon < \frac{\tau - \tau_p(F)}{2}$) and sufficiently large r , we have

$$T(r, F) \leq \exp_{p-1} \{ (\tau_p(F) + \varepsilon) r^{\mu_p(A_l)} \}. \quad (4.9)$$

First, we suppose that $b = \max \{ \rho_p(A_j) : j = 0, 1, \dots, k, j \neq l \} < \mu_p(A_l) = \mu$. We may choose sufficiently small ε satisfying

$$0 < \varepsilon < \min \left\{ \frac{\mu - \lambda_p \left(\frac{1}{A_l} \right)}{2}, \frac{\mu - b}{2}, \frac{\tau - \tau_p(F)}{2} \right\},$$

by substituting the assumptions (4.4), (4.5), (4.8) and (4.9) into (4.1) for sufficiently large $r \notin E_2$, we obtain

$$\exp_{p-1} \{ (\tau - \varepsilon) r^\mu \} \leq k \exp_{p-1} \{ r^{b+\varepsilon} \} + \exp_{p-1} \{ (\tau_p(F) + \varepsilon) r^\mu \}$$

$$+ \exp_{p-1} \left\{ r^{\lambda_p \left(\frac{1}{A_l} \right) + \varepsilon} \right\} + 3 (T(2r, f(z)))^2.$$

So

$$(1 - o(1)) \exp_{p-1} \{(\tau - \varepsilon) r^\mu\} \leq 3 (T(2r, f(z)))^2,$$

which implies $\mu_p(f) \geq \mu_p(A_l)$.

Now, we suppose that

$$\max \{ \rho_p(A_j) : j = 0, 1, \dots, k, j \neq l \} = \mu_p(A_l) = \mu,$$

$$\tau_1 = \max \{ \tau_p(A_j) : \rho_p(A_j) = \mu_p(A_l), (j \neq l) \} < \underline{\tau}_p(A_l) = \tau.$$

We may choose sufficiently small ε satisfying

$$0 < \varepsilon < \min \left\{ \frac{\mu - \lambda_p \left(\frac{1}{A_l} \right)}{2}, \frac{\tau - \tau_1}{2}, \frac{\tau - \tau_p(F)}{2} \right\},$$

by substituting the assumptions (4.5), (4.6), (4.7), (4.8) and (4.9) into (4.1) for sufficiently large $r \notin E_2$, we obtain

$$\begin{aligned} \exp_{p-1} \{(\tau - \varepsilon) r^\mu\} &\leq O(\exp_{p-1} \{(\tau_1 + \varepsilon) r^\mu\}) + O(\exp_{p-1} \{r^{\mu_0}\}) \\ &+ \exp_{p-1} \{(\tau_p(F) + \varepsilon) r^\mu\} + \exp_{p-1} \left\{ r^{\lambda_p \left(\frac{1}{A_l} \right) + \varepsilon} \right\} + 3 (T(2r, f(z)))^2. \end{aligned}$$

So

$$(1 - o(1)) \exp_{p-1} \{(\tau - \varepsilon) r^\mu\} \leq 3 (T(2r, f(z)))^2,$$

which implies $\mu_p(f) \geq \mu_p(A_l)$.

If $\mu_p(F) = \mu_p(A_l) = \mu$ and $\underline{\tau}_p(F) > \underline{\tau}_p(A_l) = \tau$, then for any given ε ($0 < \varepsilon < \frac{\underline{\tau}_p(F) - \tau}{2}$) and sufficiently large r , we have

$$T(r, F) > \exp_{p-1} \{(\underline{\tau}_p(F) - \varepsilon) r^{\mu_p(A_l)}\}. \quad (4.10)$$

By Lemma 2.2.10, for the above ε there exists a subset E_3 with infinite logarithmic measure such that for all $r \in E_3$, we have

$$T(r, A_l) \leq \exp_{p-1} \{(\underline{\tau}_p(A_l) + \varepsilon) r^{\mu_p(A_l)}\}. \quad (4.11)$$

First, we suppose that $b = \max \{ \rho_p(A_j) : j = 0, 1, \dots, k, j \neq l \} < \mu_p(A_l) = \mu$. We may choose sufficiently small ε satisfying $0 < \varepsilon < \min \left\{ \frac{\mu - b}{2}, \frac{\underline{\tau}_p(F) - \tau}{2} \right\}$, by substituting (4.4), (4.10) and (4.11) into (2.57) for sufficiently large $r \in E_3$, we obtain

$$\begin{aligned} \exp_{p-1} \{(\underline{\tau}_p(F) - \varepsilon) r^\mu\} &\leq k \exp_{p-1} \{r^{b+\varepsilon}\} \\ &+ \exp_{p-1} \{(\tau + \varepsilon) r^\mu\} + (2k + 1) T(2r, f(z)). \end{aligned}$$

So

$$(1 - o(1)) \exp_{p-1} \{(\underline{\tau}_p(F) - \varepsilon) r^\mu\} \leq (2k + 1) T(2r, f(z)),$$

which implies $\mu_p(f) \geq \mu_p(A_l)$.

