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Dedication

I dedicate this work:

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Contents

Glossary	xi
General Introduction	1
1 Fundamental concepts and preliminaries	3
1 Introduction	3
2 Fundamentals of differential equations	3
2.1 First-order ordinary differential equation	4
3 Methods of solving differential equations	4
3.1 A one-step methods	4
4 Fractional calculus	7
4.1 Special fractional functions	7
4.2 Fractional derivatives	8
5 Integral transforms	9
5.1 Laplace transform	9
5.2 Sumudu transform	11
6 Conclusion	13
2 Epidemic models	15
1 Introduction	15
2 Introduction to epidemiology	15
3 The compartmental models in epidemiology	16
3.1 Compartments and rules	16
3.2 Basic examples of epidemiology models	17
4 Conclusion	25
3 Characterization of SEIR model stability	27
1 Introduction	27
2 Preliminary stability of the model	28
2.1 Method for calculating the basic reproduction number $\mathcal{R}_0$. .	28

2.2	Lyapunov stability	29
3	Stability analysis of the SEIR ODE model	30
3.1	Problem statement	30
3.2	Positivity of solutions	30
3.3	Determination of equilibrium points	31
3.4	Calculation of the Basic Reproduction Number	32
3.5	Global Asymptotic Stability of the Disease-Free Equilibrium	33
3.6	Global Asymptotic Stability of the Disease Endemic Equilibrium	34
4	Stability Analysis of the SEIR Conformable Model	35
4.1	Problem Statement	35
4.2	Positivity of solutions	36
4.3	Determination of Equilibrium points	36
4.4	Calculation of the Basic Reproduction Number	37
4.5	Global stability of the DFE	39
4.6	Global stability of the DEE	40
5	Conclusion	41
4	Positivity and Stability Analysis of Conformable Fractional COVID-19 Models	43
1	Introduction	43
2	Model formulation	43
3	Non-negativity and boundness of solutions	45
4	Equilibrium points and basic reproduction number of the model	46
5	Stability analysis	47
5.1	Local stability	47
5.2	Global stability	48
6	Numerical results and discussion	51
7	Conclusion	53
5	Introduction to Optimal Control Theory	55
1	Introduction	55
2	Optimal Control Problem Statement	55
3	Numerical Methods in Optimal Control	56
3.1	Direct Method	56
3.2	Indirect Method	62
4	Conclusion	64
6	Nonsmooth Newton's Method for Discretized SEIR Optimal Control Problem	65
1	Introduction	65
2	Formulation of the model	65
2.1	Model without control	67
2.2	A Model with an unlimited supply of the vaccine	68
2.3	A model with a limited supply of vaccines at each instant of time	68
3	Numerical Analysis of the Model	69
3.1	Model without control	69

3.2	Model with unlimited supply of vaccine	72
3.3	Model with limited supply of vaccines at each instant of time	77
4	Numerical Results and Discussion	81
5	Conclusion	87
	General Conclusion	89
	Bibliography	91

List of Figures

2.1	SI epidemiological model diagram in the short term	17
2.2	SI epidemiological model diagram in the long term	18
2.3	SIS epidemiological model diagram in the short term	18
2.4	SIS epidemiological model diagram in the long term	18
2.5	SEI epidemiological model diagram in the short term	19
2.6	SEI epidemiological model diagram in the long term	19
2.7	SEIS epidemiological model diagram in the short term	20
2.8	SEIS epidemiological model diagram in the long term	21
2.9	SIR epidemiological model diagram in the short term	21
2.10	SIR epidemiological model diagram in the long term	21
2.11	SIRS epidemiological model diagram in the short term	22
2.12	SIRS epidemiological model diagram in the long term	22
2.13	SEIR epidemiological model diagram in the short term	23
2.14	SEIR epidemiological model diagram in the long term	24
2.15	SEIRS epidemiological model diagram in the short term	24
2.16	SEIRS epidemiological model diagram in the long term	25
4.1	Diagram of a compartmental model for COVID-19 transmission. . . .	44
4.2	Impact of α on System Dynamics	52
6.1	SEIR compartmental model with vaccination.	66
6.2	The dynamics of S, I and N when no vaccines are used (without control)	82
6.3	The optimal trajectories and optimal vaccination rate with unlimited vaccines (P_{UV}).	85
6.4	The optimal trajectories and optimal vaccination rate for (P_{mixed}) . .	86

List of Tables

4.1	Epidemiological Parameters of the Model	45
4.2	Initial model parameters	51
4.3	The initial values for the model variables.	51
6.1	Description of model parameters.	66
6.2	Parameters values and constants [67]	82
6.3	Comparative Analysis of the Methods (Iterations and Cost)	83
6.4	Initial values of the multipliers	83

Glossary

DEE	Disease-endemic equilibrium.
DFE	Disease-free equilibrium.
DOCP	Discretized optimal control problem.
GAS	Globally asymptotically stable.
IVP	Initial value problem.
KKT	Karush-Kuhn-Tucker conditions.
LAS	Locally asymptotically stable.
NNM	Nonsmooth newton method.
OCP	Optimal control problem.
ODE	Ordinary differential equation.
PMP	Pontryagin's maximum principle.
RK4	Runge-Kutta 4th Order method.
SEI	Susceptible-Exposed-Infected model.
SEIR	Susceptible-Exposed-Infected-Recovered model.
SEIRS	Susceptible-Exposed-Infected-Recovered-Susceptible model.
SEIS	Susceptible-Exposed-Infected-Susceptible model.
SI	Susceptible-Infected model.

SIR	Susceptible-Infected-Recovered model.
SIRS	Susceptible-Infected-Recovered-Susceptible model.
SIS	Susceptible-Infected-Susceptible model.

General introduction

In December 2019, the emergence of the COVID-19 virus in Wuhan, China, marked the beginning of a global pandemic that rapidly spread and became a worldwide public health emergency. With fever, cough, tiredness, and respiratory difficulty as symptoms, its high contagiousness necessitated prompt control interventions [1, 2, 3, 4]. In such a situation, mathematical modeling has become essential to infectious disease outbreak knowledge and control.

Mathematical modeling is vital in epidemiology to forecast epidemic behaviors and evaluate intervention policies. Dynamics of the disease are generally too complicated such that simplified models need to be applied to investigate transmission behaviors and to incorporate public health policies [5, 6]. The SIR model, which Kermack and McKendrick first introduced in 1927 [7], was one significant initial beginning of epidemic forecasting. Subsequent work has adapted this model to include local and global stability of SEIR and SIR models [8, 9, 10, 11, 12, 13]. More recent research has been focused on modeling COVID-19, its dynamics of spread, basic reproduction number, and stability [14, 15, 16, 17, 18], with some also incorporating vaccination effects [19, 20].

Fractional models are particularly effective at capturing epidemiological systems' long-term memory and non-local dynamics. The stability of fractional SEIR models for COVID-19 has been investigated by researchers [21, 22, 23, 24], with conformable fractional derivatives offering computational simplicity without sacrificing analytical tractability [25, 26, 27]. These derivatives offer simplicity of stability analysis without sacrificing the compatibility with classical derivative conditions. The Sumudu transform has also been employed in fractional conformable models to achieve analytical solutions [28, 29].

