Int. Journal of Math. Analysis, Vol. 1, 2007, no. 18, 851 - 860

An Example of Unbounded Fourier Integral Operator on L^2 with Symbol in $\bigcap_{0<\rho<1} S^0_{\rho,1}$

Mohamed Aïboudi, Bekkai Messirdi

Université d'Oran Es-sénia, Faculté des Sciences
Département de Mathématiques
B.P 1524 El Mnaouer, Oran 31000, Algeria

Abderrahmane Senoussaoui

Université Libre de Bruxelles, Faculté des Sciences
Département de Mathématiques
Boulevard du Triomphe, Campus Plaine CP 214, B1050 Brussels, Belgium

Abstract

In this paper, we give an example of Fourier integral operator with a symbol belongs to $\bigcap_{0<\rho<1} S^0_{\rho,1}$ that cannot be extended as a bounded operator on $L^2(\mathbb{R}^n)$.

Mathematics Subject Classification: 35S30, 35S05

Keywords: Fourier integral operators, amplitude, symbol and phase.

1 Introduction

A Fourier integral operator is a singular integral operator of the form

$$I(a,\phi)u(x) = \int\int e^{i\phi(x,y,\theta)} a(x,y,\theta)\, u(y)\, dy d\theta$$

defined under certain assumptions on the regularity and asymptotic properties of the phase function ϕ and the amplitude function a. Here θ plays the role of the covariable.

Fourier integral operators are more general than pseudodifferantial operators, where the phase function is of the form $\langle x-y,\theta\rangle$.

Let us denote by $S^m_{\rho,\delta}\left(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}^N\right)$ the space of $a(x,y,\theta)\in C^\infty\left(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}^N\right)$, satisfying

$$\left|\partial_x^\alpha\partial_y^\beta\partial_\theta^\gamma a(x,y,\theta)\right|\le C_{\alpha,\beta,\gamma}\lambda^{m-\rho|\gamma|+\delta(|\alpha|+|\beta|)}(\theta),\ \forall(\alpha,\beta,\gamma)\in\mathbb{N}^{n_1}\times\mathbb{N}^{n_2}\times\mathbb{N}^N,$$

where $\lambda(\theta)=(1+|\theta|)$.

The phase function $\phi(x,y,\theta)$ is assumed to be a-$C^\infty\left(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}^N,\mathbb{R}\right)$ real function, homogeneous in θ of degree 1.

Since 1970, many efforts have been made by several authors in order to study this type of operators (see, e.g., [1, 3, 6, 7, 8]).

For the Fourier integral operators, an interesting question is under which conditions on a and ϕ these operators are bounded on L^2 or on the Sobolev spaces H^s.

It was proved in [10] that all pseudodifferential operators with symbol in $S^0_{\rho,\delta}$ are bounded on L^2 if $\delta<\rho$. When $0<\delta=\rho<1$, Calderon and Vaillancourt [2] have proved that all pseudodifferantial operators with symbol in $S^0_{\rho,\rho}$ are bounded on L^2. On the other hand, Kumano-Go [11] has given a pseudodifferential operator with symbol belonging to $\bigcap\limits_{0<\rho<1}S^0_{\rho,1}$ which is not bounded on $L^2(\mathbb{R})$.

For Fourier integral operators, it has been proved in [1] that the operator $I(a,\phi):L^2\longrightarrow L^2$ is bounded if $\delta=m=\rho=0$. Recently, M. Hasanov [6] constructed a class of unbounded Fourier integral operators on $L^2(\mathbb{R})$ with an amplitude in $S^0_{1,1}$.

For $u\in C_0^\infty(\mathbb{R}^n)$, the integral operators

$$I(a,S)\varphi(x)=\int e^{iS(x,\theta)}a(x,y,\theta)\mathcal{F}\varphi(\theta)\,d\theta \tag{1.1}$$

appear naturally in the expression of the solutions of hyperbolic partial differential equations (see [4, 5, 12]).