Now, we suppose that

$$\begin{aligned} \max \{ \rho_p(A_j) : j = 0, 1, \dots, k, j \neq l \} &= \mu_p(A_l) = \mu, \\ \tau_1 = \max \{ \tau_p(A_j) : \rho_p(A_j) = \mu_p(A_l), (j \neq l) \} &< \tau_p(A_l) = \tau. \end{aligned}$$

We may choose sufficiently small ε satisfying $0 < \varepsilon < \frac{\tau_p(F) - \tau}{2}$, by substituting (4.6), (4.7), (4.10) and (4.11) into (2.57), for sufficiently large $r \in E_3$, we obtain

$$\begin{aligned} \exp_{p-1} \{ (\tau_p(F) - \varepsilon) r^\mu \} &\leq O(\exp_{p-1} \{ (\tau_1 + \varepsilon) r^\mu \}) + O(\exp_{p-1} \{ r^{\mu_0} \}) \\ &+ \exp_{p-1} \{ (\tau + \varepsilon) r^\mu \} + (2k + 1) T(2r, f(z)). \end{aligned}$$

So

$$(1 - o(1)) \exp_{p-1} \{ (\tau_p(F) - \varepsilon) r^\mu \} \leq (2k + 1) T(2r, f(z)),$$

which implies $\mu_p(f) \geq \mu_p(A_l)$.

(ii) Next, we consider the case $\mu_p(F) > \mu_p(A_l) = \mu$. Let f be a meromorphic solution of (2.3). Then, for any given ε ($0 < 2\varepsilon < \mu_p(F) - \mu$) and sufficiently large r , we have

$$T(r, F) \geq \exp_{p-1} \{ r^{\mu_p(F) - \varepsilon} \}. \quad (4.12)$$

By Lemma 2.2.9, for the above ε , there exists a subset E_4 with infinite logarithmic measure such that for all $r \in E_4$, we have

$$T(r, A_l) \leq \exp_{p-1} \{ r^{\mu_p(A_l) + \varepsilon} \}. \quad (4.13)$$

First, we suppose that $b = \max \{ \rho_p(A_j) : j = 0, 1, \dots, k, j \neq l \} < \mu_p(A_l) = \mu$. We may choose sufficiently small ε satisfying $0 < 2\varepsilon < \mu_p(F) - \mu$, by substituting (4.4), (4.12) and (4.13) into (2.57) for sufficiently large $r \in E_4$, we obtain

$$\exp_{p-1} \{ r^{\mu_p(F) - \varepsilon} \} \leq k \exp_{p-1} \{ r^{b + \varepsilon} \} + \exp_{p-1} \{ r^{\mu + \varepsilon} \} + (2k + 1) T(2r, f(z)).$$

So

$$(1 - o(1)) \exp_{p-1} \{ r^{\mu_p(F) - \varepsilon} \} \leq (2k + 1) T(2r, f(z)),$$

that is, $\mu_p(f) \geq \mu_p(F) - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we get $\mu_p(f) \geq \mu_p(F)$.

Now, we suppose that

$$\begin{aligned} \max \{ \rho_p(A_j) : j = 0, 1, \dots, k, j \neq l \} &= \mu_p(A_l) = \mu, \\ \tau_1 = \max \{ \tau_p(A_j) : \rho_p(A_j) = \mu_p(A_l), (j \neq l) \} &< \tau_p(A_l) = \tau. \end{aligned}$$

Now, we may choose sufficiently small ε satisfying $0 < 2\varepsilon < \mu_p(F) - \mu$, by substituting (4.6), (4.7), (4.11) and (4.12) into (2.57) for sufficiently large $r \in E_3$, we get

$$\begin{aligned} \exp_{p-1} \{ r^{\mu_p(F) - \varepsilon} \} &\leq O(\exp_{p-1} \{ (\tau_1 + \varepsilon) r^\mu \}) + O(\exp_{p-1} \{ r^{\mu_0} \}) \\ &+ \exp_{p-1} \{ (\tau + \varepsilon) r^\mu \} + (2k + 1) T(2r, f(z)). \end{aligned}$$

It follows that

$$(1 - o(1)) \exp_{p-1} \{ r^{\mu_p(F) - \varepsilon} \} \leq (2k + 1) T(2r, f(z)),$$

that is, $\mu_p(f) \geq \mu_p(F) - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we get $\mu_p(f) \geq \mu_p(F)$