Optimal control theory offers a potent framework for controlling infectious disease models, minimizes transmission, and optimizes the allocation of resources. Through the inclusion of constraints, policymakers can assess intervention strategies more accurately. A fundamental method for solving unconstrained optimal control problems is Pontryagin's maximum principle [30, 31], allowing analytical solutions

and real-world interpretations. Since its early applications in biomedical engineering [32], optimal control has been widely applied in epidemic modeling, with and without vaccination strategies [33, 34, 35, 36, 37, 38, 39, 40].

The Nonsmooth Newton method is perfectly appropriate for constrained optimal control problems, whether control, state, or mixed constrained. It is also well suited to large-scale mathematics optimization, decision science, and engineering optimization owing to its speed of convergence and stability. Whereas indirect techniques like Pontryagin's maximum principle, that solve the problem analytically first and then discretize (e.g., via RK4 or forward-backward sweep), the Nonsmooth Newton method is a direct technique that first discretizes the problem. The technique has been successfully used in some works [41, 42, 43].

This thesis is situated in the field of mathematical epidemiology, specifically the SEIR model. It is organized into six chapters, each addressing a basic or advanced topic. Chapter One provides the theoretical background by presenting differential equations and dynamical systems, emphasizing solutions, existence, uniqueness, stability, and properties of fractional equations. Building on this foundation, Chapter Two introduces classical epidemiological models, particularly the SEIR model, and presents examples of disease spread modeling. This leads to Chapter Three, which is dedicated to the stability analysis of the SEIR model, focusing on determining the basic reproduction number and analyzing the local and global asymptotic stability of the model's equilibria, including a detailed numerical analysis. Chapter Four then examines the stability analysis of COVID-19 fractional-order models, where fractional derivatives are discussed, classical mathematical methods are extended to memory-dependent systems, and parameter sensitivity is analyzed. Following this, Chapter Five explores optimal control theory, describing classical methods such as Pontryagin's Maximum Principle and their application to control problems in the epidemiological context. Finally, Chapter Six proposes a novel neural network-based method for solving an optimal control problem for a discrete SEIR model, demonstrating a convergence between classical mathematical analysis and contemporary artificial intelligence methodology. Through this structure, the thesis seeks to provide a profound, mathematically rigorous, and methodologically diverse insight into the dynamics of epidemic spread and optimal control measures.

Fundamental concepts and preliminaries

1 Introduction

This chapter provides a significant theoretical foundation for understanding the tools and techniques employed in this thesis. It begins with an exposition of the fundamental principles governing ordinary differential equations (ODEs) and the methods of *résolution*, which form the foundation for modeling the concerned dynamical systems. From this context, we then explore the advanced mathematical theory of fractional calculus, a rich extension of classical calculus that enables the description of refined phenomena inadequately described by integer-order derivatives. Finally, we introduce the Laplace and Sumudu transform, integral transformation techniques playing a central role in the analytical solution of continuous functions.

2 Fundamentals of differential equations

In mathematics, there are two types of differential equations that are normally distinguished: ordinary differential equations and partial differential equations, which can be first- or second-order, linear, or nonlinear. Differential equations are of great significance in areas such as physics, chemistry, biology, etc. However, exact solutions to these equations are not always feasible, and numerical methods are required to find approximate solutions.

2.1 First-order ordinary differential equation

Let U be an open subset of $\mathbb{R} \times \mathbb{R}^m$ and $f : U \rightarrow \mathbb{R}^m$ be a continuous function. We consider the relation

$$y' = f(t, y) \quad (1.1)$$

where $(t, y) \in U$, $t \in \mathbb{R}$, $y \in \mathbb{R}^m$

The relation (1.1) is called a first-order ODE.

A solution of (1.1) on an interval $I = [0, T] \subset \mathbb{R}$ is a differentiable function $y : I \rightarrow \mathbb{R}^m$ such that:

- $\forall t \in I : (t, y(t)) \in U$
- $\forall t \in I : y'(t) = f(t, y(t))$.

The unknown of equation (1.1) is a function of t .

Cauchy problem

Given a point $(0, y_0) \in U$, the Cauchy problem involves of finding a solution $y : I \rightarrow \mathbb{R}^m$ of (1.1) in an interval I containing 0, such that $y(0) = y_0$.

The formulation of the problem is given as follows:

$$\begin{cases} y' = f(t, y(t)) \\ y(0) = y_0, \end{cases} \quad t \in \mathbb{R} \quad (1.2)$$

Practically, the variable t denotes time, and y signifies the state of a particular material system. Equation (1.1) physically expresses the law of evolution of the system under consideration as a function of time.

3 Methods of solving differential equations

3.1 A one-step methods

The one-step method is a class of numerical methods designed to approximate the solution of an ordinary differential equation (ODE) at a point within a given interval. A one-step method is generally used for initial value problems (IVPs), in which the solution is advanced step-by-step from an initial condition.

The one-step method's most important feature is its reliance on information from a single previous point to compute the solution at the subsequent point in order. In numerical analysis, a one-step method refers to an iterative formula that computes the solution at the next time step t_{n+1} based on information at time step t_n .

Some typical examples of one-step methods are:

Euler's method

The objective of the Euler method is to provide approximations to solve a well-posed initial value problem.

$$\frac{dy}{dt} = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha \quad (1.3)$$

The solution $y(t)$ cannot be expressed as a continuous function. Instead, it will be computed at discrete points within the interval $[a, b]$, called **mesh points**.

$$a = t_0 < t_1 < \dots < t_n < t_{n+1} < \dots < t_N = b.$$

Once the approximate solution is obtained at the given points, it is possible to extend this approximation to the whole interval by interpolation ie:

$$t_i = a + ih, \text{ for each } i = 0, 1, 2, \dots, N.$$

The **step size** is the regular distance 'h' between points.

$$h = \frac{(b - a)}{N} = t_{i+1} - t_i$$

For $0 \leq n \leq N$, the approximate solution to the problem (1.1) is given by:

$$y(t_{n+1}) = y(t_n) + \int_{t_n}^{t_{n+1}} f(s, y(s)) ds. \quad (1.4)$$

By recurrence, we construct an approximation y_n of $y(t_n)$ by replacing (1.4) with:

$$y_{n+1} = y_n + h_n f(t_n, y_n), \quad \text{for each } n = 0, 1, \dots, N - 1. \quad (1.5)$$

Thus, the Euler method is defined by the relation (1.5).

Runge kutta method

The numerical method (**RK4**) is a common reference for approximating ODE solutions. The method has become increasingly popular since it provides a reasonable compromise between computational effort and increased accuracy for the numerical solution of ODEs.