If we write formally the expression of the Fourier transformation $\mathcal{F}u(\theta)$ in (1.1), we obtain the following Fourier integral operators

$$I(a,S)u(x)=\iint e^{i(S(x,\theta)-y\theta)}a(x,y,\theta)u(y)\,dyd\theta \tag{1.2}$$

in which the phase function has the form $\phi(x,y,\theta)=S(x,\theta)-y\theta$. We note that in [13], we have studied the L^2-boundedness and L^2-compactness of a class of Fourier integral operator of the form (1.2).

In this article we give an example of a Fourier integral operator, in higher dimension, of the form (1.1) with symbol $a(x,\theta)\in\bigcap\limits_{0<\rho<1}S^0_{\rho,1}$ independent on y, that cannot be extended to a bounded operator in $L^2(\mathbb{R}^n)$, $n\ge1$. Here we take the phase function in the form of separate variable $S(x,\theta)=\varphi(x)\psi(\theta)$.

2 The boundedness on $C_0^\infty(\mathbb{R}^n)$ and on $D'(\mathbb{R}^n)$

If $\varphi \in C_0^\infty(\mathbb{R}^n)$, we consider the following integral transformations

$$\begin{aligned}(I(a,S)\varphi)(x) &= \int_{\mathbb{R}^N} e^{iS(x,\theta)} a(x,\theta)\,\mathcal{F}\varphi(\theta)\,d\theta \\ &= \int_{\mathbb{R}^N\times\mathbb{R}^n} e^{i(S(x,\theta)-y\theta)} a(x,\theta)\,\varphi(y)\,d\theta dy \qquad (2.3)\end{aligned}$$

for $x \in \mathbb{R}^n$ and $N \in \mathbb{N}$.

In general the integral (2.3) is not absolutely convergent, so we use the technique of the oscillatory integral developed by L.Hörmander in [8]. The phase function S and the amplitude a are assumed to satisfy the hypothesis

$(H1)$ $S \in C^\infty(\mathbb{R}^n_x \times \mathbb{R}^N_\theta, \mathbb{R})$ (S real function)

$(H2)$ $\forall \beta \in \mathbb{N}^N$, $\exists C_\beta > 0$;

$$\left|\partial_\theta^\beta S(x,\theta)\right| \le C_\beta(x)\,\lambda^{(1-|\beta|)_+}(\theta)\ ,\ \forall (x,\theta) \in \mathbb{R}^n_x \times \mathbb{R}^N_\theta$$

where $\lambda(\theta) = (1+|\theta|)$ and $(1-|\beta|)_+ = \max(1-|\beta|, 0)$.

$(H3)$ S satisfies

$$\left(\frac{\partial S}{\partial x}, \frac{\partial S}{\partial \theta} - y\right) \neq 0,\ \forall (x,\theta) \in \mathbb{R}^n_x \times \left(\mathbb{R}^N_\theta \setminus \{0\}\right).$$

Remark 2.1 *If the phase function $S(x,\theta)$ is homogeneous in θ of degree 1, then it satisfies $(H2)$.*

For any open Ω of $\mathbb{R}^n_x \times \mathbb{R}^N_\theta$, $m \in \mathbb{R}$, $\rho > 0$ and $\delta \ge 0$ we set

$$S^m_{\rho,\delta}(\Omega) = \left\{ \begin{array}{ll} a \in C^\infty(\Omega); & \forall(\alpha,\beta) \in \mathbb{N}^n \times \mathbb{N}^N,\ \ \exists C_{\alpha,\beta} > 0; \\ \left|\partial_x^\alpha \partial_\theta^\beta a(x,\theta)\right| \le & C_{\alpha,\beta}\lambda^{m-\rho|\beta|+\delta|\alpha|}(\theta). \end{array} \right\}$$

Theorem 2.2 *If S satisfies $(H1), (H2), (H3)$ and if $a \in S^m_{\rho,\delta}(\mathbb{R}^n_x \times \mathbb{R}^N_\theta)$, then $I(a,\phi)$ is a continuous operator from $C_0^\infty(\mathbb{R}^n)$ to $C^\infty(\mathbb{R}^n)$ and from $\mathcal{E}'(\mathbb{R}^n)$ to $D'(\mathbb{R}^n)$, where $\rho > 0$ and $\delta < 1$.*