Proof of Theorem 4.1.4 Let f be a meromorphic solution of (3.4). By according to (3.54) and Lemma 2.2.7, for any given $\varepsilon (> 0)$, there exists a subset $E_3 \subset (1, +\infty)$ of finite logarithmic measure such that for all $r \notin E_3$, we have

$$\begin{aligned}
T(r, A_{l_0}(z)) &= m(r, A_{l_0}(z)) + N(r, A_{l_0}(z)) \leq \sum_{i=0, i \neq l}^n \sum_{j=0}^m m(r, A_{ij}(z)) + \sum_{j=1}^m m(r, A_{lj}(z)) \\
&+ \sum_{i=0}^n \sum_{j=1}^m m\left(r, \frac{f^{(j)}(z + c_i)}{f(z + c_i)}\right) + \sum_{i=0, i \neq l}^n m\left(r, \frac{f(z + c_i)}{f(z + c_l)}\right) + m(r, F(z)) + m\left(r, \frac{1}{f(z + c_l)}\right) \\
&+ N(r, A_{l_0}(z)) + O(1) \leq \sum_{i=0, i \neq l}^n \sum_{j=0}^m T(r, A_{ij}(z)) + \sum_{j=1}^m T(r, A_{lj}(z)) \\
&+ \sum_{i=0}^n \sum_{j=1}^m m\left(r, \frac{f^{(j)}(z + c_i)}{f(z + c_i)}\right) + \sum_{i=0, i \neq l}^k o\left(\frac{(T(r + |c_i - c_l| + |c_l|, f))^{1+\varepsilon}}{r^\delta}\right) \\
&+ T(r, F(z)) + T\left(r, \frac{1}{f(z + c_l)}\right) + N(r, A_{l_0}(z)) + O(1) \\
&\leq \sum_{i=0, i \neq l}^n \sum_{j=0}^m T(r, A_{ij}(z)) + \sum_{j=1}^m T(r, A_{lj}(z)) + \sum_{i=0}^n \sum_{j=1}^m m\left(r, \frac{f^{(j)}(z + c_i)}{f(z + c_i)}\right) \\
&+ \sum_{i=0, i \neq l}^k o\left(\frac{(T(r + |c_i - c_l| + |c_l|, f))^{1+\varepsilon}}{r^\delta}\right) + T(r, F(z)) + 2T(2r, f) + N(r, A_{l_0}(z)) + O(1) \\
&\leq \sum_{i=0, i \neq l}^n \sum_{j=0}^m T(r, A_{ij}(z)) + \sum_{j=1}^m T(r, A_{lj}(z)) + \sum_{i=0}^n \sum_{j=1}^m m\left(r, \frac{f^{(j)}(z + c_i)}{f(z + c_i)}\right) \\
&+ T(r, F(z)) + 3(T(2r, f))^2 + N(r, A_{l_0}(z)) + O(1). \tag{4.14}
\end{aligned}$$

(i) If $\rho_p(F) < \mu_p(A_{l_0}) = \mu$, then for any given ε ($0 < 2\varepsilon < \mu - \rho_p(F)$) and sufficiently large r , we have

$$T(r, F) \leq \exp_{p-1} \{r^{\rho_p(F)+\varepsilon}\}. \tag{4.15}$$

We suppose that $\alpha = \max \{\rho_p(A_{ij}) : (i, j) \neq (l, 0)\} < \mu_p(A_{l_0}) = \mu$. Then, for any given ε ($0 < 2\varepsilon < \mu - \alpha$) and sufficiently large r , we have

$$T(r, A_{l_0}) \geq \exp_{p-1} \{r^{\mu-\varepsilon}\} \tag{4.16}$$

and

$$T(r, A_{ij}) \leq \exp_{p-1} \{r^{\alpha+\varepsilon}\}, \quad (i, j) \neq (l, 0). \tag{4.17}$$

It follows by the definition of $\lambda_p\left(\frac{1}{A_{l_0}}\right)$, for any given ε ($0 < 2\varepsilon < \mu - \lambda_p\left(\frac{1}{A_{l_0}}\right)$) and sufficiently large r , we obtain

$$N(r, A_{l_0}) \leq \exp_{p-1} \left\{ r^{\lambda_p\left(\frac{1}{A_{l_0}}\right)+\varepsilon} \right\}. \tag{4.18}$$

By using Lemma 3.2.2 and substituting the assumptions (4.15), (4.16), (4.17) and (4.18) into (4.14), for any given ε ($0 < 2\varepsilon < \min\left\{\mu - \alpha, \mu - \lambda_p\left(\frac{1}{A_{l0}}\right), \mu - \rho_p(F)\right\}$) and sufficiently large $r \notin E_1 \cup E_3$, we get

$$\begin{aligned} \exp_{p-1}\{r^{\mu-\varepsilon}\} &\leq \sum_{i=0, i \neq l}^n \sum_{j=0}^m \exp_{p-1}\{r^{\alpha+\varepsilon}\} + \sum_{j=1}^m \exp_{p-1}\{r^{\alpha+\varepsilon}\} + S(r, f) \\ &\quad + \exp_{p-1}\{r^{\rho_p(F)+\varepsilon}\} + 3(T(2r, f(z)))^2 + \exp_{p-1}\{r^{\lambda_p(\frac{1}{A_{l0}})+\varepsilon}\} + O(1) \\ &= O(\exp_{p-1}\{r^{\alpha+\varepsilon}\}) + \exp_{p-1}\{r^{\rho_p(F)+\varepsilon}\} + \exp_{p-1}\{r^{\lambda_p(\frac{1}{A_{l0}})+\varepsilon}\} + 3(T(2r, f(z)))^2 + S(r, f). \end{aligned}$$

So

$$(1 - o(1)) \exp_{p-1}\{r^{\mu-\varepsilon}\} \leq 3(T(2r, f(z)))^2 + O(\log T(r, f) + \log r),$$

that is $\rho_p(f) \geq \mu - \varepsilon$, since $\varepsilon > 0$ arbitrary number, then $\rho_p(f) \geq \mu = \mu_p(A_{l0})$.