The following differential equation describes the Cauchy problem:

$$\begin{cases} y'(t) = f(t, y(t)), \\ y(a) = \alpha, \end{cases} \quad t \in I = [a, b], \quad (1.6)$$

The interval $[a, b]$ is divided as follows:

$$a = t_0 < t_1 < \dots < t_n < t_{n+1} < \dots < t_N = b.$$

Let's set:

$$h = t_{n+1} - t_n.$$

The approximation of $y'(t)$ is defined as follow:

$$\frac{y(t+h) - y(t)}{h} \approx f(t, y(t)).$$

we get the iteration rule of the so-called explicit classical Runge-Kutta method

$$\begin{aligned} k_1 &= f(t_i, y_i), \\ k_2 &= f\left(t_i + \frac{h}{2}, y_i + \frac{1}{2}k_1\right), \\ k_3 &= f\left(t_i + \frac{h}{2}, y_i + \frac{1}{2}k_2\right), \\ k_4 &= f(t_i + h, y_i + k_3). \end{aligned} \tag{1.7}$$

leading to

$$y_{i+1} = y_i + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4). \tag{1.8}$$

Midpoint's approximation method

The method of approximating the midpoint is a numerical technique used to estimate the definite integral of a function. To calculate the area under a curve, divide it into rectangles and take the value of the curve at the middle of each rectangle to determine its height. The sum of all rectangular areas approximates the total area under the curve. It is a specific type of Riemann sum, without using the left or right edge of rectangles but with the middle. This type often leads to a better balance between over- and under-estimating the area below the curve.

Given a function $f(x)$ defined on the interval $[a, b]$, its definite integral is given by:

$$\int_a^b f(x) dx \tag{1.9}$$

can be approximated using the Midpoint Rule as follows:

1. Divide the interval $[a, b]$ into n subintervals of equal width:

$$\Delta x = \frac{b - a}{n}$$

The endpoints of the subintervals are:

$$x_0 = a, x_1 = a + \Delta x, x_2 = a + 2\Delta x, \dots, x_n = b.$$

2. For each subinterval $[x_{i-1}, x_i]$, compute the midpoint:

$$c_i = \frac{x_{i-1} + x_i}{2}$$

3. Evaluate the function $f(x)$ at each midpoint c_i .

4. Multiply each function value by the width of the subinterval Δx and sum them up:

$$\int_a^b f(x) dx \approx \sum_{i=1}^n f(c_i) \Delta x \quad (1.10)$$

The Midpoint Rule formula is defined by the relation (1.10).

4 Fractional calculus

Fractional calculus represents an essential extension of the usual differentiation and integration to arbitrary orders, providing a powerful mathematical technique for describing complex phenomena and dynamical systems that conventional calculus cannot handle. For 300 years, mathematicians like Riemann, Liouville, and Caputo have developed it, and this has resulted in different definitions of fractional derivatives and integrals, which, although not always giving the same results, are taken as equivalent for a broad class of functions. It has witnessed massive interest in applied sciences, control theory, and engineering. At the same time, more recent developments, such as the conformable derivative operator introduced by Khalil et al., have led it further into applications in finance, biology, medicine, and physics. From its inception in the 17th century with Leibniz and L'Hôpital and onward through recent computational developments, fractional calculus has evolved as an effective technique of solving fractional differential equations and modeling real-world issues, placing itself firmly within applied mathematics.

4.1 Special fractional functions

Gamma function

Definition 4.1. [44]

The Gamma function, denoted by $\Gamma(z)$ is a generalization of the factorial function $k!$, i.e.,

$$\Gamma(k) = (k-1)! \quad \forall k \in \mathbb{N}. \quad (1.11)$$

For complex arguments with positive real part it is defined as

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad \operatorname{Re}(z) > 0. \quad (1.12)$$

This function has the following essential results:

Proposition 4.2. [44]

For a complex argument z with positive real part $\operatorname{Re}(z) > 0$. So we have the following result:

$$\Gamma(z+1) = z\Gamma(z). \quad (1.13)$$

Some of the most important values are

$$\begin{aligned}
\Gamma(1) &= \Gamma(2) = 1, \\
\Gamma\left(\frac{1}{2}\right) &= \sqrt{\pi}, \\
\Gamma\left(k + \frac{1}{2}\right) &= \frac{\sqrt{\pi}(2k-1)!}{2^k}, \quad \forall k \in \mathbb{N}.
\end{aligned} \tag{1.14}$$

The Mittag-Leffler function

While the Gamma function is a generalization of the factorial function, the Mittag-Leffler function is a generalization of the exponential function

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(n+1)}. \tag{1.15}$$

First introduced as a one parameter function by the series [44]

$$E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad z, \alpha \in \mathbb{C}, \operatorname{Re}(\alpha) > 0. \tag{1.16}$$

Later, the two parameter generalization is introduced by Agarwal

$$E_{\alpha, \beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad z, \alpha, \beta \in \mathbb{C}, \operatorname{Re}(\alpha) > 0, \operatorname{Re}(\beta) > 0, \tag{1.17}$$

which is of great importance for the fractional calculus. It is called two parameter function of Mittag-Leffler type. Some of its interesting values are [44]

$$\begin{aligned}
E_{1,1}(z) &= e^z, \\
E_{2,1}(z^2) &= \cosh(z), \\
E_{2,2}(z^2) &= \frac{\sinh(z)}{z}, \\
E_{\alpha,1}(z) &= E_{\alpha}(z).
\end{aligned} \tag{1.18}$$

4.2 Fractional derivatives

Conformable derivative

Definition 4.3. [45]

Let a function $x : [0, +\infty) \rightarrow \mathbb{R}$. The conformable derivative of this function of order α , with $\alpha \in (0, 1]$ is defined as follows

$$\mathbf{T}^{\alpha}(x)(t) = \lim_{\epsilon \rightarrow 0} \frac{x(t + \epsilon t^{1-\alpha}) - x(t)}{\epsilon}, \quad \forall t > 0. \tag{1.19}$$

A function x is called α -differentiable if its conformable derivative of order α exists for every $t > 0$.

Theorem 4.4. [45]

Let $\alpha \in (0, 1]$ and $x_1, x_2 : \mathbb{R}_+ \rightarrow \mathbb{R}$ be α -differentiable functions. Then, $\forall t > 0$

1. $\mathbf{T}^\alpha(ax_1(t) + bx_2(t)) = a\mathbf{T}^\alpha(x_1)(t) + b\mathbf{T}^\alpha(x_2)(t)$, for all $a, b \in \mathbb{R}$;
2. $\mathbf{T}^\alpha(t^p) = pt^{p-\alpha}$, for all $p \in \mathbb{R}$;
3. $\mathbf{T}^\alpha(\lambda) = 0$, for all constant function $x_1(t) = \lambda$;
4. $\mathbf{T}^\alpha(x_1(t)x_2(t)) = x_1(t)\mathbf{T}^\alpha(x_2)(t) + x_2(t)\mathbf{T}^\alpha(x_1)(t)$;
5. $\mathbf{T}^\alpha\left(\frac{x_1(t)}{x_2(t)}\right) = \frac{x_2(t)\mathbf{T}^\alpha(x_1)(t) - x_1(t)\mathbf{T}^\alpha(x_2)(t)}{x_2^2(t)}$;
6. If x_1 is differentiable, then, $\mathbf{T}^\alpha(x_1)(t) = t^{1-\alpha} \frac{dx_1(t)}{dt}$.