Proof. See [6], [4, pages 50-51]. ∎

Corollary 2.3 *Let $\varphi(x), \psi(\theta) \in C^\infty(\mathbb{R}^n, \mathbb{R})$ two functions, $\psi(\theta)$ is homogeneous of degree 1 $(\psi(\theta) \neq 0)$ and $\varphi(x)$ satisfies*

$$\varphi'(x) \neq 0,\ \forall x \in \mathbb{R}^n. \qquad (2.4)$$

Then the operator

$$(Fu)(x) = \int_{\mathbb{R}^n} e^{i\varphi(x)\psi(\theta)} a(x,\theta)\,\mathcal{F}u(\theta)\,d\theta, \quad u \in \mathcal{S}(\mathbb{R}^n) \tag{2.5}$$

maps continuously $C_0^\infty(\mathbb{R}^n)$ *to* $C^\infty(\mathbb{R}^n)$ *and* $\mathcal{E}'(\mathbb{R}^n)$ *to* $D'(\mathbb{R}^n)$ *for every* $a \in S^m_{\rho,\delta}(\mathbb{R}_x \times \mathbb{R}_\theta)$, *where* $\rho > 0$ *and* $\delta < 1$.

Proof. For the phase function $S(x,\theta) = \varphi(x)\psi(\theta)$, $(H1)$, $(H2)$ and $(H3)$ are satisfied. ■

3 The unboundedness of the operator F on $L^2(\mathbb{R}^n)$

In this section we shall construct a symbol $a(x,\theta)$ in the Hörmander space $\bigcap_{0<\rho<1} S^0_{\rho,1}(\mathbb{R}^n_x \times \mathbb{R}^n_\theta)$, such that the Fourier integral operator F can not be extended as a bounded operator in $L^2(\mathbb{R}^n)$.

Lemma 3.1 *(Kumano-Go [11]). Let* $f_0(t)$ *be a continuous function on* $[0,1]$ *such that*

$$f_0(0) = 0, \; f_0(t) > 0 \textit{ in }]0,1]. \tag{3.1}$$

Then, there exists a continuous function $b(t)$ *on* $[0,1]$ *such that* $b(t)$ *satisfies the conditions*

$$\begin{cases} f_0(t) \le b(t) \quad on \; [0,1], \\ b \in C^\infty(]0,1]), \; b(0) = 0, \; b'(t) > 0 \; in \;]0,1], \\ \left|b^{(n)}(t)\right| \le C_n t^{-n} \; in \;]0,1], \; n \in \mathbb{N}^*, \; C_n > 0. \end{cases} \tag{3.2}$$

Definition 3.2 *It is obvious that an operator* A *is extended as a bounded operator in* $L^2(\mathbb{R}^n)$ *if and only if there exists a constant* $C > 0$ *such that*

$$\|Au\|_{L^2} \le C\|u\|_{L^2} \textit{ for any } u \in \mathcal{S}(\mathbb{R}^n). \tag{3.3}$$

Theorem 3.3 *Let* A *be an operator, given at least for* $x \in]0,\beta[^n$ $(\beta < 1)$, *by*

$$(Au)(x) = \int_{\mathbb{R}^n} u(g(x)z)\,\mathcal{F}\Psi(z)\,dz, \quad u \in \mathcal{S}(\mathbb{R}^n),$$

we denote here by $]0,\beta[^n = \prod_{j=1}^n]0,\beta[$.

If $\Psi \in \mathcal{S}(\mathbb{R}^n)$, $\Psi(0) = 1$ *and the function* $g \in C^0\left(]0,\beta[^n, \mathbb{R}_+\right)$ *satisfies*

$$\begin{cases} \lim\limits_{|x|\to 0^+} \frac{g(x)}{|x|} = 0, \\ \forall i \in \{1,...,n\}\,;\; x_i \longrightarrow g(x_1,..,x_i,...,x_n) \text{ is increasing on }]0,\beta[\,. \end{cases} \tag{3.4}$$

Then the operator A *cannot be extended to a bounded operator in* $L^2(\mathbb{R}^n)$.