Assume that

$$\max\{\rho_p(A_{ij}) : (i, j) \neq (l, 0)\} = \mu_p(A_{l0}) = \mu,$$

$$\tau_1 = \max\{\tau_p(A_{ij}) : \rho_p(A_{ij}) = \mu_p(A_{l0}), (i, j) \neq (l, 0)\} < \tau_p(A_{l0}) = \tau.$$

Then, there exists a set $\Gamma \subseteq I \times J \setminus \{(l, 0)\}$ such that for $(i, j) \in \Gamma$, we have $\rho_p(A_{ij}) = \mu$ with $\tau_1 = \max\{\tau_p(A_{ij}) : (i, j) \in \Gamma\} < \tau_p(A_{l0}) = \tau$, and for $(i, j) \in I \times J \setminus (\Gamma \cup \{(l, 0)\})$ we have $\rho_p(A_{ij}) < \mu$. Hence, for any given ε ($0 < \varepsilon < \frac{\tau - \tau_1}{2}$) and sufficiently large r , we have

$$T(r, A_{ij}) \leq \exp_{p-1}\{(\tau_1 + \varepsilon)r^\mu\}, \quad (i, j) \in \Gamma \quad (4.19)$$

and

$$T(r, A_{ij}) \leq \exp_{p-1}\{r^{\mu-\varepsilon}\}, \quad (i, j) \in I \times J \setminus (\Gamma \cup \{(l, 0)\}). \quad (4.20)$$

By the definition of lower p -type, for the above ε and sufficiently large r , we have

$$T(r, A_{l0}) \geq \exp_{p-1}\{(\tau - \varepsilon)r^\mu\}. \quad (4.21)$$

By substituting the assumptions (4.15), (4.18), (4.19), (4.20) and (4.21) into (4.14), for any given ε ($0 < 2\varepsilon < \min\left\{\mu - \lambda_p\left(\frac{1}{A_{l0}}\right), \mu - \rho_p(F), \tau - \tau_1\right\}$) and sufficiently large $r \notin E_1 \cup E_3$, we obtain

$$\begin{aligned} \exp_{p-1}\{(\tau - \varepsilon)r^\mu\} &\leq O(\exp_{p-1}\{(\tau_1 + \varepsilon)r^\mu\}) + O(\exp_{p-1}\{r^{\mu-\varepsilon}\}) \\ &\quad + \exp_{p-1}\{r^{\rho_p(F)+\varepsilon}\} + \exp_{p-1}\{r^{\lambda_p(\frac{1}{A_{l0}})+\varepsilon}\} + 3(T(2r, f(z)))^2 + S(r, f). \end{aligned}$$

So

$$(1 - o(1)) \exp_{p-1}\{(\tau - \varepsilon)r^\mu\} \leq 3(T(2r, f(z)))^2 + O(\log T(r, f) + \log r),$$

which implies $\rho_p(f) \geq \mu = \mu_p(A_{l0})$.

If $\rho_p(F) = \mu_p(A_{l0}) = \mu$ and $\tau_p(F) < \tau_p(A_{l0}) = \tau$, then for any given ε ($0 < 2\varepsilon < \tau - \tau_p(F)$) and sufficiently large r , we have

$$T(r, F) \leq \exp_{p-1}\{(\tau_p(F) + \varepsilon)r^{\mu_p(A_{l0})}\}. \quad (4.22)$$

We suppose that $\alpha = \max \{\rho_p(A_{ij}) : (i, j) \neq (l, 0)\} < \mu_p(A_{l0}) = \mu$. We may choose sufficiently small ε satisfying

$$0 < 2\varepsilon < \min \left\{ \mu - \lambda_p \left(\frac{1}{A_{l0}} \right), \mu - \alpha, \tau - \tau_p(F) \right\},$$

by substituting the assumptions (4.17), (4.18), (4.21) and (4.22) into (4.14) for sufficiently large $r \notin E_1 \cup E_3$, we obtain

$$\begin{aligned} & \exp_{p-1} \{(\tau - \varepsilon) r^\mu\} \leq O(\exp_{p-1} \{r^{\alpha+\varepsilon}\}) \\ & + \exp_{p-1} \{(\tau_p(F) + \varepsilon) r^\mu\} + \exp_{p-1} \left\{ r^{\lambda_p \left(\frac{1}{A_{l0}} \right) + \varepsilon} \right\} + 3(T(2r, f(z)))^2 + S(r, f). \end{aligned}$$

So

$$(1 - o(1)) \exp_{p-1} \{(\tau - \varepsilon) r^\mu\} \leq 3(T(2r, f(z)))^2 + O(\log T(r, f) + \log r),$$

which implies $\rho_p(f) \geq \mu_p(A_{l0})$.