5 Integral transforms

Integral transforms are essential in solving engineering and differential equations problems [46, 47, 48]. The current focus is on fractional transforms [49, 50] to solve fractional differential equations in physics and engineering. Their algebraic transmutation of differential operators enables computationally tractable solutions. This section will list interesting properties and rules for continuous functions, including Laplace and Sumudu, along with their integral transforms in the conformable fractional case.

5.1 Laplace transform

Laplace transformation is a fundamental concept in engineering mathematics and mathematical sciences. The French mathematician Pierre-Simon Laplace, working in the 18th century, originated the Laplace transform. Mathematically, it provides a robust tool for simplifying the solution of differential equations and converting functions dependent on time into a corresponding representation in complex frequency space.

Definition 5.1. [51] A function f is causal if it satisfies the condition:

$$f(t) = 0, \quad \text{for } t < 0. \quad (1.20)$$

Definition 5.2. [51] Let $f : [0, +\infty) \rightarrow \mathbb{R}$, the Laplace transform of a causal function $f(t)$ is a function of the complex variable s , represented by $F(s)$. The Laplace transform of f is

$$\mathcal{L}[f(t)] := F(s) = \int_0^{+\infty} e^{-st} f(t) dt. \quad (1.21)$$

A necessary condition for the existence of the Laplace transform of a function $f(t)$ is the convergence of the integral.

Conditions for the Existence of the Laplace Transform

A function f must satisfy the following conditions for its Laplace transform to be defined.

- f must be defined for $t \geq 0$.
- f must be of exponential growth; that is, a constant M and a real number σ such that $|f(t)| \leq Me^{\sigma t}$ for all $t \geq 0$.
- $f(t)$ must be integrable over $[0, \infty)$, i.e.,

$$\int_0^\infty |f(t)|e^{-\nu t} dt < \infty \quad \text{for some } \nu.$$

Some Main Properties of the Laplace Transform

Let f, g be causal functions for $\alpha, \beta \in \mathbb{R}$, $a \in \mathbb{R}^*$, and $s > 0$. Then, the important properties of the Laplace transform are defined as follows:

• Linearity

$$\begin{aligned} \mathcal{L}[\alpha f(t) + \beta g(t)] &= \alpha \mathcal{L}[f(t)] + \beta \mathcal{L}[g(t)] \\ &= \alpha F(s) + \beta G(s). \end{aligned} \quad (1.22)$$

• Derivative

The Laplace transform of the derivative $f'(t)$ is given by:

$$\mathcal{L}[f'(t)] = sF(s) - f(0). \quad (1.23)$$

• Integration

The Laplace transform of the integral is:

$$\mathcal{L}\left[\int_0^t f(\tau) d\tau\right] = \frac{F(s)}{s}. \quad (1.24)$$

• Convolution

$$\mathcal{L}[(f \star g)(t)] = \mathcal{L}[f(t)]\mathcal{L}[g(t)] = F(s)G(s) \quad (1.25)$$

where $(f \star g)(t) = \int_0^t f(s)g(t-s)ds$.

• Time Shift

$$\mathcal{L}[f(t-a)] = e^{-as}F(s). \quad (1.26)$$

• Multiplication by a scalar

$$\mathcal{L}[f(at)] = \frac{F(s)}{a}. \quad (1.27)$$

• Dirac impulse

$$\mathcal{L}[\delta(t)] = 1. \quad (1.28)$$

Conformable Laplace transform

Theorem 5.3. [52] Let $f : [0, +\infty) \rightarrow \mathbb{R}$ be a causal function and $0 < \alpha \leq 1$. Thus, the conformable fractional Laplace transform of $\mathcal{T}_\alpha f$ is given by

$$\mathcal{L}_\alpha [\mathbf{T}^\alpha f(t)](s) = sF_\alpha(s) - F(0), \quad s > 0. \quad (1.29)$$

Theorem 5.4. [52] Let $f : [0, +\infty) \rightarrow \mathbb{R}$ be a causal function and $0 < \alpha \leq 1$. Hence, the following relation holds between the conformable Laplace transform and the conventional Laplace transform:

$$F_\alpha(s) = \mathcal{L} \left[f(\alpha t)^{\frac{1}{\alpha}} \right](s). \quad (1.30)$$

Theorem 5.5. [52] Consider $c, a, p \in \mathbb{R}$ and for $0 < \alpha \leq 1$. Thus, the Laplace transform of some usual functions is defined as follows:

1.

$$\mathcal{L}_\alpha[c](s) = \frac{c}{s}, \quad s > 0; \quad (1.31)$$

2.

$$\mathcal{L}_\alpha[t^p](s) = \alpha^{\frac{p}{\alpha}} \frac{\Gamma(1 + \frac{p}{\alpha})}{s^{1 + \frac{p}{\alpha}}}, \quad s > 0; \quad (1.32)$$

3.

$$\mathcal{L}_\alpha[e^{a \frac{t^\alpha}{\alpha}}](s) = \frac{1}{s - a}, \quad s > a. \quad (1.33)$$

5.2 Sumudu transform

The Sumudu transform is a substantial integral transform that emerged in the 1990s as a less complicated alternative to the Laplace transform for solving differential equations. Unlike conventional approaches, it maintains units and scaling, rendering it highly intuitive for applied sciences and engineering. The transform naturally accommodates initial conditions without alteration, simplifying solutions to ODEs, PDEs, and difference equations. Its applications are spread over control theory, signal processing, and fractional calculus; thus, it is pervasive. The following subsection covers necessary terminology and key theorems for the Sumudu transform. Then, the fractional Sumudu transform is introduced.

Definition 5.6. [53] Let $\mathcal{A}$ denote the set of functions of exponential order, defined as follows:

$$\mathcal{A} = \{f(t) | \exists M, \tau_1, \tau_2 > 0, |f(t)| < M e^{-\frac{|t|}{\tau_j}}, \text{ if } t \in (-1)^j \times [0, \infty)\}, \quad (1.34)$$

We define the Sumudu transform of a continuous function as follows:

$$\mathcal{S}[f(t)] = F(v) = \int_0^\infty f(vt) e^{-t} dt, \quad v \in (-\tau_1, \tau_2), \quad (1.35)$$

or a similar alternative

$$\mathcal{S}[f(t)] = F(v) = \frac{1}{v} \int_0^\infty f(t) e^{-\frac{t}{v}} dt, \quad v > 0. \quad (1.36)$$

Some Main Properties of the Sumudu Transform

Theorem 5.7. [54]

We state the fundamental properties of the Sumudu transform for real parameters λ, β, a and given functions f and g .

