UNIVERSITY OF ABDELHAMID IBN BADIS - MOSTAGANEM
Faculty of Exact Sciences and Computer Science

Department of Mathematics and Computer Science

Course Handouts

Measure and Integration

Course and Application Exercises

Prepared by:

MENAD Abdallah

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Introduction

This course, designed for third-year undergraduate students in Mathematics (LMD system), covers the core of the material on Measure and Integration. It includes lectures with examples, and application exercises with solutions are provided at the end of each chapter to allow students to test their knowledge and prepare for tests and final exams.

Based on my experience teaching this subject for several years, I decided to prepare this course pack, which contains all the fundamental concepts related to this topic.

It should be noted that the content of this course pack is exactly the same as that offered in the official curriculum, following the framework provided by the Ministry and currently applied in all departments of Algerian universities.

We assume that the reader has a good knowledge of the usual topology of $\mathbb{R}$, the first principles of set theory and the concept of integration in the Riemann sense.

Since this course material is a lecture, we have chosen to prove almost all results completely, that is, without referring back to a well-known result or admitting a difficult auxiliary result. We have also included a considerable number of solved exercises as they were tested in tutorials or used in reflective assignments or knowledge assessments. It goes without saying that the reader will benefit from trying to solve the problems without first reading the solution. The chapters of this course material conclude with solved exercises drawn from the tutorial series prepared by the teaching staff of the Mathematics Department at Abdelhamid Ibn Badis University of Mostaganem.

The originality of this handout lies in its content, shamelessly inspired by existing literature.

Let's now turn to a more precise description of what you will find in this handout.

In the first chapter, we will briefly present the useful properties concerning operations on sets, countability, limits of sets, and characteristic functions of sets. We will then introduce the notion of a sigma-algebra, particularly the Borel sigma-algebra. We will offer a detailed study of positive measure, exterior measure, and in particular, Lebesgue measure on the Borel sigma-algebra.

The second chapter contains the general properties of measurable functions, notably measurable numerical maps, which will be denoted by $\mathcal{L}^0$. We will study convergence almost everywhere and convergence in measure.

In the third chapter, we will address and discuss the concept of integration with respect to a positive measure. First, we will study it for measurable real-valued functions and present the Monotone Convergence Theorem (or Beppo-Levi Theorem) and its consequences. We will then examine the integral of a measurable real-valued function and conclude with a comparison of the Lebesgue integral with the Riemann integral. Finally, we will provide a general overview of the construction of the L^1 space and the Dominated Convergence Theorem in this space.

In the fourth chapter, we devote ourselves to the study of product measure, including Fubini's Theorems and some applications.

Finally, given the frequent and recurring errors in exam papers for this subject, I've noticed that most students don't pay enough attention to the lecture material and instead work on exercises based solely on the solutions. I therefore advise students to first read the lecture notes carefully, then complete all the examples provided after each solution, and finally, proceed to solve the exercises without referring back to the solutions. The solutions are only useful for assessing the student's level of effort.

Ultimately, I hope this document will help students who want to master this area ofmathematics.

Chapitre 1

Tribes and measures

1.1 Reminders on Set Theory

Throughout the following, we consider a basic set E. We recall that $\mathcal{P}(E)$ denotes the family of all subsets of E. For all subsets A and B of E, we have

$$A^c = \{x \in E : x \notin A\},$$

the complement of A in E.

$$\begin{aligned} A \setminus B &= \{x \in E : x \in A \text{ and } x \notin B\} \\ &= A \cap B^c = A \setminus (A \cap B) \end{aligned}$$

$$A \Delta B = B \Delta A = (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B).$$

1.1.1 Countability

It is essential, in everything concerning measurement theory, to be able to distinguish what is countable from what is not.

Definition 1.1 *The set X is said to be countable if there exists a bijection between X and $\mathbb{N}$. In other words, we can write*

$$X = \{x_n : n \in \mathbb{N}\} = \{x_0, x_1, ..., x_n, ...\},$$

that is, any countable set that can be indexed by $\mathbb{N}$ (or if we can enumerate all its elements).

Remark 1.1 *1. Any finite set is countable.*

2. *$\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ are countable, but $\mathbb{R}$ is not countable.*
3. *Any subset of a countable set is countable.*
4. *If for every integer $n \in \mathbb{N}$, the set A_n is countable, then $\bigcup_{n=1}^{+\infty} A_n$ is countable.*
5. *Property 4) remains true if we replace the sequence $(A_n)_{n \geqslant 1}$ by the countable family $(A_i)_{i \in I}$, that is, with I countable.*
6. *Any product of a finite number of countable sets is countable.*

In general, countable families or properties that are expressed in terms of countability are denoted with the prefix σ to indicate their countability (examples : σ-algebra, σ-additivity).

1.1.2 Limits of sets

Definition 1.2 *Let E a non-empty set and $(A_n)_{n\geqslant 1}$ a sequence of subsets of E, we define the upper and lower limits of the sequence $(A_n)_{n\geqslant 1}$ by :*

$$\overline{\lim} A_n = \limsup_n A_n := \bigcap_{n\geqslant 1} \bigcup_{k\geqslant n} A_k = \bigcap_{n=1}^{+\infty} \bigcup_{k=n}^{+\infty} A_k.$$

$$\underline{\lim} A_n = \liminf_n A_n := \bigcup_{n\geqslant 1} \bigcap_{k\geqslant n} A_k = \bigcup_{n=1}^{+\infty} \bigcap_{k=n}^{+\infty} A_k.$$

Remark 1.2 *Note that*

$$\underline{\lim} A_n \subset \overline{\lim} A_n$$

Indeed : we have

$$\bigcap_{k\geqslant n} A_k \subset A_k,\ \forall k \geqslant n,$$

then

$$\bigcap_{k\geqslant n} A_k \subset \bigcap_{n\geqslant 1} \bigcup_{k\geqslant n} A_k = \overline{\lim} A_n,$$

therefore

$$\bigcup_{n\geqslant 1} \bigcap_{k\geqslant n} A_k \subset \overline{\lim} A_n.$$

Proposition 1.1 *(Convergent sequence of sets)*

1. *If $(A_n)_n$ is increasing, that is*

$$A_n \subset A_{n+1}, \forall n \geqslant 1,$$

then

$$\overline{\lim} A_n = \underline{\lim} A_n = \bigcup_{n=1}^{+\infty} A_n.$$

2. *If $(A_n)_n$ is decreasing that is*

$$A_{n+1} \subset A_n, \forall n \geqslant 1,$$

then

$$\overline{\lim} A_n = \underline{\lim} A_n = \bigcap_{n=1}^{+\infty} A_n.$$

In both cases, we say that the sequence $(A_n)_n$ is convergent.

For real sequences, recall that if $(a_n)_n$ is an sequence in $\mathbb{R}$, we define

$$\limsup_n a_n = \inf_{n\geqslant 1} \sup_{k\geqslant n} a_k.$$

$$\liminf_n a_n = \sup_{n\geqslant 1} \inf_{k\geqslant n} a_k.$$

These two numbers always exist in $\overline{\mathbb{R}} = [-\infty, +\infty]$.

Similarly if $(f_n)_n$ is a sequence of functions of a set E in $\mathbb{R}$, $f_n : E \longrightarrow \mathbb{R}$, we define $\lim_n \sup f_n$ and $\lim_n \inf f_n$ as functions from E in $\overline{\mathbb{R}}$

$$\left(\lim_n \sup f_n\right)(x) = \inf_{n\geqslant 1}\sup_{k\geqslant n} f_k(x).$$

$$\left(\lim_n \inf f_n\right)(x) = \sup_{n\geqslant 1}\inf_{k\geqslant n} f_k(x).$$

1.2 Characteristic functions of sets

Recall that the characteristic (or indicator) functionχ_A of a part A of X is the function $\chi_A : X \longrightarrow \mathbb{R}$ defined by

$$\chi_A(x) = \begin{cases} 1, \text{ if } x \in A \\ 0, \text{ if } x \notin A. \end{cases}$$

This function therefore only takes two values :1 or 0, depending on whether it is evaluated on A or not.

The characteristic function satisfies the following properties :

Proposition 1.2 *Let X a non empty set and A,$B \subset X$*

1. *If $A \cap B = \emptyset$ then $\chi_{A\cup B} = \chi_A + \chi_B$.*
2. *$\chi_{A^c} = 1 - \chi_A$ and if $B \subset A$ then $\chi_{A\setminus B} = \chi_A - \chi_B$.*
3. *If $(A_n)_n$ a sequence of subsets of X then we have*

$$\chi_{\underline{\lim} A_n} = \lim_n \inf \chi_{A_n} \text{ and } \chi_{\overline{\lim} A_n} = \lim_n \sup \chi_{A_n}.$$

Moreover, if the A_n are pairwise disjoint, then

$$\chi_{\bigcup_{n=1}^{+\infty} A_n} = \sum_{n=1}^{+\infty} \chi_{A_n}$$

1.3 Algebras and Tribes

Definition 1.3 *(Boolean algebra) : Let E any non-empty set and $\mathcal{P}(E)$ the set of subsets of the set E.*

$\mathcal{A} \subset \mathcal{P}(E)$ is a (Boolean) algebra if for all $A, B \in \mathcal{A}$:

1. *$\{\emptyset, X\} \in \mathcal{A}$*
2. *$A^c = E \setminus A \in \mathcal{A}$*
3. *$A \cap B \in \mathcal{A}$*
4. *$A \cup B \in \mathcal{A}$*

It is a semi-algebra (denoted δ) if conditions 1) and 3) are satisfied and the complement of an element of δ is a finite union of elements of δ.

Remark 1.3 *The algebra $\mathcal{A}$ generated by a semialgebra consists of the finite unions of subsets of δ*

Definition 1.4 *(Tribe or $\sigma-$algebra) : Let E any set and $T \subset \mathcal{P}(E)$ a set of parts of E. T is tribe or $\sigma-$algebra on E if :*

1. $E \in T$
2. T is stable by the transition to the complement

$$\forall A \in T,\ A^c \in T.$$

3. T is stable by the countable union

$$\forall \left(A_n\right)_{n\in\mathbb{N}} \subset T,\ \bigcup_{n\in\mathbb{N}} A_n \in T.$$

Example .1

1. $T = \{\emptyset, E\}$ Rough tribe.
2. $T = P(E)$ Discrete tribe.

Remark 1.4 $\varnothing \in T$, $\varnothing = E^c$

T is stable by countable intersection

$$\forall \left(A_n\right)_{n\in\mathbb{N}} \ \subset \ T,\ \bigcap_{n\in\mathbb{N}} A_n \in T.$$

$$\bigcap_{n\in\mathbb{N}} A_n \ = \ \left(\bigcup_{n\in\mathbb{N}} A_n^c\right)^c.$$

In general, the tribe is stable under any countable sequence of operations on sets.

Remark 1.5 *If T is a tribe on E then*

i) *$E \in T$ because $E = \emptyset^C$*

ii) *If $\left(A_n\right)_{n\geqslant 1}$ is a sequence of T then $\bigcap_{n\in\mathbb{N}} A_n \in T$.*

iii) *If $A, B \in T$ then $A \diagdown B$, $A\Delta B \in T$ because*

$$A\diagdown B = A \cap B^c \in T \text{ and } A\Delta B = (A\diagdown B) \cup (B\diagdown A) \in T.$$

Proposition 1.3 *T is a tribe on E if and only if*

1. *$\emptyset \in T$.*
2. *T is stable under complementation and finite intersection.*
3. *If $\left(A_n\right)_{n\geqslant 1}$ is a sequence of T pairwise disjoint(i.e $A_i \cap A_j = \emptyset$, $\forall i \neq j$), then*

$$\bigcup_{n\in\mathbb{N}} A_n \in T.$$

Proof. *The necessary condition is obvious. To show that it is sufficient, it suffices to show that T is stable under countable union, so let $((A_n)_{n\geqslant 1}$ be a sequence of T and let*

$$\begin{cases} B_1 = A_1 \\ B_n = A_n \cap \left(\bigcup\limits_{i=1}^{n-1} A_i\right)^c, \ n \geqslant 2, \end{cases}$$

according to the hypothesis (2) of the proposition we have $B_n \in T$ for all $n \geqslant 1$. For all $n \neq m$, if $n > m$ we have

$$B_m \cap B_n \subset A_m \cap B_n = A_m \cap \left(A_n \cap A_1^c \cap ... \cap A_m^c \cap ... \cap A_{n-1}^c\right) = \emptyset,$$

then

$$B_m \cap B_n = \emptyset, \ \forall n \neq m,$$

on the other hand , it is clear that

$$\bigcup_{n=1}^{+\infty} B_n \subset \bigcup_{n=1}^{+\infty} A_n.$$

For inverse inclusion, if $x \in \bigcup\limits_{n=1}^{+\infty} A_n$, threre exists $n \geqslant 1$ such that $x \in A_n$. We put r the smallest $n \geqslant 1$ which verifies that $x \in A_n$ i.e

$$r = \min\{n \in \mathbb{N}, x \in A_n\},$$

then

$$x \in A_r, x \notin A_{r-1}, x \notin A_{r-2}, ... and\ x \notin A_1,$$

which gives $x \in B_r$, then

$$x \in \bigcup_{n=1}^{+\infty} B_n.$$

We have shown that the B_n are pairwise disjoint and

$$\bigcup_{n=1}^{+\infty} A_n = \bigcup_{n=1}^{+\infty} B_n,$$

then according (3) of the proposition, we have

$$\bigcup_{n=1}^{+\infty} A_n \in T.$$

■

1.3.1 Generated Tribe

Proposition 1.4 *Let $(T_i)_i$ a family of tribes on the set E. Then $\bigcap\limits_{i\in I} T_i$ is a tribe on E.*

Proof. *We have* $E \in T_i$, $\forall i \in I$ *then* $E \in \bigcap_{i \in I} T_i$.*Now let*

$$\begin{aligned} A &\in \bigcap_{i \in I} T_i \Longrightarrow A \in T_i, \forall i \in I \\ &\Longrightarrow A^c \in T_i, \forall i \in I \text{ (because } T_i \text{ is aTribe)} \\ &\Longrightarrow A^c \in \bigcap_{i \in I} T_i. \end{aligned}$$

let

$$\{A_n\}_{n \in \mathbb{N}} \subset \bigcap_{i \in I} T_i,$$

then $\bigcap_{i \in I} T_i$ *is a tribe on* E. ■

Remark 1.6 *The union of a family of tribes does not necessarily constitute a tribe.*

Example .2 *Let*

$$E = \{a, b, c\}$$

$$\begin{aligned} T_1 &= \{\varnothing, \{a\}, \{b, c\}, E\} \\ T_2 &= \{\varnothing, \{b\}, \{a, c\}, E\} \end{aligned}$$

T_1 *and* T_2 *are tribes, but*

$$T_1 \cup T_2 = \{\varnothing, \{a\}, \{b\}, \{a, c\}, \{b, c\}, E\}$$

is not a tribe because

$$\{a\} \cup \{b\} = \{a, b\} \notin T_1 \cup T_2.$$

Remark 1.7 *If* $A \in T$ *and* $B \subset A \nRightarrow B \in T$

$$\{b, c\} \in T_1 \text{ but } \{b\} \subset \{b, c\}$$

doses not belong to T_1.

Definition 1.5 *(Generated tribe) Let* E *be a set and* C *a subset of* $P(E)$. *We call a tribe generated by* C *noted* $\sigma(C)$, *the smallest tribe containing* C. *Therefore, it is the intersection tribe of all the tribes containing* C.

$$\sigma(C) = \cap T, \ T \text{ tribe, } C \subset T$$

Example .3

1. $\sigma(\{\varnothing\}) = \sigma(E) = \{E, \varnothing\}$.
2. *Let* $A \in P(E)$ *such that* $A \neq E$ *and* $A \neq \varnothing$, *then*

$$\sigma(A) = \{\varnothing, E, A, A^c\}.$$

3. $E = \mathbb{N}$, $A = \{\{n\}, n \in \mathbb{N}\}$, $\sigma(A) = P(E)$.

1.3.2 Direct image tribe, reciprocal image tribe

Theorem 1.1 *Let $f : E \longrightarrow F$ an application. We have*

1. *If T a tribe on E, then*
$$T' = \left\{B \subset F : f^{-1}(B) \in T\right\},$$
is a tribe on F. (Direct image tribe).
2. *If T' is a tribe on F, then*
$$f^{-1}(T') = \left\{f^{-1}(B) : B \in T'\right\},$$
is a tribe on E. (reciprocal image tribe).

Proof. Let $f : E \longrightarrow F$ an application ■

1. We have
$$T' = \left\{B \subset F : f^{-1}(B) \in T\right\},$$
and
$$F \subset F \text{ and } f^{-1}(F) = E \in T \ ,$$
therefore
$$F \in T'.$$
Let
$$B \in T' \Longrightarrow B \subset F \text{ et } f^{-1}(B) \in T,$$
and we have
$$B^c \subset F,$$
on the other hand, we have
$$f^{-1}(B^c) = \left(f^{-1}(B)\right)^c \in T \text{ (because } T \text{ stable by complementarity)},$$
therefore
$$B^c \in T'.$$
Now, let
$$\begin{aligned}(B_n)_{n\in\mathbb{N}} &\subset T' \Longrightarrow \forall n \in \mathbb{N}, B_n \in T' \\ &\Longrightarrow \forall n \in \mathbb{N}, B_n \subset F \text{ and } f^{-1}(B_n) \in T,\end{aligned}$$
therefore, we obtain
$$\bigcup_{n\in\mathbb{N}} B_n \subset F \text{ and } f^{-1}(\bigcup_{n\in\mathbb{N}} B_n) = \bigcup_{n\in\mathbb{N}} f^{-1}(B_n) \in T,$$
(because T stable by $\cup$ countable), therefore, we get
$$\bigcup_{n\in\mathbb{N}} B_n \in T',$$
then
$$T' = \left\{B \subset F : f^{-1}(B) \in T\right\},$$
is a tribe on F.

2. We have
$$f^{-1}(T') = \left\{ f^{-1}(B) : B \in T' \right\},$$
is a tribe on E

a) we have
$$E = f^{-1}(F) \text{ and } F \in T' \Longrightarrow E \in f^{-1}(T').$$

b) Let
$$\begin{aligned} A &\in f^{-1}(T') \Longrightarrow A = f^{-1}(B),\ B \in T' \\ &\Longrightarrow A^c = \left(f^{-1}(B)\right)^c = f^{-1}(B^c), \end{aligned}$$
and we have
$$B^c \in T'$$
terefore
$$A^c \in f^{-1}(T').$$

c) Let $(A_n)_{n\in\mathbb{N}} \subset f^{-1}(T')$, then
$$\begin{aligned} (A_n)_{n\in\mathbb{N}} &\subset f^{-1}(T') \Longrightarrow \forall n \in \mathbb{N}, A_n \in f^{-1}(T') \\ &\Longrightarrow \forall n \in \mathbb{N},\ \exists B_n \in T' : A_n = f^{-1}(B_n), \end{aligned}$$
on the other hand, we have
$$\bigcup_{n\in\mathbb{N}} A_n = \bigcup_{n\in\mathbb{N}} f^{-1}(B_n) = f^{-1}(\bigcup_{n\in\mathbb{N}} B_n), \text{ with } \bigcup_{n\in\mathbb{N}} B_n \in T',$$
therefore
$$\bigcup_{n\in\mathbb{N}} A_n \in f^{-1}(T').$$

1.3.3 The Borelian tribe or Borel tribe

Definition 1.6 *(E, τ) is said to be a topological space if τ it is a family of subsets of E called open subsets containing $\varnothing$ and E, stable by any union and stable by finite intersection.*

Example .4 *$(E, P(E))$ is a topological space.*

Definition 1.7 *(Borel tribe) Let (E, τ) a topological space. The Borel tribe or Borel tribe is the tribe generated by the set of open sets of τ. This tribe will be denoted $\mathcal{B}(E)$. The elements of $\mathcal{B}(E)$ are called Borel elements of E.*

Example .5 *If $E = \mathbb{R}$, $\mathcal{B}(\mathbb{R}) = \sigma(C)$, where C is any of the families of intervals,*
$$\begin{aligned} &\{]a, b[\,, a, b \in \mathbb{R}, a < b\},\ \{]-\infty, a[\,, a \in \mathbb{R}, \} \\ &\{]a, +\infty[\,, a \in \mathbb{R}\}. \end{aligned}$$

Remark 1.8 *$\mathcal{B}(E)$ is also the tribe generated by the set of closed sets of E. The closed sets are the complements of the open sets.*

$$\begin{aligned}\mathcal{B}(\mathbb{R}) &= \sigma(C) \text{ with } C = \{[a,b]\,, a,b \in \mathbb{R}, a<b\} \\ &\quad \text{or } \{]-\infty,a]\,, a \in \mathbb{R},\} \text{ or } \{[a,+\infty[\,, a \in \mathbb{R},\}\,.\end{aligned}$$

Proposition 1.5 *Let T a tribe and $C \subset P(E)$. Then*

$$C \subset T = \sigma(C) \subset T.$$

Proposition 1.6 *The Borel tribe $\mathcal{B}(E)$ is also generated by the closed sets of the topological space E.*
Proof. *Let $\mathcal{F}$ the family of all closed sets of E. Since the tribes are stable by transition to the complement and the closed sets are the complements of the open sets, we have*

$$\mathcal{F} \subset \mathcal{B}(E),$$

then

$$\sigma(\mathcal{F}) \subset \mathcal{B}(E).$$

Conversely, let $\theta \in \tau$ an open of E, then

$$\theta^c \in \mathcal{F} \subset \sigma(\mathcal{F}),$$

therefore $\theta \in \sigma(\mathcal{F})$. This shows that $\tau \subset \sigma(\mathcal{F})$ and since $\mathcal{B}(E) = \sigma(\tau)$ we have $\mathcal{B}(E) \subset \sigma(\mathcal{F})$.
■

Lemma 1.1 *Every non-empty open set of $\mathbb{R}$ is a countable union of intervals $]a,b[$.*
Proof. *Let θ a non-empty open set of $\mathbb{R}$. We put the part $A \subset \mathbb{Q}^2$ dfined by*

$$A = \left\{(p,q) \in \mathbb{Q}^2, p<q :]p,q[\subset \theta\right\}.$$

If $x \in \theta$, there exists $\varepsilon_x > 0$ such that $]x-\varepsilon_x, x+\varepsilon_x[\subset \theta$. By the density of $\mathbb{Q}$ in $\mathbb{R}$ we can take $p_x, q_x \in \mathbb{Q}$ verifying

$$x - \varepsilon_x \leqslant p_x \leqslant x \text{ and } x \leqslant q_x \leqslant x + \varepsilon_x,$$

it results that

$$x \in]p_x, q_x[\subset]x-\varepsilon_x, x+\varepsilon_x[\subset \theta,$$

then $p_x, q_x \in A$, therefore

$$x \in]p_x, q_x[\subset \bigcup_{(p,q)\in A}]p,q[\,,$$

that is to say

$$\theta \subset \bigcup_{(p,q)\in A}]p,q[\,.$$

Conversely, if

$$x \in \bigcup_{(p,q)\in A}]p,q[\,,$$

there exists $p_x, q_x \in A$ with $x \in]p_x, q_x[\subset \theta$ then $x \in \theta$. We deduce that

$$\theta = \bigcup_{(p,q)\in A}]p,q[\,.$$

This union is countable because the part $A \subset \mathbb{Q}^2$ is countable. ■

Theorem 1.2 *(The tribe $\mathcal{B}(\mathbb{R})$) Th etribe $\mathcal{B}(\mathbb{R})$ is generated by the intervals $]a,+\infty[$ for $a \in \mathbb{R}$.*
Proof. *Let $\mathcal{S} = \{]a,+\infty[\,, a \in \mathbb{R}\}$ the family of all intervals $]a,+\infty[$ and τ the set of open sets of $\mathbb{R}$. It is clear that $\mathcal{S} \subset \tau$ then $\sigma(\mathcal{S}) \subset \mathcal{B}(\mathbb{R})$ because by definition we have $\sigma(\tau) = \mathcal{B}(\mathbb{R})$. Now, we show that $\tau \subset \sigma(\mathcal{S})$, let $\theta \in \tau$ then according to the previous lemma, θ is a countable union of intervals of the form $]a,b[$ then $\theta = \bigcup_{n=1}^{+\infty}]a_n, b_n[$. For all $a,b \in \mathbb{R}$ with $a < b$ we have*

$$]a,b[\, = \,]-\infty, b[\, \cap \,]a,+\infty[\;\text{ and }\; [b,+\infty[\, = \bigcap_{n=1}^{+\infty} \left]b - \frac{1}{n}, +\infty\right[.$$

For all $n \geqslant 1$ we have

$$\left]b - \frac{1}{n}, +\infty\right[\in \mathcal{S} \subset \sigma(\mathcal{S}),$$

therefore stability by intersection guarantees that $[b,+\infty[\, \in \sigma(\mathcal{S})$ and with complementary

$$]-\infty, b[\, = ([b,+\infty[)^c \in \sigma(\mathcal{S}),$$

and

$$]a,+\infty[\, \in \mathcal{S} \subset \sigma(\mathcal{S}),$$

which gives $]a,b[\, \in \sigma(\mathcal{S})$. Then

$$\theta = \bigcup_{n=1}^{+\infty}]a_n, b_n[\, \in \sigma(\mathcal{S}).$$

Finally we get inclusion $\tau \subset \sigma(\mathcal{S})$ which implies that $\mathcal{B}(\mathbb{R}) \subset \sigma(\mathcal{S})$. ■

1.4 Positive measure

Definition 1.8 *"Measurable space - Measurable part" Let E be a set and T a tibe on E. The pair (E,T) is called the measurable space. The parts of E that are elements of T are called the measurable parts of E.*

Proposition 1.7 *Let E be a set , (E',T') is a measurable space and $f : E \longrightarrow E'$ an application.*

$$T = f^{-1}(T') = \left\{f^{-1}(B),\; B \in T'\right\}$$

is a tribe on E called the reciprocal image tribe of T' byr f.

Proposition 1.8 *Let (E,T) is a measurable space E is a set and $f : E \longrightarrow E'$ an application,*

$$T' = \left\{B \subset E' : f^{-1}(B) \in T\right\}$$

is a tribe on E called the direct image tribe of T by f.

Definition 1.9 *Let (E,T) a measurable space, a positive measure on (E,T) (or more simply on E) is an application of set $\mu : T \longrightarrow [0,+\infty]$ verifying the following properties :*

(i) $\mu(\emptyset) = 0$.

(ii) *For all sequence* $(A_n)_{n\geqslant 1} \subset T$ *of pairwise disjoint measurable parts, we have*

$$\mu\left(\bigcup_{n=1}^{+\infty} A_n\right) = \sum_{n=1}^{+\infty} \mu(A_n).$$

We say that (E, T, μ) *is a measured space.*

Remark 1.9 *1. In the previous definition, condition (i) can be replaced by the condition*

$$\exists A \in T : \mu(A) < \infty, \ \textit{(i.e } \mu(A) \textit{ is finite).}$$

2. The $\sigma-$*additivity (condition (ii)) contains, in particular the simple additivity property for all* $n \geq 1$*, if the* n *measurable sets* $A_1, ..., A_n$ *are pairwise disjoint , then*

$$\mu(A_1 \cup ... \cup A_n) = \mu(A_1) + ... + \mu(A_n),$$

it suffices to take $A_{n+1} = A_{n+2} = ... = \emptyset$.

Definition 1.10 *"Measure" Let* (E, T) *a measurable space. We call measure on* (E, T) *an application* $m : T \longrightarrow \overline{\mathbb{R}_+}$ *verifying*

1. $m(\varnothing) = 0$.
2. m *is* $\sigma-$*additive, that is, for any family* $(A_n)_{n\in\mathbb{N}} \subset T$ *of pairwise disjoint parts* ($A_n \cap A_m = \varnothing$ *for* $n \neq m$)*, we have*

$$m\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} m(A_n).$$

The measure m *is called finite if* $m(E) < +\infty$*. The measure* m *is a probability on* E *if* $m(E) = 1$.

Definition 1.11 *Let* E *is a set and* T *a tribe on* E*. We call probability is a measure* P *on* T *such that* $P(E) = 1$*. We say that* (E, T, P) *is a probability space and the elements of* T *are called events.*

Example .6 *1. Dirac mass : Let* E *is a set,* T *is a tribe on* E *and* $a \in E$*. We define on* T *the application* δ_a *for* $A \in T$ *by*

$$\delta_a(A) = \begin{cases} 0, \textit{ if } a \notin A \\ 1, \textit{ if } a \in A, \end{cases}$$

we verify that δ_a *is a measure on* E*. We have*

(a) $\delta_a(\varnothing) = 0$, $a \notin \varnothing$.

(b) *Let* $(A_n)_{n\in\mathbb{N}} \subset T$ *such that* $A_n \cap A_m = \varnothing$ *for* $n \neq m$.

We show that

$$\delta_a(\bigcup_{n\in\mathbb{N}} A_n) = \sum_{n\in\mathbb{N}} \delta_a(A_n).$$

We distinguish two cases

1) *If* $a \in \bigcup\limits_{n\in\mathbb{N}} A_n$,*we have* $\delta_a(\bigcup\limits_{n\in\mathbb{N}} A_n) = 1$. *Moreover*

$$a \in \bigcup_{n\in\mathbb{N}} A_n \Longrightarrow \exists ! n_0 \in \mathbb{N} : a \in A_{n_0},\ \ (A_n)_{n\in\mathbb{N}} \text{ are disjoint,}$$

then

$$\delta_a(A_n) = 0,\ \forall n \neq n_0,$$

therefore

$$\delta_a(\bigcup_{n\in\mathbb{N}} A_n) = 1 = \delta_a(A_{n_0}) = \sum_{n\in\mathbb{N}} \delta_a\left(A_n\right).$$

2) *If* $a \notin \bigcup\limits_{n\in\mathbb{N}} A_n$, *we have* $\delta_a(\bigcup\limits_{n\in\mathbb{N}} A_n) = 0$. *Moreover*

$$a \notin \bigcup_{n\in\mathbb{N}} A_n \Longrightarrow \forall n \in \mathbb{N} : a \notin A_n,\ \ \delta_a(A_n) = 0,\ \ \forall n \in \mathbb{N},$$

then

$$\delta_a(\bigcup_{n\in\mathbb{N}} A_n) = \sum_{n\in\mathbb{N}} \delta_a\left(A_n\right) = 0.$$

From 1) and 2), δ_a *is a measure on* E *and it is a probability,* $\delta_a(E) = 1$.