Now, we suppose that

$$\max \{\rho_p(A_{ij}) : (i, j) \neq (l, 0)\} = \mu_p(A_{l0}) = \mu,$$

$$\tau_1 = \max \{\tau_p(A_{ij}) : \rho_p(A_{ij}) = \mu_p(A_{l0}), (i, j) \neq (l, 0)\} < \tau_p(A_{l0}) = \tau.$$

We may choose sufficiently small ε satisfying

$$0 < 2\varepsilon < \min \left\{ \mu - \lambda_p \left(\frac{1}{A_{l0}} \right), \tau - \tau_1, \tau - \tau_p(F) \right\},$$

by substituting the assumptions (4.18) – (4.22) into (4.14) for sufficiently large $r \notin E_1 \cup E_3$, we obtain

$$\begin{aligned} & \exp_{p-1} \{(\tau - \varepsilon) r^\mu\} \leq O(\exp_{p-1} \{(\tau_1 + \varepsilon) r^\mu\}) + O(\exp_{p-1} \{r^{\mu-\varepsilon}\}) \\ & + \exp_{p-1} \{(\tau_p(F) + \varepsilon) r^\mu\} + \exp_{p-1} \left\{ r^{\lambda_p \left(\frac{1}{A_{l0}} \right) + \varepsilon} \right\} + 3(T(2r, f(z)))^2 + S(r, f). \end{aligned}$$

So

$$(1 - o(1)) \exp_{p-1} \{(\tau - \varepsilon) r^\mu\} \leq 3(T(2r, f(z)))^2 + O(\log T(r, f) + \log r),$$

which gives $\rho_p(f) \geq \mu_p(A_{l0})$.

If $\mu_p(F) = \mu_p(A_{l0}) = \mu$ and $\tau_p(F) > \tau_p(A_{l0}) = \tau$, then for any given ε ($0 < 2\varepsilon < \tau_p(F) - \tau$) and sufficiently large r , we have

$$T(r, F) > \exp_{p-1} \{(\tau_p(F) - \varepsilon) r^\mu\}. \quad (4.23)$$

By Lemma 2.2.10, for the above ε there exists a subset E_4 with infinite logarithmic measure such that for all $r \in E_4$, we have

$$T(r, A_{l0}) \leq \exp_{p-1} \{(\tau + \varepsilon) r^\mu\}. \quad (4.24)$$

First, we suppose that $\alpha = \max\{\rho_p(A_{ij}) : (i, j) \neq (l, 0)\} < \mu_p(A_{l0}) = \mu$. We may choose sufficiently small ε satisfying $0 < 2\varepsilon < \min\{\mu - \alpha, \underline{\tau}_p(F) - \tau\}$, by substituting (4.17), (4.23) and (4.24) into (3.60), for sufficiently large $r \in E_4 \setminus E_6$, we obtain

$$\begin{aligned} \exp_{p-1}\{(\underline{\tau}_p(F) - \varepsilon)r^\mu\} &\leq O(\exp_{p-1}\{r^{\alpha+\varepsilon}\}) + \exp_{p-1}\{(\tau + \varepsilon)r^\mu\} \\ &\quad + O(T(2r, f(z))) + S(r, f). \end{aligned}$$

So

$$(1 - o(1)) \exp_{p-1}\{(\underline{\tau}_p(F) - \varepsilon)r^\mu\} \leq O(T(2r, f(z))) + o(T(r, f)),$$

which implies $\rho_p(f) \geq \mu = \mu_p(A_{l0})$.

Now, we suppose that

$$\max\{\rho_p(A_{ij}) : (i, j) \neq (l, 0)\} = \mu_p(A_{l0}) = \mu,$$

$$\tau_1 = \max\{\tau_p(A_{ij}) : \rho_p(A_{ij}) = \mu_p(A_{l0}), (i, j) \neq (l, 0)\} < \underline{\tau}_p(A_{l0}) = \tau.$$

We may choose sufficiently small ε satisfying $0 < \varepsilon < \frac{\tau_p(F) - \tau}{2}$, by substituting (4.19), (4.20), (4.23) and (4.24) into (3.60), for sufficiently large $r \in E_4 \setminus E_6$, we obtain

$$\begin{aligned} \exp_{p-1}\{(\underline{\tau}_p(F) - \varepsilon)r^\mu\} &\leq O(\exp_{p-1}\{(\tau_1 + \varepsilon)r^\mu\}) + O(\exp_{p-1}\{r^{\mu-\varepsilon}\}) \\ &\quad + \exp_{p-1}\{(\tau + \varepsilon)r^\mu\} + O(T(2r, f(z))) + o(T(r, f)). \end{aligned}$$

So

$$(1 - o(1)) \exp_{p-1}\{(\underline{\tau}_p(F) - \varepsilon)r^\mu\} \leq O(T(2r, f(z))) + o(T(r, f))$$

which implies $\rho_p(f) \geq \mu = \mu_p(A_{l0})$.