- **Linearity property**

$$\mathcal{S}[\alpha f + \beta g](v) = \lambda F(v) + \beta G(v), \quad v \in (-\tau_1, \tau_2); \quad (1.37)$$

- **Integral function**

$$\mathcal{S} \left[\int_0^t f(\tau) d\tau \right] = v F(v); \quad (1.38)$$

- **Convolution product**

$$\mathcal{S}[(f * g)(v)] = v \mathcal{S}[f](v) \mathcal{S}[g](v), \quad v \in (-\tau_1, \tau_2); \quad (1.39)$$

where

$$(f * g)(t) = \int_0^t f(\tau) g(t - \tau) d\tau;$$

- **Dirac impulsion**

$$\mathcal{S}[\delta(t)](v) = v^{-1}; \quad (1.40)$$

- **Multiplication by a scalar**

$$\mathcal{S}[f(at)](v) = F(av). \quad (1.41)$$

Conformable Sumudu transform

Theorem 5.8. [52] Let $f : [0, \infty) \rightarrow \mathbb{R}$ be a given function, $0 < \alpha \leq 1$. Then we have the following property

$$\mathcal{S}_\alpha[\mathbf{T}^\alpha f(t)](v) = \frac{1}{v} [\mathcal{S}_\alpha[f(t)](v) - f(0)], \quad \forall t > 0, \quad (1.42)$$

Theorem 5.9. [52] Let c and $a \in \mathbb{R}$ and $0 < \alpha \leq 1$. Then

1.

$$\mathcal{S}_\alpha[e^{-a \frac{t^\alpha}{\alpha}} f(t)] = \frac{\mathcal{S}_\alpha[f(t)](\frac{1}{v} + a)}{v}, \quad v > 0; \quad (1.43)$$

2.

$$\mathcal{S}_\alpha[c](v) = c; \quad (1.44)$$

3.

$$\mathcal{S}_\alpha \left[\frac{t^{n\alpha}}{\alpha^n} \right] (v) = \frac{\Gamma(1+n)^n}{v}, \quad v > 0; \quad (1.45)$$

4.

$$\mathcal{S}_\alpha \left[e^{-\frac{at^\alpha}{\alpha}} \right] (v) = \frac{1}{1+av}, \quad v > \frac{1}{a}. \quad (1.46)$$

Although the Laplace and Sumudu transforms are both integral tools used to solve differential equations, their main difference lies in their kernel and their handling of scales. The Laplace transform uses an exponential kernel e^{-st} , which makes it particularly effective for dealing with initial condition problems and linear time-invariant systems. In contrast, the Sumudu transform is distinguished by its kernel e^{-tu} and its scale invariance property (u -scale invariance), which allows it to preserve the dimensional homogeneity of equations and facilitate the transition between a function and its derivatives. This characteristic makes the Sumudu transform often more direct for manipulating initial conditions and more suitable for certain control and physics problems where scale properties are crucial.

6 Conclusion

This chapter's work provided a theoretical basis for this thesis's computational and analytical strategy. We established classical solution techniques required for dynamical system modeling from the fundamentals of ordinary differential equations (ODEs). From there, we introduced fractional calculus as a successful extension of integer-order calculus to model complex memory-dependent and non-local behaviors, which classical methods cannot model. Furthermore, we discussed the Laplace and Sumudu transforms as basic instruments for solving differential and fractional differential equations, highlighting their advantages in simplifying and unifying analytical techniques. The Sumudu transform, in particular, demonstrated its utility by preserving scaling properties and handling initial conditions with ease. These mathematical tools provide this work's subsequent analytical and numerical investigations with a complete foundation, brokering between theoretical rigor and practical applicability in dynamical systems analysis.

Epidemic models

1 Introduction

Compartmentalized epidemiological models, essential mathematical tools in epidemiology (the study of population health and its causes), divide a population into compartments (such as SIR or SEIR) to model the spread of infectious diseases. These models, which date back to the early 20th century with the pioneering work of En'ko, Hamer, Ross, Kermack, and McKendrick, describe transitions between states of health (susceptible, exposed, infected, recovered) using differential equations to predict epidemic dynamics, assess response strategies and understand disease behavior at the population level based on microscopic data and assumptions.

2 Introduction to epidemiology

Epidemiology is the scientific study of the distribution of health issues within a population and the factors that determine it. Its main objective is to understand the reasons leading to diseases and hence enhance the treatment and preventive techniques for diseases. Modeling and mathematics are key components of epidemiology. Mathematical models in epidemiology describe the progression of a disease over time, which depends on various parameters, including:

1. Disease stages.
2. Mobility of individuals.
3. Disease type.

The usefulness of mathematical models in epidemiology can be summarized as follows:

- Mathematical models help us understand the disease and its development.
- They serve as decision-support tools for health authorities and policymakers.

3 The compartmental models in epidemiology

Compartment models appeared in 1920. These compartmental models are deterministic models where the population is subdivided into several categories according to different characteristics (such as age, sex, particular genetic character) and their disease status (susceptible, infected but not contagious (Exposed), infectious, immune, deceased, etc.). A set of differential equations governs the transition of individuals between these categories.

3.1 Compartments and rules

A compartment model describes disease spread using two key components:

The rules

The rules define how individuals move between compartments, specifying the proportions transitioning from one state to another.

The compartments

The following compartments represent common disease states, with each compartment indicating the number of people in that condition at time t :

- Susceptibles $S(t)$: Individuals that can contract the disease.
- Exposed $E(t)$: Infected individuals who have already been infected but not yet infectious in the latent period.
- Infected $I(t)$: Individuals who can infect others.
- Recovered $R(t)$: People recovered from the infection.
- Deceased $D(t)$: The number of individuals who have died from the disease.
- Quarantine $Q(t)$: Individuals in this state are still potentially infectious but are quarantined to prevent the spread and isolated from the general population through further disease spread.
- Massive Immunity $M(t)$: This can be divided into two compartments. It represents people who have immunity from birth, probably through maternal means. It also arouses people who have the disease but do not show overt symptoms, contributing to the disease's latent or asymptomatic phase.
- Total Population $N(t)$: Represents the overall size of individuals in the model.

3.2 Basic examples of epidemiology models

The value of β plays a central role in modeling disease transmission and may vary depending on the context and specific parameters relevant to the situation under investigation. It is essential to understand that this parameter depends on the host population and the disease itself.

The most commonly used expression for β is as follows:

$$\beta = -c \log(1 - p);$$

In this equation:

- "c" represents the rate of contact between individuals, which depends on the characteristics of the population and how individuals interact with one another.
- "p" corresponds to the probability of infection during an infectious contact, which depends on the specific properties of the disease, such as its virulence and its ability to spread.

All the rates mentioned in the following models are favorable and expressed in units such as $[\text{time unit}]^{-1}$ so that both sides of the ODE systems are consistent with units of $[\text{number of individuals}]$ and $[\text{time unit}]^{-1}$.

SI Model:

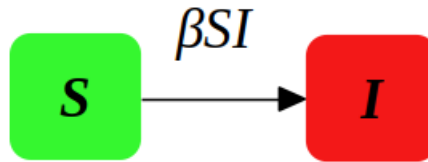


Fig. 2.1: SI epidemiological model diagram in the short term

$$\begin{aligned} \frac{dS}{dt} &= -\beta SI \\ \frac{dI}{dt} &= \beta SI \end{aligned} \tag{2.1}$$

In the long term the number of births and deaths is not negligible,

$$\begin{aligned} \frac{dS}{dt} &= \mu_B N - \beta SI - \mu_D S \\ \frac{dI}{dt} &= \beta SI - \mu_D I \end{aligned} \tag{2.2}$$

Where: $S(t) + I(t) = N(t)$.