2. **Counting Measure :** Let (E, T) is a measurable space and m the application defined by

$$m(A) = \begin{cases} card(A), \text{ if } A \text{ is finite} \\ +\infty, \text{ if not,} \end{cases}$$

we can verify that m is indeed a measure on E.

Definition 1.12 *(Measured space) We call that measured space every triple* (E, T, m) *where* (E, T) *is a measurable space and* m *a measure on* (E, T).

Definition 1.13 *"measure* $\sigma-$*finie" Let* (E, T, m) *a mesured space . We say that* m *is* $\sigma-$*finite if*

$$\exists \left(A_n\right)_{n\in\mathbb{N}} \subset T \text{ such that } E = \bigcup_{n\in\mathbb{N}} A_n \text{ and } \forall n \in \mathbb{N}, m\left(A_n\right) < +\infty.$$

Proposition 1.9 *Let* (E, T, μ) *a mesured space . The measure* μ *has the following properties*

1. Monotonicity : If $A, B \in T$ with $A \subset B$ then $\mu(A) \leq \mu(B)$.
2. If $A, B \in T$ with $A \subset B$ and $\mu\left(A\right) < \infty$ then $\mu\left(B \diagdown A\right) = \mu\left(B\right) - \mu\left(A\right)$.
3. Subadditivity : For any sequence $\left(A_n\right)_{n\geqslant 1}$ in T we have

$$\mu\left(\bigcup_{n=1}^{+\infty} A_n\right) \leqslant \sum_{n=1}^{+\infty} \mu\left(A_n\right).$$

Proof.

1. If $A, B \in T$ with $A \subset B$ then $B = A \cup (B \diagdown A)$, since $A \cap (B \diagdown A) = \emptyset$, by the additivity of the measure we have
$$\mu(B) = \mu(A) + \mu(B \diagdown A) \geqslant \mu(A).$$
2. If moreoever $\mu(A) < \infty$ then $\mu(B \diagdown A) = \mu(B) - \mu(A)$.
3. From the sequence $(A_n)_{n \geqslant 1}$, we construct the sequence $(B_n)_{n \geqslant 1}$ defined by
$$\begin{cases} B_1 = A_1 \\ B_n = A_n \cap \left(\bigcup_{i=1}^{n-1} A_i \right)^c, \ n \geqslant 2, \end{cases}$$
we have $B_n \subset A_n$ for $n \geqslant 1$, the sets B_n are disjoint pairwise in T and
$$\bigcup_{n=1}^{+\infty} A_n = \bigcup_{n=1}^{+\infty} B_n.$$
The monotony of the measure gives $\mu(B_n) \leqslant \mu(A_n)$ for all $n \geqslant 1$. On the other hand , according to the $\sigma-$additivity of the measure we have
$$\mu\left(\bigcup_{n=1}^{+\infty} A_n \right) = \mu\left(\bigcup_{n=1}^{+\infty} B_n \right) = \sum_{n=1}^{\infty} \mu(B_n) \leqslant \sum_{n=1}^{\infty} \mu(A_n).$$

■

Definition 1.14 *We say that a positive measure μ is finite if it has finite values, i.e.*
$$\mu(A) < \infty \text{ for all } A \in T.$$
In other words, $\mu(E) < \infty$.

Definition 1.15 *Let μ a measure on (E, T). We say that it is $\sigma-$finite if ther exists a sequence of measurable parts $(E_n)_{n \geqslant 1}$ such that*
$$E = \bigcup_{n=1}^{+\infty} E_n$$
and $\mu(E_n) < \infty$ for all $n \geqslant 1$.

Example .7 *1. Dirac's measure δ_x is finite because $\delta_x(E) = 1 < \infty$*

2. The counting measure on E is :

i) *finite if and only if E is finite*

ii) *$\sigma-$finite if and only if E is countable.*

1.4.1 Properties of measures, external measure, complete measure

The property of measure continuity will be the basis of one of the most important and most used theorems for the Lebesgue integral and the monotone convergence theorem.

Theorem 1.3 *Let (E, T, μ) a measured space, then*

1. ***Increasing continuity :*** *If $(A_n)_{n\geqslant 1}$ is an increasing sequence of measurable parts, we have*
$$\mu\left(\bigcup_{n=1}^{+\infty} A_n\right) = \lim_{n\longrightarrow\infty} \mu(A_n).$$
2. ***Decreasing continuity :*** *If $(A_n)_{n\geqslant 1}$ is a decreasing sequence of measurable parts with*
$$\mu(A_1) < \infty,$$
then, we have
$$\mu\left(\bigcap_{n=1}^{+\infty} A_n\right) = \lim_{n\longrightarrow\infty} \mu(A_n).$$

Proof.

1. *Putting*
$$\begin{cases} B_1 = A_1 \\ B_n = A_n \setminus A_{n-1}, \ n \geqslant 2, \end{cases}$$
the sets B_n are mesurables and $A_n = \bigcup_{k=1}^{n} B_k$ for all $n \geqslant 1$, wich is impies that $\bigcup_{n=1}^{+\infty} B_n = \bigcup_{n=1}^{+\infty} A_n$. Moreover, the sets B_n, $n \geqslant 1$ are pairwise disjoint, therefore
$$\begin{aligned} \mu\left(\bigcup_{n=1}^{+\infty} A_n\right) &= \mu\left(\bigcup_{n=1}^{+\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n) = \lim_{n\longrightarrow\infty} \sum_{n=1}^{\infty} \mu(B_k) \\ &= \lim_{n\longrightarrow\infty} \mu\left(\bigcup_{k=1}^{n} B_k\right) = \lim_{n\longrightarrow\infty} \mu(A_n). \end{aligned}$$
2. *For all $n \geqslant 1$ we put $B_n = A_1 \setminus A_n = A_1 \cap A_n^c$. Since $(A_n)_{n\geqslant 1}$ is decreasing, the sequence $(B_n)_{n\geqslant 1}$ is increasing, using 1) we obtain*
$$\mu\left(\bigcup_{n=1}^{+\infty} B_n\right) = \lim_{n\longrightarrow\infty} \mu(B_n) = \mu(A_1) - \lim_{n\longrightarrow\infty} \mu(A_n),$$
on the other hand, we have
$$\bigcup_{n=1}^{+\infty} B_n = A_1 \cap \left(\bigcup_{n=1}^{+\infty} A_n^c\right) = A_1 \setminus \bigcap_{n=1}^{+\infty} A_n,$$

it results that

$$\mu\left(\bigcup_{n=1}^{+\infty} A_n\right) = \mu(A_1) - \mu\left(\bigcap_{n=1}^{+\infty} A_n\right).$$

Finally, we can simplify by $\mu(A_1)$ *since this last quantity is finite,*

$$\mu\left(\bigcap_{n=1}^{+\infty} A_n\right) = \lim_{n\longrightarrow\infty} \mu(A_n).$$

■

Remark 1.10 *The condition* $\mu(A_1) < \infty$ *in 2) of the previous theorem is necessary as shown in the following example*

Example .8 *Let us consider the measured space* $(\mathbb{N}, \mathcal{P}(\mathbb{N}), card)$ *and the sequence of measurable parts* $(A_n)_{n\geqslant 1}$ *such that*

$$A_n = \{n, n+1, n+2, ...\},$$

let us show that the condition $\mu(A_1) < \infty$ *is necessary for the decreasing continuity of the measure*

Proof. *The sequence* $(A_n)_{n\geqslant 1}$ *is decreasing*

$$A_{n+1} = \{n+1, n+2, n+3, ...\} \subset A_n = \{n, n+1, n+2, ...\},$$

and

$$\mu(A_1) = card(\{1, 2, 3, ...\}) = +\infty.$$

If $x \in \bigcap_{n=1}^{+\infty} A_n$ *then* $x \geqslant n$*, for all* $n \geqslant 1$*. Therefore* $\mathbb{N}$ *is bounded, which is a contradiction. Therefore*

$$\bigcap_{n=1}^{+\infty} A_n = \emptyset.$$

On the other hand, we have

$$0 = \mu\left(\bigcap_{n=1}^{+\infty} A_n\right) \neq \lim_{n\longrightarrow\infty} \mu(A_n) = \lim_{n\longrightarrow\infty} card(\{n, n+1, n+2, ...\}) = +\infty.$$

■

1.4.2 Negligible set and complete measure

Definition 1.16 *Let* (E, T, μ) *a measured space and* $N \subset E$*. The set* N *is said negligible in* (E, T, μ) *if there exists* $A \in T$ *such that* $N \subset A$ *and* $\mu(A) = 0$

Remark 1.11 *If* $(A_n)_{n\geqslant 1}$ *a sequence of negligible parts in* (E, T, μ) *then* $\bigcup_{n=1}^{+\infty} A_n$ *is nigligible.*

Indeed : for all $n \geqslant 1$ *there exists* $B_n \in T$ *such that* $A_n \subset B_n$ *and* $\mu(B_n) = 0$.
Since

$$\bigcup_{n=1}^{+\infty} A_n \subset \bigcup_{n=1}^{+\infty} B_n \text{ and } \mu\left(\bigcup_{n=1}^{+\infty} B_n\right) \leqslant \sum_{n=1}^{+\infty} \mu(B_n) = 0,$$

therefore $\bigcup_{n=1}^{+\infty} A_n$ *is nigligible.*

Definition 1.17 *a measured space* (E, T, μ) *is said to be complete if every negligible part is measurable (and therefore of zero measure). In this case we say that the measure* μ *is complete.*

1.4.3 External measure

Definition 1.18 *Let* X *a set. We call an exterior measure on* X *an* $\mu^* : \mathcal{P}(X) \longrightarrow [0, +\infty]$ *possessing the following properties*

1. $\mu^*(\emptyset) = 0$.
2. *If* $A \subset B \subset X$ *then* $\mu^*(A) \leqslant \mu^*(B)$.
3. *For all sequence* $(A_n)_n$ *of parts of* X *we have*

$$\mu^*\left(\bigcup_{n=1}^{+\infty} A_n\right) \leqslant \sum_{n=1}^{+\infty} \mu^*(A_n).$$

Remark 1.12 *It is clear that any positive measure on* $(X, \mathcal{P}(X))$ *is an exterior measure on* X*, but the converse is not true in general, as the following example shows*

Example .9 *Let* X *a non-empty set. The application* $\mu^* : \mathcal{P}(X) \longrightarrow \{0, 1\}$ *defined by*

$$\begin{cases} \mu^*(\emptyset) = 0 \\ \mu^*(A) = 1, \text{ if } A \neq \emptyset, \end{cases}$$

is an exterior measure on X*.Moreoever if* $card(X) > 1$*, the application* μ^* *is not a positive measure on* $(X, \mathcal{P}(X))$.
Proof. *Let* $A, B \in \mathcal{P}(X)$ *with* $A \subset B$

i) *If* $A = \emptyset$*, then* $\mu^*(A) = 0 \leqslant \mu^*(B)$

ii) *If* $A \neq \emptyset$ *then* $B \neq \emptyset$ *and therefore* $\mu^*(A) = 1 = \mu^*(B)$

Now, let $(A_n)_n$ *a sequence of parts of* X*.If all the sets* A_n *are empty, we have*

$$\mu^*\left(\bigcup_{n=1}^{+\infty} A_n\right) = \mu^*(\emptyset) = 0 = \sum_{n=1}^{+\infty} \mu^*(A_n).$$

Conversely, if there exists $j \in \mathbb{N}$ such that $A_j \neq \emptyset$ we have $\bigcup_{n=1}^{+\infty} A_n \neq \emptyset$ and then

$$\mu^*\left(\bigcup_{n=1}^{+\infty} A_n\right) = 1 = \mu^*(A_j) \leqslant \sum_{n=1}^{+\infty} \mu^*(A_n),$$

which means that μ^* is an exterior measure on X.

Since $\mathrm{card}(X) > 1$, we can choose $a, b \in X$ with $a \neq b$. We put $A = \{a\}$ and $B = \{b\}$. In this case μ^* is not additive because

$$\mu^*(A \cup B) = 1 \neq \mu^*(A) + \mu^*(B) = 2,$$

then μ^* is not a positive measure on $(X, \mathcal{P}(X))$. ■

Proposition 1.10 *Any additional external measure on X is a positive measure on $(X, \mathcal{P}(X))$.*

Proof. *We verify the $\sigma-$additivity. Let $(A_n)_n$ a sequence of $\mathcal{P}(X)$ pairwise disjoint.*

First, note that

$$\bigcup_{n=1}^{p} A_n \subset \bigcup_{n=1}^{+\infty} A_n,$$

for all $p \geqslant 1$. By the additivity hypothesis and 2. of the previous definition, we have

$$\sum_{n=1}^{p} \mu^*(A_n) = \mu^*\left(\bigcup_{n=1}^{p} A_n\right) \leqslant \mu^*\left(\bigcup_{n=1}^{+\infty} A_n\right),$$

and by taking the limit when $p \longrightarrow +\infty$ we obtain

$$\sum_{n=1}^{+\infty} \mu^*(A_n) \leqslant \mu^*\left(\bigcup_{n=1}^{+\infty} A_n\right),$$

this last inequality and 3) of the previous definition give the $\sigma-$additivity of μ^*. ■

Definition 1.19 *Let X a non-empty set and let μ^* be an exterior measure on X. A part E of X is called μ^*-measurable if for all $A \subset X$ we have*

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c).$$

We also say that E is measurable in the Carathéodory sense (with respect to μ^*). We denote by $\mathcal{M}(\mu^*)$ the family of μ^*-measurable subsets of X.

Remark 1.13

1. *The measurability of E does not involve $\mu^*(E)$ but $\mu^*(A)$ where A is called the test set.*
2. *For all $A \subset X$ we can write*

   $$A = A \cap (E \cup E^c) = (A \cap E) \cup (A \cap E^c),$$

   by the subadditivity of the exterior measure (3 in definition 1.19) we always

   $$\mu^*(A) \leqslant \mu^*(A \cap E) + \mu^*(A \cap E^c),$$

   then to show that a subset $E \subset X$ is μ^*-measurable, it suffices to show that

   $$\mu^*(A) \geqslant \mu^*(A \cap E) + \mu^*(A \cap E^c),$$

   for all $A \subset X$.

Example .10 *Let X a non-empty set*

1. *X and $\emptyset$ are μ^*-measurable for all exterior measure.*
2. *If μ^* is an exterior measure on X and $E \subset X$ such that $\mu^*(E) = 0$, then E is μ^*-measurable.*

Proof. *2) it suffices to show that*

$$\mu^*(A) \geqslant \mu^*(A \cap E) + \mu^*(A \cap E^c),$$

is true for all $A \subset X$. By the inclusions

$$(A \cap E) \subset E \text{ and } (A \cap E^c) \cap A,$$

we have

$$\mu^*(A \cap E) \leqslant \mu^*(E) = 0 \quad \text{and} \quad \mu^*(A \cap E^c) \leqslant \mu^*(A),$$

which implies that

$$\mu^*(A \cap E) + \mu^*(A \cap E^c) = \mu^*(A \cap E^c) \leqslant \mu^*(A).$$

■

Theorem 1.4 *Let μ^* be an exterior measure on a non-empty set X. Then $\mathcal{M}(\mu^*)$ is a sigma-algebra (tribe) on X and the restriction of μ^* to $\mathcal{M}(\mu^*)$ is a measure.*
Proof. *$\emptyset$, $X \in \mathcal{M}(\mu^*)$, by the previous example. Immediately, starting from the equation*

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c),$$

we have $E \in \mathcal{M}(\mu^*)$ if and only if $E \in \mathcal{M}(\mu^*)^c$. It therefore remains to be seen that $\mathcal{M}(\mu^*)$ is closed under countable unions.

We begin by establishing this for a finite union.

Let $E_1, E_2 \in \mathcal{M}(\mu^*)$, for all $A \subset X$

$$\mu^*(A) = \mu^*(A \cap E_1) + \mu^*(A \cap E_1^c), \tag{1.1}$$

we test the μ^*-mesurability of E_2 by the set $A \cap E_1^c$

$$\mu^*(A \cap E_1^c) = \mu^*(A \cap E_1^c \cap E_2) + \mu^*(A \cap E_1^c \cap E_2^c), \tag{1.2}$$

by replacing 1.2 in 1.1 we obtain

$$\begin{aligned}\mu^*(A) &= \mu^*(A \cap E_1) + \mu^*(A \cap E_1^c \cap E_2) + \mu^*(A \cap E_1^c \cap E_2^c) \\ &\geqslant \mu^*[(A \cap E_1) \cup (A \cap E_1^c \cap E_2)] + \mu^*(A \cap E_1^c \cap E_2^c),\end{aligned}$$

but

$$(A \cap E_1) \cup (A \cap E_1^c \cap E_2) = A \cap [E_1 \cup (E_2 \diagdown E_1)] = A \cap (E_1 \cup E_2),$$

and also

$$A \cap E_1^c \cap E_2^c = A \cap (E_1 \cup E_2)^c,$$

therefore we obtain

$$\mu^*(A) \geqslant \mu^*[A \cap (E_1 \cup E_2)] + \mu^*(A \cap (E_1 \cup E_2)^c),$$

then

$$E_1 \cup E_2 \in \mathcal{M}(\mu^*).$$

To complete the proof that $\mathcal{M}(\mu^*)$ is a sigma-algebra (tribe), let's show its closure with respect to countable unions. We consider a family $(E_n)_n$ of pairwise disjoint elements of $\mathcal{M}(\mu^*)$ if $(A_n)_{n\geqslant 1}$ is a family of elements of $\mathcal{M}(\mu^*)$, we can always writeest $U_n A_n = U_n E_n$, with the elements E_n are pairwise disjoint. We

$$F_n = \bigcup_{k=1}^{n} E_n$$

for all $n \geqslant 1$and we show by induction on n that for any subset $A \subset X$ we have

$$\mu^*(A \cap F_n) = \sum_{k=1}^{n} \mu^*(A \cap E_k), \tag{1.3}$$

the property is true for $n = 1$ and if we assume that it is true for an index n, and since $F_n \in \mathcal{M}(\mu^*)$ is a finite union of the elements in $\mathcal{M}(\mu^*)$, we test its measurability by $A \cap F_{n+1}$

$$\mu^*(A \cap F_{n+1}) = \mu^*(A \cap F_{n+1} \cap F_n) + \mu^*(A \cap F_{n+1} \cap F_n^c), \tag{1.4}$$

on the other hand in fact that

$$F_{n+1} \cap F_n = F_n \text{ and } F_{n+1} \cap F_n^c = E_{n+1},$$

the equality (1.4) gives

$$\begin{aligned}
\mu^*(A \cap F_{n+1}) &= \mu^*(A \cap F_n) + \mu^*(A \cap E_{n+1}) \\
&= \sum_{k=1}^{n} \mu^*(A \cap E_k) + \mu^*(A \cap E_{n+1}) \\
&= \sum_{k=1}^{n+1} \mu^*(A \cap E_k).
\end{aligned}$$

Now, if we put $F = \bigcup_{k=1}^{+\infty} E_k$ we have $A \cap F_n \subset A \cap F$ for all $n \geqslant 1$. Therefore, according to the monotonicity of the exterior measure and (1.3) we have

$$\mu^*(A \cap F) \geqslant \mu^*(A \cap F_n) = \sum_{k=1}^{n} \mu^*(A \cap E_k),$$

we taking the limit whene $n \longrightarrow +\infty$, we obtain

$$\mu^*(A \cap F) \geqslant \sum_{k=1}^{+\infty} \mu^*(A \cap E_k),$$

conversely and by (3) of the definition 1.19

$$\mu^*(A\cap F)=\mu^*\left[\bigcup_{k=1}^{+\infty}(A\cap E_k)\right]\leqslant\sum_{k=1}^{+\infty}\mu^*(A\cap E_k),$$

therefore for all $A\subset X$

$$\mu^*(A\cap F)=\sum_{k=1}^{+\infty}\mu^*(A\cap E_k). \tag{1.5}$$

We have a $F^c\subset F_n^c$ for all $n\geqslant 1$ and

$$\begin{aligned}\mu^*(A) &= \mu^*(A\cap F_n)+\mu^*(A\cap F_n^c)\\ &= \sum_{k=1}^{n}\mu^*(A\cap E_k)+\mu^*(A\cap F_n^c)\\ &\geqslant \sum_{k=1}^{n}\mu^*(A\cap E_k)+\mu^*(A\cap F^c),\end{aligned}$$

and by taking the limit as $n\longrightarrow+\infty$ and by (1.5), we get

$$\mu^*(A)\geqslant\mu^*(A\cap F)+\mu^*(A\cap F^c),$$

therefore

$$F=\bigcup_{k=1}^{+\infty}E_k\in\mathcal{M}(\mu^*),$$

the $\sigma-$additivity of μ^* on $\mathcal{M}(\mu^*)$ resulte of the formula (1.5) taking as the test set $A=X$.
■

1.5 Lebesgue's measurement of the Borelian tribe

1.5.1 Measure of Lebesgue on $\mathbb{R}$

Theorem 1.5 *There is a unique positive measure on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$, noted λ, such that*

$$\lambda(]b,a[)=b-a,$$

for all $a,b\in\mathbb{R}$, $a<b$. It is called the Lebesgue measure on $\mathbb{R}$.

Remark 1.14 *1. It is clear that the measure λ is $\sigma-$finite since*

$$\lambda([-n,n])=2n<+\infty \text{ and } \mathbb{R}=\bigcup_{n=1}^{+\infty}[-n,n].$$

2. For all $x\in\mathbb{R}$ we have $\lambda(\{x\})=0$ and consequently

$$\lambda(]b,a[)=\lambda(]b,a])=\lambda([b,a[)=\lambda([b,a])=b-a.$$

Indeed :

$$\{x\} = \bigcap_{n=1}^{+\infty} \left] x - \frac{1}{n}, x + \frac{1}{n} \right[,$$

therefore, by decreasing continuity, (2) in Theorem 1.3, we have

$$\lambda(\{x\}) = \lim_{n \longrightarrow +\infty} \lambda\left(\left] x - \frac{1}{n}, x + \frac{1}{n} \right[\right) = \lim_{n \longrightarrow +\infty} \frac{2}{n} = 0.$$

It immediately follows that

Proposition 1.11 *Every countable set D of $\mathbb{R}$ has a zero Lebesgue measure $\lambda(D) = 0$.*
Proof. *Since*

$$D = \bigcup_{n=1, x_n \in \mathbb{R}}^{+\infty} \{x_n\},$$

we have

$$\lambda(D) \leqslant \sum_{n=1}^{+\infty} \lambda(\{x_n\}) = 0,$$

which implies that $\lambda(D) = 0$. ■

Proposition 1.12 *The Lebesgue measure λ is invariant under translation and invariant under symmetry. That is, for all $a \in \mathbb{R}$, we have*

$$\lambda(\alpha + A) = \lambda(A) \text{ and } \lambda(-A) = \lambda(A),$$

for all Borel A of $\mathbb{R}$, then

$$\alpha + A = \{\alpha + a, a \in A\} \text{ and } -A = \{-a, a \in A\}.$$

1.5.2 Lebesgue measure on $\mathbb{R}^m$

Recall that a cuboid P of $\mathbb{R}^m$ is a product of bounded intervals $P = I_1 \times I_2 \times ... \times I_m$, with $I_j \subset \mathbb{R}$ $(j = 1, ..., m)$ is a bounded interval. The measure of cuboid P is given by

$$m(P) = |I_1| \times |I_2| \times ... \times |I_m|,$$

with $|I_j|$ is the length of the segment I_j.

Definition 1.20 *For any part A of $\mathbb{R}^m$, we define*

$$\lambda^*(A) = \inf\left\{\sum_{n=1}^{+\infty} m(P_i) : A \subset \bigcup_{n=1}^{+\infty} P_i, P_i \text{ open cuboid of } \mathbb{R}^m\right\},$$

the infimum is taken over all countable coverings of A by open tilings

Theorem 1.6 *We have the following assertions*

1. *λ^* is an exterior measure on $\mathbb{R}^m$.*
2. *The tribe $\mathcal{M}(\mu^*)$ contains the Borel sigma-algebra$\mathcal{B}(\mathbb{R})$.*
3. *$\lambda^*(P) = m(P)$, for any cuboid $P \subset \mathbb{R}^m$.*

1.6 Corrected Exercises

Exercise 1.1 *Let E a non-empty set and $(A_i)_{i\in I}$ a family of subsets of E. Show that*

1. $E-\left(\bigcup_{i\in I} A_i\right)=\bigcap_{i\in I}(E-A_i)$
2. $E-\left(\bigcap_{i\in I} A_i\right)=\bigcup_{i\in I}(E-A_i)$

Solution 1.1 1. *Let* $x\in E-\left(\bigcup_{i\in I} A_i\right)$, *then*

$$\begin{aligned} x &\in E-\left(\bigcup_{i\in I} A_i\right)\Longleftrightarrow x\in E,\, x\notin\bigcup_{i\in I} A_i\\ &\Longleftrightarrow x\in E,\, x\notin A_i\ \forall i\in I\\ &\Longleftrightarrow x\in(E-A_i)\,,\ \forall i\in I\\ &\Longleftrightarrow x\in\bigcap_{i\in I}(E-A_i)\,, \end{aligned}$$

then

$$E-\left(\bigcup_{i\in I} A_i\right)=\bigcap_{i\in I}(E-A_i)$$

2. *In the same way, we show that*

$$E-\left(\bigcap_{i\in I} A_i\right)=\bigcup_{i\in I}(E-A_i)\,.$$

Exercise 1.2 *Let E a non-empty set and $(A_i)_{i\in I}$ a family of subsets of E. Show that*

1. $f^{-1}\left(\bigcup_{i\in I} A_i\right)=\bigcup_{i\in I} f^{-1}(A_i)\,.$
2. $f^{-1}\left(\bigcap_{i\in I} A_i\right)=\bigcap_{i\in I} f^{-1}(A_i)\,.$

Solution 1.2 1. *Let* $x\in f^{-1}\left(\bigcup_{i\in I} A_i\right)$, *then*

$$\begin{aligned} x &\in f^{-1}\left(\bigcup_{i\in I} A_i\right)\Longleftrightarrow f(x)=\bigcup_{i\in I} A_i\\ &\Longleftrightarrow \exists k\in I,\ f(x)\in A_k\\ &\Longleftrightarrow x\in f^{-1}(A_k),\ \ k\in I\\ &\Longleftrightarrow x\in\bigcup_{i\in I} f^{-1}(A_i)\,, \end{aligned}$$

therefore

$$f^{-1}\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} f^{-1}\left(A_i\right).$$

2. *In the same way for*

$$f^{-1}\left(\bigcap_{i\in I} A_i\right) = \bigcap_{i\in I} f^{-1}\left(A_i\right).$$

Exercise 1.3 *Let* $(A_i)_{i\in I}$ *a family of non-empty subsets of a set* E. *Show that*

1. $f\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} f\left(A_i\right)$
2. $f\left(\bigcap_{i\in I} A_i\right) \subset \bigcap_{i\in I} f\left(A_i\right)$, *and in this case the equality is true if and only if the function* f *is injective.*

Solution 1.3 1. *Let*

$$\begin{aligned} y &\in f\left(\bigcup_{i\in I} A_i\right) \Longleftrightarrow \exists x \in \bigcup_{i\in I} A_i : y = f(x) \\ &\Longleftrightarrow \exists k \in I, x \in A_k \\ &\Longleftrightarrow y \in f(A_k) \\ &\Longleftrightarrow y \in \bigcup_{i\in I} f\left(A_i\right) \\ &\Longrightarrow f\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} f\left(A_i\right). \end{aligned}$$

2. *Let*

$$\begin{aligned} y &\in f\left(\bigcap_{i\in I} A_i\right) \Longleftrightarrow \exists x \in \bigcap_{i\in I} A_i : y = f(x) \\ &\Longleftrightarrow x \in A_i \ \forall i \in I \\ &\Longrightarrow y \in f(A_i), \forall i \in I \\ &\Longleftrightarrow y \in \bigcap_{i\in I} f\left(A_i\right) \\ &\Longrightarrow f\left(\bigcap_{i\in I} A_i\right) \subset \bigcap_{i\in I} f\left(A_i\right). \end{aligned}$$

Now, let

$$\begin{aligned} y &= f(x) \in \bigcap_{i\in I} f\left(A_i\right) \Longleftrightarrow y \in f\left(A_i\right), \forall i \in I \\ &\Longrightarrow \exists x_i \in A_i : y = f(x_i), \end{aligned}$$

therefore

$$\begin{aligned} x_1 &= x_2 = \dots = x_n = x \in A_i \Longrightarrow x \in \bigcap_{i\in I} A_i \\ &\Longrightarrow y = f(x) \in f\left(\bigcap_{i\in I} A_i\right). \end{aligned}$$

Example .11 *Let*

$$\begin{array}{rccc} f: & [-2,2] & \longrightarrow & [0,4] \\ & x & \longmapsto & x^2 \end{array}$$

we have

$$f(\{-2\} \cap \{2\}) \neq f(\{-2\}) \cap f(\{2\}),$$

therefore, the inclusion operation is verified, but the equality is not verified because f is not injective.