(ii) Next, we consider the case $\mu_p(F) > \mu_p(A_{l0}) = \mu$. Then, for any given ε ($0 < 2\varepsilon < \mu_p(F) - \mu$) and sufficiently large r , we have

$$T(r, F) \geq \exp_{p-1}\{r^{\mu_p(F)-\varepsilon}\}. \quad (4.25)$$

By Lemma 2.2.9, for the above ε , there exists a subset E_5 with infinite logarithmic measure such that for all $r \in E_5$, we have

$$T(r, A_{l0}) \leq \exp_{p-1}\{r^{\mu+\varepsilon}\}. \quad (4.26)$$

First, we suppose that $\alpha = \max\{\rho_p(A_{ij}) : (i, j) \neq (l, 0)\} < \mu_p(A_{l0}) = \mu$. We may choose sufficiently small ε satisfying $0 < 2\varepsilon < \mu_p(F) - \mu$, by substituting (4.17), (4.25) and (4.26) into (3.60) for sufficiently large $r \in E_5 \setminus E_6$, we obtain

$$\exp_{p-1}\{r^{\mu_p(F)-\varepsilon}\} \leq O(\exp_{p-1}\{r^{\alpha+\varepsilon}\}) + \exp_{p-1}\{r^{\mu+\varepsilon}\} + O(T(2r, f(z))) + o(T(r, f)).$$

So

$$(1 - o(1)) \exp_{p-1}\{r^{\mu_p(F)-\varepsilon}\} \leq O(T(2r, f(z))) + o(T(r, f)),$$

that is, $\rho_p(f) \geq \mu_p(F) - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we get $\rho_p(f) \geq \mu_p(F)$.

Now, we suppose that

$$\max\{\rho_p(A_{ij}) : (i, j) \neq (l, 0)\} = \mu_p(A_{l0}) = \mu,$$

$$\tau_1 = \max\{\tau_p(A_{ij}) : \rho_p(A_{ij}) = \mu_p(A_{l0}), (i, j) \neq (l, 0)\} < \underline{\tau}_p(A_{l0}) = \tau.$$

Now, we may choose sufficiently small ε satisfying $0 < 2\varepsilon < \mu_p(F) - \mu$, by substituting (4.19), (4.20), (4.24) and (4.25) into (3.60) for sufficiently large $r \in E_4 \setminus E_6$, we get

$$\begin{aligned} \exp_{p-1} \{r^{\mu_p(F)-\varepsilon}\} &\leq O(\exp_{p-1} \{(\tau_1 + \varepsilon)r^\mu\}) + O(\exp_{p-1} \{r^{\mu-\varepsilon}\}) \\ &+ \exp_{p-1} \{(\tau + \varepsilon)r^\mu\} + O(T(2r, f(z))) + o(T(r, f)). \end{aligned}$$

It follows that

$$(1 - o(1)) \exp_{p-1} \{r^{\mu_p(F)-\varepsilon}\} \leq O(T(2r, f(z))) + o(T(r, f)),$$

which implies $\rho_p(f) \geq \mu_p(F) - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we get $\rho_p(f) \geq \mu_p(F)$. The proof of Theorem 4.1.4 is complete.

4.3 Examples

Example 4.3.1. (i) Consider the non-homogeneous difference equation with meromorphic coefficients

$$A_2(z) f(z + 2\pi i) + A_1(z) f(z + i\pi) + A_0(z) f(z) = F(z). \quad (4.27)$$

Case 1. $\rho_2(F) < \mu_2(A_l)$. In (4.27), for

$$\begin{aligned} A_0(z) &= -ze^z, \quad A_1(z) = \frac{(z + i\pi)e^{-\cosh(2z) + \cosh(z) + z}}{z}, \\ A_2(z) &= (z + 2i\pi)e^z, \quad F(z) = \frac{e^z}{z}, \end{aligned}$$

we have

$$\lambda_2 \left(\frac{1}{A_1} \right) = 0 < \mu_2(A_1) = 1,$$

$$\max \{ \rho_2(A_j) \ (j = 0, 2), \rho_2(F) \} = 0 < \mu_2(A_1) = 1.$$

As we see, the conditions of Theorem 4.1.3 (i) are verified. The meromorphic function

$$f(z) = \frac{e^{\cosh(2z) + \cosh(z)}}{z}$$

is a solution of equation (4.27) and f satisfies

$$\mu_2(f) = 1 \geq \mu_2(A_1) = 1.$$

Case 2. $\rho_2(F) = \mu_2(A_l)$ and $\tau_2(F) < \tau_2(A_l)$. In (4.27), for

$$\begin{aligned} A_0(z) &= -ze^{\sin(\frac{z}{\pi}) + z}, \quad A_1(z) = \frac{(z + i\pi)e^{-\cosh(2z)}}{z}, \\ A_2(z) &= (z + 2i\pi)e^{\sin(\frac{z}{\pi}) + z}, \quad F(z) = \frac{e^{-\cosh(z)}}{z}, \end{aligned}$$

we have

$$\lambda_2 \left(\frac{1}{A_1} \right) = 0 < \mu_2(A_1) = 1, \quad \max \{ \rho_2(A_j) \ (j = 0, 2), \rho_2(F) \} = \mu_2(A_1) = 1,$$