Proof. Using the Fourier inversion formula in $\mathcal{S}(\mathbb{R}^n)$, we have

$$(Au)(x) = \int_{\mathbb{R}^n} u(g(x)z)\,\mathcal{F}\Psi(z)\,dz, \quad u \in \mathcal{S}(\mathbb{R}^n).$$

Then there exists a constant $N_0 > 0$ such that

$$\left|(2\pi)^{-\frac{n}{2}} \int_{[-N,N]^n} \mathcal{F}\Psi(z)\,dz\right| \geq \beta \quad \text{for any } N \geq N_0. \tag{3.5}$$

Setting for $\varepsilon > 0$

$$u_\varepsilon(z) = \begin{cases} (2\pi)^{-\frac{n}{2}}, & \text{for } z \in [-\varepsilon,\varepsilon]^n \\ 0, & \text{for } z \notin [-\varepsilon,\varepsilon]^n \end{cases}.$$

Then, using the density of $\mathcal{S}(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$, we see that $(Au_\varepsilon)(x)$ must be

$$(Au_\varepsilon)(x) = (2\pi)^{-\frac{n}{2}} \int_{\left[-\frac{\varepsilon}{g(x)},\frac{\varepsilon}{g(x)}\right]^n} \mathcal{F}\Psi(z)\,dz \quad \text{for } x \in]0,\beta[^n. \tag{3.6}$$

By (3.4) for any $p \in \mathbb{N}^*$ there exists a small $\varepsilon_p \geq 0$ such that

$$\frac{\varepsilon_p}{g(p\varepsilon_p,...,p\varepsilon_p)} \geq N_0 \text{ and } p\varepsilon_p \leq \beta.$$

It follows from the condition (3.4) that

$$\frac{\varepsilon_p}{g(x)} \geq \frac{\varepsilon_p}{g(p\varepsilon_p,...,p\varepsilon_p)} \geq N_0, \quad \text{holds for } x \in]0, p\varepsilon_p]^n,$$

so that, using (3.5) and (3.6), we have

$$\left\|Au_{\varepsilon_p}\right\|_{L^2}^2 = \int_{\mathbb{R}^n} \left|Au_{\varepsilon_p}(x)\right|^2 dx \geq \int_{[0,p\varepsilon_p]^n} \left|Au_{\varepsilon_p}(x)\right|^2 dx \geq \beta^2 (p\varepsilon_p)^n. \tag{3.7}$$

Assume that A is bounded on $L^2(\mathbb{R}^n)$. According to (3.3) there exists $C > 0$ such that:

$$\beta^2 (p\varepsilon_p)^n \leq \left\|Au_{\varepsilon_p}\right\|_{L^2}^2 \leq C^2 (2\varepsilon_p)^n \text{ for any } p.$$

Which is a contradiction. ■

Let $K(t)$ be a function from $\mathcal{S}(\mathbb{R})$ such that $K(t) = 1$ on $[-\delta, \delta]$ ($\delta < 1$), $b(t) \in C^0([0,1])$ be a continuous function satisfying conditions (3.2) and $\varphi(x), \psi(\theta) \in C^\infty(\mathbb{R}^n, \mathbb{R})$ with $\psi(\theta)$ homogeneous of degree 1 ($\psi(\theta) \neq 0$). We assume that $\varphi(x)$ satisfies

$$|\varphi(x)| \leq C|x| \text{ for } |x| \leq 1. \tag{3.8}$$

We remark that if the function $\varphi(x)$ is homogeneous of degree 1, then it satisfies (3.8).

For $x = (x_1, ..., x_n), \theta = (\theta_1, ...\theta_n) \in \mathbb{R}^n$, set

$$q(x,\theta) = e^{-i\varphi(x)\psi(\theta)} \prod_{j=1}^{n} K(b(|x|)|x|\theta_j)$$

Lemma 3.4 *The function $q \in C^\infty([-1,1]^n \times \mathbb{R}^n_\theta)$ and the following estimate holds:*

$$\forall(\alpha,\beta) \in \mathbb{N}^n \times \mathbb{N}^n, \ \exists C_{\alpha,\beta} > 0;$$
$$\left|\partial_x^\alpha \partial_\theta^\beta q(x,\theta)\right| \leq C_{\alpha,\beta} \lambda^{|\alpha|-|\beta|}(\theta) \left\{b\left(\lambda^{-1}(\theta)\right)\right\}^{-|\beta|} \ \text{on } [-1,1]^n \times \mathbb{R}^n_\theta. \tag{3.9}$$

Proof. We adopt here the same strategy of Kumano-Go [11] lemma 2.