Exercise 1.4 *(Oprations on tribes)*

1. Let $\mathcal{F}$ a tribe of Ω and B an element of $\mathcal{F}$. Show that the set
$$\mathcal{F}_B = \{A \cap B, A \in \mathcal{F}\},$$
is a tribe of B.
2. Let $(X \times Y, \mathcal{F})$ a measured product space and $\pi : X \times Y \longrightarrow X$ the canonical projection. The set
$$\mathcal{F}_X = \{\pi(F), F \in \mathcal{F}\},$$
is a tribe ?
3. We consider on $\mathbb{N}$, for each $n \geqslant 0$, the tribe
$$\mathcal{F}_n = \sigma(\{0\}, \{1\}, \{2\}, ..., \{n\}).$$
Show that the sequence of sigma-algebras $(\mathcal{F}_n)_{n\geqslant 0}$ is increasing, but that $\bigcup_{n\geqslant 0} \mathcal{F}_n$ is not a sigma-algebra. (Hint : you can reason by contradiction and use the subset $2\mathbb{N}$.
4. Let $(E, \mathcal{A})$ a measurable space . Let C a une family of parts of E, and let $B \in \sigma(C)$, then necessarily there exists a countable family $\mathcal{D} \subset C$ such that $B \in \sigma(C)$. Is this true ?

Solution 1.4 *1. i)We have*

$$B = \Omega \cap B \in \mathcal{F}_B$$

ii) Let $C = B \cap D \in \mathcal{F}_B$ with $D \in \mathcal{F}$. Then $D^c \in \mathcal{F}$ and $C_B^c = B \cap D^c \in \mathcal{F}_B$.

iii) Let $C_n = B \cap D_n \in \mathcal{F}_B$ with $D_n \in \mathcal{F}$. Then $\bigcup_n D_n \in \mathcal{F}$ and

$$\bigcup_n B_n = B \cap \bigcup_n D_n \in \mathcal{F}_B.$$

Therefore the set $\mathcal{F}_B$ is a tribe on B.

2. *We conside for $X = Y = \{0, 1\}$ the tribe $\mathcal{F}$ generated by the element $(0, 0) \in X \times Y$. It is clear that*
$$\mathcal{F} = \{\emptyset, X \times Y, \{(0,0)\}, X \times Y \backslash \{(0,0)\}\}.$$
We verify that
$$\mathcal{F}_X = \{\emptyset, \{0\}, X\},$$
which is not a tribe.

3. *We put*
$$\mathcal{F} = \bigcup_{n \in \mathbb{N}} \mathcal{F}_n,$$
and we suppose that $\mathcal{F}$ is a tibe . We have
$$\{2n\} \in \mathcal{F}_{2n} \subset \mathcal{F} \text{ and } 2\mathbb{N} = \bigcup_{n \geqslant 0} \{2n\}.$$
Therefore , $2\mathbb{N} \in \mathcal{F}$ i.e there exists $n_0 \in \mathbb{N}$ such that $2\mathbb{N} \in \mathcal{F}_{n_0}$. Now, the only elements of infinite cardinality of $\mathcal{F}_{n_0}$ are of the form $\mathbb{N} \backslash A$, where A is a part of $\{0, 1, ..., _{n_0}\}$. We therefore obtain a contradiction.

4. *Yes is true . Indeed , let*
$$\mathcal{G} = \{B \in \sigma(C); \exists \mathcal{D} \subset C \text{ countable such that } B \in \sigma(\mathcal{D})\}.$$
We show that $\mathcal{G}$ is a tribe on E.

i) *It is clear that $E \in \mathcal{G}$*

ii) *If $A \in \mathcal{G}$, then there exists $\mathcal{D} \subset C$ countable such that $A \in \sigma(\mathcal{D})$, and then $A^c \in \sigma(\mathcal{D})$: we have $A^c \in \mathcal{G}$.*

iii) *If $(A_n) \subset \mathcal{G}$, then for all n there exists $\mathcal{D}_n \subset C$ countable such that $A_n \in \sigma(\mathcal{D}_n)$, then*
$$\bigcup_{n \in \mathbb{N}} A_n \in \sigma(\mathcal{D}), \text{ where } \mathcal{D} = \bigcup_{n \in \mathbb{N}} D_n \subset C \text{ is countable },$$
(being a countable union of countable sets), we have
$$\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{G}.$$
In conclusion, $\mathcal{G}$ is a tribe . Or $C \subset \mathcal{G}$, which implies that
$$\sigma(C) \subset \sigma(\mathcal{G}) = \mathcal{G} \subset \sigma(C),$$
hence the result.

Exercise 1.5 1. *Is the set of open sets of $\mathbb{R}$ a sigma-algebra (tribe) ?*

2. *If we denote by λ The Lebesgue measure, recall why $\lambda(\{x\}) = 0$ for all $x \in \mathbb{R}$, then*
$$\lambda(\mathbb{R}) = \lambda\left(\bigcup_{x \in \mathbb{R}} \{x\}\right) = \sum_{x \in \mathbb{R}} \lambda(\{x\}) = \sum_{x \in \mathbb{R}} 0 = 0.$$
Where is the problem ?

3. *If $\mathcal{F}$ and $\mathcal{G}$ are two tribes , Is F■G always a sigma-algebra (tribe) ?*
4. *If $(a_n)_{n\geqslant 1}$ and $(b_n)_{n\geqslant 1}$ are two sequences of real numbers, do we always have*

$$\lim_{n\longrightarrow\infty}\sup\left(a_n+b_n\right)=\lim_{n\longrightarrow\infty}\sup a_n+\lim_{n\longrightarrow\infty}\sup b_n?$$

And if the two sequences are bounded ? And if b_n converges ?

Solution 1.5 1. *No, because the complement of $]-\infty,0[$ is not open.*

2. *No, not always. For example, yes, if we note*

$$\Omega=\{1,2,3\},$$

we take

$$\mathcal{F}=\{\emptyset,\{1\},\{2,3\},\Omega\},$$

and

$$\mathcal{G}=\{\emptyset,\{2\},\{1,3\},\Omega\}?$$

Indeed, we have

$$\mathcal{F}\cup\mathcal{G}=\{\emptyset,\{1\},\{2\},\{2,3\},\{1,3\},\Omega\},$$

which is not closed under union

$$\{1\}\cup\{2\}\notin\mathcal{F}\cup\mathcal{G}.$$

3. *We have*

$$\lambda\left(\{x\}\right)=\lim_{n\longrightarrow\infty}\lambda\left([x-1/n,x+1/n]\right)=0.$$

Furthermore, the equality

$$\lambda\left(\bigcup_{x\in\mathbb{R}}\{x\}\right)=\sum_{x\in\mathbb{R}}\lambda\left(\{x\}\right),$$

is not verified because $\mathbb{R}$ is uncountable.

4. *This is not always true (for example, $a_n=-b_n=n$). On the other hand, the left-hand side is less than or equal to the right-hand side when both sequences are bounded, and the equality is verified whenb_n converges.*

Exercise 1.6 *(Borel-Cantelli Lemma) $(E,\mathcal{A},\mu)$ is a measured space (μ is a positive measure) and $(A_n)_{n\geqslant 1}$ is a sequence of elements of $\mathcal{A}$. Recall that we denote*

$$\lim_{n\longrightarrow\infty}\inf A_n=\bigcup_{n\geqslant 1}\bigcap_{k\geqslant n}A_k\quad\text{and}\quad\lim_{n\longrightarrow\infty}\sup A_n=\bigcap_{n\geqslant 1}\bigcup_{k\geqslant n}A_k.$$

1. *Show that*

$$\mu\left(\lim_{n\longrightarrow\infty}\inf A_n\right)\leqslant\lim_{n\longrightarrow\infty}\inf\mu\left(A_n\right),$$

and if

$$\mu\left(\bigcup_{n\geqslant 1}A_n\right)<\infty,$$

then

$$\mu\left(\lim_{n\longrightarrow\infty}\sup A_n\right)\geqslant\lim_{n\longrightarrow\infty}\sup\mu\left(A_n\right).$$

What happens if

$$\mu\left(\bigcup_{n\geqslant1}A_n\right)=\infty?$$

2. *(Borel-Cantelli Lemma.) We assume that* $\sum_{n\geqslant1}\mu\left(A_n\right)<\infty$. *Show that*

$$\mu\left(\lim_{n\longrightarrow\infty}\sup A_n\right)=0.$$

3. *(An application of the Borel-Cantelli Lemma.) Let* $\epsilon>0$.*Show that for almost* $x\in[0,1]$ *(for the measure of Lebesgue), there exists only a finite number of rational numbers* p/q *with* p *and* q *coprime such that*

$$\left|x-\frac{p}{q}\right|<\frac{1}{q^{2+\epsilon}},$$

i.e., almost every x *is "poorly approximated by rational numbers of order* $2+\epsilon$*".*

Solution 1.6 1. *We note that, for all* $n\geqslant0$ *and for all* $k\geqslant n$,

$$\mu\left(\bigcap_{p\geqslant n}A_n\right)\leqslant\mu\left(A_k\right).$$

Hence,

$$\mu\left(\bigcap_{k\geqslant n}A_k\right)\leqslant\inf_{k\geqslant n}\mu\left(A_k\right). \tag{1.6}$$

Or the sequence $\left(\bigcap_{k\geqslant n}A_k\right)_{n\geqslant0}$ *is increasing. The result is therefore obtained by taking the limit when* $n\longrightarrow\infty$ *in (1.6). Similarly, we have*

$$\mu\left(\bigcup_{k\geqslant n}A_k\right)\geqslant\sup_{k\geqslant n}\mu\left(A_k\right). \tag{1.7}$$

Or the sequence $\left(\bigcup_{k\geqslant n}A_k\right)_{n\geqslant0}$ *is decreasing and* $\mu\left(\bigcup_{n\geqslant0}A_n\right)<+\infty$. *The result is therefore obtained by taking the limit when* $n\longrightarrow\infty$ *in (1.7). We can also use the previous result and reason by taking the complement. Indeed , we put* $F=\bigcup_{n\geqslant0}A_n$, *then we have*

$$F\backslash\lim_{n\longrightarrow\infty}\sup A_n=\lim_{n\longrightarrow\infty}\inf\left(F\backslash A_n\right),$$

then

$$\mu\left(F\backslash \lim_{n\longrightarrow\infty} \sup A_n\right) \leqslant \lim_{n\longrightarrow\infty} \inf\left(F\backslash A_n\right),$$

that is to say

$$\mu\left(F\right) - \mu\left(\lim_{n\longrightarrow\infty} \sup A_n\right) \leqslant \mu\left(F\right) - \lim_{n\longrightarrow\infty} \sup \mu\left(A_n\right),$$

since $\mu\left(F\right) < \infty$, that implies the result.

2. *We have for all $n \geqslant 0$,*

$$\mu\left(\bigcup_{k\geqslant n} A_k\right) \leqslant \sum_{k\geqslant n} \mu\left(A_k\right),$$

or

$$\mu\left(\lim_{k\longrightarrow\infty} \sup A_k\right) \leqslant \mu\left(\bigcup_{k\geqslant n} A_k\right), \text{ for all } n \geqslant 0$$

and $\sum_{k\geqslant n} \mu\left(A_k\right)$ is the remainder of a convergent series and therefore tends towards 0 when $n \longrightarrow \infty$. Hence we obtain the result.

3. *For all $q \geqslant 1$, we denote*

$$A_q = [0,1] \cap \bigcup_{p=0}^{q} \left[\frac{p}{q} - \frac{1}{q^{2+\epsilon}}, \frac{p}{q} + \frac{1}{q^{2+\epsilon}}\right],$$

hence , $\lambda\left(A_q\right) \leqslant 2/q^{1+\epsilon}$, consequently

$$\sum_{q\geqslant 1} \lambda\left(A_q\right) < +\infty.$$

According to the Borel-Cantelli lemma, $\lambda\left(\lim_{q\longrightarrow\infty} \sup A_q\right) = 0$, or the set $\lim_{q\longrightarrow\infty} \sup A_q$ contains the set of real numbers well approximated by rational numbers of order $2+\epsilon$.

Exercise 1.7 *Consider E a set and T a part of $\mathcal{P}(E)$ stable under countable union, stable under passage to complement with $\emptyset \in T$.*

1. *Prove that T is atribe on E*
2. *Is the set of finite subsets of E a sigma-algebra (tribe) ?*

Solution 1.7 1. *We have $E \in T$ since $E = \emptyset^c$ and T is stable under complementation. On the other hand, T is stable under countable intersection because, if*

$$\left(A_n\right) \subset T,$$

we have

$$\left(\bigcap_{n\in\mathbb{N}} A_n\right)^c = \bigcup_{n\in\mathbb{N}} A_n^c \in T,$$

(because T is stable under complement and under countable union) therefore

$$\bigcap_{n\in\mathbb{N}} A_n \in T,$$

(because T is stable under complement).

2. We distinguish two cases :

a) *If E is finite, the set of finite subsets of E is a sigma-algebra, it is the sigma-algebra $\mathcal{P}(E)$.*

b) *If E is infinite, the set of finite subsets of E is not a sigma-algebra, because it is not closed under the complement (the complement of a finite subset is infinite...).*

Exercise 1.8 *Let E an infinite set and*

$$S = \{\{x\}, x \in E\}.$$

Determine the sigma-algebra tribe) generated by S (distinguishing between countable and uncountable cases of E).

Solution 1.8 *We denote by $T(S)$ the tribe generated by S.*

i) *We suppose that E E is at most countable (that is, finite or countable). According to the closure of $T(S)$ under countable unions, the sigma-algebra (tribe) $T(S)$ must contain all at most countable subsets. Since all subsets of E are at most countable, we deduce that $T(S) = \mathcal{P}(E)$.*

ii) *Now, we suppose that E is infinite and uncountable. We denote by $\mathcal{A}$ the set of parts of E at most countable and*

$$\mathcal{B} = \{A^c,\ A \in \mathcal{A}\}.$$

According to the stability of $T(S)$ under countable unions, the sigma-algebra $T(S)$ must contain $\mathcal{A}$. By the stability of $T(S)$ under complementation, $T(S)$ must also contain $\mathcal{B}$. We will now show that $\mathcal{A} \cup \mathcal{B}$ is a sigma-algebra (it follows that $T(S) = \mathcal{A} \cup \mathcal{B}$). We have

$$\emptyset \in \mathcal{A} \subset \mathcal{A} \cup \mathcal{B},$$

and it is clear that $\mathcal{A} \cup \mathcal{B}$ is stable under complementation because

$$A \in \mathcal{A} \Longrightarrow A^c \in \mathcal{B},$$

and

$$A \in \mathcal{B} \Longrightarrow A^c \in \mathcal{A}.$$

Finally , if $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A} \cup \mathcal{B}$, we distinguish two cases :

1) *First case if*

$$A_n \in \mathcal{A}, \forall n \in \mathbb{N},$$

then, we have

$$\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A} \subset \mathcal{A} \cup \mathcal{B}.$$

2) *Second case : if there exists $n \in \mathbb{N}$ such that $A_n \in \mathcal{B}$ then, we have $A_n^c \in \mathcal{A}$, therefore A_n^c is at most countable and*

$$\left(\bigcup_{p\in\mathbb{N}} A_p\right)^c = \bigcap_{p\in\mathbb{N}} A_p^c \subset A_n^c,$$

is also almost countable , which gives

$$\left(\bigcup_{p\in\mathbb{N}} A_p\right)^c \in \mathcal{A},$$

and

$$\bigcup_{p\in\mathbb{N}} A_p \in \mathcal{B} \subset \mathcal{A}\cup\mathcal{B}.$$

We have already shown that

$$\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}\cup\mathcal{B}.$$

This proves the stability under countable unions of $\mathcal{A}\cup\mathcal{B}$*.*

Finally , $\mathcal{A}\cup\mathcal{B}$ *is a tribe containing* S *and contents in* $T(S)$*, which gives*

$$T(S) = \mathcal{A}\cup\mathcal{B}.$$

Exercise 1.9 *Consider* A *a measurable set (measure of Lebesgue) such that* $A \subset [0,1]$*, and let*

$$f(x) = \mu(A\cap[0,x])$$

1. Prove that f is continuous.
2. We put $\mu(A) = \alpha$, $(\alpha > 0)$. Show that

$$\forall\beta,\ 0 \leqslant \beta \leqslant \alpha;\ \exists B \subset A;\ \mu(B) = \beta$$

Solution 1.9

1. We show that f is continuous.

$$\forall\epsilon > 0; \exists\delta > 0; |x - x_0| < \delta \Longrightarrow |f(x) - f(x_0)| < \epsilon$$

$$\begin{aligned} |f(x) - f(x_0)| &= |\mu(A\cap[0,x]) - \mu(A\cap[0,x_0])| \\ &= |\mu(A\cap]\inf(x,x_0);\sup(x,x_0)])| \\ &\leqslant |\mu(]\inf(x,x_0);\sup(x,x_0)])| = |x - x_0| \end{aligned}$$

$\Longrightarrow f$ is continuous.

2. We have

$$f(0) = 0\ \ ;\ \ f(1) = \mu(A\cap[0,1]) = \mu(A) = \alpha$$

f is continuous and by the Intermediate Value Theorem there exists $c \in [0,1]$ such that $f(c) = \beta$ with $\beta \in [0,\alpha]$ which means the existence of a set of the form $B = A\cap[0,c]$ which is included in the set A and also

$$f(c) = \mu(A\cap[0,c]) = \mu(B) = \beta.$$

Exercise 1.10 *Let* (E,T,m) *a measured space .*

Prove that

$$\forall A,B \in T : m(A\cup B) + m(A\cap B) = m(A) + m(B).$$

Solution 1.10 *We have*

$$A \cup B = (A \diagdown B) \cup B \text{ and } (A \diagdown B) \cap B = \varnothing,$$

By using the $\sigma-$additivity of m, we obtain

$$\begin{aligned} m(A \cup B) &= m(A \diagdown B) + m(B) \\ &\Longrightarrow m(A \cup B) + m(A \cap B) = m(A \diagdown B) + m(B) + m(A \cap B) \\ &\Longrightarrow m(A \cup B) + m(A \cap B) = m(A) + m(B), \end{aligned}$$

because

$$\begin{aligned} m(A \diagdown B) + m(A \cap B) &= m((A \diagdown B) \cup (A \cap B)) \\ &= m(A). \end{aligned}$$

Exercise 1.11 *Consider (E, T, m) a measured space where m is a measure verifying*

$$m(E) = 1.$$

Show that the set

$$\mathcal{F} = \{A \in T; m(A) = 0 \vee m(A) = 1\},$$

is a tribe on E.

Solution 1.11

1. We have $E \in T$ and $m(E) = 1$, therefore $E \in \mathcal{F}$
2. Let $A \in \mathcal{F} \Longrightarrow A \in T : m(A) = 0 \curlyvee m(A) = 1$, on the other hand, we have $A^c \in T$, (car T is stable by par complement) and

$$\begin{aligned} m(A^c) &= m(E - A) = m(E) - m(A) \\ &= 1 - m(A) = \begin{cases} 1 & \text{if } m(A) = 0 \\ 0 & \text{if } m(A) = 1 \end{cases} \\ &\Longrightarrow m(A^c) \in \mathcal{F} \Longrightarrow A^c \in \mathcal{F} \end{aligned}$$

3. Let

$$\begin{aligned} (A_n)_{n \in \mathbb{N}} &\subset \mathcal{F} \Longrightarrow \forall n \in \mathbb{N}, A_n \in \mathcal{F} \\ &\Longrightarrow \forall n \in \mathbb{N}, A_n \in T \text{ and } m(A_n) = 0 \curlyvee m(A_n) = 1, \end{aligned}$$

therefore, we have

$$\bigcup_{n \in \mathbb{N}} A_n \in T \text{ (because } T \text{ is stable by } \cup \text{ countable)},$$

We distinguish two cases. :

i) If

$$\forall n \in \mathbb{N},\ m(A_n) = 0,$$

then

$$\begin{aligned} 0 &\leqslant m\left(\bigcup_{n\in\mathbb{N}} A_n\right) \leqslant \sum_{n\in\mathbb{N}} m(A_n) = 0 \\ &\Longrightarrow m\left(\bigcup_{n\in\mathbb{N}} A_n\right) = 0. \end{aligned}$$

ii) If

$$\exists n_0 \in \mathbb{N},\ m(A_{n_0}) = 1,$$

then

$$\begin{aligned} 1 &= m(A_{n_0}) \leqslant m\left(\bigcup_{n\in\mathbb{N}} A_n\right) \leqslant m(E) = 1 \\ &\Longrightarrow m\left(\bigcup_{n\in\mathbb{N}} A_n\right) = 1, \end{aligned}$$

therefore, we get

$$\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{F}.$$

Exercise 1.12 *Let (E, T, m) a measured space , we put*

$$\overline{T} = \{A \cup N : A \in T,\ N \in N_m\},$$

where N_m is a set of negligible parts.

Prove that $\overline{T}$ is a tribe on E containing $T\cup\ N_m$.

Solution 1.12 *Let*

$$\overline{T} = \{A \cup N : A \in T,\ N \in N_m\},$$

We prove that $\overline{T}$ is a tribe on E :

1. *We have*

$$E \in \overline{T} \text{ because } E = E \cup \varnothing \text{ with } E \in T \text{ and } \varnothing \in N_m, (m(\varnothing) = 0).$$

2. *Let*

$$C \in \overline{T} \Longrightarrow \exists A \in T, \exists N \in N_m : C = A \cup N,$$

since

$$N \in N_m, \text{ then } \exists B \in T : \ N \subset B \text{ and } m(B) = 0,$$

on the other hand, we have

$$\begin{aligned} C^c &= (A \cup N)^c \\ &= A^c \cap N^c \\ &= A^c \cap (B^c \cup (N^c \setminus B^c)) \\ &= (A^c \cap B^c) \cup (A^c \cap (N^c \setminus B^c)) \\ &= (A^c \cap B^c) \cup (A^c \cap N^c \cap B), \end{aligned}$$

with

$$(A^c \cap B^c) \in T \text{ and } (A^c \cap N^c \cap B) \in N_m,$$

because

$$A^c \cap N^c \cap B \subset B,$$

we deduce that

$$C^c \in \overline{T}.$$

Now, let

$$\begin{aligned} C_n \quad \subset \quad & \overline{T} \Longrightarrow \forall n \in \mathbb{N}, C_n \in \overline{T} \\ \Longrightarrow \quad & \forall n \in \mathbb{N}, \exists A_n \in T, \exists N_n \in N_m : C_n = A_n \cup N_n, \end{aligned}$$

on the other hand, we have

$$N_n \in N_m \Longrightarrow \exists B_n \in T : \ N_n \subset B_n \text{ and } m(B_n) = 0,$$

then

$$\bigcup_{n\in\mathbb{N}} C_n = \bigcup_{n\in\mathbb{N}} (A_n \cup N_n) = \left(\bigcup_{n\in\mathbb{N}} A_n\right) \cup \left(\bigcup_{n\in\mathbb{N}} N_n\right),$$

with

$$\left(\bigcup_{n\in\mathbb{N}} A_n\right) \in T \quad \text{and} \quad \left(\bigcup_{n\in\mathbb{N}} N_n\right) \subset \left(\bigcup_{n\in\mathbb{N}} B_n\right),$$

and

$$m\left(\bigcup_{n\in\mathbb{N}} B_n\right) \leqslant \sum_{n\in\mathbb{N}} m(B_n) = 0,$$

therefore

$$\bigcup_{n\in\mathbb{N}} C_n \in \overline{T}.$$

Hence $\overline{T}$ is a tribe on E.

We show that $T \cup N_m \subset \overline{T}$

i) If $A \in T$, then

$$A = A \cup \varnothing \text{ and } \varnothing \in N_m,$$

therefore

$$A \in \overline{T}, \text{ hence } T \subset \overline{T}.$$

ii) If $N \subset N_m$, then
$$N = \varnothing \cup N \text{ and } \varnothing \in T \Longrightarrow N \subset \overline{T},$$
therefore
$$N_m \subset \overline{T},$$
finally, we obtain
$$T \cup N_m \subset \overline{T}.$$

Exercise 1.13 *Let E be an uncountable set. For any subset A of E, we set*
$$m(A) = \begin{cases} 0 \text{ if } A \text{ is atmost countable} \\ +\infty, \text{ otherwise} \end{cases}$$
m is a measure on $\mathcal{P}(E)$?

Solution 1.13 *1. We have $m(\varnothing) = 0$ because $\varnothing$ is almost countable*

2. Let
$$(A_n)_{n\in\mathbb{N}} \subset \mathcal{P}(E) : A_n \cap A_m = \varnothing \ \forall n \neq m,$$
We distinguish two cases :

i) *If $\forall n \in \mathbb{N}, A_n$ is almost countable, then $\bigcup\limits_{n\in\mathbb{N}} A_n$ is almost countable, therefore*
$$m\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n=0}^{+\infty} m(A_n) = 0.$$

ii) *If $\exists p \in \mathbb{N} : A_p$ is infinite and uncountable, then $\bigcup\limits_{n\in\mathbb{N}} A_n$ is infinite and uncountable, then we have :*
$$\begin{aligned} m\left(\bigcup_{n\in\mathbb{N}} A_n\right) &= +\infty = \sum_{n\in\mathbb{N}} m(A_n) \\ &= \sum_{n\neq p} m(A_n) + m(A_p). \end{aligned}$$
Therefore m is indeed a measure on $\mathcal{P}(E)$.

Exercise 1.14 *(Lemma of Borel-Cantelli) : Let (E, T, m) a measured space and $(A_n)_{n\in\mathbb{N}} \subset T$ such that $\sum\limits_{n\in\mathbb{N}} m(A_n) < \infty$.*

Show that $m\left(\overline{\lim\limits_n} A_n\right) = 0$.

Solution 1.14 *We have*
$$m(A_n) < \infty \text{ therefore } \sum_{k=n}^{+\infty} m(A_k) \underset{n\longrightarrow\infty}{\longrightarrow} 0,$$

on the other hand, we have

$$m\left(\bigcup_{n\geqslant n} A_p\right) \leqslant \sum_{p=n}^{+\infty} m\left(A_p\right),$$

therefore

$$m\left(\bigcup_{p\geqslant n} A_p\right) \underset{n\longrightarrow\infty}{\longrightarrow} 0,$$

the sequence $\left(\bigcup_{p\geqslant n} A_p\right)_{n\in\mathbb{N}}$ *is decreasing, because if we put*

$$\begin{aligned} B_n &= \bigcup_{n\geqslant n} A_p,\ B_n = A_n \cup A_{n+1} \cup A_{n+2} \cup \ldots. \\ B_{n+1} &= A_{n+1} \cup A_{n+2} \cup A_{n+3} \cup \ldots. \\ &\Longrightarrow B_{n+1} \subset B_n, \forall n \in \mathbb{N}, \end{aligned}$$

on the other hand, we have

$$\begin{aligned} m\left(\bigcap_{n\in\mathbb{N}} B_n\right) &= m\left(\bigcap_{n\in\mathbb{N}} \bigcup_{p\geqslant n} A_p\right) = \lim_{n\longrightarrow\infty} m(B_n) \\ &= \lim_{n\longrightarrow\infty} m(\bigcup_{n\geqslant n} A_p) = 0, \end{aligned}$$

hence

$$m\left(\overline{\lim_n} A_n\right) = 0.$$

Exercise 1.15 *Consider* (E, T, m) *a measured space and* $(A_n)_{n\in\mathbb{N}} \subset T$ *such that* $m(A_n) = m(E)$, $\forall n \in \mathbb{N}$.