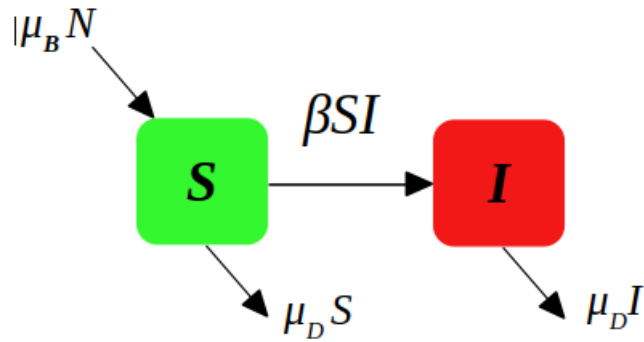


Fig. 2.2: SI epidemiological model diagram in the long term

SIS Model:

In the short term:

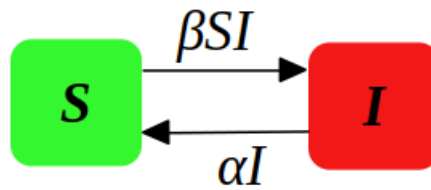


Fig. 2.3: SIS epidemiological model diagram in the short term

$$\begin{aligned}
 \frac{dS}{dt} &= -\beta SI + \alpha I \\
 \frac{dI}{dt} &= \beta SI - \alpha I
 \end{aligned}
 \tag{2.3}$$

In the Long-term:

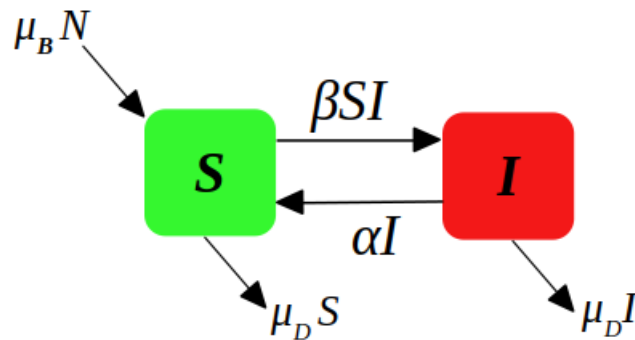


Fig. 2.4: SIS epidemiological model diagram in the long term

$$\begin{aligned}\frac{dS}{dt} &= \mu_B N - \beta SI + \alpha I - \mu_D S \\ \frac{dI}{dt} &= \beta SI - \alpha I - \mu_D I\end{aligned}\tag{2.4}$$

Where: $S(t) + I(t) = N(t)$.

SEI Model:

In the short term:

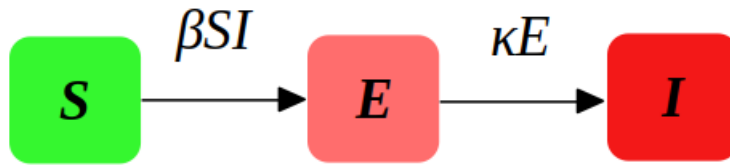


Fig. 2.5: SEI epidemiological model diagram in the short term

$$\begin{aligned}\frac{dS}{dt} &= -\beta SI \\ \frac{dE}{dt} &= \beta SI - \kappa E \\ \frac{dI}{dt} &= \kappa E\end{aligned}\tag{2.5}$$

In the Long-term:

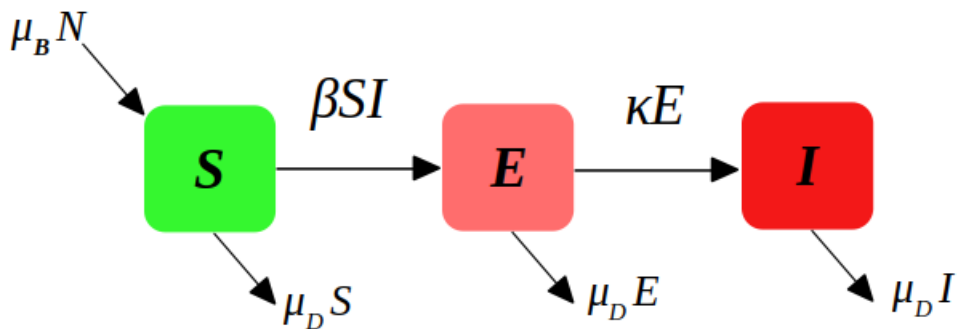


Fig. 2.6: SEI epidemiological model diagram in the long term

$$\begin{aligned}
\frac{dS}{dt} &= \mu_B N - \beta SI - \mu_D S \\
\frac{dE}{dt} &= \beta SI - \kappa E - \mu_D E \\
\frac{dI}{dt} &= \kappa E - \mu_D I
\end{aligned} \tag{2.6}$$

Where: $S(t) + E(t) + I(t) = N(t)$.

SEIS Model:

In the short term:

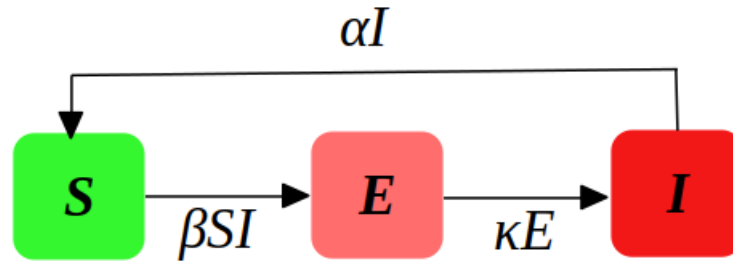


Fig. 2.7: SEIS epidemiological model diagram in the short term

$$\begin{aligned}
\frac{dS}{dt} &= -\beta SI + \alpha I \\
\frac{dE}{dt} &= \beta SI - \kappa E \\
\frac{dI}{dt} &= \kappa E - \alpha I
\end{aligned} \tag{2.7}$$

In the Long-term:

$$\begin{aligned}
\frac{dS}{dt} &= \mu_B N - \beta SI + \alpha I - \mu_D S \\
\frac{dE}{dt} &= \beta SI - \kappa E - \mu_D E \\
\frac{dI}{dt} &= \kappa E - \alpha I - \mu_D I
\end{aligned} \tag{2.8}$$

Where: $S(t) + E(t) + I(t) = N(t)$.