Since $\psi(\theta)$ is homogeneous of degree 1 ($\psi(\theta) \neq 0$), it will be sufficient to check the estimate for $n = 1$ on $[-1,1] \times \mathbb{R}_\theta$, i.e.

$$\forall(j,k) \in \mathbb{N} \times \mathbb{N}, \ \exists C_{j,k} > 0;$$
$$\left|\partial_x^j \partial_\theta^k \left[e^{-i\varphi(x)\theta} K(b(|x|)x\theta)\right]\right| \leq C_{j,k} \lambda^{j-k}(\theta) \left\{b\left(\lambda^{-1}(\theta)\right)\right\}^{-k}. \tag{3.10}$$

Since $K(t) \in \mathcal{S}(\mathbb{R})$ and $K^{(n)}(t) = 0$ on $[-\delta, \delta]$, $n \in \mathbb{N}^*$, then

$$\left|t^l K(t)\right| \leq C_l, \quad \forall l \in \mathbb{N} \tag{3.11}$$
$$\left|t^l K^{(n)}(t)\right| \leq C_{l,n}, \ \forall n \in \mathbb{N}^*, \forall l \in \mathbb{Z}. \tag{3.12}$$

By Leibnitz's formula we have

$$\partial_x^j \partial_\theta^k \left[e^{-i\varphi(x)\theta} K(b(|x|)x\theta)\right] =$$
$$\sum_{\substack{j_1+j_2=j \\ k_1+k_2=k}} C^{j,k}_{j_1,j_2,k_1,k_2} \partial_x^{j_1} \left[K^{(k_1)}(b(|x|)x\theta)(b(|x|)x)^{k_1}\right] \partial_x^{j_2} \left[(-i\varphi(x))^{k_2} e^{-i\varphi(x)\theta}\right],$$

where $C^{j,k}_{j_1,j_2,k_1,k_2} = \frac{j!k!}{j_1!j_2!k_1!k_2!}$. Then, by means of (3.8) and (3.2), we have for constants $C'_{j,k}$

$$\left|\partial_x^j \partial_\theta^k \left[e^{-i\varphi(x)\theta} K(b(|x|)x\theta)\right]\right| \leq C'_{j,k} K_{j+k}(b(|x|)x\theta)$$
$$\sum_{\substack{j_1+j_2=j \\ k_1+k_2=k}} \max_{\substack{s_1+s_2+s_3=j_1 \\ s_3 \leq k_1}} \left\{|\theta|^{s_1} |x|^{-s_2} (b(|x|)x)^{k_1-s_3}\right\} \max_{\substack{l_1+l_2=j_2 \\ l_2 \leq k_2}} \left\{|\theta|^{l_1} |x|^{k_2-l_2}\right\}, \tag{3.13}$$

where $K_p(t)$, $p \in \mathbb{N}$ are defined by

$$K_0(t) = |K(t)|, \ K_p(t) = \max_{1 \le p' \le p} \left| K^{(p')}(t) \right|, \ p \in \mathbb{N}^*.$$

Writing $|b(|x|)x| = |b(|x|)x\theta| |\theta|^{-1}$ and $|x|^{-1} = |b(|x|)x\theta|^{-1} b(|x|) |\theta|$, then there exists a constant C,

$$\begin{cases} |b(|x|)x| \le C\lambda(b(|x|)x\theta)\lambda^{-1}(\theta), \\ |x|^{-1} \le C|b(|x|)x\theta|^{-1}\lambda(\theta) \end{cases} \text{ on } [-1,1] \times \mathbb{R}_\theta. \tag{3.14}$$