Show that

$$m\left(\bigcap_{n\in\mathbb{N}} A_n\right) = m(E).$$

Solution 1.15 *We have* $m(E) < \infty$

$$\begin{aligned} \forall n &\in \mathbb{N} : m(A_n^c) = m(E \setminus A_n) \\ &= m(E) - m(A_n) = 0. \end{aligned}$$

By $\sigma-$*additivity of* m*, we have*

$$m\left(\bigcup_{n\in\mathbb{N}} A_n^c\right) = m\left(\left(\bigcap_{n\in\mathbb{N}} A_n\right)^c\right) = 0,$$

hence

$$m\left(\bigcap_{n\in\mathbb{N}} A_n\right) = m(E) - m\left(\left(\bigcap_{n\in\mathbb{N}} A_n\right)^c\right) = m(E).$$

Exercise 1.16 *We consider (E,T,m) a measured space and $(A_n)_{n\in\mathbb{N}} \subset T$. We suppose that*

$$\exists n_0 \in \mathbb{N} : \ m\left(\bigcup_{p\geqslant n_0} A_p\right) < \infty.$$

We prove that

$$m\left(\underline{\lim}_n A_n\right) \leqslant \underline{\lim}_n m\left(A_n\right) \leqslant \overline{\lim}_n m(A_n) \leqslant m\left(\overline{\lim}_n A_n\right).$$

Solution 1.16 *We have*

$$m\left(\underline{\lim}_n A_n\right) = m\left(\bigcup_{n\in\mathbb{N}}\bigcap_{p\geqslant n} A_p\right).$$

The sequence $\left(\bigcap_{p\geqslant n} A_p\right)_{n\in\mathbb{N}}$ *is decreasing and by increasing monotonicity, we have*

$$m\left(\bigcup_{n\in\mathbb{N}}\bigcap_{p\geqslant n} A_p\right) = \lim_n m\left(\bigcap_{p\geqslant n} A_p\right).$$

Moreover

$$m\left(\bigcap_{p\geqslant n} A_p\right) \leqslant m\left(A_q\right), \forall q \geqslant n,$$

because

$$\bigcap_{p\geqslant n} A_q \subset A_q, \forall q \geqslant n$$

therefore

$$\begin{aligned} m\left(\bigcap_{p\geqslant n} A_p\right) \quad &\leqslant \quad \inf_{q\geqslant n} m\left(A_q\right) \\ &\Longrightarrow \quad \lim_n m\left(\bigcap_{p\geqslant n} A_p\right)_n \leqslant \underline{\lim}_n m\left(A_n\right), \end{aligned}$$

and as

$$\inf_{p\geqslant n} m\left(A_p\right) \leqslant \sup_{p\geqslant n} m\left(A_p\right),$$

then

$$\underline{\lim}_n m\left(A_n\right) \leqslant \overline{\lim}_n m\left(A_n\right),$$

we have also

$$m\left(\overline{\lim}_n A_n\right) = m\left(\bigcap_{n\in\mathbb{N}}\bigcup_{p\geqslant n} A_p\right),$$

and since $\left(\bigcup_{p\geqslant n} A_p\right)_{n\in\mathbb{N}}$ *is decreasing and*

$$\exists n_0 \in \mathbb{N} : m\left(\bigcup_{p\geqslant n} A_p\right) < +\infty,$$

and by increasing monotonicity

$$m\left(\bigcap_{n\in\mathbb{N}}\bigcup_{p\geqslant n} A_p\right) = \lim_n m\left(\bigcup_{p\geqslant n} A_p\right),$$

moreover

$$m\left(\bigcup_{p\geqslant n} A_p\right) \geqslant m\left(A_q\right), \forall q \geqslant n \text{ \textit{because} } A_q \subset \bigcup_{p\geqslant n} A_p \ \forall q \geqslant n,$$

therefore

$$m\left(\bigcup_{p\geqslant n} A_p\right) \geqslant \sup_{q\geqslant n} m\left(A_q\right),$$

therefore

$$\lim_n m\left(\bigcup_{p\geqslant n} A_p\right) \geqslant \overline{\lim_n} m\left(A_n\right),$$

hence

$$m\left(\overline{\lim_n} A_n\right) \geqslant \overline{\lim_n} m\left(A_n\right).$$

Chapitre 2

Measurable Functions "Random Variables"

2.1 Mesurable functions

Definition 2.1 *Let (E, T) and (F, T') two measurable spaces and $f : E \to F$ an pplication. f is said to be measurable if*

$$\forall A \in T', \; f^{-1}(A) \in T,$$

which amounts to saying that

$$f^{-1}(T') \subset T.$$

When E and F are topological spaces equipped with their Borel tribes, we say that f is bounded.

Example .12 1. *If we equip E with the tribe $P(E)$, then any application f of E in the measurable space (F, T') is measurable.*

2. *Let (E, T) a measurable space and $f : E \to \mathbb{R}$ such that $f(x) = a = cste, \forall x \in E$. Then f is* ***measurable****.*
   Indeed : Let $A \in B(\mathbb{R})$

If $a \in A$, $f^{-1}(A) = E \in T$.
If $a \notin A$, $f^{-1}(A) = \varnothing \in T$.

Theorem 2.1 *Let $(E, T), (F, T')$ and (G, T'') three measurable spaces and $f : E \longrightarrow F$, $g : F \longrightarrow G$ two measurable applications. Then $g \circ f$ is a measurable application.*

Proof. We have for all $C \in T''$, $g^{-1}(C) \in F$ because g is measurable, and then $f^{-1}\left(g^{-1}\left(C\right)\right) \in T$ since f is also measurable, therefore

$$\begin{aligned}
(gof)^{-1}(C) &= \{x \in E : (gof)(x) \in C\} \\
&= \{x \in E : g(f(x)) \in C\} \\
&= \left\{x \in E : f(x) \in g^{-1}(C)\right\} \\
&= \left\{x \in E : x \in f^{-1}\left(g^{-1}(C)\right)\right\} \\
&= f^{-1}\left(g^{-1}(C)\right) \in T.
\end{aligned}$$

which proves that gof is measurable. ■

Proposition 2.1 *Let* (E, T) *a measurable space and* f, g *two applications from* E *in* $\mathbb{R}$. *Then the application*

$$h : (f, g) : \begin{array}{ccc} (E, T) & \to & (\mathbb{R}^2, B(\mathbb{R}^2)) \\ x & \mapsto & (f(x), g(x)) \end{array}$$

is measurable if f *and* g *are measurable.*

Proposition 2.2 *If* E *and* F *are two toplogical spacesque equipped with their Borelian tribes and* $f : E \to F$ *a continuous application, then* f *is measurable.*

Remark 2.1 *The converse is generally false.*

Example .13 *Let*

$$\begin{array}{ccc} f : \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & f(x) = \left\{ \begin{array}{ll} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{array} \right. \end{array}$$

f *is non est non continuous but measurable. Indeed : Let* $a \in \mathbb{R}$, *we have :*

- *If* $]a, +\infty[\cap \{0, 1\} = \varnothing$ *then* $f^{-1}(]a, +\infty[) = \varnothing \in B(\mathbb{R})$
- *If* $]a, +\infty[\cap \{0, 1\} \neq \varnothing$ *then* $f^{-1}(]a, +\infty[) \in \{\mathbb{Q}, \mathbb{R} - \mathbb{Q}, \mathbb{R}\} \subset B(\mathbb{R})$.

Remark 2.2 *An application can be measurable relative to one tribe and not be for other tribes.*
Proof. *Let the tribe*

$$\mathcal{D}(\mathbb{R}) = \{A \subset \mathbb{R} : A \text{ countable or } A^c \text{ countable}\},$$

the identical application

$$id : (\mathbb{R}, \mathcal{B}(\mathbb{R})) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R})),$$

is obviously measurable. However

$$id : (\mathbb{R}, \mathcal{D}(\mathbb{R})) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R})),$$

is not measurable according to (2) in the previous example because $\mathcal{B}(\mathbb{R}) \nsubseteq \mathcal{D}(\mathbb{R})$ *as* $[a, b] \in \mathcal{B}(\mathbb{R})$ *but* $[a, b] \notin \mathcal{D}(\mathbb{R})$. ■

2.1.1 Characterization of the measurability and stability of $\mathcal{L}^0(E)$

Proposition 2.3 *(Measurability criterion)*

Let (E, T) *and* (F, T') *two measurable space and* $f : E \to F$ *an application and let* $G \subset \mathcal{P}(F)$ *generating the tribe* $T' = \sigma(G)$. *Then* f *is measurable if and only if :*

$$\forall C \subset G : f^{-1}(C) \in T. \tag{2.1}$$

Proof. *If the function* f *is measurable, then the condition 2.1 is evident.*

Conversely, suppose that $f^{-1}(C) \in T$, *for* $C \subset G$ *and let* $\widetilde{T}'$ *the image tribe of* T *with* f *defined by*

$$T' = \left\{C \subset F : f^{-1}(C) \in T\right\},$$

then $G \subset \widetilde{T}'$ *and since* $T' = \sigma(G)$ *we have* $T' \subset \widetilde{T}'$, *and in particular* f *is measurable.* ■

Example .14 **a)** *If* $(F, T') = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, *f is masurable if and only if* $f^{-1}(]a, b[) \in T, \forall a, b \in$ T *such that* $a < b$.

b) $(F, T') = \left(\overline{\mathbb{R}}, \mathcal{B}(\overline{\mathbb{R}})\right)$, *f is measurable if and only if* $f^{-1}(]a, +\infty]) \in T, \forall a \in \mathbb{R}$.

Corollary 2.1 *Let E and F two topological spaces equipped with their Borelian tribes. If* $f : (E, \mathcal{B}(E)) \longrightarrow (F, \mathcal{B}(F))$ *is continuous , then it is measurable.*

Remark 2.3 *(Measurability of a numerical function)*

Since the tribe $\mathcal{B}(\mathbb{R})$ *is generated by the family of intervals of* $\mathbb{R}$ *of the form* $]a, +\infty[$, *then the function* $f : (E, T) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ *is measurable if and only if*

$$f^{-1}(]a, +\infty]) \in T, \forall a \in \mathbb{R}.$$

We denote by $\mathcal{L}^0(E)$ *the set of the functions* $f : (E, T) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ *measurable numerically.*

Proposition 2.4 *Let* (E, T) *a measurable space and ,* (F, τ) *a topological space ,* $f_1, f_2 \in \mathcal{L}^0(E)$ *two numerical measurables applications and* $\Phi : \mathbb{R}^2 \longrightarrow (F, \tau)$ *an continuous application. Then the application* $h : (E, T) \longrightarrow (F, \tau)$ *defined by*

$$h(x) = \Phi\left(f_1(x), f_2(x)\right), \text{ for all } x \in E,$$

is measurable. In other words, a continuous combination of two measurable applications is measurable.

Proof. *We can write* $h = \Phi \circ F$ *with* $F : (E, T) \longrightarrow (\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$ *is defined by* $F(x) = \left(f_1(x), f_2(x)\right)$. *Since* Φ *is measurable (because is continuous), il suffices to show that F is maesurable. If I and J two intervals of* $\mathbb{R}$ *we have*

$$F^{-1}(I \times J) = f_1^{-1}(I) \cap f_2^{-1}(J) \in T.$$

By the proposition 2.3 and since $\mathcal{B}(\mathbb{R}^2)$ *is generated by the rectangles of the form* $I \times J$ *the application F is measurable.* ■

2.1.2 Operations on measurable functions

Proposition 2.5 *Let* (E, T) *a measurable space and* $f, g : E \to \mathbb{R}$ *two applications. Then if* f, g *are measurables the functions :* $f + g,\ f \cdot g,\ \lambda f\ (\lambda \in \mathbb{R}),\ \inf(f, g),\ \sup(f, g),\ \dfrac{f}{g}\ (g \neq 0)$ *are measurable.*

Remark 2.4 $|f|$ *can be measurable without f being measurable.*

Example .15 *Let* $A \notin T$ *and* $f = \chi_A - \chi_{A^c}$ *that is to say*

$$f(x) = \begin{cases} 1, \text{ if } x \in A \\ -1, \text{ if } x \in A^c, \end{cases}$$

then $|f| = 1$ *is measurable, but f non measurable because for example*

$$\{1\} \in \mathcal{B}(\mathbb{R}) \text{ et } f^{-1}(\{1\}) = A \notin T.$$

Proposition 2.6 *Let $(f_n)_{n\in\mathbb{N}}$ is a sequence of measurable applications from (E,T) in $\left(\overline{\mathbb{R}},\mathcal{B}(\overline{\mathbb{R}})\right)$. Then the applications $\sup\limits_n f_n$, $\inf\limits_n f_n$, $\limsup\limits_{n\longrightarrow+\infty} f_n$, $\liminf\limits_{n\longrightarrow+\infty} f_n : E \longrightarrow \overline{\mathbb{R}}$ are also measurables. Moreover if the sequence $(f_n)_{n\in\mathbb{N}}$ converges simply to f, then f is measurable.*

More generally, the set

$$\left\{x\in E,\ \lim_{n\longrightarrow+\infty} f_n(x)\ \textit{exists}\right\},$$

is measurable.

Proof. *Soit $g=\sup\limits_n f_n$. Pour tout $a\in\mathbb{R}$, si $x\in g^{-1}\left(]a,+\infty[\right)$ alors il existe $m\geqslant 1$tel que $f_m(x)\geqslant a$, donc $x\in\bigcup\limits_{n=1}^{+\infty} f_n^{-1}\left(]a,+\infty[\right)$.*

Inversement, si $x\in\bigcup\limits_{n=1}^{+\infty} f_n^{-1}\left(]a,+\infty[\right)$ il est clair que $g(x)=\sup\limits_n f_n(x)\geqslant a$, d'où

$$g^{-1}\left(]a,+\infty[\right)=\bigcup_{n=1}^{+\infty} f_n^{-1}\left(]a,+\infty[\right)\in T.$$

Ainsi g est mesurable. Il en va de même de $\inf\limits_n f_n=-\sup\limits_n\left(-f_n\right)$. Par définition on a

$$\limsup_{n\longrightarrow+\infty} f_n=\inf_{n\geqslant1}\sup_{k\geqslant n} f_k,$$

et

$$\liminf_{n\longrightarrow+\infty} f_n=\sup_{n\geqslant1}\inf_{k\geqslant n} f_k,$$

On est donc ramené aux résultats précédents. De plus si $(f_n)_n$ converge simplement vers f alors

$$f=\lim_{n\longrightarrow+\infty} f_n=\limsup_{n\longrightarrow+\infty} f_n=\liminf_{n\longrightarrow+\infty} f_n,$$

d'où le résultat. Finalement pour montrer la dernière affirmation, on définit l'application

$$F:(E,T)\longrightarrow\left(\overline{\mathbb{R}}^2,\mathcal{B}(\overline{\mathbb{R}}^2)\right),\ F(x)=\left(\liminf_{n\longrightarrow+\infty} f_n,\limsup_{n\longrightarrow+\infty} f_n\right),$$

et on remarque que

$$\left\{x\in E,\ \lim_{n\longrightarrow+\infty} f_n(x)\ \textit{existe}\right\}=F^{-1}(\Delta),$$

où

$$\Delta=\left\{(x,x):x\in\overline{\mathbb{R}}\right\},$$

l'ensemble Δ est fermé alors il appartient à $\mathcal{B}(\overline{\mathbb{R}}^2)$ et donc $F^{-1}(\Delta)\in T$ puisque F est mesurable. ■

Proposition 2.7 *Let (E,T) a espace measurable space. Any positive measurable numerical $f:E\longrightarrow[0,+\infty]$ is a simple limit of an increasing sequence of $(f_n)_n$ (measurable) stepped functions.*

We give the proof of this proposition in the form of a corrected exercise (see exercise 2.2.)

2.2 Characteristic or indicator function

Definition 2.2 *Let (E,T) a espace measurable space and $A \subset E$. The characteristic or indicator function of the set A, denoted 1_A or χ_A, is the function defined by*

$$1_A(x) = \chi_A(x) = \begin{cases} 1 \text{ , if } x \in A \\ 0 \text{ , Otherwise.} \end{cases}$$

Remark 2.5 *The characteristic or indicator function 1_A is measurable if and only if A is measurable.*

Indeed : Let $a \in \mathbb{R}$, we have

$$1_A^{-1}(]a,+\infty[) = \begin{cases} \emptyset \text{ , if } a \geqslant 1 \\ A \text{ , if } 0 \leqslant a < 1 \\ E, \text{ if } a < 1. \end{cases}$$

Since $\emptyset$, $E \in T$, it suffices that $A \in T$.

2.2.1 Stepped functions

Definition 2.3 *Let (E,T) a measurable space and f a function from E in $\mathbb{R}$. We say that f is step-function if it takes only a finite number of values (i.e : Card$f(E) < +\infty$). Noting*

$$f(E) = \{\alpha_1, \alpha_2, ..., \alpha_n\},$$

then we have

$$f(x) = \sum_{i=1}^{n} \alpha_i 1_{A_i}(x),$$

where for each $i \in \{1,2,...,n\}$, $A_i = f^{-1}(\{\alpha_i\})$. This notation is called the canonical notation.
f is said to be positive step-oriented if $\alpha_i \in \mathbb{R}_+$, $\forall i = 1,2,...,n$.

Remark 2.6 *f is a finite linear combination of characteristic functions, therefore f is measurable if and only if $\forall i = \overline{1,n}$, A_i is measurable.*

Example .16

$$\begin{array}{cccc} f & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & f(x) = \begin{cases} 1 \text{ , if } x \in \mathbb{Q} \\ 2 \text{ , Otherwise} \end{cases} \end{array}$$

f is step-dependent because $f(\mathbb{R}) = \{1,2\}$ and we have

$$\begin{array}{ccl} f & : & 1_{\mathbb{Q}} + 2 \cdot 1_{\mathbb{R}-\mathbb{Q}} \\ \forall x & \in & \mathbb{R} : \ f(x) = 1_{\mathbb{Q}}(x) + 2 \cdot 1_{\mathbb{R}-\mathbb{Q}}(x). \end{array}$$

Proposition 2.8 *If $f, g : E \longrightarrow \mathbb{R}$ are two stepped functions and $\lambda \in \mathbb{R}$. Then $f + \lambda g, f \cdot g, \sup(f,g)$, and $\inf(f,g)$ are stepped functions.*

Proof. *We write f and g in the form*

$$f = \sum_{i=1}^{n} a_i \cdot \chi_{A_i} \quad \text{and} \quad g = \sum_{j=1}^{m} b_j \cdot \chi_{B_j},$$

where $n, m \in \mathbb{N}$. Since families $(A_i)_{1\leqslant i\leqslant n}$ and $(B_j)_{1\leqslant j\leqslant m}$form partitions of E we can write

$$f = \sum_{i=1}^{n}\sum_{j=1}^{m} a_i \cdot \chi_{A_i\cap B_j} \quad \text{and} \quad g = \sum_{j=1}^{m}\sum_{i=1}^{n} b_j \cdot \chi_{A_i\cap B_j},$$

then we have

$$f+g = \sum_{i=1}^{n}\sum_{j=1}^{m} (a_i+b_j) \cdot \chi_{A_i\cap B_j} \quad \text{and} \quad f\cdot g = \sum_{j=1}^{m}\sum_{i=1}^{n} (a_i b_j) \cdot \chi_{A_i\cap B_j}$$

and also

$$\sup(f,g) = \sum_{i=1}^{n}\sum_{j=1}^{m} \sup(a_i,b_j) \cdot \chi_{A_i\cap B_j} \quad \text{and} \quad \inf(f,g) = \sum_{i=1}^{n}\sum_{j=1}^{m} (a_i,b_j) \cdot \chi_{A_i\cap B_j}.$$

■

2.2.2 Second characterization of measurability

Definition 2.4 *Let (E,T) a measurable space and $f : E \longrightarrow \overline{\mathbb{R}}$ an application. We put for all $x \in E$:*

$$\begin{aligned} f^+(x) &= \max(f(x),0) \\ f^-(x) &= \min(f(x),0) = \max(-f(x),0) \\ |f|(x) &= |f(x)| \end{aligned}$$

Proposition 2.9 *Under the same conditions of the previous definition we have :*

$$\begin{aligned} f &= f^+ - f^- , \qquad |f| = f^+ + f^- \\ f^+ &= \frac{|f|+f}{2} , \qquad f^- = \frac{|f|-f}{2}, \end{aligned}$$

f is measurable if and only if f^+ and f^- are measurable.

Proposition 2.10 *Let (E,T) a measurable space and $f : E \longrightarrow \mathbb{R}$ a measurable function.Then there exists a sequence of functions $(f_n)_{n\in\mathbb{N}}$ (measurable) stepped simply converges to f.*
Proof. *The applications f^+ and f^- are measurables then by using proposition 2.7 there exist two sequences of function increasing $(h_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ an applications (measurables) stepped such that $h_n \longrightarrow f^+$ and $g_n \longrightarrow f^-$ simply as $n \longrightarrow +\infty$. We put $f_n = h_n - g_n$, such that $f_n(x) \longrightarrow f^+(x) - f^-(x) = f(x)$, as $n \longrightarrow +\infty$, for all $x \in E$. On the other hand , $(f_n)_{n\in\mathbb{N}}$ is stepped (see proposition 2.8).* ■

2.3 Some properties of measurable applications

Proposition 2.11 *let $f, g : (E, T) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ two measurable application and $a \in \mathbb{R}$. Then the following parts*

$$\begin{aligned}(f = a) : &= \{x \in E : f(x) = a\} \\ (f = g) : &= \{x \in E : f(x) = g(x)\} \\ (f \neq g) : &= \{x \in E : f(x) \neq g(x)\} \\ (f > g) : &= \{x \in E : f(x) > g(x)\},\end{aligned}$$

are measurable, that means belong to T.

Proof. *We have directly*

$$\begin{aligned}(f = a) &= f^{-1}(\{a\}) \in T \\ (f = g) &= (f - g)^{-1}(\{a\}) \in T \\ (f \neq g) &= (f = g)^c \in T \\ (f > g) &= (f - g)^{-1}(]0, +\infty[) \in T.\end{aligned}$$

■

2.3.1 Properties true almost everywhere

Definition 2.5 *Let (E, T, m) a measured space . A property $p(x)$ concerning $x \in E$ is said to be true almost everywhere if the set*

$$\{x \in E : p(x) \text{ is not true}\},$$

is negligible.

2.3.2 Equality almost everywhere

Definition 2.6 *Let (E, T, m) a measured space and $f, g : E \longrightarrow F$ two functions ($F = \mathbb{R} \vee \overline{\mathbb{R}} \vee \mathbb{R}_+ \vee \ldots$). We say that $f = g$ almost everywhere and we denote $f = g$ a.e if the set*

$$\{x \in E : f(x) \neq g(x)\}$$

is negligible that means

$$\exists A \in T \text{ such that } m(A) = 0 \text{ and } f(x) = g(x)\ \forall x \in A^c.$$

Example .17 *If the functions $f, g : E \longrightarrow \mathbb{R}$ are equal almost everywhere, then there exists $A \in T$ such that*

$$(f \neq g) \subset A \text{ and } m(A) = 0,$$

therefore

$$f(x) = g(x), \forall x \in A^c \text{ and } m(A) = 0,$$

we can note that if f and g are measurable, then $(f \neq g) \in T$ and therefore $f = g$ almost everywhere if and only if $m(f \neq g) = 0$.

Example .18 *Let* $(E,T,m) = (\mathbb{R}, B(\mathbb{R}), \lambda)$ *and let* $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ *such that*

$$f(x) = \begin{cases} \dfrac{1}{x} \text{ , if } x \neq 0 \\ 1 \text{ ,if } x = 0 \end{cases} \quad , \qquad g(x) = \begin{cases} \dfrac{1}{x} \text{ , if } x \neq 0 \\ -1 \text{ ,if } x = 0, \end{cases}$$

we have

$$\{x \in \mathbb{R} : f(x) \neq g(x)\} = \{0\}\,, \text{ and } \lambda\left(\{0\}\right) = 0,$$

and

$$\forall x \in \mathbb{R} - \{0\}\,,\ f(x) \neq g(x),$$

therefore

$$f = g \quad a.e.$$

Theorem 2.2 *Let* (E,T,m) *a measured space and* $f, g : (E,T) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ *two applications such that* f *is measurable and* $f = g$ *almost everywhere. Then* g *is measurable.*
Proof. *There exists* $A \in T$ *such that* $m(A) = 0$ *and*

$$f(x) = g(x), \forall x \in A^c.$$

Let $a \in \mathbb{R}$, *if we put*

$$(g > a) = g^{-1}\left(]a, +\infty[\right),$$

we have

$$\begin{aligned} (g > a) &= (g > a) \cap (A \cup A^c) \\ &= [(g > a) \cap A] \cup [(g > a) \cap A^c] \\ &= [(g > a) \cap A] \cup [(f > a) \cap A^c]\,, \end{aligned}$$

on the other hand, since f *is measurable we have*

$$(f > a) \cap A^c \in T$$

and on the other hand the set

$$(g > a) \cap A$$

is negligible because

$$(g > a) \cap A \subset A \text{ and } m(A) = 0,$$

then

$$(g > a) \cap A \in T,$$

since the measure m *is complet. We have*

$$(g > a) \in T,$$

therefore we obtain the measurability of g. ■

2.4 Convergence almost everywhere and convergence in measure

Definition 2.7 *Let (E,T,m) a measured space , $f_n : E \longrightarrow \mathbb{R}$ or $[0,+\infty]$ a sequence of functions and $f : E \longrightarrow \mathbb{R}$ or $[0,+\infty]$ a function . We say that the sequence $(f_n)_{n\in\mathbb{N}}$ converges almost everywhere to f on E if there exists $A \subset E$ negligible such that for all $x \in A^c$, the sequence $(f_n(x))_{n\in\mathbb{N}}$ converges to $f(x)$. In this case we write $f_n \longrightarrow f$ a.e.*

Remark 2.7 1. *IF the functions f_n and f are measurable, then the sequence $(f_n)_{n\in\mathbb{N}}$ converges almost everywhere to f if*

$$m\left(\left\{x \in E : \lim_{n\longrightarrow+\infty} f_n(x) \neq f(x)\right\}\right) = 0.$$

2. *Simple convergence implies convergence almost everywhere because if $f_n \longrightarrow f$ simply we have*

$$m\left(\lim_{n\longrightarrow+\infty} f_n \neq f\right) = m(\emptyset) = 0.$$

Example .19 *Let $E = [0,1]$, T the Borelian tribe on $[0,1]$ equipped with the Lebesgue measure λ and $(f_n)_{n\in\mathbb{N}}$ the sequence of functions defined by $f_n(x) = (-x)^n$. For all $x \in [0,1]$ we have $\lim\limits_{n\longrightarrow+\infty} f_n(x) = 0$ and*

$$\lambda\left(\left\{x \in [0,1] : \lim_{n\longrightarrow+\infty} f_n(x) \neq 0\right\}\right) = \lambda(\{1\}) = 0,$$

then $f_n \longrightarrow f$ a.e.

Definition 2.8 *Let (E,T,m) a espace mesured space and $(f_n)_{n\in\mathbb{N}}$ a measurable sequence of functions of E in $\overline{\mathbb{R}}$, $f : E \longrightarrow \overline{\mathbb{R}}$ measurable. We say that f_n converges **in measure** to f if :*

$$\forall\varepsilon > 0, \ \lim_{n\to+\infty} m(\{x \in E : |f_n(x) - f(x)| \geqslant \varepsilon_0\}) = 0,$$

that means :

$$\forall\varepsilon > 0, \ \exists n_0 \in \mathbb{N}, \forall n \geqslant n_0 \ m(\{x \in E : |f_n(x) - f(x)| \geqslant \varepsilon\}) < \varepsilon,$$

or

$$\lim_{n\longrightarrow+\infty} m(\{x \in E : |f_n(x) - f(x)| \geqslant \varepsilon\}) = 0.$$

Remark 2.8 *Convergence almost everywhere does not entail convergence in measure.*

Example .20 *Let $E = \mathbb{R}$, $T = \mathcal{B}(\mathbb{R})$, $m = \lambda$ the measure of Lebesgue and $f_n = \chi_{[n,n+1]}$,*

$$f_n(x) = \begin{cases} 1 & \text{if } \ n \leqslant x \leqslant n+1 \\ 0 & \text{if } \ x < n \text{ or } x > n+1, \end{cases}$$

there exists $n_0 = [x] + 1$ such that for all $n \geqslant n_0$ we have $f_n(x) = 0$. Then $(f_n)_n$ converges simply to 0 on E, which gives $f_n \longrightarrow f$ a.e. On the other hand if we put $\varepsilon = 1$, we have

$$\begin{aligned} \lambda(\{x \in E : |f_n(x) - 0| \geqslant 1\}) &= \lambda\left(\left\{x \in E : \chi_{[n,n+1]} \geqslant 1\right\}\right) \\ &= \lambda([n, n+1]) = 1 \neq 0, \end{aligned}$$

then f_n does not tend towards to 0 in measure.

Proposition 2.12 *In a finite measured space $m(E) < \infty$, Convergence almost everywhere leads to convergence in measure.*

Proposition 2.13 *Suppose that the sequence of functions $(f_n)_n$ converges in mesure to f. Then there exists a subsequence $(f_{n_k})_k$ of $(f_n)_n$ converges almost everywhere to f.*

Proposition 2.14 *Let (E, T, m) a measured space and $f_n : E \longrightarrow \mathbb{R}$ or $[0, +\infty]$ a measurable sequence of functions and $f, g : E \longrightarrow \mathbb{R}$ or $[0, +\infty]$ two measurable functions. If $f_n \longrightarrow f$ in measure and $f_n \longrightarrow g$ in measure, then $f = g$ almost everywhere. That is, the limit is unique almost everywhere.*

Proof. We show that $f = g$ a.e .

Let $\delta > 0, \forall x \in E$ and $n \in \mathbb{N}^*$

$$\begin{aligned} |f(x) - g(x)| &= |f(x) - g(x) - f_n(x) + f_n(x)| \\ &\leqslant |f(x) - f_n(x)| + |f_n(x) - g(x)|, \end{aligned}$$

then

$$\left\{|f - f_n| \leqslant \frac{\delta}{2}\right\} \cap \left\{|f_n - g| \leqslant \frac{\delta}{2}\right\} \subset \{|f - g| \leqslant \delta\},$$

therefore

$$\{|f - g| > \delta\} \subset \left\{|f - f_n| > \frac{\delta}{2}\right\} \cup \left\{|f_n - g| > \frac{\delta}{2}\right\},$$

then

$$m(\{|f - g| > \delta\}) \leqslant m\left(\left\{|f - f_n| > \frac{\delta}{2}\right\}\right) + m\left(\left\{|f_n - g| > \frac{\delta}{2}\right\}\right).$$

Passing to the limit when $n \longrightarrow +\infty$, we obtain that

$$m(\{|f - g| > \delta\}) = 0,$$

and we have

$$\begin{aligned} \{x \in E : f(x) \neq g(x)\} &= \{|f - g| > 0\} \\ &= \bigcup_{n \in \mathbb{N}^*} \left\{|f - g| > \frac{1}{n}\right\}, \end{aligned}$$

then

$$m(\{x \in E : f(x) \neq g(x)\}) \leqslant \sum_{n=1}^{+\infty} m\left(\left\{|f - g| > \frac{1}{n}\right\}\right) = 0,$$

therefore

$$f = g \quad \text{a.e.}$$

■

2.5 Random variable

Definition 2.9 *Let (E,T) a a probability space, a random variable is a function $X : E \longrightarrow \mathbb{R}$ measurable. that means :*

$$X^{-1}(A) \in T,\ \forall A \in \mathcal{B}(\mathbb{R}),$$

or any measurable function of (E,T) in $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

Example .21

1. *Any application $f : (X, \mathcal{P}(X)) \longrightarrow (Y, \mathcal{P}(Y))$ is measurable.*
2. *If T and T' are two tribes on E then the identity on E*

   $$id : (E,T) \longrightarrow (E,T'),\ id(x) = x,$$

   is measurable if and only if $T' \subset T$.
3. *If (E,T) and (F,T') are two measurables space and $f : T \longrightarrow T'$ a constant function (that is say there exists $y_0 \in F$ such that for all $x \in E$ we have $f(x) = y_0$) then f is measurable.*
4. *A stepped function f is always measurable.*
5. *Let (E,T) a measurable space and $A \in T$. The indicator function χ_A of the set A is an application from E in $\mathbb{R}$ equipped with the Borel sigma-algebra $\mathcal{B}(\mathbb{R})$ is measurable if and only if $A \in T$. For this reason, the elements of T are called measurable sets.*

Proof. *We prove only (4) and (5).*

For (4) indeed if f is a function of (E,T) in $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, there exists $(A_i)_{1 \leqslant i \leqslant n}$ a partition of E and $\{a_1, ..., a_n\} \subset \mathbb{R}$ such that $f = \sum_{i=1}^{n} a_i \cdot \chi_{A_i}$. for all $B \in \mathcal{B}(\mathbb{R})$, therefore, we have

$$f^{-1}(B) = \left(\bigcup_{a_i \in A} A_i \right) \in T,$$

which proves that f is measurable.