$$\tau_2(F) = \frac{1}{\pi} < \frac{2}{\pi} = \tau_2(A_1).$$

Thus, the conditions of Theorem 4.1.3 (i) are verified. The meromorphic function

$$f(z) = \frac{e^{\cosh(2z) + \cosh(z)}}{z}$$

is a solution of equation (4.27) and f satisfies

$$\mu_2(f) = 1 \geq \mu_2(A_1) = 1.$$

Case 3. $\mu_2(F) = \mu_2(A_1)$ and $\tau_2(F) > \tau_2(A_1)$. In (4.27), for

$$A_0(z) = -ze^{\sin(\frac{z}{\pi}) + z}, \quad A_1(z) = \frac{(z + i\pi)e^{\cosh(z)}}{z},$$

$$A_2(z) = (z + 2i\pi)e^{\sin(\frac{z}{\pi}) + z}, \quad F(z) = \frac{e^{\cosh(2z)}}{z},$$

we have

$$\lambda_2 \left(\frac{1}{A_1} \right) = 0 < \mu_2(A_1) = 1,$$

$$\max \{ \rho_2(A_j) \ (j = 0, 2), \mu_2(F) \} = \mu_2(A_1) = 1,$$

$$\tau_2(F) = \frac{2}{\pi} > \frac{1}{\pi} = \tau_2(A_1).$$

Hence, the conditions of Theorem 4.1.3 (i) are verified. The meromorphic function

$$f(z) = \frac{e^{\cosh(2z) + \cosh(z)}}{z}$$

is a solution of equation (4.27) and f satisfies

$$\mu_2(f) = 1 \geq \mu_2(A_1) = 1.$$

(ii) $\mu_2(F) > \mu_2(A_1)$. In (4.27), for

$$A_0(z) = -ze^z, \quad A_1(z) = \frac{(z + i\pi)e^{\cosh(\sqrt{z})}}{z},$$

$$A_2(z) = (z + 2i\pi)e^z, \quad F(z) = \frac{e^{\cosh(2z) - \cosh(z) + \cosh(\sqrt{z})}}{z},$$

we have

$$\lambda_2 \left(\frac{1}{A_1} \right) = 0 < \mu_2(A_1) = \frac{1}{2}, \quad \max \{ \rho_2(A_j) : j = 0, 2 \} = 0 < \mu_2(A_1) = \frac{1}{2},$$

$$\mu_2(F) = 1 > \mu_2(A_1) = \frac{1}{2}.$$

Obviously, the conditions of Theorem 4.1.3 (ii) are verified. The meromorphic function

$$f(z) = \frac{e^{\cosh(2z)+\cosh(z)}}{z}$$

is a solution of equation (4.27) and f satisfies

$$\mu_2(f) = 1 \geq \mu_2(F) = 1.$$

Example 4.3.2. (i) Consider the non-homogeneous differential-difference equation with meromorphic coefficients

$$A_{12}f''(z + 2021) + A_{20}f(z + 2020) + A_{11}f'(z + 2021) + A_{00}f(z) = F(z). \quad (4.28)$$

Case 1. $\rho_p(F) < \mu_p(A_{10})$. In (4.28), for

$$A_{00}(z) = \pi(\tan(\pi z) - 1), \quad A_{11}(z) = \pi^2(\sin(2\pi z) - \sin(\pi z) - \cos(\pi z) - 1),$$

$$A_{20}(z) = \pi(\cos(\pi z) - \sin(\pi z))(\sec(\pi z) - e^{-(\cos(\pi z) + \sin(\pi z))}),$$

$$A_{12}(z) = \pi(\sin(\pi z) - \cos(\pi z)), \quad F(z) = \pi(\sin(\pi z) - \cos(\pi z)),$$

we have

$$\lambda_2\left(\frac{1}{A_{20}}\right) = 0 < \mu_2(A_{20}) = 1,$$

$$\max\{\rho_2(F), \rho_2(A_{ij}), (i, j) \neq (2, 0)\} = 0 < \mu_2(A_{20}) = 1.$$

As we see, the conditions of Theorem 4.1.4 (i) are satisfied. The function

$$f(z) = e^{\cos(\pi z) + \sin(\pi z)}$$

is a solution of equation (4.28) and f satisfies

$$\rho_2(f) = 1 \geq \mu_2(A_{20}) = 1.$$

Case 2. $\rho_p(F) = \mu_p(A_{10})$ and $\tau_p(F) < \tau_p(A_{10})$. In (4.28), for

$$A_{00}(z) = \tan(\pi z), \quad A_{11}(z) = \pi^2(\sin(2\pi z) - \sin(\pi z) - \cos(\pi z) - 1),$$

$$A_{20}(z) = e^{\cos(\frac{z}{\pi}) - \cos(\pi z) - \sin(\pi z)} - \tan(\pi z), \quad A_{12}(z) = \pi(\sin(\pi z) - \cos(\pi z)),$$

$$F(z) = e^{\cos(\frac{z}{\pi})},$$

we have

$$\lambda_2\left(\frac{1}{A_{20}}\right) = 0 < \mu_2(A_{20}) = 1, \quad \max\{\rho_2(A_{ij}) : (i, j) \neq (2, 0)\} = 0 < \mu_2(A_{20}) = 1,$$

$$\rho_2(F) = \mu_2(A_{20}) = 1, \quad \tau_2(F) = \frac{1}{\pi^2} < \tau_2(A_{20}) = 1.$$