We have $b^{-1}(|x|) \le b^{-1}(\lambda^{-1}(\theta))$ when $|x| \ge \lambda^{-1}(\theta)$ (because b is increasing). Then,

$$\begin{aligned} |x| = |b(|x|)x\theta| (b(|x|)|\theta|)^{-1} &\le |b(|x|)x\theta| |\theta|^{-1} b^{-1}\left(\lambda^{-1}(\theta)\right) \\ &\le C\lambda(b(|x|)x\theta)\lambda^{-1}(\theta) b^{-1}\left(\lambda^{-1}(\theta)\right). \end{aligned}$$

Bearing in mind the other case when $|x| \le \lambda^{-1}(\theta)$, we obtain for a constant $\widetilde{C}$

$$|x| \le \widetilde{C}\lambda(b(|x|)x\theta)\lambda^{-1}(\theta) b^{-1}\left(\lambda^{-1}(\theta)\right). \tag{3.15}$$

Finally from (3.11) to (3.15), we obtain (3.10). ■

Lemma 3.5 *For any continuous function $b_0(t)$ on $[1,+\infty[$ such that*

$$b_0(t) > 0, \qquad \lim_{t \to +\infty} b_0(t) = +\infty, \tag{3.16}$$

then there exists a continuous function $b(t)$ on $[0,1]$ witch satisfies conditions (3.2) such that we have on $[-1,1]^n \times \mathbb{R}^n_\theta$

$$\left|\partial_x^\alpha \partial_\theta^\beta q(x,\theta)\right| \le C_{\alpha,\beta} \lambda^{|\alpha|-|\beta|}(\theta) \{b_0(\lambda(\theta))\}^{|\beta|}, \ \alpha, \beta \in \mathbb{N}^n. \tag{3.17}$$

Proof. Setting

$$\begin{cases} f_0(t) = \{b_0(t^{-1})\}^{-1} \text{ on }]0,1] \\ f_0(0) = 0, \end{cases}$$

then f_0 is a continuous function on $[0,1]$ which verifies condition (3.1). Then, by lemma 3.1, there exists a continuous function $b(t)$ which satisfies (3.2). Noting that

$$\left\{b\left(\lambda^{-1}(\theta)\right)\right\}^{-1} \le \left\{f_0\left(\lambda^{-1}(\theta)\right)\right\}^{-1} = b_0(\lambda(\theta))$$

this gives (3.17). ■

Lemma 3.6 *Let $\{b_l(t)\}_{l\in\mathbb{N}^*}$ be a sequence of continuous functions on $[1,+\infty[$ which satisfy (3.16). Then, there exists a continuous function $b_0(t)$ verifiying (3.16), such that, for any l_0,*

$$b_l(t) \geq b_0(t) \ \text{ on } \ [t_{l_0},+\infty[\,, \ l=1,...,l_0$$

Finally, our but is to give an unbounded Fourier integral operator of the form (2.5) with symbol in $\bigcap\limits_{0<\rho<1} S^0_{\rho,1}(\mathbb{R}^n_x\times\mathbb{R}^n_\theta)$.

Theorem 3.7 *There exist a Fourier integral operator F of the form (2.5), with symbol $a\in\bigcap\limits_{0<\rho<1} S^0_{\rho,1}(\mathbb{R}^n_x\times\mathbb{R}^n_\theta)$, which cannot be extended to be a bounded operator on $L^2(\mathbb{R}^n)$.*

Proof. Let $\phi(s)$ be a-$C_0^\infty(\mathbb{R})$ function such that

$$\begin{cases} \phi(s)=1 \ \text{ on } \ [-\beta,\beta] \quad (\beta<1) \\ \operatorname{supp}\phi\subset[-1,1]. \end{cases}$$

Define a C^∞-symbol $a(x,\theta)$ by

$$a(x,\theta)=e^{-i\varphi(x)\psi(\theta)}\prod_{j=1}^n\phi(x_j)\,K(b(|x|)\,|x|\,\theta_j) \ \text{ in } \mathbb{R}^n_x\times\mathbb{R}^n_\theta$$

where $K(t)$ and $b(t)$ are the functions of lemma 3.4. Let $b_l(t)=\overbrace{\log...\log}^{l}(C_l+t)$ defined on $[1,+\infty[$ and C_l some large constant, then by lemmas 3.5 and 3.6 we have $,\forall(\alpha,\gamma)\in\mathbb{N}^n\times\mathbb{N}^n$, $\forall l\in\mathbb{N}^*$

$$\left|\partial_x^\alpha\partial_\theta^\gamma a(x,\theta)\right|\leq C_{\alpha,\gamma,l}\lambda^{|\alpha|-|\gamma|}(\theta)\left\{b_l(\lambda(\theta))\right\}^{|\gamma|}, \tag{3.18}$$