Now, we show that (5), according to if $A \in T$ the stepped function χ_A is measurable.

Conversely, if χ_A is measurable, then

$$A = (\chi_A)^{-1}(\{1\}) \in T,$$

because $\{1\} \in \mathcal{B}(\mathbb{R})$ like a closed set in $\mathbb{R}$. ■

Definition 2.10 *Let (E,T) and (F,T') two probability spaces, a function $X : E \longrightarrow F$ is a random element if it is measurable, that is to say :*

$$X^{-1}(A) \in T,\ \forall A \in T'.$$

When F is a vector space, we say that X is a vector random variable or a random vector.

2.5.1 Almost uniform convergence

Definition 2.11 *Let (E, T, m) a measured space and $(f_n)_{n\in\mathbb{N}}$ a measurable sequence of functions of E in $\overline{\mathbb{R}}$, $f : E \longrightarrow \overline{\mathbb{R}}$ measurable. We say that f_n converges almost uniform to f if :*

$$\forall\varepsilon > 0,\ \exists A \in T \text{ such that } m(A) \leqslant \varepsilon \text{ and } f_n \overset{unif}{\longrightarrow} f \quad \text{on } A^c.$$

$$\text{Conver } A.U \Longrightarrow \ \text{Conv} \quad A.E.$$

Definition 2.12 *(Essential sup) Let (E, T, m) a measured space and $f \in M$, f is essentially convergent if*

$$\exists C \in \mathbb{R}_+ \text{ such that } \quad |f| \leqslant C \quad a.e.$$

We call Essential sup of $|f|$ and we denote $\|f\|_\infty$

$$\|f\|_\infty = \inf\{C \quad \text{such that} \quad |f| \leqslant C \quad a.e\},$$

if f is not essentially bounded, we set $\|f\|_\infty = \infty$.

2.5.2 Essentially uniform convergence

Definition 2.13 *Let (E, T, m) a measured space and $(f_n)_{n\in\mathbb{N}}$ a measurable sequence of functions from E in $\overline{\mathbb{R}}$, $f : E \longrightarrow \overline{\mathbb{R}}$ measurable. We say that f_n converges essentialy uniforme to f if : $\|f - f\|_\infty \underset{n\to\infty}{\longrightarrow} 0$.*

2.6 Corrected exercises

Exercise 2.1 *Consider the function $f : (\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2)) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ defined by*

$$f(x, y) = \begin{cases} \dfrac{1}{(1+x)^2} \text{ if } x > 0 \text{ and } x < y \leqslant 2x \\ -\dfrac{1}{(1+x)^2} \text{ if } x > 0 \text{ and } 2x < y \leqslant 3x \\ 0 \text{ if } x > 0 \text{ and } y \notin]x, 3x[\\ 0 \text{ if } x \leqslant 0. \end{cases}$$

1. *For all $n \in \mathbb{N}^*$, we put*

$$A_n = \left\{(x, y) \in \mathbb{R}^2 : x > 0 \text{ and } x < y < 2x + \frac{1}{n}\right\},$$

and

$$B_n = \left\{(x, y) \in \mathbb{R}^2 : x > 0 \text{ and } 2x < y < 3x + \frac{1}{n}\right\}.$$

Determine

$$A = \bigcap_{n=1}^{+\infty} A_n \quad \text{and} \quad B = \bigcap_{n=1}^{+\infty} B_n.$$

2. *Deduce that f is measurable.*

Solution 2.1 1. *We have*

$$A = \left\{(x,y) \in \mathbb{R}^2 : x > 0 \text{ and } x < y < 2x\right\}$$

and

$$B = \left\{(x,y) \in \mathbb{R}^2 : x > 0 \text{ and } 2x < y < 3x\right\},$$

now, we put

$$g(x,y) = \frac{1}{(1+|x|)^2} \text{ for all } (x,y) \in \mathbb{R}^2.$$

The function $g : (\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2)) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is measurable because it is continuous. We now notice that

$$f = g\chi_A - g\chi_B.$$

For all $n \in \mathbb{N}^*$ the sets A_n and B_n are open sets of $\mathbb{R}^2$ then belong to $\mathcal{B}(\mathbb{R}^2)$.

2. *We deduce that $A, B \in \mathcal{B}(\mathbb{R}^2)$ and then the functions χ_A and χ_B are measurables. Finally the function f is measurable as a sum of products of measurable functions. (Recall that the Borel sigma-algebra $\mathcal{B}\left(\overline{\mathbb{R}}\right)$ on $\mathbb{R}$ is generated by the intervals $]a, +\infty[\,, a \in \mathbb{R}$).*

Exercise 2.2 *For all $n \geqslant 1$, we define the stepped function $\varphi_n : [0, +\infty] \longrightarrow [0, +\infty[$ by*

$$\varphi_n(t) = \begin{cases} 2^{-n}E(2^n t) & \text{if } 0 \leqslant t < n \\ n & \text{if } t \geqslant n, \end{cases}$$

where $E(x)$ denotes the integer part of $x \in \mathbb{R}$.

1. *Show that $0 \leqslant \varphi_n(t) \leqslant \varphi_{n+1}(t)$ for all $t \in [0, +\infty]$ and $n \geqslant 1$.*
2. *We put $f_n = \varphi_n \circ f$. Verify that f_n is stepped and prove that the sequence $(f_n)_n$ is increasing.*
3. *Prove that if $f(x) < \infty$ then for n sufficiently large we have*

$$f(x) - 2^{-n} \leqslant f_n(x) \leqslant f(x).$$

Conclude.

Solution 2.2 1. *We have three cases*

i) *For $t \geqslant n+1$ (or also $t = +\infty$). Then*

$$\varphi_{n+1}(t) = n + 1 > n = \varphi_n(t).$$

ii) *For $n \leqslant t < n+1$. According to the growth of the floor (integer part) function we have*

$$E\left(2^{n+1}t\right) \geqslant E\left(2^{n+1}n\right) = 2^{n+1}n,$$

which implies that

$$\varphi_{n+1}(t) = \frac{1}{2^{n+1}}E\left(2^{n+1}t\right) \geqslant n = \varphi_n(t).$$

iii) *For* $0 \leqslant t < n$. *Since*

$$2^{n+1}t \geqslant 2E\left(2^n t\right) \in \mathbb{N},$$

we have

$$E\left(2^{n+1}t\right) \geqslant 2E\left(2^n t\right),$$

then

$$\varphi_{n+1}(t) = \frac{1}{2^{n+1}}E\left(2^{n+1}t\right) \geqslant \frac{1}{2^n}E\left(2^n t\right) = \varphi_n(t).$$

Therefore in all theses cases and for all $t \in [0, +\infty]$ *and* $n \geqslant 1$ *we have*

$$\varphi_n(t) \leqslant \varphi_{n+1}(t).$$

2. *It is clear that the functions* φ_n *are stepped for all* $n \geqslant 1$, *so the* f_n *are also stepped because*

$$f_n(E) = \varphi_n(f(E)) = \varphi_n([0, +\infty]),$$

is finite . On the other hand the growth of the sequence $(\varphi_n)_n$ *we can write*

$$f_n(x) = \varphi_n(f(x)) \leqslant \varphi_{n+1}(f(x)) = f_{n+1}(x), \forall n \geqslant 1, \forall x \in E.$$

3. *For sufficiently large* n, *let's choose* $n > f(x)$, *in this case we have*

$$f_n(x) = \frac{1}{2^n}E\left(2^n f(x)\right),$$

by the properties of the integer part

$$2^n f(x) - 1 \leqslant E\left(2^n f(x)\right) \leqslant 2^n f(x),$$

therefore

$$f(x) - \frac{1}{2^n} \leqslant f_n(x) \leqslant f(x),$$

finally by taking the limit as $n \to +\infty$ *we conclude that* $f_n \longrightarrow f$ *simply.*

Exercise 2.3 *Let* $X = [0, 1[$ *equipped with the Borelian tribe and the Lebesgue measure* λ. *Let the sequence* $(f_n)_n$ *of functions defined by* $f_n = \chi_{\left[\frac{1}{n}, \frac{2}{n}\right[}$.

Prove that $f_n \longrightarrow 0$ *in measure, but that* $(f_n)_n$ *not converges almost everywhere to* 0 *.*

Solution 2.3 *For all* $\alpha > 0$ *, we have*

$$\{x \in X : |f_n(x) - 0| \geqslant \alpha\} \subset \left[\frac{1}{n}, \frac{2}{n}\right[,$$

therefore

$$\lambda\left(\{x \in X : |f_n(x) - 0| \geqslant \alpha\}\right) \leqslant \lambda\left(\left[\frac{1}{n}, \frac{2}{n}\right[\right) = \frac{1}{n} \longrightarrow 0,$$

it results that $f_n \longrightarrow 0$ *in measure.*

On the other hand, we have

$$\limsup_{n \longrightarrow +\infty} f_n = 1 \neq \liminf_{n \longrightarrow +\infty} f_n = 0,$$

then for all $x \in [0,1[$, *we have* $\lim\limits_{n \longrightarrow +\infty} f_n(x) \neq 0$. *Hence*

$$\lambda\left(\left\{x \in X : \lim_{n \longrightarrow +\infty} f_n(x) \neq 0\right\}\right) = \lambda\left([0,1[\right) = 1 \neq 0,$$

therefore $(f_n)_n$ *cannot converge almost everywhere to* 0.

Exercise 2.4 *Consider* $f, g, h : (\Omega, \mathcal{M}, \mu) \longrightarrow \mathbb{R}$ *any three functions.*

Show that if $f = g$ *almost everywhere and* $g = h$ *almost everywhere, then* $f = h$ *almost everywhere.*

Solution 2.4 *There exists* $A, B \in \mathcal{M}$ *such that*

$$(f \neq g) \subset A, \; (g \neq h) \subset B \quad \textit{and} \quad \mu(A) = \mu(B) = 0.$$

Since

$$(f = g) \cap (g = h) \subset (f = h),$$

we have

$$(f \neq h) \subset (f \neq g) \cup (g \neq h) \subset A \cup B,$$

and we have

$$\mu(A \cup B) \leqslant \mu(A) + \mu(B) = 0,$$

hence

$$(f \neq h) \subset A \cup B \quad \textit{with} \quad \mu(A \cup B) = 0.$$

Finally we get $f = g$ *almost everywhere.*

Exercise 2.5 *Consider* (E, T) *a measurable space and* $f : E \longrightarrow \mathbb{R}$ *a measurable function, for* $a > 0$, *we put*

$$f_a(x) = \begin{cases} a & \textit{if} \;\; f(x) > a \\ f(x) & \textit{if} \;\; |f(x)| \leqslant a \\ -a & \textit{if} \;\; f(x) < -a \end{cases}$$

Prove that f_a *is measurable.*

Solution 2.5 *We define the function* $T_a : \mathbb{R} \longrightarrow \mathbb{R}$ *by*

$$T_a(s) = \begin{cases} a & \textit{if}\; s > a \\ s & \textit{if}\; |s| \leqslant a \\ -a & \textit{if} \;\; s < -a, \end{cases}$$

T_a *is continuous from* $\mathbb{R}$ *in* $\mathbb{R}$, *therefore measurabele, and moreover* $f_a = T_a o f$, *then* f_a *is measurable as a composition of two measurable applications.*

Exercise 2.6 *Additivity of the Lebesgue integral over positive functions*

Let (E, τ, μ) *a measured space . Consider* f, g *two positives mesurable function .*

Prove that

$$\int_E (f+g)\, d\mu = \int_E f d\mu + \int_E g d\mu.$$

We can begin by assuming f *and* g *are step functions. Then use a theorem for approximating positive measurable functions with step functions, and finally the monotone convergence theorem.(Beppo-Levi).*

Solution 2.6 *Let's say that f and g are step functions and taking the values a_i, $i = 1, ..., n$ and b_j, $j = 1, ..., p$ on the sets A_i and B_j.*

Note that $(A_i)_i$ et $(B_j)_j$ form partitions of E. On the other hand, we have :

$$\begin{aligned}
\int_E f d\mu + \int_E g d\mu &= \sum_{i=1}^{n} a_i \mu(A_i) + \sum_{j=1}^{p} b_j \mu(B_J) \\
&= \sum_{i=1}^{n} a_i \left(\sum_{j=1}^{p} \mu(A_i \cap B_j) \right) + \sum_{j=1}^{p} b_j \left(\sum_{i=1}^{n} \mu(B_j \cap A_i) \right) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{p} (a_i + b_j) \mu(A_i \cap B_j).
\end{aligned}$$

On the other hand :

$$\begin{aligned}
f(x) + g(x) &= \sum_{i=1}^{n} a_i 1(A_i) + \sum_{j=1}^{p} b_j 1(B_j) \\
&= \sum_{i=1}^{n} a_i \left(\sum_{j=1}^{p} 1(A_i \cap B_j) \right) + \sum_{j=1}^{p} b_j \left(\sum_{i=1}^{n} 1(B_j \cap A_i) \right) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{p} (a_i + b_j) 1(A_i \cap B_j),
\end{aligned}$$

in fact that

$$\int_E (f + g) d\mu = \sum_{i=1}^{n} \sum_{j=1}^{p} (a_i + b_j) \mu(A_i \cap B_j) = \int_E f d\mu + \int_E g d\mu.$$

To move to any positive measurable functions f, g, it suffices to consider (u_n) and (v_n) as increasing sequences of positive step functions such that $(u_n) \longrightarrow f(x)$ and $(v_n) \longrightarrow g(x)$ (increasing, therefore) for all $x \in E$. For all n, then, we have

$$\int (u_n + v_n) d\mu = \int u_n d\mu + \int v_n d\mu.$$

Taking the limit in this equality and using Beppo-Levi, we find

$$\int (f + g) d\mu = \int f d\mu + \int g d\mu.$$

Exercise 2.7 *Let f_n a sequence of positive measurable functions on (E, τ, μ). We define the function F for all $x \in E$ by r*

$$F(x) = \sum_{n=0}^{\infty} f_n(x).$$

Show that the function F is positive measurable and that

$$\int_E F d\mu = \sum_{n=0}^{\infty} \int_E f_n d\mu.$$

Solution 2.7 *Let F_p the partial sum*

$$F_p(x) = \sum_{n=0}^{p} f_n(x),$$

from the previous exercise we have

$$\int F d\mu = \sum_{n=0}^{p} \int f_n d\mu,$$

Furthermore, the sequence of functions $(F_p)_p$ is an increasing sequence (since the f_n are positive) of positive measurable functions such that $F_p(x) \longrightarrow F(x)$ for all $x \in E$. The monotone convergence theorem then gives that F is measurable (it is also positive) and that :

$$\int F d\mu = \lim_{p \longrightarrow \infty} \int_E F_p d\mu = \sum_{n=0}^{\infty} \int f_n d\mu.$$

Exercise 2.8 *We are working on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ equipped with the counting measure μ : for all $A \subset \mathbb{N}$, $\mu(A)$ is the cardinality of A if A is finite, $\mu(A) = +\infty$ otherwise.*

$$\mu(A) = \begin{cases} Card(A), & \text{if } A \text{ finite} \\ +\infty, & \text{otherwise.} \end{cases}$$

1. *Let $f : \mathbb{N} \longrightarrow [0, +\infty]$ a positive function on $\mathbb{N}$.*
   We can also see f as a sequence of positive real numbers $u_n = f(n)$. By noting that f can be written as
   $$f = \sum_{n=0}^{\infty} f(n) 1_{\{n\}},$$
   explain the values of $\int_{\mathbb{N}} f d\mu$.
2. *Let $(u_{n,p})_{n,p \in \mathbb{N}}$ a double sequence of positive real numbers.*
   Prove that
   $$\sum_{n=0}^{\infty} \left(\sum_{p=0}^{\infty} u_{n,p} \right) = \sum_{p=0}^{\infty} \left(\sum_{n=0}^{\infty} u_{n,p} \right).$$
3. *Calculate*
   $$\sum_{p=2}^{\infty} \left(\sum_{n=2}^{\infty} \frac{1}{n^p} \right).$$

Solution 2.8 1. *By term-by-term integration of a series of positive functions, we obtain :*

$$\int_{\mathbb{N}} f d\mu = \sum_{n=0}^{\infty} \int_{\mathbb{N}} f(n) 1_{\{n\}}(x) d\mu(x) = \sum_{n=0}^{\infty} f(n) \mu(\{n\}) = \sum_{n=0}^{\infty} f(n).$$

The classic series are complete editions by Lebesgue.

2. *Let f_n the sequence of positive functions on $\mathbb{N}$ defined by $f_n(p) = u_{n,p}$ for all $p \in \mathbb{N}$. The term-by-term integration theorem gives :*

$$\int_{\mathbb{N}} \sum_{n=0}^{\infty} f_n d\mu = \sum_{n=0}^{\infty} \int_{\mathbb{N}} f_n d\mu.$$

The first question, applied to each side of this equation, yields the result.

3. *The terms are positive, so we interchange :*

$$\begin{aligned} \sum_{p=2}^{\infty} \left(\sum_{n=2}^{\infty} \frac{1}{n^p} \right) &= \sum_{n=2}^{\infty} \left(\sum_{p=2}^{\infty} \frac{1}{n^p} \right) \\ &= \sum_{n=2}^{\infty} \frac{1}{n^2 - n} \\ &= \sum_{n=2}^{\infty} -\frac{1}{n} + \frac{1}{n-1} = 1, \end{aligned}$$

by telescoping. We don't know much about the values of the Riemann sums $\sum 1/n^p$ for integer $p \geq 2$, but we can calculate the sum of these sums!

Exercise 2.9 *Let (E, τ, μ) a measured space , and $(f_n)_n$ a sequence of integrable functions that converges almost everywhere to an integrable function f.*

1. *We suppose that*

$$\lim_{n \longrightarrow \infty} \int_E |f_n - f| \, d\mu = 0.$$

Prove that

$$\lim_{n \longrightarrow \infty} \int_E f_n d\mu = \int_E f d\mu \text{ and } \lim_{n \longrightarrow \infty} \int_E |f_n| \, d\mu = \int_E |f| \, d\mu.$$

2. *Conversely, we assume that*

$$\lim_{n \longrightarrow \infty} \int_E |f_n| \, d\mu = \int_E |f| \, d\mu$$

Show that

$$\lim_{n \longrightarrow \infty} \int_E |f_n - f| \, d\mu = 0,$$

we can use the sequence of functions

$$g_n = |f| + |f_n| - |f - f_n|$$

and apply to it the only convergence theorem whose hypotheses it satisfies.

3. *Summarize the results of the previous questions into an equivalenc*

4. *We consider* $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$.
   Give an example of a sequence (f_n) *of integrable functions that converges to an integrable function* f*, such that*

$$\lim_{n \longrightarrow \infty} \int_E f_n d\mu = \int_E f d\mu,$$

   and such that, however, $(\int_E |f_n - f| \, d\mu)$ *does not tend towards* 0.

Solution 2.9 1. *It suffices to write :*

$$\left| \int_E f_n d\mu - \int_E f d\mu \right| \leqslant \int_E |f_n - f| \, d\mu,$$

and

$$\left| \int_E |f_n| \, d\mu - \int_E |f| \, d\mu \right| = \left| \int_E |f_n| - |f| \, d\mu \right| \leqslant \int_E ||f_n| - |f|| \, d\mu \leqslant \int_E |f_n - f| \, d\mu,$$

(inverted triangle inequality).

2. *The functions* g_n *are positive, we can only do Fatou. Note that* g_n *converges pointwise to* $2|f|$ *almost everywhere by hypothesis. We obtain*

$$\int_E \lim_{n \longrightarrow +\infty} \inf g_n d\mu \leqslant \lim_{n \longrightarrow +\infty} \inf \int_E g_n d\mu,$$

in other words :

$$\int_E 2|f| \, d\mu \leqslant \lim_{n \longrightarrow +\infty} \inf \int_E |f| + |f_n| - |f - f_n| \, d\mu = 2 \int_E |f| \, d\mu - \lim_{n \longrightarrow +\infty} \sup \int_E |f_n - f| \, d\mu.$$

The second equality comes from the hypothesis of this question. Therefore

$$0 \leqslant \lim_{n \longrightarrow +\infty} \sup \int_E |f_n - f| \, d\mu \leqslant 0.$$

Consequently

$$\lim_{n \longrightarrow +\infty} \sup \int_E |f_n - f| \, d\mu = 0.$$

3. *We have just shown that, under the hypotheses of the exercise :*

$$\lim_{n \longrightarrow +\infty} \int_E |f_n - f| \, d\mu = 0 \iff \lim_{n \longrightarrow +\infty} \int_E |f_n| \, d\mu = \lim_{n \longrightarrow +\infty} \int_E |f| \, d\mu.$$

4. *It will be necessary to choose functions that change sign, otherwise the previous results ensure that such functions do not exist. Consider*

$$f_n = \frac{1}{n} \left(1_{[0,n]} - 1_{[-n,0]} \right)$$

The f_n *are integrable, converge to* 0 *(which is integrable). We have also :*

$$\int_{\mathbb{R}} f_n d\mu = 0 \longrightarrow 0 = \int_{\mathbb{R}} 0 d\mu,$$

moreover

$$\int_{\mathbb{R}} |f_n - 0| \, d\mu = 2.$$

Exercise 2.10 *Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ two continuous functions.*

1. If λ is the measure of Lebesgue, prove that

$$f = g \ \text{ a.e} \quad \Longleftrightarrow \ f = g.$$

2. If δ_0 is the measure of Dirac :

$$\forall A \in B(\mathbb{R}) : \delta_0(A) = \begin{cases} 1 & \text{if } \ 0 \in A \\ 0 & \text{otherwise} \end{cases},$$

show that

$$f = g \ \text{ a.e} \Longleftrightarrow f(0) = g(0)$$

Solution 2.10

1. We show that

$$f = g \ \text{ a.e} \quad \Longleftrightarrow \ f = g$$

i) $\Longleftarrow$ If $f = g$ i.e :$f(x) = g(x)$ $\forall x \in \mathbb{R}$, and we have $f = g$ a.e because $f = g$ on $\varnothing^c$ and $\lambda(\varnothing) = 0$

ii) $\Longrightarrow$ If $f = g$ a.e , then $\exists A \in B(\mathbb{R}) : \lambda(A) = 0$ and $f = g$ on A^c,then we have

$$\{f(x) \neq g(x)\} \subset A$$

$$\{f(x) = g(x)\} = (f - g)^{-1}(\mathbb{R}^*),$$

is an open since $f - g$ is continuous from $\mathbb{R}$ to $\mathbb{R}$, therefore

$$\{f(x) \neq g(x)\} \in B(\mathbb{R}),$$

moreover

$$\lambda(\{f(x) \neq g(x)\}) \leqslant \lambda(A) = 0,$$

therefore

$$\{f(x) \neq g(x)\} = \varnothing,$$

because a non-empty open set always has strictly positive Lebesgue measure, hence

$$f = g \ \text{ everywhere.}$$

2. We prove that

$$f = g \ \text{ a.e} \quad \Longleftrightarrow \ f(0) = g(0)$$

i) $\Longleftarrow$ If $f(0) = g(0)$, we take $A = \{0\}^c$ then, we have $A \in B(\mathbb{R})$, and $\delta_0(A) = 0$ then

$$f = g \text{ on } A^c \ \text{ therefore } f = g \ \text{ a.e.}$$

ii) $\Longrightarrow$ If $f = g$ a.e, therefore

$$\exists A \in B(\mathbb{R}) : f = g \text{ on } A^c \text{ and } \delta_0(A) = 0,$$

therefore

$$0 \notin A \text{ i.e} : 0 \in A^c \text{ hence } \ f(0) = g(0).$$

Chapitre 3

Integrable Functions

In this chapter (E, T, m) denotes a measure space. We will subsequently study the Lebesgue integral over E with respect to the measure m.

We denote by $\mathcal{E}_+$: the set of measurable and positive step functions from (E, T) to $\mathbb{R}$■ equipped with the Borel sigma-algebra.

3.1 Integral of a positive step function

Definition 3.1 *Let $f \in \mathcal{E}_+$, with canonical decomposition $f = \sum\limits_{i=1}^{n} a_i \cdot \chi_{A_i}$ We put*

$$\int_E f dm = \sum_{i=1}^{n} a_i m(_{A_i}) \in \overline{\mathbb{R}}_+.$$

This quantity is called the integral on E of the function f with respect to the measure m.

The integral $\int f dm$ is an element of $[0, +\infty]$. Although the a_i are all real, it can very well be equal to $+\infty$, if for $a_i > 0$, the corresponding $m(A_i)$ is equal to $+\infty$. Recall the convention $0 \times (+\infty) :=$ which is very useful here when $m\left(f^{-1}\left(\{0\}\right)\right) = +\infty$.

Example .22 *1. If f is a constant function, then its canonical decomposition is written $f = a\chi_E$, with $a \geqslant 0$. the formula $\int_E f dm = \sum\limits_{i=1}^{n} a_i m(_{A_i})$ which then gives us*

$$\int_E a dm = a \cdot m(E).$$

2. If $f = a\chi_A$ with $a > 0$, $A \in T$ and $A \neq E$, its canonical decomposition is

$$f = a\chi_A + 0\chi_{A^c},$$

hence

$$\int a \cdot \chi_A = a \cdot m(A) + 0 \cdot m(A^c) = a \cdot m(A).$$

We denote

$$\int_E f dm = \int_{x \in E} f(x) dm$$

3. *If $\exists i : a_i = 0$ and $m(A_i) = +\infty$, Therefore, we have $0 \times (+\infty) = 0$. The Lebesgue integral is independent of the expression of the stepped function, so if :*

$$f = \sum_{i=1}^{n} a_i 1_{A_i} = f = \sum_{j=1}^{p} b_j 1_{Bj},$$

then

$$\sum_{i=1}^{n} a_i m(_{A_i}) = \sum_{j=1}^{p} b_j m(_{B_J}).$$

4. *If $A \in T$,*

$$\int_E 1_A dm = m(A).$$

Proposition 3.1 *Let f and g two measurable, and positive, step functions. Then*

$$\int_E (f+g)\, dm = \int_E f dm + \int_E g dm.$$

1. $\forall \alpha \in \mathbb{R}_+ : \int_E (\alpha f)\, dm = \alpha \int_E f dm.$
2. *If $f \geqslant g$ then $\int_E f dm \geqslant \int_E g dm$.*

Proof.

1. *We put*

$$f = \sum_{i=1}^{n} a_i 1_{A_i} \quad \text{and} \quad g = \sum_{j=1}^{p} b_j 1_{Bj}.$$

We have

$$f + g = \sum_{k=1}^{n+p} \gamma_k 1_{C_k},$$

with

$$\gamma_k = \begin{cases} a_k \ , & 1 \leqslant k \leqslant n \\ b_{k_n} \ , & n+1 \leqslant k \leqslant n+p \end{cases} \quad \text{and} \quad C_k = \begin{cases} A_k \ , & 1 \leqslant k \leqslant n \\ B_{k_n} \ , & n+1 \leqslant k \leqslant n+p. \end{cases}$$

Therefore

$$\begin{aligned} \int_E (f+g)\, dm &= \sum_{k=1}^{n+p} \gamma_k m(C_{_k}) = \sum_{k=1}^{n} \gamma_k m(C_{_k}) + \sum_{k=n+1}^{n+p} \gamma_k m(C_{_k}) \\ &= \sum_{k=1}^{n} a_k m(A_k) + \sum_{k=n+1}^{n+p} b_{k-n} m(B_{k-n}) \\ &= \sum_{k=1}^{n} a_k m(A_k) + \sum_{k=1}^{p} b_k m(B_k) \\ &= \int_E f dm + \int_E g dm. \end{aligned}$$

2. *We have*

$$\alpha f = \alpha \sum_{i=1}^{n} a_i 1_{A_i} = \sum_{i=1}^{n} (\alpha a_i) 1_{A_i}.$$

Therefore

$$\begin{aligned}\int_E (\alpha f) dm &= \sum_{i=1}^{n} (\alpha a_i) m(A_i) \\ &= \alpha \sum_{i=1}^{n} a_i m(A_i) = \alpha \int_E f dm.\end{aligned}$$

■

Proposition 3.2 *Let $f, g \in \mathcal{E}_+$ ($\mathcal{E}_+$: Set of measurable and positive step functions) such that $f \geqslant g$: we have $(f-g) \in \mathcal{E}_+$ (because $\mathbb{R}_-$ is vector space) and*

$$\int_E f dm = \int_E [(f-g)+g]\, dm = \int_E (f-g)\, dm + \int_E g dm,$$

because $\int_E (f-g)\, dm \geqslant 0$.