SIR Model:

In the short term:

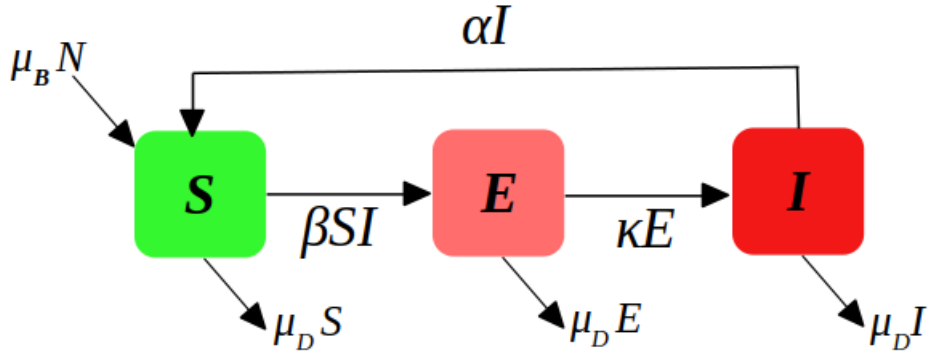


Fig. 2.8: SEIS epidemiological model diagram in the long term

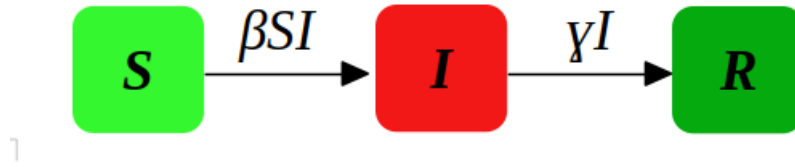


Fig. 2.9: SIR epidemiological model diagram in the short term

$$\begin{aligned}
 \frac{dS}{dt} &= -\beta SI \\
 \frac{dI}{dt} &= \beta SI - \gamma I \\
 \frac{dR}{dt} &= \gamma I
 \end{aligned} \tag{2.9}$$

In the Long-term:

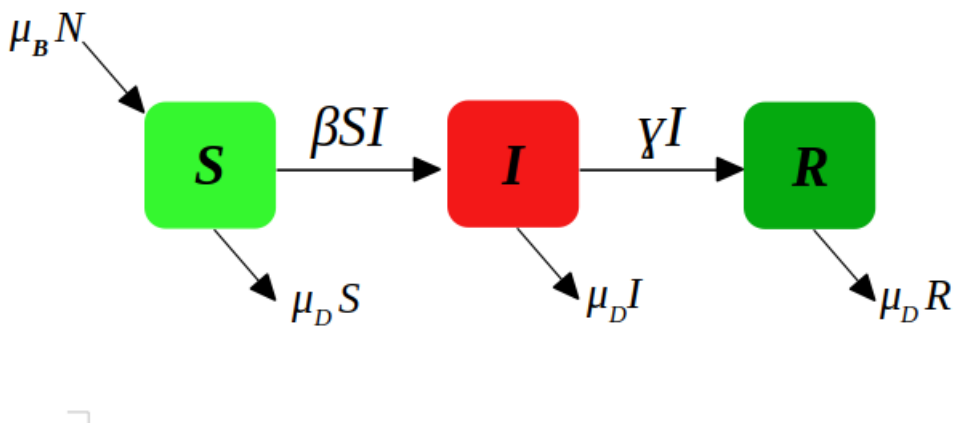


Fig. 2.10: SIR epidemiological model diagram in the long term

$$\begin{aligned}
\frac{dS}{dt} &= \mu_B N - \beta SI - \mu_D S \\
\frac{dI}{dt} &= \beta SI - \gamma I - \mu_D I \\
\frac{dR}{dt} &= \gamma I - \mu_D R
\end{aligned} \tag{2.10}$$

Where: $S(t) + I(t) + R(t) = N(t)$.

SIRS Model:

In the short term:

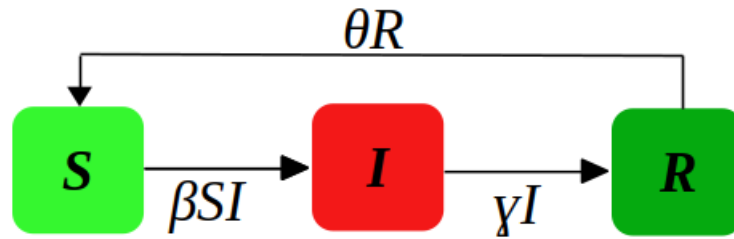


Fig. 2.11: SIRS epidemiological model diagram in the short term

$$\begin{aligned}
\frac{dS}{dt} &= -\beta SI + \theta R \\
\frac{dI}{dt} &= \beta SI - \gamma I \\
\frac{dR}{dt} &= \gamma I - \theta R
\end{aligned} \tag{2.11}$$

In the Long-term:

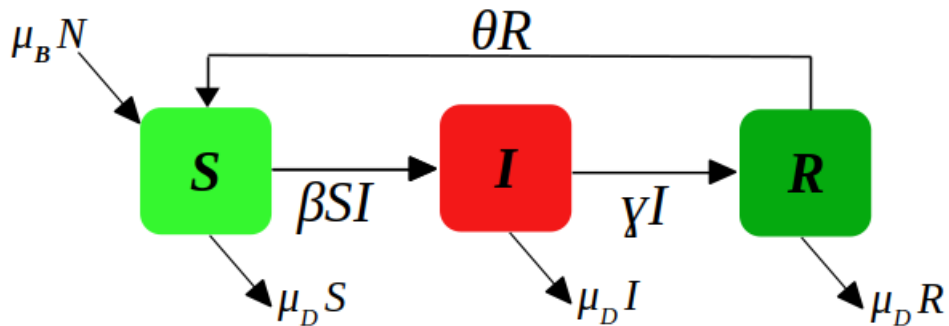


Fig. 2.12: SIRS epidemiological model diagram in the long term

$$\begin{aligned}
\frac{dS}{dt} &= \mu_B N - \beta SI + \theta R - \mu_D S \\
\frac{dI}{dt} &= \beta SI - \gamma I - \mu_D I \\
\frac{dR}{dt} &= \gamma I - \theta R - \mu_D R
\end{aligned} \tag{2.12}$$

Where: $S(t) + I(t) + R(t) = N(t)$.

SEIR Model:

In the short term:

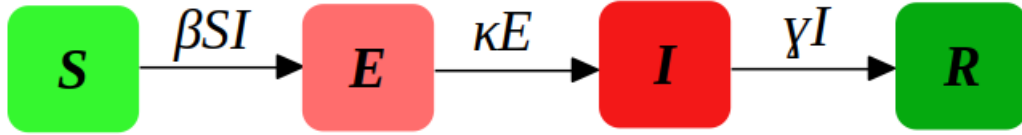


Fig. 2.13: SEIR epidemiological model diagram in the short term

$$\begin{aligned}
\frac{dS}{dt} &= -\beta SI \\
\frac{dE}{dt} &= \beta SI - \kappa E \\
\frac{dI}{dt} &= \kappa E - \gamma I \\
\frac{dR}{dt} &= \gamma I
\end{aligned} \tag{2.13}$$

In the Long-term:

$$\begin{aligned}
\frac{dS}{dt} &= \mu_B N - \beta SI - \mu_D S \\
\frac{dE}{dt} &= \beta SI - \kappa E - \mu_D E \\
\frac{dI}{dt} &= \kappa E - \gamma I - \mu_D I \\
\frac{dR}{dt} &= \gamma I - \mu_D R
\end{aligned} \tag{2.14}$$

Where: $S(t) + E(t) + I(t) + R(t) = N(t)$.