$C_{\alpha,\gamma,l}$ are constants, so that $a(x,\theta)\in\bigcap\limits_{0<\rho<1} S^0_{\rho,1}(\mathbb{R}^n_x\times\mathbb{R}^n_\theta)$. Furthermore the corresponding Fourier integral F is

$$\begin{aligned}(Fu)(x) &= \int_{\mathbb{R}^n_\theta} e^{i\varphi(x)\psi(\theta)}a(x,\theta)\,\mathcal{F}u(\theta)\,d\theta \\ &= \prod_{j=1}^n\phi(x_j)\int_{\mathbb{R}^n_\theta}\prod_{j=1}^n K(b(|x|)\,|x|\,\theta_j)\,\mathcal{F}u(\theta)\,d\theta, \quad u\in\mathcal{S}(\mathbb{R}^n)\end{aligned} \tag{3.19}$$

We consider $(Fu)(x)$ in $]0,\beta]^n$. Then, using an adequate change of variable in the integral (3.19), we have

$$(Fu)(x)=\int_{\mathbb{R}^n_z} u(b(|x|)\,|x|\,z)\prod_{j=1}^n\mathcal{F}K(z_j)\,dz$$

which has the form of A in theorem 3.3. In addition the function $g(x) = b(|x|)|x|$ satisfies (3.4). Consequently the operator F cannot be extended as a bounded operator on $L^2(\mathbb{R}^n)$. ■

ACKNOWLEDGEMENTS. We would like to thank Professors J.-P. Gossez and all the member of Analysis Service at ULB for severals discussions.

References

[1] Asada, K and Fujiwara, D.: *On some oscillatory transformation in* $L^2(\mathbb{R}^n)$. *Japan. J. Math. vol 4 (2), 1978, p299-361.*

[2] Caldéron, A. P. and Vaillancourt, R.: *On the boundedness of pseudodifferential operators. J. Math. Soc. Japan 23, 1972, p374-378.*

[3] Duistermaat, J.J.: *Fourier integral operators. Courant Institute Lecture Notes, New-York 1973.*

[4] Egorov, Yu. V. and Shubin, M. A.: *Partial differential equations. Vol II, Springer-Verlag, Berlin, 1994.*

[5] Egorov, Yu. V. and Shubin, M. A.: *Partial differential equations. Vol IV, Springer-Verlag, Berlin, 1994.*

[6] Hasanov, M., *A class of unbounded Fourier integral operators. J. Math. Analysis and application 225, 1998, p641-651.*

[7] Helffer, B.: *Théorie spectrale pour des opérateurs globalement elliptiques. Société Mathématiques de France, Astérisque 112, 1984.*

[8] Hörmander, L.: *Fourier integral operators I, Acta Math. vol 127, 1971, p33-57.*

[9] Hörmander, L.: *Pseudo-differential operators of type (1.1). Comm. Pure Appl. Math. 33 (1988), p1085-1111.*

[10] Hörmander, L.: *On the* L^2 continuity *of pseudodifferential operators. Comm. Pure Appl. Math. 24 (1971), p529-535.*

[11] Kumano-Go, A.: *A problem of Nirenberg on pseudo-differential operators. Comm. Pure Appl. Math. 23 (1970), p115-121.*

[12] Messirdi, B. and Senoussaoui, A.: *Parametrix du problème de Cauchy* C^∞ *pour un opérateur différentiel matriciel fortement hyperbolique muni des ordres de Leray-Volevič. Journal for Analysis and its Applications. Vol 24, (3), 2005, 581-592.*

[13] Messirdi, B. and Senoussaoui, A.: *On the L^2-boundedness and L^2-compactness of a class of Fourier integral Operators. Electronic J. Diff. Equa. Vol 2006, (26), 2006, 1–12.*

Received: March 12, 2007