Remark 3.1 *If $f, g \in E$ with $f \geqslant g$ and $\int_E g dm < +\infty$, then*

$$\int_E (f-g)\, dm = \int_E f dm - \int_E g dm$$

3.2 Integral of a positive measurable function

Definition 3.2 *Let (E, T, m) a measured space and $f \in \mathcal{M}_+$ (The set of positive measurable functionsof E to $\mathbb{R}_+$). We define the integral of f by :*

$$\int_E f dm = \sup \left\{ \int_E g dm;\ g \in M_+ : g \leqslant f \right\}.$$

3.2.1 Monotone convergence theorem

Theorem 3.1 *(of the monotone convergence or de Beppo-Levi) Let (E, T, m) a measured space and $(f_n)_{n \in \mathbb{N}}$ an increasing sequence of positive measurable functions converging to a function f. Then $f \in \mathcal{M}_+$ and*

$$\int_E f dm = \lim_{n \to +\infty} \int_E f_n dm = \sup_n \int_E f_n dm.$$

Proof. *The sequence $\left(\int_E f_n dm\right)_n$ is increasing in $[0, +\infty]$, then converges to $L \in [0, +\infty]$*

$$L := \lim_{n \to +\infty} \int_E f_n dm = \sup_n \int_E f_n dm.$$

For all $n \in \mathbb{N}$ we have $f_n \leqslant f$ and by the increasing integral we obtain

$$\int_E f_n dm \leqslant \int_E f dm,$$

then by taking the supremum on $n \in \mathbb{N}$

$$L \leqslant \int_E f dm.$$

Furthermore, let $s \in \mathcal{E}_+$ *such that* $s \leqslant f$ *and let* $\alpha \in]0,1[$. *We define*

$$A_n = \{x \in E : f_n(x) \geqslant \alpha \cdot s(x), \}$$

as

$$A_n = (f_n - \alpha \cdot s)^{-1} ([0, +\infty]),$$

and the function $x \longmapsto f_n(x) - \alpha \cdot s(x)$ *is measurable, then* $A_n \in T$ *for all tout* $n \in \mathbb{N}$. *On the other hand, the sequence* $(A_n)_n$ *is increasing because if* $x \in A_n$, *we have* $\alpha \cdot s(x) \leqslant f_n(x) \leqslant f_{n+1}(x)$ *by increasing of* $(f_n)_n$, *then* $x \in A_{n+1}$ *and also we have* $E = \bigcup_{n=1}^{+\infty} A_n$. *Thanks to the definition of* A_n *we can write an inequality between positive measurable functions*

$$\alpha \cdot s \cdot \chi_{A_n} \leqslant f_n \chi_{A_n} \leqslant f_n.$$

We deduce by increasing of the intgral and $\int_A f dm = \int_A f \chi_A dm$

$$\int_{A_n} (\alpha \cdot s) \, dm \leqslant \int_{A_n} f_n dm \leqslant \int f_n dm. \tag{3.1}$$

Moreover, if $s = \sum_{i=1}^{m} b_i \cdot \chi_{B_i}$, *then* $s \cdot \chi_{A_n} = \sum_{i=1}^{m} b_i \cdot \chi_{(B_i \cap A_n)}$ *therefore we have*

$$\int_{A_n} s dm = \sum_{i=1}^{m} b_i \cdot m (B_i \cap A_n). \tag{3.2}$$

For all $i = 1, ..., m$, *the sequence* $(B_i \cap A_n)_n$ *is increasing and*

$$\bigcup_{n=1}^{+\infty} (B_i \cap A_n) = B_i \cap \left(\bigcup_{n=1}^{+\infty} A_n \right) = B_i \cap E = B_i,$$

in (3.2), we can take the limit as n *tends to* $+\infty$, *by applying the increasing continuity of measure* m

$$\lim_{n \to +\infty} \int_{A_n} s dm = \sum_{i=1}^{m} b_i \cdot \left(\lim_{n \to +\infty} m (B_i \cap A_n) \right) = \sum_{i=1}^{m} b_i \cdot m (B_i) = \int s dm.$$

Making tender n *to* $+\infty$ *in 3.1 we obtain , for all* $\alpha \in]0,1[$ *and all* $s \in \mathcal{E}_+$ *with* $s \leqslant f$, *we have*

$$\alpha \cdot \int s dm \leqslant L.$$

■

Corollary 3.1 *Let (E, T, m) a measured space and $(f_n)_{n\in\mathbb{N}} \subset \mathcal{M}_+$, then*

$$\int_E \left(\sum_{n=0}^{+\infty} f_n\right) dm = \sum_{n=0}^{+\infty} \int_E f_n dm.$$

Proof. *We apply the monotone convergence theorem to the sequence $(g_n)_{n\in\mathbb{N}}$ defined by*

$$g_n = \sum_{p=0}^{n} f_p.$$

We have $g_n \in M_+$ and

$$g_n \longrightarrow f = \sum_{p=0}^{+\infty} f_p \quad , \quad n \longrightarrow +\infty,$$

moreover

$$g_{n+1} \geqslant g_n \quad \forall n \in \mathbb{N}, \text{ then } f \in \mathcal{M}_+,$$

and

$$\lim_{n\to+\infty} \int_E g_n dm = \int_E f dm,$$

that means

$$\begin{aligned} \lim_{n\to+\infty} \int_E \left(\sum_{p=0}^{n} f_p\right) dm &= \int_E \left(\sum_{p=0}^{+\infty} f_p\right) dm \\ &\Longrightarrow \sum_{p=0}^{+\infty} \int_E f_p dm = \int_E \left(\sum_{p=0}^{+\infty} f_p\right) dm. \end{aligned}$$

■

Proposition 3.3 *Let $f, g \in M_+$ and $\alpha \in \mathbb{R}_+$. Then*

1. $$\int_E (f+g)\, dm = \int_E f dm + \int_E g dm.$$

2. $$\int_E (\alpha f)\, dm = \alpha \int_E f dm.$$

3. *If $f \geqslant g$ then*
$$\int_E f dm \geqslant \int_E g dm.$$

4. *If $m(E) = 0$, then*
$$\int_E f dm = 0.$$

Proof.

1. *Since $f, g \in \mathcal{M}_+, \exists (\theta_n)_{n\in\mathbb{N}}, (\varphi_n)_{n\in\mathbb{N}} \subset \mathcal{M}_+$ and increasing such that*

$$f = \lim_{n\to+\infty} \theta_n \text{ and } g = \lim_{n\to+\infty} \varphi_n.$$

We have

$$f+g=\lim_{n\to+\infty}(\theta_n+\varphi_n),$$

according to the Monotone Convergence Theorem

$$\begin{aligned}\int_E (f+g)\,dm &= \lim_{n\to+\infty}\int_E(\theta_n+\varphi_n)dm=\lim_{n\to+\infty}\left(\int_E\theta_n dm+\int_E\varphi_n dm\right)\\ &= \lim_{n\to+\infty}\int_E\theta_n dm+\lim_{n\to+\infty}\int_E\varphi_n dm\\ &= \int_E fdm+\int_E gdm.\end{aligned}$$

2. *We have*

$$\alpha f=\alpha\lim_{n\to+\infty}\theta_n=\lim_{n\to+\infty}(\alpha\theta_n),$$

then

$$\int_E(\alpha f)\,dm=\lim_{n\to+\infty}\int_E\alpha\theta_n dm=\alpha\lim_{n\to+\infty}\int_E\theta_n dm=\alpha\int_E fdm.$$

3. *Let $f,g\in M_+$ such that $f\geqslant g$, we have $f-g\in M_+$, according to we have :*

$$\int_E fdm=\int_E[(f-g)+g]\,dm=\int_E(f-g)\,dm+\int_E gdm\geqslant\int_E gdm,$$

because

$$\int_E(f-g)\,dm\geqslant 0.$$

Consequence : *If $f,g\in M_+$ such that $f\geqslant g$ and $\int_E gdm<+\infty$, then*

$$\int_E(f-g)\,dm=\int_E fdm-\int_E gdm.$$

4. *For any positive step function $\varphi=\sum_{i=1}^{n}a_i 1_{A_i}$ we have*

$$\int_E\varphi dm=\sum_{i=1}^{n}a_i m(A_i)=0 \text{ because } A_i\subset E \text{ and } m(E)=0,$$

hence

$$\int_E fdm=\sup\left\{\int_E gdm;\ g\in M_+:g\leqslant f\right\}=0.$$

■

Definition 3.3 *Let (E,T,m) a measured space and $f\in M_+$, $A\in T$, we define the integral of Lebesgue of f on A by*

$$\int_A fdm=\int_A f1_A dm \text{ where } (f1_A)(x)=\begin{cases} f(x) \text{ if } x\in A\\ 0 \text{ , otherwise}\end{cases},f1_A\in M_+,$$

if $m(A)=0$, then

$$\int_A fdm=0.$$

Proposition 3.4 *Let (E,T,m) a measured space, $A,B \in T$ such that $A \cap B = \varnothing$ and $f,g \in M_+$. Then :*

1. $$\int_{A\cup B} fdm = \int_A fdm + \int_B fdm.$$

2. *If $f = g$ a.e then*
$$\int_E fdm = \int_E gdm.$$

Proof.

1. *We have :*
$$\begin{aligned}\int_{A\cup B} fdm &= \int_E f1_{A\cup B}dm = \int_E (f1_A + f1_B)\,dm, \text{ (because } 1_{A\cup B} = 1_A + 1_B\text{)}\\ &= \int_E f1_A dm + \int_E f1_B dm = \int_A fdm + \int_B fdm.\end{aligned}$$

2. *Let $A \in T$ such that $m(A) = 0$ and $f1_{A^c} = g1_{A^c}$ (i.e : $f(x) = g(x)$ on A), we have*
$$f1_{A^c} \text{ and } g1_{A^c} \in M_+ \text{ and } \int_E f1_{A^c}dm = \int_E g1_{A^c}dm.$$
Moreover
$$\int_E fdm = \int_A fdm + \int_{A^c} fdm = \int_E f1_{A^c}dm,$$
$$\begin{aligned}\int_E gdm &= \int_A gdm + \int_{A^c} gdm = \int_E g1_{A^c}dm\\ &= \int_E f1_{A^c}dm = \int_E fdm.\end{aligned}$$
Consequence : *If $f \in \mathcal{M}_+$ such that $f = 0$ a.e then :*
$$\int_E fdm = 0.$$

■

3.2.2 Fatou's Lemma

Lemma 3.1 *(Fatou's Lemma) Let (E,T,m) a measured space and $(f_n)_{n\in\mathbb{N}} \subset \mathcal{M}_+$ then*
$$\int_E \underline{\lim}_n f_n dm \leqslant \underline{\lim}_n \int_E f_n dm = \underline{\lim}_n \left(\inf_{p \geqslant n} \int_E f_p dm \right).$$

Proof. *We have*
$$\underline{\lim}_n f_n = \lim_n \inf_{p\geqslant n} f_p,$$

we put $g_n = \inf_{p \geqslant n} f_p$ *then* $(g_n)_{n\in\mathbb{N}} \subset \mathcal{M}_+$*, and* $(g_n)_{n\in\mathbb{N}} \longrightarrow \underline{\lim}_n f_n$ *,* $n \longrightarrow \infty$*, according to the monotone convergence theorem*

$$\underline{\lim}_n f_n \in \mathcal{M}_+ \text{ and } \lim_{n\to\infty} \int_E g_n dm = \int_E \lim_{n\to\infty} g_n dm,$$

that means

$$\lim_{n\to\infty} \int_E \inf_{p\geqslant n} f_p = \int_E \underline{\lim}_n f_n dm.$$

On the other hand for all $p \geqslant n$*, we have* $g_n \leqslant f_n$ *therefore* $p \geqslant n$

$$\int_E g_n dm \leqslant \int_E f_p dm \Longrightarrow \int_E g_n dm \leqslant \inf_{p\geqslant n} \int_E f_p dm.$$

Taking the limit when $n \longrightarrow +\infty$*, it results that :*

$$\begin{aligned} \lim_{n\to+\infty} \int_E g_n dm &= \int_E \underline{\lim}_n f_n dm \leqslant \lim_{n\to\infty} \inf_{p\geqslant n} \int_E f_p dm \\ &= \underline{\lim}_n \int_E f_n dm. \end{aligned}$$

■

3.3 Density Measure and Probabilities

3.3.1 Density Measure

Definition 3.4 *Let* (E, T, m) *a measured space and* $f \in \mathcal{M}_+$*,* $\int_A f dm = \int f 1_A dm$*, we define* $\mu : T \longrightarrow \overline{\mathbb{R}_+}$ *by*

$$\mu(A) = \int f 1_A dm = \int_A f dm \ , \forall A \in T,$$

μ *is a measure on* T *called* ***density measure*** *of* f *with respect to* m *and denoted by* $\mu = fm$.

Example .23 *"Density Probability"*

Definition 3.5 *Let* P *aune probability on* $B(\mathbb{R})$*.* P *is a density probability (with respect to Lebesgue) if*

$$\exists f \in \mathcal{M}_+ \text{ such that } \int f d\lambda = 1 \text{ and } P(A) = \int f 1_A d\lambda = \int_A f d\lambda \ , \forall A \in B(\mathbb{R}),$$

Reminder : A probability law is by definition a probability on $B(\mathbb{R})$.

<u>Uniform Law $U(a, b)$</u> : Let $a, b \in \mathbb{R}$, $a < b$: $\dfrac{1}{a-b} 1_{[a,b]}$

$$P(A) = \frac{1}{a-b} \int 1_{[a,b]} 1_A d\lambda, \ \forall A \in T.$$

Exponential law $\varepsilon(\tau)$: Let $\tau > 0$, the exponential distribution is defined by the density f defined by :

$$f(x) = \begin{cases} 0 \ , \ \text{if} \ x < 0 \\ \tau e^{-\tau x} \ , \ \text{if} \ x \geqslant 0. \end{cases}$$

Gaussian distribution (Law of Guauss) $\mathcal{N}(\mu, \sigma^2)$: Let $(\mu, \sigma) \in \mathbb{R} \times \mathbb{R}_+^*$. The law of Gauss of the parameter (μ, σ) is defined by the density f

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

3.4 The space L^1 of integrable functions

Remark 3.2 *If $f \in \mathcal{M}$ (The set of measurable functions), $|f|$, f^+, $f^- \in \mathcal{M}$. The monotony of the integral on $\mathcal{M}_+$ gives*

$$\int_E f^+ dm \leqslant \int_E |f|\, dm \ \text{and} \ \int_E f^- dm \leqslant \int_E |f|\, dm,$$

with

$$f^+(x) = \max(f(x), 0) \ \text{and} \ f^-(x) = \min(f(x), 0) = \max(-f(x), 0), \ (|f|(x) = |f(x)|\,).$$

Definition 3.6 *Let (E, T, m) a measured space and $f : E \longrightarrow \overline{\mathbb{R}}$ measurable. We say that f is integrable in the Lebesgue sense if*

$$\int_E |f|\, dm < +\infty.$$

In the case

$$\int_E f^+ dm < +\infty \quad \text{and} \quad \int_E f^- dm < +\infty,$$

we put then

$$\int_E |f|\, dm = \int_E f^+ dm - \int_E f^- dm \in \mathbb{R}.$$

We denote by $\mathcal{L}^1$ the set of integrable functions.

Proposition 3.5 *f is integrable if and only if f^+ and f^- are integrable.*
Proof. *"$\Longrightarrow$" if f is integrable , we have $\int_E |f|\, dm < +\infty$*

$$0 \leqslant f^+ \leqslant |f| \Longrightarrow 0 \leqslant \int_E f^+ dm \leqslant \int_E |f|\, dm < +\infty,$$

$$0 \leqslant f^- \leqslant |f| \Longrightarrow 0 \leqslant \int_E f^- dm \leqslant \int_E |f|\, dm < +\infty,$$

hence f^+ and f^- are integrable.

"$\Longleftarrow$" if f^+ and f^- are integrable, then

$$\int_E f^+ dm < +\infty \quad \text{and} \quad \int_E f^- dm < +\infty,$$

then

$$\int_E |f|\, dm = \int_E f^+ dm + \int_E f^- dm < +\infty,$$

therefore we get :

$$f \in \mathcal{L}^1 \Leftrightarrow \int_E f^+ dm < +\infty \quad \text{and} \quad \int_E f^- dm < +\infty.$$

■

Example .24 *1.* $\theta_n = \dfrac{1}{2n} \cdot 1_{[0,n]}$, $n \in \mathbb{N}^*$ *is integrable because*

$$\int_{\mathbb{R}} \theta_n d\lambda = \frac{1}{2n}\lambda([0,n]) = \frac{1}{2} \quad , \forall n \in \mathbb{N}^*.$$

2. $\theta = 1_{[0,+\infty[}$ *is non integrable because*

$$\int_{\mathbb{R}} \theta d\lambda = \int_{\mathbb{R}} 1_{[0,+\infty[} d\lambda = \lambda(_{[0,+\infty[}) = +\infty.$$

3.4.1 The L^1 space (Complex case)

If $f : E \longrightarrow \mathbb{C}$ a measurable function , then $(\mathrm{Re}(f), \mathrm{Im}(f)$ both are measurable), we say that f is integrable and we denote $f \in \mathcal{L}^1_{\mathbb{C}}(E, T, m)$ if

$$\int_E |f|\, dm = \int_E \sqrt{(\mathrm{Re}\, f)^2 + (\mathrm{Im}\, f)^2} dm < \infty,$$

then we put

$$\int_E f dm = \int_E \mathrm{Re}(f) dm + i \int_E \mathrm{Im}(f) dm.$$

Proposition 3.6 *Let* (E, T, m) *a measured space. Then*

1. $\mathcal{L}^1$ *is a* $\mathbb{R}$ *vector space .*
2. *The application :* $f \longmapsto \int_E f dm$ *is linear from* $\mathcal{L}^1$ *to* $\mathbb{R}$.
3. *Let* $f, g \in \mathcal{L}^1$ *such that* $f \leqslant g$ *then* $\int_E f dm \leqslant \int_E g dm$.
4. $\forall f \in \mathcal{L}^1 : \left|\int_E f dm\right| \leqslant \int_E |f|\, dm$.

Proof.

1. Let $f, g \in \mathcal{L}^1$ and $\alpha, \beta \in \mathbb{R}$. We have

$$\int_E |\alpha f + \beta g|\, dm \leqslant |\alpha| \int_E |f|\, dm + |\beta| \int_E |g|\, dm < +\infty,$$

   therefore $(\alpha f + \beta g) \in \mathcal{L}^1$.

2. We begin with linearity **:** Let $f, g \in \mathcal{L}^1$ and $\alpha \in \mathbb{R}$. We show that

   i) $\int_E (\alpha f)\, dm = \alpha \int_E f dm$.

ii) $\int_E (f+g)\,dm = \int_E f dm + \int_E g dm.$

i) We have :

$$(\alpha f)^+ = \begin{cases} \alpha f^+ & \text{if } \alpha \geqslant 0 \\ -\alpha f^- & \text{if } \alpha < 0 \end{cases}, \quad (\alpha f)^- = \begin{cases} \alpha f^- & \text{if } \alpha \geqslant 0 \\ -\alpha f^+ & \text{if } \alpha < 0, \end{cases}$$

therefore

$$\begin{aligned} \int_E (\alpha f)\,dm &= \int_E (\alpha f)^+\,dm - \int_E (\alpha f)^-\,dm \\ &= \begin{cases} \int_E \alpha f^+ dm - \int_E \alpha f^- dm & \text{if } \alpha \geqslant 0 \\ \int_E (-\alpha f^-)\,dm - \int_E (-\alpha f^+)\,dm & \text{if } \alpha < 0 \end{cases} \\ &= \begin{cases} \alpha \left(\int_E f^+ dm - \int_E f^- dm\right) & \text{if } \alpha \geqslant 0 \\ -\alpha \left(\int_E f^- dm - \int_E f^+ dm\right) & \text{if } \alpha < 0 \end{cases} \\ &= \begin{cases} \alpha \int_E f dm & \text{if } \alpha \geqslant 0 \\ \alpha \int_E f dm & \text{if } \alpha < 0. \end{cases} \end{aligned}$$

ii) We have

$$(f+g)^+ - (f+g)^- = f^+ - f^- + g^+ - g^-$$

$$\Longrightarrow (f+g)^+ + f^- + g^- = (f+g)^- + f^+ + g^+$$

$$\Longrightarrow \int_E (f+g)^+\,dm + \int_E f^- dm + \int_E g^- dm = \int_E (f+g)^-\,dm + \int_E f^+ dm + \int_E g^+ dm$$

$$\Longrightarrow \int_E (f+g)^+\,dm - \int_E (f+g)^-\,dm = \int_E f^+ dm - \int_E f^- dm + \int_E g^+ dm - \int_E g^- dm$$

$$\Longrightarrow \int_E (f+g)\,dm = \int_E f dm + \int_E g dm.$$

3. Let $f, g \in \mathcal{L}^1$ such that $f \leqslant g$

$$\Longrightarrow f^+ - f^- \leqslant g^+ - g^-$$

$$\begin{aligned} &\Longrightarrow f^+ + g^- \leqslant f^- + g^+ \\ &\Longrightarrow \int_E \left(f^+ + g^-\right) dm \leqslant \int_E \left(f^- + g^+\right) dm \\ &\Longrightarrow \int_E f^+ dm + \int_E g^- dm \leqslant \int_E f^- dm + \int_E g^+ dm \\ &\Longrightarrow \int_E f^+ dm - \int_E f^- dm \leqslant + \int_E g^+ dm - \int_E g^- dm \\ &\Longrightarrow \int_E f dm \leqslant \int_E g dm. \end{aligned}$$

4. Let $f \in \mathcal{L}^1$, we have

$$\begin{aligned} \left| \int_E f dm \right| &= \left| \int_E f^+ dm - \int_E f^- dm \right| \\ &\leqslant \int_E f^+ dm + \int_E f^- dm \\ &= \int_E f^+ dm - \int_E |f|\,dm. \end{aligned}$$

■

Consequence : If $f, g \in \mathcal{L}^1$ such that $f = g$ a.e then $\int_E f dm = \int_E g dm$

Proof. We have

$$\begin{aligned}\left|\int_E f dm - \int_E g dm\right| &= \left|\int_E (f-g)\, dm\right| \\ &\leqslant \int_E |f-g|\, dm = 0,\end{aligned}$$

because

$$|f-g| = 0 \text{ a.e (if } f \in M_+,\ f = 0 \text{ a.e} \Longrightarrow \int_E f dm = 0,$$

hence

$$\int_E f dm = \int_E g dm.$$

■

3.4.2 Dominated convergence theorem

Let (E, T, m) a measured space and $(f_n)_{n\in\mathbb{N}}$ a sequence of integrable functions converging to a function$f$ almost everywhere and $g \in \mathcal{L}^1$ such that

$$\forall n \in \mathbb{N}, \quad |f_n| \leqslant g \text{ a.e,}$$

then

$$f \in \mathcal{L}^1 \text{ and } \lim_{n\longrightarrow\infty} \int_E f_n dm = \int_E f dm.$$

Proposition 3.7 *Let (E, T, m) a measured space, $A \in T$ such that $m(A) < \infty$. If $(f_n)_{n\in\mathbb{N}}$ is a sequence of bounded integrable functions converging to f, then*

$$\lim_{n\longrightarrow\infty} \int_A f_n dm = \int_A f dm \text{ and } \lim_{n\longrightarrow\infty} \int_E |f_n - f|\, dm = 0.$$

Proposition 3.8 *"Corollary" Let (E, T, m) a measured space, $g \in \mathcal{L}^1, (f_n)_{n\in\mathbb{N}} \subset \mathcal{L}^1$. If $\sum_{n=0}^{+\infty} f_n$ is convergent such that*

$$\forall n \in \mathbb{N}, \quad \left|\sum_{p=0}^{n} f_p\right| \leqslant g,$$

then

$$\int_E \left(\sum_{n=0}^{+\infty} f_n\right) dm = \sum_{n=0}^{+\infty} \int_E f_n dm.$$

3.4.3 Applications of the Dominated Convergence Theorem

Integrals depending on a parameter

Let (E, T, m) a measured space and $f : E \times \mathbb{R} \longrightarrow \mathbb{R}$ a function. For $t \in \mathbb{R}$ fixed, we define the application

$$f(.,t) \quad \begin{array}{ccc} E & \longrightarrow & \mathbb{R} \\ x & \longmapsto & f(x,t) \end{array}$$

and we suppose that $f(.,t) \in \mathcal{L}^1, \forall t \in \mathbb{R}$, we denote F the application from $\mathbb{R}$ to $\mathbb{R}$ by

$$F(t) = \int_E f(.,t)dm = \int_E f(x,t)dm(x).$$

Theorem 3.2 *(Continuity under the integral) : If*

1. $\forall t \in \mathbb{R}$, $|f(x,t)| \leqslant g(x)$ *a.e where* $g \in \mathcal{L}^1$
2. *The application* $t \longmapsto f(x,t)$ *is continuous in* t_0 *for almost everywhere* $x \in E$*. Then* F *is continuous in* t_0

Theorem 3.3 *(Differentiability under the integral) : If* $t \longmapsto f(x,t)$ *is differentiable a.e and that on this set*

$$\left|\frac{\partial}{\partial t}\left(f(x,t)\right)\right| \leqslant g(x) \quad \textit{where } g \in \mathcal{L}^1.$$

Then F *is differentiable on* $\mathbb{R}$ *an*

$$F'(t) = \int_E \frac{\partial}{\partial t}\left(f(x,t)\right) dm(x).$$

3.5 Comparison between the Lebesgue integral and the Riemann integral

3.5.1 Riemann integrability

Let $f : [a,b] \longrightarrow \mathbb{R}$ a bounded function and let $x_0, x_1, ..., x_n$ a finite set of points such that $a = x_0 <, x_1 <, ..., x_n = b$. In each interval $[x_k, x_{k+1}]$ we chooses ζ_k and we put

$$S_n = \sum_{k=0}^{n-1} y_k \left(x_{k+1} - x_k\right) \quad \text{such that} \quad y_k = f(x_k).$$

If $S_n \underset{n\longrightarrow\infty}{\longrightarrow} S$ (independent of the choice of x_k) then S is called the Riemann integral on $[a,b]$ and denoted $\int\limits_a^b f(x)dx$.

Lebesgue's idea : to approximate the area under the graph of the function by a union of rectangles (like Riemann's method).

In Riemann's method, we are interested in the variation of the function over its domain of definition.

In Lebesgue's method, we define the rectangle in terms of the function's values.

Definition 3.7 *Apart A of $[a,b]$ is Riemannian measurable if 1_A is est Riemannian integrable.*

Example .25 $1_{\mathbb{Q}\cap[0,1]}$ *is not Riemannian integrable on $[0,1]$ then $\mathbb{Q}\cap[0,1]$ is not Riemann measurable.*

Proposition 3.9 *Let $f:[a,b]\longrightarrow\mathbb{R}$ a function Riemannian integrable. Then f is measurable for the tribe $\widetilde{B}\left([a,b]\right)$.*

Proposition 3.10 *Let $f:[a,b]\longrightarrow\mathbb{R}$ a bounded function . Then f is Riemannian integrable if and only if f is continuous a.e (the set of discontinuities of f is negligible for the Lebesgue measure).*

Proposition 3.11 *Let $f:[a,b]\longrightarrow\mathbb{R}$ a bounded function defined on the bounded interval $[a,b]$ $(b>a)$. If f is Riemannian integrable, then f is Lebesgue integrable.*

$$\int_{[a,b]} f d\lambda = \int_a^b f(x)dx.$$

Remark 3.3 *The converse is generally false.*

Example .26 *The Dirichlet function defined on $[0,1]$ by*

$$f(x)=\begin{cases} 0 & if\ \ x\in\mathbb{Q} \\ 1 & if\ \ x\notin\mathbb{Q} \end{cases}$$

$f=1$ a.e , f is Lebesgue integrable and its integral is equal to 1, so its Riemannian integral is undefined.