Thus, the conditions of Theorem 4.1.4 (i) are verified. The function

$$f(z) = e^{\cos(\pi z) + \sin(\pi z)}$$

is a solution of equation (4.28) and f satisfies

$$\rho_2(f) = 1 \geq \mu_2(A_{20}) = 1.$$

Case 3. $\mu_p(F) = \mu_p(A_{10})$ and $\tau_p(F) > \tau_p(A_{10})$. In (4.28), for

$$\begin{aligned} A_{00}(z) &= \tan(\pi z), & A_{11}(z) &= \pi^2 (\sin(2\pi z) - \sin(\pi z) - \cos(\pi z) - 1) e^{\sin(\frac{z}{2\pi})}, \\ A_{20}(z) &= e^{\cos(\frac{z}{\pi})} - \tan(\pi z), & A_{12}(z) &= \pi (\sin(\pi z) - \cos(\pi z)) e^{\sin(\frac{z}{2\pi})}, \\ F(z) &= e^{\cos(\frac{z}{\pi}) + \cos(\pi z) + \sin(\pi z)}, \end{aligned}$$

we have

$$\begin{aligned} \lambda_2\left(\frac{1}{A_{20}}\right) &= 0 < \mu_2(A_{20}) = 1, & \max\{\rho_2(A_{ij}), (i, j) \neq (2, 0)\} &= \mu_2(A_{20}) = 1, \\ \max\{\tau_2(A_{ij}) : \rho_2(A_{ij}) = \mu_2(A_{20}), (i, j) \neq (2, 0)\} &= \frac{1}{2\pi^2} < \tau_2(A_{20}) = \frac{1}{\pi^2}, \\ \mu_2(F) = \mu_2(A_{20}) &= 1, & \tau_2(F) = 1 > \tau_2(A_{20}) &= \frac{1}{\pi^2}. \end{aligned}$$

Hence, the conditions of Theorem 4.1.4 (i) are satisfied. The function

$$f(z) = e^{\cos(\pi z) + \sin(\pi z)}$$

is a solution of equation (4.28) and f satisfies

$$\rho_2(f) = 1 \geq \mu_2(A_{20}) = 1.$$

(ii) $\mu_p(F) > \mu_p(A_{10})$. In (4.28), for

$$\begin{aligned} A_{00}(z) &= \tan(\pi z), & A_{11}(z) &= \pi^2 (\sin(2\pi z) - \sin(\pi z) - \cos(\pi z) - 1), \\ A_{20}(z) &= e^{\cos(\sqrt{\pi z})} - \tan(\pi z), & A_{12}(z) &= \pi (\sin(\pi z) - \cos(\pi z)), \\ F(z) &= e^{\cos(\sqrt{\pi z}) + \cos(\pi z) + \sin(\pi z)}, \end{aligned}$$

we have

$$\begin{aligned} \lambda_2\left(\frac{1}{A_{20}}\right) &= 0 < \mu_2(A_{20}) = \frac{1}{2}, & \max\{\rho_2(A_{ij}) : (i, j) \neq (2, 0)\} &= 0 < \mu_2(A_{20}) = \frac{1}{2}, \\ \mu_2(F) &= 1 > \mu_2(A_{20}) = \frac{1}{2}. \end{aligned}$$

Obviously, the conditions of Theorem 4.1.4 (ii) are verified. The function

$$f(z) = e^{\cos(\pi z) + \sin(\pi z)}$$

is a solution of equation (4.28) and f satisfies

$$\rho_2(f) = 1 \geq \mu_2(F) = 1.$$

Conclusion and perspectives

In this thesis, we were interested in the study of certain problems related to the order of growth of complex functional equations, pointing out here that throughout this study, we have adopted the theory of Nevanlinna on \mathbb{C} , more precisely the first main theorem of Nevanlinna, lemma of the logarithmic derivative, and the associated properties of Nevanlinna characteristic of $f(z+\eta)$, where $\eta \in \mathbb{C}$. Important estimates were obtained in chapters two and three concerning the lower bound of the order and also of the lower order of growth of higher order linear difference equations, and differential-difference equations with entire or meromorphic coefficients in the complex plane.

In the last chapter, we generalized some of our results obtained in the two previous chapters, where we used the concept of p -iterated order instead of the usual growth. Finally, we think that this work constitutes a contribution in this field of research, and the problem of extensions and generalization remains open under the following natural questions :

Q1 : Can we generalize the results of this thesis, when we use the concept $[p, q]$ -order or more general $[p, q] - \varphi$?

Q2 : Can we extend the present results analogously to the unit disk?

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