Proposition 3.12 *Let $f:\,]a,b[\,\longrightarrow\mathbb{R}$ $(-\infty\leqslant a<b\leqslant+\infty)$ a measurable function such that f is bounded on any compact $[\alpha,\beta]\subset\,]a,b[$ and integrable Riemannian. If the generalized Riemannian $\int_a^b f(x)dx$ is absolutely convergent, then f is Lebesgue integrable on $]a,b[$ and then on $[a,b]$ and we have*

$$\int_{[a,b]} f d\lambda = \int_{]a,b[} f d\lambda = \int_a^b f(x)dx = \lim_{\substack{d\longrightarrow b \\ c\longrightarrow a}} \int_c^d f(x)dx.$$

Conséequence *: If $f:\,]a,b[\,\longrightarrow\mathbb{R}$ is continuous and has a generalized Riemann integral $\int_a^b f(x)dx$ absolutely convergent, then f is Lebesgue integrable on $]a,b[$ and*

$$\int_{]a,b[} f d\lambda = \int_a^b f(x)dx.$$

Remark 3.4 *If the generalized Riemannian integral* $\int\limits_a^b f(x)dx$ *is not absolutely convergent, then*

$$\int_{]a,b[} |f|\, d\lambda = +\infty$$

therefore f *is not Lebesgue integrable.*

Example .27 *The function* $x \longmapsto \dfrac{\sin x}{x}$ *is not Lebesgue integrable on* $\mathbb{R}_+$ *because*

$$\lim_{A \longrightarrow \infty} \int\limits_0^A \left|\frac{\sin x}{x}\right| dx = +\infty.$$

3.6 The L^p spaces $(1 \leqslant p \leqslant +\infty)$

3.6.1 The L^p spaces $(1 \leqslant p < +\infty)$

Definition 3.8 *Let* (E, T, m) *a measured space ,* $1 \leqslant p < +\infty$ *and* $f :]a, b[\longrightarrow \mathbb{R}$ *a measurable function. We say that*

$$f \in \mathcal{L}^p_{\mathbb{R}}(E, T, m) = \mathcal{L}^p$$

if

$$\int_E |f|^p\, dm < +\infty.$$

Therefore $f \notin \mathcal{L}^p$ *if*

$$\int_E |f|^p\, dm = +\infty.$$

Proposition 3.13 *Let* (E, T, m) *a measured space and* $1 \leqslant p < +\infty$*, then* $\mathcal{L}^p$ *is a* $\mathbb{R}$*-vector space .*

Proof.

i) *Let* $\alpha \in \mathbb{R}$ *and* $f \in \mathcal{L}^p$*. We have* $\alpha f \in \mathcal{M}$ *and*

$$\int_E |\alpha f|^p\, dm = |\alpha|^p \int_E |f|^p\, dm < +\infty,$$

therefore $\alpha f \in \mathcal{L}^p$

ii) *Let* $f, g \in \mathcal{L}^p$*, we have* $f + g \in M$ *and*

$$\begin{aligned} |f+g|^p &\leqslant [|f| + |g|]^p \leqslant [2\sup(|f|, |g|)]^p \\ &\leqslant 2^p \left[|f|^p + |g|^p\right], \end{aligned}$$

therefore

$$\begin{aligned} \int_E |f+g|^p\, dm &\leqslant 2^p \int_E |f|^p\, dm + 2^p \int_E |g|^p\, dm < +\infty \\ &\Longrightarrow (f+g) \in \mathcal{L}^p. \end{aligned}$$

■

3.6.2 Young's inequality

Let $a, b \in \mathbb{R}_+$ and $p, q \in]1, +\infty[$ such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$ (p and q are said to be conjugates). Then

$$ab \leqslant \frac{a^p}{p} + \frac{b^q}{q}.$$

3.6.3 Hölder's inequality

Let (E, T, m) a measured space and $p, q \in]1, +\infty[$ such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$ and let $f, g \in \mathcal{L}^p$. Then $fg \in \mathcal{L}^1$ and

$$\int_E |fg|\, dm \leqslant \left(\int_E |f|^p\, dm\right)^{1/p} \left(\int_E |g|^q\, dm\right)^{1/q}.$$

i) We show that $fg \in \mathcal{L}^1$. We have $fg \in \mathcal{M}$ and according to the inequality of Young, we have

$$\begin{aligned} |fg| &\leqslant \frac{|f|^p}{p} + \frac{|g|^q}{q} \\ &\Longrightarrow \int_E |fg|\, dm \leqslant \frac{1}{p}\int_E |f|^p\, dm + \frac{1}{q}\int_E |g|^q\, dm < +\infty, \end{aligned}$$

therefore $fg \in \mathcal{L}^1$.

ii) We show that the inequality of Holder

a) If $\int_E |f|^p\, dm = 0$ or $\int_E |g|^q\, dm = 0$ (i.e : $f = 0$ a.e or $g = 0$ a.e)

$$fg = 0 \text{ a.e and } \int_E |fg|\, dm = 0.$$

b) If $\int_E |f|^p\, dm \neq 0$ and $\int_E |g|^q\, dm \neq 0$. We put

$$\widetilde{f} = \frac{f}{\left(\int_E |f|^p\, dm\right)^{1/p}} \text{ and } \widetilde{g} = \frac{g}{\left(\int_E |g|^q\, dm\right)^{1/q}},$$

we have

$$\begin{aligned} \left|\widetilde{f}\widetilde{g}\right| &\leqslant \frac{1}{p}\left|\widetilde{f}\right|^p + \frac{1}{q}\left|\widetilde{g}\right|^q \\ &\Longrightarrow \frac{|fg|}{\left(\int_E |f|^p\, dm\right)^{1/p}\left(\int_E |g|^q\, dm\right)^{1/q}} \leqslant \frac{1}{p}\frac{|f|^p}{\int_E |f|^p\, dm} + \frac{1}{q}\frac{|g|^q}{\int_E |g|^q\, dm}, \end{aligned}$$

taking the integral, we obtain :

$$\begin{aligned} \frac{\int_E |fg|\, dm}{\left(\int_E |f|^p\, dm\right)^{1/p}\left(\int_E |g|^q\, dm\right)^{1/q}} &\leqslant \frac{1}{p}\frac{\int_E |f|^p\, dm}{\int_E |f|^p\, dm} + \frac{1}{q}\frac{\int_E |g|^q\, dm}{\int_E |g|^q\, dm} \\ &\leqslant \frac{1}{p} + \frac{1}{q} = 1 \end{aligned}$$

$$\Longrightarrow \int_E |fg|\, dm \leqslant \left(\int_E |f|^p\, dm\right)^{1/p} \left(\int_E |g|^q\, dm\right)^{1/q}.$$

3.6.4 Inequality of Minkowski

Let (E, T, m) a measured space and $1 \leqslant p < +\infty$ and let $f, g \in \mathcal{L}^p$. Then $(f+g) \in \mathcal{L}^p$ and

$$\left(\int_E |f+g|^p \, dm\right)^{1/p} \leqslant \left(\int_E |f|^p \, dm\right)^{1/p} + \left(\int_E |g|^p \, dm\right)^{1/p}.$$

Proof. Let q the conjugate exotrope of p (i.e : $\frac{1}{p} + \frac{1}{q} = 1$).

We have

$$|f+g|^{p-1} \in \mathcal{L}^q \text{ because } \left(|f+g|^{p-1}\right)^q = |f+g|^{(p-1)q} = |f+g|^p.$$

By using the inequality of Holder, we obtain

$$\int_E |f| \, |f+g|^{p-1} \, dm \leqslant \left(\int_E |f|^p \, dm\right)^{1/p} \left(\int_E |f+g|^p \, dm\right)^{1/q}$$

$$\int_E |g| \, |f+g|^{p-1} \, dm \leqslant \left(\int_E |g|^p \, dm\right)^{1/p} \left(\int_E |f+g|^p \, dm\right)^{1/q}$$

therefore

$$\int_E |f+g|^p \, dm = \int_E |f+g| \, |f+g|^{p-1} \, dm \leqslant \int_E |f| \, |f+g|^{p-1} \, dm + \int_E |g| \, |f+g|^{p-1} \, dm,$$

on the other hand

$$\begin{aligned} \int_E |f+g|^p \, dm &\leqslant \left[\left(\int_E |f|^p \, dm\right)^{1/p} + \left(\int_E |g|^p \, dm\right)^{1/p}\right] \left(\int_E |f+g|^p \, dm\right)^{1/q} \\ &\Longrightarrow \left(\int_E |f+g|^p \, dm\right)^{1/p} \leqslant \left(\int_E |f|^p \, dm\right)^{1/p} + \left(\int_E |g|^p \, dm\right)^{1/p} \\ \text{since } \frac{1}{q} &= 1 - \frac{1}{p} \end{aligned}$$

■

Proposition 3.14 *Let (E, T, m) a measured space and $1 \leqslant p < +\infty$. Then the application :*

$$f \longmapsto \|f\|_p = \left(\int_E |f|^p \, dm\right)^{1/p}$$

is a semi-norm on $\mathcal{L}^p$.

Proof. *We have*

i) $\|f\|_p \in \mathbb{R}_+, \forall f \in \mathcal{L}^p$ *and*

$$\|f\|_p = 0 \Longleftrightarrow f = 0 \ a.e.$$

ii) *We have $\forall \alpha \in \mathbb{R}, \forall f \in \mathcal{L}^p$*

$$\begin{aligned} \|\alpha f\|_p &= \left(\int_E |\alpha f|^p \, dm\right)^{1/p} = |\alpha| \left(\int_E |f|^p \, dm\right)^{1/p} \\ &= |\alpha| \, \|f\|_p . \end{aligned}$$

iii) *We have* $\forall f, g \in \mathcal{L}^p$:

$$\|f+g\|_p \leqslant \|f\|_p + \|g\|_p \ \ \textit{(Inaquality of Minkowski).}$$

■

Definition 3.9 *Let* (E, T, m) *a measured space and* $1 \leqslant p < +\infty$. *We define the space* $L^p(E, T, m) = L^p$ *as the set of equivalence classes of functions* $\mathcal{L}^p$ *for the relation equivalence "$=$ a.e". Therefore*

$$L^p = \left\{ \mathring{f} / f \in \mathcal{L}^p \right\} \ \textit{with } \mathring{f} = \{ g \in \mathcal{L}^p / g = f \ \textit{a.e}\} .$$

If $F \in L^p$ *and* $f \in F$. *We put* $F = \mathring{f}$

$$\|F\|_p = \|f\|_p = \left(\int_E |f|^p \, dm \right)^{1/p} .$$

Remark 3.5 *All the previous inequalities remain true for the space* L^p.

Proposition 3.15 *The application :* $f \longmapsto \|f\|_p$ *is a norm on* L^p.

Theorem 3.4 *(Theorem of Fischer-Riesz)* $\left(L^p, \|.\|_p \right)$ *is a Banach space (i.e : a complete normed vector space).*

3.7 The space $\mathbf{L}^\infty$

Definition 3.10 *Let* (E, T, m) *a measured space and* $f : E \longrightarrow \mathbb{R}$ *measurable. We say that* f *is essentially bounded and we denote*

$$f \in \mathcal{L}^\infty(E, T, m) = \mathcal{L}^\infty,$$

if there exists $c \in \mathbb{R}_+$ *such that* $|f| \leqslant c$ *a.e.*

Proposition 3.16 *Let* (E, T, m) *a measured space . Then* $\mathcal{L}^\infty$ *is a* $-\mathbb{R}$ *vector space .*
Proof.

i) *Let* $\alpha \in \mathbb{R}$ *and* $f \in \mathcal{L}^\infty$ *then* $\exists c \in \mathbb{R}_+$ *such that* $|f| \leqslant c$ *a.e , we have* $\alpha f \in \mathcal{M}$ *and* $|\alpha f| \leqslant |\alpha| \, |f| \leqslant |\alpha| \, c$ *a.e hence* $\alpha f \in \mathcal{L}^\infty$.

ii) *Let* $f, g \in \mathcal{L}^\infty$.

$f \in \mathcal{L}^\infty \Longrightarrow \exists c \in \mathbb{R}_+$ *such that* $|f| \leqslant c$ *a.e , therefore* $\exists A \in T : m(A) = 0$ *and* $|f| \leqslant c$ *on* A^C.

$g \in \mathcal{L}^\infty \Longrightarrow \exists k \in \mathbb{R}_+$ *such that* $|g| \leqslant k$ *a.e , therefore* $\exists B \in T : m(B) = 0$ *and* $|g| \leqslant k$ *on* B^C.

We have $(f+g) \in \mathcal{M}$ *and* $\exists D = A \cup B \in T : m(D) = 0$ *and*

$$|f+g| \leqslant |f| + |g| \leqslant c + k + D^C,$$

hence

$$|f+g| \leqslant c + k \ \ \textit{a.e and } (f+g) \in \mathcal{L}^\infty.$$

■

Proposition 3.17 *If $f \in \mathcal{L}^\infty$, we put*

$$\|f\|_\infty = \inf\{c \in \mathbb{R}_+ : |f| \leqslant c \quad a.e\} \text{ and we have } |f| \leqslant \|f\|_\infty \ a.e.$$

Proof. *We suppose that $f \in \mathcal{L}^\infty$. By definition of the infimum. We have*

$$\exists (C_n)_{n\in\mathbb{N}} \subset \{c \in \mathbb{R}_+ : |f| \leqslant c \quad a.e\}$$

such that $C_n \longrightarrow \|f\|_\infty$ when $n \longrightarrow \infty$, therefore for all $n \in \mathbb{N}, \exists A_n \in T$ such that $m(A_n) = 0$ and $|f| \leqslant c_n$ on A_n^C, we put $A = \bigcup\limits_{n\in\mathbb{N}} A_n$, than we have $m(A) = 0$ and $|f| \leqslant c_n \ \forall n \in \mathbb{N}$ on A^C. as $C_n \longrightarrow \|f\|_\infty$ when $n \longrightarrow \infty$. We deduce that

$$|f| \leqslant \|f\|_\infty \text{ on } A^C \quad \text{hence } |f| \leqslant \|f\|_\infty \ a.e.$$

■

Proposition 3.18 *Let (E, T, m) a measured space. Then the application $f \longmapsto \|f\|_p$ is a semi-norm on $\mathcal{L}^\infty$.*
Proof. *We have*

i) $\|f\|_\infty, \forall \ f \in \mathcal{L}^\infty$

ii) *We have*

$$\begin{aligned} \|f\|_\infty &= 0 \Longleftrightarrow \inf\{c \in \mathbb{R}_+ : |f| \leqslant c \quad a.e\} = 0 \\ &\Longleftrightarrow |f| \leqslant \|f\|_\infty = 0 \ a.e \\ &\Longleftrightarrow f = 0 \ a.e\ . \end{aligned}$$

iii) *If $\alpha \in \mathbb{R}$ and $f \in \mathcal{L}^\infty$, then $\alpha f \in \mathcal{L}^\infty$ and*

$$\begin{aligned} \|\alpha f\| &= \inf\{c \in \mathbb{R}_+ : |\alpha f| \leqslant c \quad a.e\} \\ &= \inf\{c \in \mathbb{R}_+ : |\alpha|\,|f| \leqslant c \quad a.e\} \\ &= |\alpha|\,\|f\|_\infty\,. \end{aligned}$$

iv) *If $f, g \in \mathcal{L}^\infty$, we have*

$$\begin{aligned} (f+g) &\in \mathcal{L}^\infty \text{ and } |f+g| \leqslant |f| + |g| \leqslant \|f\|_\infty + \|g\|_\infty \ a.e \\ &\Longrightarrow \|f+g\|_\infty \leqslant \|f\|_\infty + \|g\|_\infty\,. \end{aligned}$$

■

Definition 3.11 *Let (E, T, m) a measured space . We define the space $L^\infty_{\mathbb{R}}(E, T, m) = L^\infty$, as the set of equivalence classes of $\mathcal{L}^\infty$ functions for the relation "$=$ a.e ". Therefore*

$$L^\infty = \left\{\mathring{f}/f \in \mathcal{L}^\infty\right\} \text{ with } \mathring{f} = \{g \in \mathcal{L}^\infty : g = f \ a.e\}$$

If $F \in L^\infty$ and $f \in F$. We put $F = \mathring{f}$

$$\|F\|_\infty = \|f\|_\infty = \inf\{c \in \mathbb{R}_+ : |f| \leqslant c \quad a.e\}\,.$$

Proposition 3.19 *The application :* $f \longmapsto \|f\|_p$ *is a norm on* L^∞.

Theorem 3.5 $(L^\infty, \|.\|_\infty)$ *is a Banach space .*

Proposition 3.20 *"Comparison of* L^p $(1 \leqslant p \leqslant +\infty)$ *Let* (E, T, m) *a finite measured space* $(m(E) < +\infty)$ *and* $p, q \in \mathbb{R}_+$ *such that* $1 \leqslant p \leqslant q \leqslant +\infty$. *Then* $L^q \subset L^p$.
Proof.

- *If* $q = +\infty$ *and* $1 \leqslant p < +\infty$. *We have* $|f|^p \leqslant \|f\|_\infty^p$ *, therefore*

$$\int_E |f|^p \, dm \leqslant \int_E \|f\|_\infty^p \, dm \leqslant \|f\|_\infty^p \, m(E) < +\infty,$$

hence

$$f \in L^p \text{ and } \|f\|_p \leqslant m(E)^{1/p} \cdot \|f\|_\infty .$$

- *We suppose that* $q < +\infty$ *and let* $1 \leqslant p \leqslant q$, $f \in L^q$, *then*

$$\begin{aligned} \int_E |f|^q \, dm &< +\infty \Longrightarrow \int_E (|f|^p)^{q/p} \, dm < +\infty \\ &\Longrightarrow |f|^p \in L^{q/p}. \end{aligned}$$

We apply Holder's inequality to the functions $|f|^p$ *and* 1 *with the conjugate exponents* $\alpha = \dfrac{q}{p}$ *and* $\beta = \dfrac{\alpha}{\alpha - 1} = \left(1 - \dfrac{p}{q}\right)^{-1}$ *, we obtain*

$$\begin{aligned} \int_E |f|^p \, dm &\leqslant \left(\int_E (|f|^p)^{q/p} \, dm \right)^{p/q} \left(\int_E 1 dm \right)^{1-\frac{p}{q}} \\ &= \|f\|_q^p \, m(E)^{1-\frac{p}{q}} < +\infty, \end{aligned}$$

therefore

$$f \in L^p \text{ and } \|f\|_p \leqslant m(E)^{\left(\frac{1}{p} - \frac{1}{q}\right)} \cdot \|f\|_q .$$

■

Remark 3.6 1. *If* (E, T, m) *is a probability space* $(m(E) = 1)$ *and if* $1 \leqslant p \leqslant q \leqslant +\infty$, *then*

$$L^\infty \subset L^q \subset L^p \subset L^1,$$

and

$$\|f\|_1 \leqslant \|f\|_p \leqslant \|f\|_q \leqslant \|f\|_\infty .$$

2. *If the condition* $m(E) < +\infty$ *is not verified,* L^p *spaces are in general incomparable.*

Example .28 *If* E *is not of finite measure,* $f = 1 \in L^\infty$ *but not in* L^1.

-
$$\frac{1}{1 + |x|} \in L^2 \left(\mathbb{R}^+, B(\mathbb{R}^+), \lambda\right) \diagdown L^1 \left(\mathbb{R}^+, B(\mathbb{R}^+), \lambda\right).$$

-
$$\frac{1}{\sqrt{x}} \in L^1 ([0, 1]) \diagdown L^2 ([0, 1]) .$$

-
$$\frac{1}{\sqrt{x}} \in L^1 ([0, 1]) \diagdown L^\infty ([0, 1]) .$$

3.8 Corrected exercises

Exercise 3.1 *Consider the sequence of functions $f_n : (\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), \lambda) \longrightarrow \mathbb{R}_+$ defined by*

$$f_n(x) = \chi_{[0,n[}(x)\frac{1}{E(x)!},$$

where $E(x)$ denotes the integer part of $x \in \mathbb{R}$.

1. *Give the simple limit of the sequence $(f_n)_n$.*
2. *Calculate $\int \frac{1}{E(x)!} d\lambda(x)$.*

Solution 3.1 1. *Since the sequence $([0,n[)_{n\geqslant 1}$ is increasing and*

$$\bigcup_{n=1}^{+\infty}[0,n[= \mathbb{R}_+,$$

on the other hand , we have

$$\chi_{\underline{\lim} A_n} = \liminf_n \chi_{A_n} \text{ and } \chi_{\overline{\lim} A_n} = \limsup_n \chi_{A_n}.$$

therfore

$$\lim_{n\longrightarrow+\infty}\chi_{[0,n[}(x) = \chi_{\bigcup\limits_{n=1}^{+\infty}[0,n[}(x) = \chi_{\mathbb{R}_+}(x) = 1, \text{ for all } x \in \mathbb{R}_+,$$

hence

$$\lim_{n\longrightarrow+\infty} f_n(x) = \frac{1}{E(x)!}, \text{ for all } x \in \mathbb{R}_+.$$

2. *The sequence $(f_n(x))_{n\geqslant 1}$ is increasing for all $x \in \mathbb{R}_+$. The positives functions $x \mapsto f_n(x)$ are decreasing then measurables. According to the monotone convergence theorem, we have*

$$\int \frac{1}{E(x)!}d\lambda(x) = \int \lim_{n\longrightarrow+\infty} f_n(x)d\lambda(x) = \lim_{n\longrightarrow+\infty}\int f_n(x)d\lambda(x),$$

on the other hand, we have

$$\begin{aligned}\int f_n(x)d\lambda(x) &= \int_{[0,n[}\frac{1}{E(x)!}d\lambda(x) = \int_{\bigcup\limits_{k=0}^{n-1}[k,k+1[}\frac{1}{E(x)!}d\lambda(x)\\ &= \sum_{k=0}^{n-1}\int_{[k,k+1[}\frac{1}{E(x)!}d\lambda(x) = \sum_{k=0}^{n-1}\int_{[k,k+1[}\frac{1}{k!}\lambda([k,k+1[)\\ &= \sum_{k=0}^{n-1}\frac{1}{k!},\end{aligned}$$

hence

$$\int \frac{1}{E(x)!}d\lambda(x) = \lim_{n\longrightarrow+\infty}\sum_{k=0}^{n-1}\frac{1}{k!} = \sum_{k=0}^{\infty}\frac{1}{k!} = e.$$

Exercise 3.2 *Let (E, T, m) a measured space and $f : E \longrightarrow \overline{\mathbb{R}}_+$ a measurable function.*

1. *Show that*

$$\lim_{m(A)\longrightarrow 0} \int_A f(x)dm = 0 \Leftrightarrow \forall\varepsilon > 0,\ \exists\delta > 0 :\ \forall A \in T,\ m(A) < \delta \Longrightarrow \int_A fdm < \varepsilon.$$

2. *Calculate*

$$\int\limits_{[0,1]} -\frac{\log(1-x)}{x}dm \quad \text{and} \quad \lim_{n\longrightarrow\infty} \int\limits_{[0,1]} \left(1-\frac{1}{n}\right)^n e^{\frac{n}{2n+x}}dm.$$

Solution 3.2 1. *We show that*

$$\lim_{m(A)\longrightarrow 0} \int_A f(x)dm = 0 \Leftrightarrow \forall\varepsilon > 0,\ \exists\delta > 0 :\ \forall A \in T,\ m(A) < \delta \Longrightarrow \int_A fdm < \varepsilon.$$

i) *The case where f is bounded : we have*

$$\forall x \in E,\ \exists M,\ f(x) \leqslant M$$

$$\int_A fdm \leqslant M.m(A) \Longrightarrow m(A) < \frac{\varepsilon}{M} \Longrightarrow \int_A fdm < \varepsilon,$$

ii) *The case where f is not bounded : In this case we define the sequence (A_n) in the following form*

$$A_n = \{x,\ f(x) \leqslant n\},$$

is an increasing measurable that means

$$A_1 \subset A_2 \subset .... \subset A_n \subset ...,\quad \bigcup_{n=1}^{\infty} A_n = E.$$

We put $f_n = f.1_{A_n}$, then we have

$$f_n \leq f_{n+1} \text{ and } \lim_{n\longrightarrow\infty} f_n = f\ 1_E = f$$

et en utilisant la convergence monotone, on déduit

$$\begin{aligned}
\lim_{n\longrightarrow\infty} \int_E f_n dm &= \int_E \lim_{n\longrightarrow\infty} f_n dm \Longrightarrow \lim_{n\longrightarrow\infty} \int_{A_n} fdm = \int_E fdm \\
&\Longrightarrow \forall\varepsilon > 0;\ \exists n_0 \in \mathbb{N},\ n \geq n_0 \left| \int_{A_n} fdm - \int_E fdm \right| < \varepsilon \\
&\Longrightarrow \int_E fdm < \varepsilon + \int_{A_n} fdm \\
&\Longrightarrow \int_{E\cap A} fdm < \varepsilon + \int_{A\cap A_n} fdm \\
&\Longrightarrow \int_A fdm < \varepsilon + n.m(A \cap A_n) \leq \varepsilon + n.m(A) < \varepsilon_1,\quad m(A) < \frac{\varepsilon}{n}.
\end{aligned}$$

2. *We calculate*

$$\int_{[0,1]} -\frac{\log(1-x)}{x} dm,$$

we have

$$\log(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} \quad ; \quad \forall x \in [0,1],$$

then

$$\int_{[0,1]} -\frac{\log(1-x)}{x} dm = \int_{[0,1]} \frac{\sum_{n=1}^{\infty} \frac{x^n}{n}}{x} dm = \int_{[0,1]} \sum_{n=1}^{\infty} \frac{x^{n-1}}{n} dm,$$

and according to the property of monotonic convergence, we have

$$\int_{[0,1]} \sum_{n=1}^{\infty} \frac{x^{n-1}}{n} dm = \sum_{n=1}^{\infty} \int_{[0,1]} \frac{x^{n-1}}{n} dm = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

We calculate

$$\lim_{n\longrightarrow\infty} \int_{[0,1]} \left(1-\frac{1}{n}\right)^n e^{\frac{n}{2n+x}} dm,$$

we have

$$\left(1-\frac{1}{n}\right)^n = e^{n\log\left(1-\frac{1}{n}\right)} \quad \textit{and} \quad \log\left(1-\frac{1}{n}\right) \leqslant -\frac{1}{n}$$

therefore

$$\left(1-\frac{1}{n}\right)^n \leqslant e^{-1}$$

$$\begin{aligned} \frac{n}{2n+x} &= \frac{2n+x}{2n+x} - \frac{n+x}{2n+x} = 1 - \frac{n+x}{2n+x} \leqslant 1 \\ &\Longrightarrow e^{\frac{n}{2n+x}} \leqslant e^1 \Longrightarrow \left(1-\frac{1}{n}\right)^n e^{\frac{n}{2n+x}} \leqslant e^{-1}.e^1 \quad , \quad m([0,1]) = 1 < \infty \end{aligned}$$

and by the bounded convergence theorem, we deduce that

$$\begin{aligned} \lim_{n\longrightarrow\infty} \int_{[0,1]} \left(1-\frac{1}{n}\right)^n e^{\frac{n}{2n+x}} dm &= \int_{[0,1]} \lim_{n\longrightarrow\infty} \left(1-\frac{1}{n}\right)^n e^{\frac{n}{2n+x}} dm \\ &= \int_{[0,1]} e^{-1}.e^{1/2} dm = e^{-\frac{1}{2}} m([0,1]) = \frac{1}{\sqrt{e}}. \end{aligned}$$

Exercise 3.3 *Consider* (E, T, m) *a measured space , where* m *is a positive measure and* $f : E \longrightarrow \mathbb{C}$ *a measurable function.*

1. Show that if $f \in \mathcal{L}^1(E,T,m)$, then

$$\lim_n n \cdot m\left(\{|f| \geqslant n\}\right) = 0,$$

is the converse true ?

2. Prove that if $f \in \mathcal{L}^1(E,T,m)$, then

$$\sum_{n\geq 1} \frac{1}{n^2} \int\limits_{|f|\leqslant n} |f|^2 \, dm < +\infty,$$

is the converse true ?

Solution 3.3

1. We prove that if $f \in \mathcal{L}^1(E,T,m)$, then

$$\lim_n n \cdot m\left(\{|f| \geqslant n\}\right) = 0,$$

We have

$$0 \leqslant n.m\left(\{|f| \geqslant n\}\right) = \int_E n 1_{\{|f|\geqslant n\}} dm \leqslant \int_E |f| \, 1_{\{|f|\geqslant n\}} dm,$$

on the other hand, we have

$$0 \leqslant g_n = |f| \, 1_{\{|f|\geqslant n\}} \leqslant |f| \in \mathcal{L}^1(m) \text{ and } \lim_{n\longrightarrow\infty} |f(x)| \, 1_{\{|f|\geqslant n\}}(x) = 0,$$

the dominated convergence theorem ensures that

$$\lim_{n\longrightarrow\infty} \int_E |f| \, 1_{\{|f|\geqslant n\}} dm = 0, \qquad \text{d'ou le résultat.}$$

hence the result.

The converse is false . The function is continuous and positive on $[0, e^{-1}]$ given by

$$g(x) = x \ln\left(x^{-1}\right)$$

has the derivative

$$\ln\left(x^{-1}\right) - 1$$

and is increasing on $[0, e^{-1}]$ of $g(0) = 0$ in $g(e^{-1}) = e^{-1}$.

$$\int\limits_0^{e^{-1}} \frac{dx}{g(x)} = \int\limits_0^{e^{-1}} \frac{dx}{x \ln\left(x^{-1}\right)} = \int\limits_e^{+\infty} \frac{du}{u \ln(u)} = \lim_{A\longrightarrow\infty} \ln\left(\ln A\right) = +\infty.$$

However, for $n \geqslant 1$

$$\left\{x \in \left[0, e^{-1}\right], \frac{1}{g(x)} \geqslant n\right\} = \left\{x \in \left[0, e^{-1}\right], g(x) \leqslant n^{-1}\right\} = [0, x_n].$$

where

$$x_n \in \left[0, e^{-1}\right]$$

is characterized by

$$x_n \ln\left(x_n^{-1}\right) = g(x_n) = n^{-1},$$

which implies that

$$n \cdot m\left(\left\{x \in \left[0, e^{-1}\right], \frac{1}{g(x)} \geqslant n\right\}\right) = n x_n = \frac{1}{|\ln x_n|} \longrightarrow 0,$$

since $x_n \longrightarrow 0_+$. The property (1) can therefore be verified without that f (here $1/g$) let $\mathcal{L}^1$.

2. For $f \in \mathcal{L}^1$, we have

$$\sum_{n\geqslant 1} \frac{1}{n^2} \int_{|f|\leqslant n} |f|^2 \, dm = \int \left(\sum_{n\geq 1} n^{-2} |f|^2 1_{\{|f| \leq n\}}\right) dm,$$

with

$$F(x) = \sum_{n\geqslant 1} n^{-2} |f(x)|^2 1_{\{|f|\leqslant n\}}(x) = \sum_{n\geqslant \max(|f(x)|,1)} n^{-2} |f(x)|^2 = |f(x)|^2 \sum_{n\geqslant \max(|f(x)|,1)} n^{-2}$$

as for $N \geqslant 1$,

$$\sum_{n\geqslant N} n^{-2} \leqslant \min\left(\frac{\pi^2}{6}, \frac{1}{N-1}\right),$$

then, we get

$$0 \leqslant F(x) \leqslant \min\left(\frac{\pi^2}{6}, \frac{1}{\max(|f(x)|, 1)}\right) |f(x)|^2 \leq \begin{cases} \dfrac{\pi^2}{6} |f(x)|^2 & \text{if } |f(x)| \leqslant 2 \\ \dfrac{|f(x)|^2}{|f(x)| - 1} & \text{if } |f(x)| > 2. \end{cases}$$

As for $|f(x)| \leqslant 2$, we have

$$\frac{\pi^2}{6} |f(x)|^2 \leqslant \frac{\pi^2}{6} |f(x)| |f(x)| \leqslant |f(x)| \frac{2\pi^2}{6} \leqslant 4 |f(x)|$$

and for $|f(x)| > 2$

$$\frac{|f(x)|^2}{|f(x)| - 1} = \frac{|f(x)|}{|f(x)| - 1} |f(x)| \leqslant 2 |f(x)|,$$

it results that

$$0 \leqslant F(x) \leqslant 4 |f(x)|,$$

which is gives the result.

the converse is false because with

$$f(x) = \frac{1}{x} 1_{[1,+\infty[}(x)$$

(which is not in $\mathcal{L}^1$), we nevertheless have

$$\sum_{n\geqslant 1} \frac{1}{n^2} \int_{|f|\leqslant n} |f|^2 \, dm = \sum_{n\geqslant 1} \frac{1}{n^2} \int_{x\geqslant 1} \frac{1}{x^2} dx = \frac{\pi^2}{6}.$$

Exercise 3.4 *1. Study the possible limit of the sequence* $(w_n)_{n\in\mathbb{N}}$*, given by*

$$w_n = \int_{\mathbb{R}_+} \frac{\sin(\pi x)}{1+x^{n+2}} dx.$$

2. Let $a, b \in]0,+\infty[$*. Justify the following equality*

$$\int_{]0,+\infty[} \frac{xe^{-ax}}{1-e^{-bx}} dx = \sum_{n\geqslant 0} \frac{1}{(a+bn)^2}.$$

Solution 3.4 *1. We put*

$$f_n(x) = \frac{\sin(\pi x)}{1+x^{n+2}}, \; x \in \mathbb{R}_+,$$

we remark that

$$\forall x \in \mathbb{R}_+, \forall n \in \mathbb{N}, 0 \leqslant |f_n(x)| \leqslant 1_{[0,1]}(x) + 1_{]1,+\infty[}(x) \frac{1}{1+x^2} = g(x).$$

We verify that g is integrable with respect to the Lebesgue measure on $\mathbb{R}_+$.Moreover for all $x \in \mathbb{R}_+$, we have

$$\lim_n f_n(x) = f(x),$$

where

$$f(x) = \sin(\pi x),$$

if $x \in [0,1[$,

$$f(1) = \frac{1}{2}\sin(\pi).$$

and

$$f(x) = 0 \text{ if } x > 1.$$

The dominated convergence theorem then implies that

$$\lim_{n\longrightarrow\infty} w_n = \int_{[0,1[} \sin(\pi x) dx = \frac{2}{\pi}.$$

2. For all $x \in]0,+\infty[$ and all $n \in \mathbb{N}$, we put

$$f_n(x) = xe^{-(a+bn)x}$$

We clearly have

$$\forall x \in \left]0,+\infty\right[\,, \frac{xe^{-ax}}{1-e^{-bx}} = \sum_{n\in\mathbb{N}} f_n(x).$$

Since these are positive functions, measurable because continuous on $]0,+\infty[$ *; the intervention of a positive series/integral implies that*

$$\int_{]0,+\infty[} \frac{xe^{-ax}}{1-e^{-bx}} dx = \sum_{n\geqslant 0} \int_{]0,+\infty[} f_n(x)dx.$$

By a linear change of variable

$$\int_{]0,+\infty[} f_n(x)dx = \frac{1}{(a+bn)^2} \int_{]0,+\infty[} xe^{-x}dx.$$

Integration by parts implies that

$$\int_{]0,+\infty[} xe^{-x}dx = 1,$$

which allows us to conclude.

Exercise 3.5 *Consider* (E,T,m) *a measured space,* $f : E \longrightarrow \mathbb{R}$ *a positive measurable function and* $\alpha \in \mathbb{R}_+^*$.

1. *Show that*

$$m\left(\{x : f(x) \geqslant \alpha\}\right) \leqslant \frac{1}{\alpha}\int_E f dm.$$

2. *Demonstrate the following equivalence :*

$$\int_E f dm = 0 \iff f = 0 \quad a.e.$$

3. *Prove that*

$$\int_E f dm < \infty \Longrightarrow f < \infty \quad a.e\ .$$

Solution 3.5 1. *We show that*

$$m\left(\{x : f(x) \geqslant \alpha\}\right) \leqslant \frac{1}{\alpha}\int_E f dm.$$

We put

$$A = \{x : f(x) \geqslant \alpha\},$$

then

$$\begin{aligned} f &\geqslant \alpha.1_A = \alpha.m(A) \\ &\Longrightarrow m(A) \leqslant \frac{1}{\alpha}\int_E f dm. \end{aligned}$$

2. *We prove that*

$$\int_E f dm = 0 \iff f = 0 \quad a.e.$$

$\Longleftarrow$: *If* $f = 0 \quad a.e$, *then*

$$\int_E f dm = \int_E 0 dm = 0$$

$\Longrightarrow$: *We suppose that*

$$\int_E f dm = 0,$$

let

$$A_n = \left\{ x : f(x) \geqslant \frac{1}{n} \right\}, \forall n \geqslant 1,$$

by using 1) , we have

$$m(A_n) \leqslant n \int_E f dm = 0,$$

therefore

$$m(A_n) = 0 \ \forall n \geqslant 1,$$

moreover

$$\begin{aligned} m\left(\{x \in E : f(x) > 0\}\right) &= m\left(\bigcup_{n \in \mathbb{N}^*} \left\{ x : f(x) \geqslant \frac{1}{n} \right\} \right) \\ &\leqslant \sum_{n \in \mathbb{N}^*} m(A_n) = 0. \end{aligned}$$

3. *We show that*

$$\int_E f dm < \infty \Longrightarrow f < \infty \quad a.e.$$

We put

$$B_n = \{x : f(x) \geqslant n\}, \forall n \geqslant 1,$$
$$B_\infty = \{x : f(x) = \infty\},$$

therfore, we have

$$\begin{aligned} m\left(B_\infty\right) &= m\left(\bigcap_{n \geqslant 1} B_n \right) = \lim_{n \longrightarrow \infty} m\left(B_n\right) \\ &\leqslant \lim_{n \longrightarrow \infty} \frac{1}{n} \int_E f dm = 0, \end{aligned}$$

because $(B_n)_{n \geqslant 1}$ *is decreasing and*

$$m\left(B_n\right) \leqslant \frac{1}{n} \int_E f dm < +\infty, \forall n \geqslant 1,$$

hence $f < \infty \quad a.e.$

Exercise 3.6 *We consider* (E, T, m) *a measured space such that* $m(E) < +\infty$, *and* $f : E \longrightarrow \overline{\mathbb{R}_+}$ *a measurable function such that , for all* $A \in T$ *we have :*

$$\int_A f dm \leqslant m(A).$$

1. *For all* $k > 1$, *we put*
$$A_k = \{x \in E : f(x) \geqslant k\}.$$
We prove that
$$km(A_k) \leqslant \int_{A_k} f dm;$$
en déduire que $m(A_k) = 0$.
2. *We show that*
$$f(x) \leqslant 1 \quad a.e.$$

Solution 3.6 1. *We prove that*

$$k.m(A_k) \leqslant \int_{A_k} f dm.$$

We have

$$A_k = f^{-1}([k, +\infty[) \in T \text{ because } f \text{ is measurable,}$$

on the other hand, we have

$$\begin{aligned} \forall x &\in A_k, \ f(x) \geqslant k \\ &\Longrightarrow \int_{A_k} f dm \geqslant \int_{A_k} k dm = km(A_k) \\ &\Longrightarrow \int_{A_k} f dm \geqslant km(A_k). \end{aligned}$$

Now, we prove that $m(A_k) = 0$: *Since* $A_k \in T$ *we have*

$$\begin{aligned} km(A_k) &\leqslant \int_{A_k} f dm \leqslant m(A_k) \\ &\Longrightarrow (k-1)\, m(A_k) \leqslant 0 \\ \text{and } k &> 1 \Longrightarrow m(A_k) = 0 \ \forall k > 1. \end{aligned}$$

2. *We show that* $f(x) \leqslant 1 \quad a.e$: *We have*

$$\begin{aligned} \{x \in E : f(x) > 1\} &= \bigcup_{n \in \mathbb{N}^*} \left\{ x \in E : f(x) \geqslant 1 + \frac{1}{n} \right\} \\ &= \bigcup_{n \in \mathbb{N}^*} A_{1+\frac{1}{n}}, \end{aligned}$$

on the other hand, the sequence $\left(A_{1+\frac{1}{n}}\right)_{n \in \mathbb{N}^*}$ *is increasing in the sense of inclusion, therefore*

$$m(\{x \in E : f(x) > 1\}) = \lim_{n \longrightarrow \infty} m\left(A_{1+\frac{1}{n}}\right) = 0,$$

because

$$\forall n \in \mathbb{N}^*,\ 1+\frac{1}{n} > 1 \text{ and } m\left(A_{1+\frac{1}{n}}\right) = 0,$$

hence the result .

Exercise 3.7 *Let $(f_n)_{n\in\mathbb{N}}$ a sequence of positives measurable function convergeant to a function f.*

Prove that

$$\exists C \geqslant 0 : \forall n \in \mathbb{N}, \int_E f_n dm \leqslant C \Longrightarrow \int_E f dm \leqslant C.$$

Solution 3.7 *By Fattou's lemma, we have*

$$\int_E f dm = \int_E \lim_{n\longrightarrow\infty} f_n dm = \int_E \underline{\lim}_n f_n dm \leqslant \underline{\lim}_n \int_E f_n dm \leqslant C.$$

Exercise 3.8 1. *For all real α and all integer $n \geqslant 1$, we put :*

$$I_n(a) = \int_0^n \left(1+\frac{x}{n}\right)^n e^{-ax} dx.$$

Determine $\lim\limits_{n\longrightarrow\infty} I_n(a)$.

2. *Determine the following limits of integrales when $n \longrightarrow \infty$*

$$\int_0^{+\infty} e^{-x} (\sin x)^n \sin(nx)\, dx, \quad \int_0^n \left(1-\frac{x}{n}\right)^n \cos x dx$$

$$\int_0^1 \frac{ne^{-x}}{nx+1} dx, \quad \int_0^1 \frac{n\sin\left(\frac{x}{n}\right)}{1+x^2} dx.$$

Solution 3.8 1. *We determine $\lim\limits_{n\longrightarrow\infty} I_n(a)$: We put*

$$f_n(x) = \left(1+\frac{x}{n}\right)^n e^{-ax} 1_{[0,n]}(x), x \in \mathbb{R}, n \in \mathbb{N}^*,$$

$(f_n)_{n\in\mathbb{N}^*}$ is a sequenc of positives countinuous measurable function and

$$|f_n(x)| \leqslant e^x e^{-ax} 1_{[0,n]}(x) \leqslant e^{(1-a)x} 1_{\mathbb{R}^+}(x) = f(x),$$

on the other hand , we have

$$\int_0^{+\infty} f(x) dx = \int_0^{+\infty} e^{(1-a)x} dx = \begin{cases} \dfrac{1}{a-1} & \text{if } a > 1 \\ +\infty & \text{if } a \leqslant 1 \end{cases}$$

where $f \in \mathcal{L}^1(\mathbb{R})$ if and only if $a > 1$,moreover

$$\lim_{n\longrightarrow\infty} f_n(x) = f(x),$$

therefore if $a > 1$ and according to the dominated convergence theorem, we have

$$\lim_{n \longrightarrow \infty} I_n(a) = \lim_{n \longrightarrow \infty} \int_{\mathbb{R}} f_n(x) = \frac{1}{a-1},$$

and if $a \leqslant 1$, we have

$$\underline{\lim}_n f_n(x) = e^{(1-a)x} 1_{\mathbb{R}^+}(x),$$

and according the lemma of Fattou, we have

$$\lim_{n \longrightarrow \infty} I_n(a) = \underline{\lim}_n \int_{\mathbb{R}} f_n(x) dx = \int_{\mathbb{R}} \underline{\lim}_n f_n(x) dx = \int_{\mathbb{R}} e^{(1-a)x} dx.$$

Therefore

$$\lim_{n \longrightarrow \infty} I_n(a) = +\infty, \ a \leqslant 1.$$

(We can also show that $(f_n)_{n \in \mathbb{N}^*}$ is increasing and est croissante et we apply the monotone convergence theorem and thus obtain the result directly.

2. *We determine the limits of integrals :*

i) *We determine*

$$\lim_{n \longrightarrow \infty} \int_0^{+\infty} e^{-x} (\sin x)^n \sin(nx)\, dx,$$

we put

$$\forall n \in \mathbb{N}, \ f_n(x) = e^{-x} (\sin x)^n \sin(nx), \ x \in \mathbb{R}^+,$$

$(f_n)_n$ is a sequence of continuous, and therefore measurable, functions ; on the other hand, we have

$$\forall x \in \mathbb{R}^+ - \left\{ \frac{\pi}{2} + n\pi \right\}, \ |\sin x| < 1,$$

and we have

$$|\sin x|^n \underset{n \longrightarrow \infty}{\longrightarrow} 0,$$

moreover, we have

$$\forall x \in \mathbb{R}^+, |f_n(x)| < |\sin x|^n,$$

therefore

$$\forall x \in \mathbb{R}^+ - \left\{ \frac{\pi}{2} + n\pi \right\}, f_n \underset{n \longrightarrow \infty}{\longrightarrow} 0,$$

and

$$\lambda \left(\left\{ \frac{\pi}{2} + n\pi : n \in \mathbb{N} \right\} \right) = 0,$$

because this set is negligible, therefore $(f_n)_n$ converges to 0 a.e and

$$\forall x \in \mathbb{R}^+, \forall n \in \mathbb{N} : |f_n(x)| \leqslant e^{-x} \quad and \quad \int_0^{+\infty} e^{-x} dx = 1 < +\infty,$$

therefore, by the dominated convergence theorem :

$$\lim_{n \longrightarrow \infty} \int_0^{+\infty} e^{-x} (\sin x)^n \sin(nx)\, dx = 0.$$

ii) *We determine*

$$\lim_{n\longrightarrow\infty}\int\limits_0^n\left(1-\frac{x}{n}\right)^n\cos x dx,$$

we put

$$f_n(x)=\left(1-\frac{x}{n}\right)^n\cos x.1_{[0,n]}(x),\ x\in\mathbb{R},\ n\in\mathbb{N}^*,$$

$(f_n)_{n\in\mathbb{N}^*}$ *is a sequence of measurables function and :*

$$|f_n(x)|\leqslant e^{-x}\left|\cos x\right|1_{[0,n]}(x)\leqslant e^{-x}1_{[0,n]}(x)=f(x),$$

with $f\in\mathcal{L}^1\left(\mathbb{R}\right)$*, moreover*

$$\lim_{n\longrightarrow\infty}f_n(x)=e^{-x}\cos x1_{[0,n]}(x),$$

therefore, by the dominated convergence theorem, we get

$$\lim_{n\longrightarrow\infty}\int\limits_0^n\left(1-\frac{x}{n}\right)^n\cos x dx=\int\limits_0^{+\infty}e^{-x}\cos x dx.$$

iii) *We determine*

$$\lim_{n\longrightarrow\infty}\int\limits_0^1\frac{ne^{-x}}{nx+1}dx,$$

$\forall n\in\mathbb{N},\forall x\in[0,1]$*, we put*

$$f_n(x)=\frac{ne^{-x}}{nx+1}=e^{-x}\left(x+\frac{1}{n}\right)^{-1},$$

the lemma of Fattou gives

$$\underline{\lim}_n\int\limits_0^1 e^{-x}\left(x+\frac{1}{n}\right)^{-1}dx=\underline{\lim}_n\int\limits_0^1 f_n(x)dx\geqslant\int\limits_0^1\underline{\lim}_n f_n(x)dx=\int\limits_0^1 e^{-x}x^{-1}dx=+\infty.$$

iv) *We determine*

$$\lim_{n\longrightarrow\infty}\int\limits_0^1\frac{n\sin\left(\frac{x}{n}\right)}{1+x^2}dx,$$

$\forall n\in\mathbb{N}^*,\forall x\in[0,1]$*, we put*

$$f_n(x)=\frac{n\sin\left(\frac{x}{n}\right)}{1+x^2},$$

$(f_n)_{n\in\mathbb{N}^*}$ *is a sequence of continuous, and therefore measurable, functions, and we have*

$$|f_n(x)|=\left|\frac{n\sin\left(\frac{x}{n}\right)}{1+x^2}\right|\leqslant\frac{x}{1+x^2},$$

and

$$x \longmapsto \frac{x}{1+x^2}$$

is intéegrable on $[0,1]$*, moreover*

$$\lim_{n\longrightarrow\infty} f_n(x) = \lim_{n\longrightarrow\infty} \frac{x\frac{n}{x}\sin\left(\frac{x}{n}\right)}{1+x^2} = \frac{x}{1+x^2}, \forall x \in \left]0,1\right],$$

the dominated convergence theorem gives

$$\begin{aligned} \lim_{n\longrightarrow\infty} \int_0^1 f_n(x)dx &= \int_0^1 \lim_{n\longrightarrow\infty} f_n(x)dx \\ &= \int_0^1 \frac{x}{1+x^2}dx = \left[\frac{1}{2}\ln\left(1+x^2\right)\right]_0^1 = \frac{1}{2}\ln 2. \end{aligned}$$

Exercise 3.9 *Let* f *a integrable function on* $\mathbb{R}$.
Show that

$$\lim_{n\longrightarrow\infty} \int_{|x|\geqslant n} f_n(x)dx = 0.$$

Solution 3.9 *We put for all* $n \in \mathbb{N}$*, and* $x \in \mathbb{R}$ *:*

$$f_n(x) = f(x).1_{]-\infty,-n]\cup[n,+\infty[}(x).$$

Then

$$\forall n \in \mathbb{N}, \text{ we have : } |f_n(x)| \leqslant |f| \in \mathcal{L}^1,$$

and

$$\forall x \in \mathbb{R}, \text{ we have } \lim_{n\longrightarrow\infty} f_n(x) = 0,$$

the dominated convergence theorem gives

$$\lim_{n\longrightarrow\infty} \int_{\mathbb{R}} f_n(x)dx = \int_{\mathbb{R}} \lim_{n\longrightarrow\infty} f_n(x)dx = 0,$$

hence the result.

Exercise 3.10 *For all* $n \in \mathbb{N}^*$*, we put*

$$f_n(x) = ne^{-nx};\ f(x) = \sum_{n=1}^{\infty} f_n(x).$$

Calculate

$$\int_1^{+\infty} f(x)dx.$$

Solution 3.10 *We calculate*

$$\int_1^{+\infty} f(x)dx,$$

we have

$$\int_1^{+\infty} f(x)dx = \int_1^{+\infty} \sum_{n=1}^{\infty} ne^{-nx}dx = \int_1^{+\infty} \sum_{n=1}^{\infty} f_n(x)dx,$$

$(f_n)_n$ *is a sequence of continuous, and therefore measurable and positive, functions ; therefore, according to the corollary of monotone convergence, we have :*

$$\begin{aligned} \int_1^{+\infty} \sum_{n=1}^{\infty} f_n(x)dx &= \sum_{n=1}^{\infty} \int_1^{+\infty} f_n(x)dx = \sum_{n=1}^{\infty} \int_1^{+\infty} ne^{-nx}dx \\ &= \sum_{n=1}^{\infty} e^{-n} = \frac{1}{1-e^{-1}} - 1 = \frac{1}{e-1}. \end{aligned}$$

Exercise 3.11 *For all* $t \in \mathbb{R}$*, we put :*

$$F(t) = \int_0^{+\infty} e^{-x^2} \cos(xt)\, dx, \ \ F(0) = \int_0^{+\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$$

1. *Show that* F *is continuous and differentiable on* $\mathbb{R}$.
2. *Show that* F *satisfies the differential equation :*

$$F'(t) + \frac{t}{2}F(t) = 0,$$

and deduce its value .

Solution 3.11 1. *We show that* F *is continuous and differentiable on* $\mathbb{R}$.

i) $\forall t \in \mathbb{R}, x \in [0, +\infty[$, *we put*

$$f(x,t) = e^{-x^2} \cos(xt).$$

ii) *We have* $\forall x \in [0, +\infty[\,, x \longmapsto f(x,t)$ *is continuous on* $\mathbb{R}$ *and* $\forall t \in \mathbb{R}, \forall x \in [0, +\infty[$

$$|f(x,t)| \leqslant e^{-x^2} \text{ and } \int_0^{+\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2} < +\infty,$$

therefore, by the continuity theorem under integrals, F *is continuous on* $\mathbb{R}$

$$\forall x \in [0, +\infty[\,, x \longmapsto f(x,t)$$

is differentiable on $\mathbb{R}$*, and we have*

$$\frac{\partial}{\partial t} f(x,t) = -xe^{-x^2} \sin(xt),$$

moreover

$$\left|\frac{\partial}{\partial t}f(x,t)\right| \leqslant xe^{-x^2},\ \forall t \in \mathbb{R},$$

therefore

$$\int_0^{+\infty} xe^{-x^2}dx = \left[-\frac{1}{2}e^{-x^2}\right]_0^{+\infty} = \frac{1}{2} < +\infty;$$

According to the differentiability theorem under the integral sign, F is differentiable on $\mathbb{R}$ and

$$F'(t) = -\int_0^{+\infty} xe^{-x^2}\sin(xt)\,dx.$$

2. *We prove that F verifies the differential equation :*

$$F'(t) + \frac{t}{2}F(t) = 0,$$

i) *We have*

$$\begin{aligned}
F'(t) &= -\int_0^{+\infty} xe^{-x^2}\sin(xt)\,dx \\
&= \left[\frac{1}{2}e^{-x^2}\sin(xt)\right]_0^{+\infty} - \frac{1}{2}\int_0^{+\infty} e^{-x^2}t\cos(xt)\,dx \\
&= -\frac{t}{2}\int_0^{+\infty} e^{-x^2}\cos(xt)\,dx = \frac{t}{2}F(t),
\end{aligned}$$

hence the result .

ii) *We deduce tha value of $F(t)$: we have*

$$\begin{aligned}
F'(t) + \frac{t}{2}F(t) &= 0 \Longrightarrow F'(t) = -\frac{t}{2}F(t) \\
&\Longrightarrow \frac{F'(t)}{F(t)} = -\frac{t}{2} \Longrightarrow \ln|F(t)| = -\frac{t^2}{4} + C \\
&\Longrightarrow F(t) = Ke^{-\frac{t^2}{4}},\ \textit{moreover}\ \ F(0) = \frac{\sqrt{\pi}}{2} = K.
\end{aligned}$$

Therefore

$$F(t) = \frac{\sqrt{\pi}}{2}e^{-\frac{t^2}{4}}.$$

Exercise 3.12 *We consider on $\mathbb{R}_+$ the function :*

$$F(t) = \int_0^{+\infty} \frac{e^{-tx^2}}{1+x^2}dx.$$

1. *Prove that F is continuous on $\mathbb{R}_+$ and calculate* $\lim\limits_{t\longrightarrow+\infty} F(t)$.
2. *Show that F is est differentiable on $\mathbb{R}_+^*$ and calculate* $\lim\limits_{t\longrightarrow 0^+} F'(t)$.

Solution 3.12 1. *We prove that F is continuous on $\mathbb{R}_+$: $\forall t \geqslant 0, \forall x \geqslant 0,$ we put*

$$f(x,t) = \frac{e^{-tx^2}}{1+x^2},$$

we have $t \longmapsto f(x,t)$ is continuous on $\mathbb{R}_+$ and

$$|f(x,t)| \leqslant \frac{1}{1+x^2}, \forall t \geqslant 0$$

and

$$\int_0^{+\infty} \frac{dx}{1+x^2} = [arc\tan x]_0^{+\infty} = \frac{\pi}{2} < +\infty,$$

Therefore, according to the continuity theorem under the integral, F is continuous on $\mathbb{R}_+$.
We calculate $\lim\limits_{n\longrightarrow+\infty} F(t)$, according to the dominated convergence theorem : $\forall (t_n)_{n\in\mathbb{N}}$ a sequence of positive real numbers that converges to $+\infty$

$$\lim_{n\longrightarrow+\infty} F(t_n) = \int_0^{+\infty} \lim_{n\longrightarrow+\infty} \frac{e^{-t_n x^2}}{1+x^2} dx = 0.$$

2. *We show that F is differentiable on $\mathbb{R}_+^*$*
We have $\forall t \geqslant 0$, $t \longmapsto f(x,t)$ is differentiable and

$$\frac{\partial}{\partial t} f(x,t) = -\frac{x^2 e^{-tx^2}}{1+x^2}.$$

Moreover, $\forall a > 0, t > 0$

$$\left|\frac{\partial}{\partial t} f(x,t)\right| \leqslant \frac{x^2 e^{-ax^2}}{1+x^2} \text{ with } x \longmapsto \frac{x^2 e^{-ax^2}}{1+x^2} \text{ integrable,}$$

$$\int_0^1 \frac{x^2 e^{-ax^2}}{1+x^2} dx < +\infty \text{ and } \int_0^{+\infty} \frac{x^2 e^{-ax^2}}{1+x^2} dx < +\infty,$$

because

$$\lim_{n\longrightarrow+\infty} \frac{x^2}{1+x^2} e^{-ax^2} = 0.$$

Therefore F is differentiable on $]a,+\infty[$ and as a is arbitrary F is differentiable $\mathbb{R}_+^*$ and we have :

$$F'(t) = -\int_0^{+\infty} \frac{x^2 e^{-tx^2}}{1+x^2} dx.$$

We calculate $\lim\limits_{t \longrightarrow 0^+} F'(t)$: *We have* $\forall (t_n)_n$ *a sequence of positive real numbers that converges to* 0, *and according to the lemma of Fattou, we obtain*

$$\begin{aligned} \lim_n (-F'(t)) &= \lim_n \int_0^{+\infty} \frac{x^2 e^{-t_n x^2}}{1+x^2} dx \geqslant \int_0^{+\infty} \lim_n \frac{x^2 e^{-t_n x^2}}{1+x^2} dx \\ &= \int_0^{+\infty} \frac{x^2}{1+x^2} dx = +\infty, \end{aligned}$$

therefore $F'(t)$ *converges to* $-\infty$ *when* $t \longrightarrow 0^+$.

Exercise 3.13 *Consider* (E, T, m) *a measured space and* f, g *two functions belonging respectively to* L^p *and* L^q *where* $p, q \in \mathbb{R}_+^*$.

Show that if $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, *we have*

$$fg \in L^r \text{ and } \|fg\|_r \leqslant \|f\|_p \|g\|_q .$$

Solution 3.13 *We have* $\frac{r}{p} + \frac{r}{q} = 1$ *therefore* $\frac{p}{r}$ *and* $\frac{q}{r}$ *are conjugate, on the other hand* f *and* g *are measurable, therefore* $|f.g|^r$ *is also measurable , then by using the inequality of Hölder, we have*

$$\begin{aligned} \int_E |f.g|^r \, dm &= \int |f|^r . |g|^r \, dm \\ &\leqslant \left(\int_E (|f|^r)^{p/r} \, dm \right)^{r/p} . \left(\int_E (|g|^r)^{q/r} \, dm \right)^{r/q} \\ &= \left(\int_E (|f|^p) \, dm \right)^{r/p} . \left(\int_E (|f|^q) \, dm \right)^{r/q} < +\infty, \end{aligned}$$

because

$$|f|^r \in L^{p/r} \text{ and } |g|^r \in L^{q/r},$$

hence

$$f.g \in L^r \text{ and } \|fg\|_r \leqslant \|f\|_p \|g\|_q .$$

Exercise 3.14 *Let* (E, T, m) *a measured space and* f, g *are two functions belonging respectively to* L^p *and* L^q *where* $p, q, r \in \mathbb{R}_+^*$, *such that*

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1, \ f \in L^p, \ g \in L^q, \ h \in L^r.$$

Prove that

$$fgh \in L^1 \text{ and } \|fgh\|_1 \leqslant \|f\|_p \|g\|_q \|h\|_r .$$

Solution 3.14 *We put*

$$\frac{1}{p}+\frac{1}{q}=\frac{1}{s},$$

then according to the previous exercise, we have

$$fg \in L^s \text{ and } \|fg\|_s \leqslant \|f\|_p \|g\|_q ,$$

moreover

$$\frac{1}{s}+\frac{1}{r}=1,$$

therefore according to the inequality of Hölder, we have

$$\begin{aligned}(fg)\,h &\in L^1 \text{ and } \|fgh\|_1 \leqslant \|fg\|_s \|h\|_r \\ &\leqslant \|f\|_p \|g\|_q \|h\|_r ,\end{aligned}$$

hence the result.

Exercise 3.15 *Let*

$$f \in L^p([0,+\infty[) ,\ g \in L^q([0,+\infty[) \text{ and } 1 \leqslant p \leqslant q \leqslant +\infty \text{ are conjugate.}$$

Calculeate

$$\lim_{t\longrightarrow\infty}\frac{1}{t}\int_0^t f(s)g(s)ds.$$

Solution 3.15 *We have*

$$\begin{aligned}\left|\int_0^t fg\right| &\leqslant \int_0^t |fg| \leqslant \left(\int_0^t |f|^p\right)^{1/p}\left(\int_0^t |g|^q\right)^{1/q} \\ &\leqslant \|f\|_p \|g\|_q ,\end{aligned}$$

therefore

$$\lim_{t\longrightarrow\infty}\frac{1}{t}\int_0^t f(s)g(s)ds = 0.$$

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