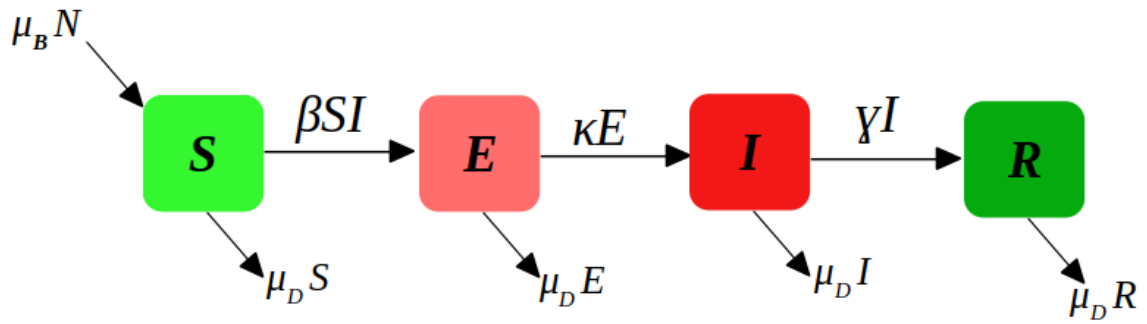


Fig. 2.14: SEIR epidemiological model diagram in the long term

SEIRS Model:

In the short term:

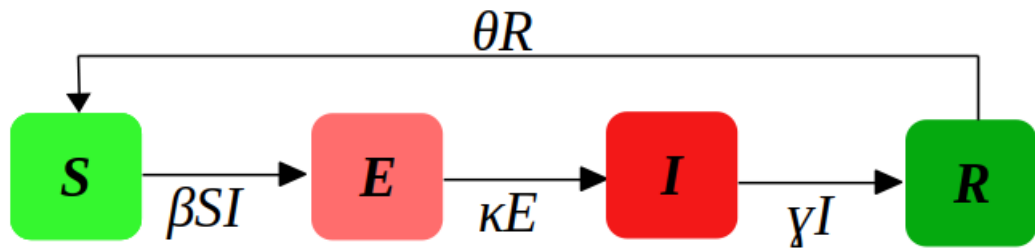


Fig. 2.15: SEIRS epidemiological model diagram in the short term

$$\begin{aligned}
 \frac{dS}{dt} &= -\beta SI + \theta R \\
 \frac{dE}{dt} &= \beta SI - \kappa E \\
 \frac{dI}{dt} &= \kappa E - \gamma I \\
 \frac{dR}{dt} &= \gamma I - \theta R
 \end{aligned}
 \tag{2.15}$$

In the Long-term:

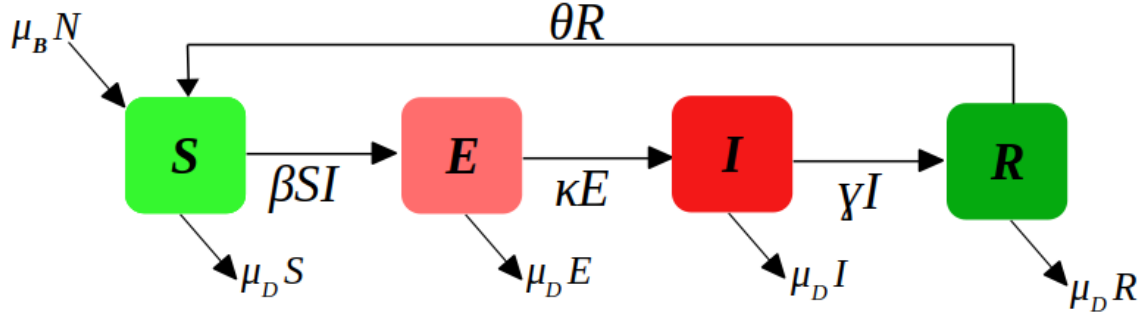


Fig. 2.16: SEIRS epidemiological model diagram in the long term

$$\begin{aligned}
 \frac{dS}{dt} &= \mu_B N - \beta SI + \theta R - \mu_D S \\
 \frac{dE}{dt} &= \beta SI - \kappa E - \mu_D E \\
 \frac{dI}{dt} &= \kappa E - \gamma I - \mu_D I \\
 \frac{dR}{dt} &= \gamma I - \theta R - \mu_D R
 \end{aligned} \tag{2.16}$$

Where: $S(t) + E(t) + I(t) + R(t) = N(t)$.

Such as all the parameters are defined as follows:

- N is the total population.
- β is the transmission rate divided by N .
- κ is the incubation rate.
- γ is the recovered rate.
- α is the transition rate from the infected compartment (I) to the susceptible category (S).
- θ is the transition rate from the recovered compartment (R) to the susceptible category (S).
- μ_B is the natural Birth rate.
- μ_D is the natural death rate.

4 Conclusion

To conclude, the compartmental models are essential to understanding and managing epidemics. They can predict disease trends, assess interventions, and optimize control strategies through extensions that consider more realistic biological and social factors.

Characterization of SEIR model stability

1 Introduction

Natural or artificial, dynamical systems operate in some regimes, often characterized by equilibrium points. These equilibria are important operating states whose stability determines the behavior of the system in the long term.

Parametric dynamical systems are at the core of epidemic modeling, particularly through compartmental models such as the SEIR model (Susceptible – Exposed – Infectious – Recovered). These models frequently exhibit a disease-free equilibrium (DFE), whose stability relies critically on epidemiological parameters.

A fundamental question is identifying the necessary and sufficient conditions under which this equilibrium is asymptotically stable. The basic reproduction number ($\mathcal{R}_0$), as defined by Van den Driessche and Watmough (2002) [55] for finite-dimensional models, is a key metric that quantifies the average number of secondary cases caused by an infected individual in a fully susceptible population.

When $\mathcal{R}_0 < 1$, the DFE is asymptotically stable, whereas for $\mathcal{R}_0 > 1$, it becomes unstable, signaling epidemic spread.

In this chapter, we consider the global stability of the SEIR model in two mathematical frameworks. The classical one is described by a system of first-order ordinary differential equations (ODEs) in which usual derivatives govern evolution. The fractional one, characterized by conformable fractional derivatives encompassing memory effects and non-local dynamics, can better depict slow or anomalous processes.

We first compute the disease-free (DFE) and endemic (DEE) equilibria by solving

both systems and the basic reproduction number. We then discuss stability conditions in each case with the help of Lyapunov functions.

2 Preliminary stability of the model

2.1 Method for calculating the basic reproduction number $\mathcal{R}_0$

This technique was first developed by Diekmann [hamadi.26] and then revisited by Van den Driessche and Watmough [55] for finite-dimensional systems.

We consider an epidemiological model comprising n compartments

$$\begin{aligned}\dot{x}_i &= f_i(x) \\ &= F_i(x) + V_i^+(x) - V_i^-(x).\end{aligned}\tag{3.1}$$

Here x_i (where $i \in \{1, \dots, n\}$) denotes the population size in each compartment. The first m compartments correspond to infected states (such as latent or infectious individuals), while the remaining compartments represent uninfected individuals (e.g., susceptible or recovered populations).

Definition 2.1. [55] Let ω be defined as the set of all disease-free states. That is

$$\omega = \{x \geq 0 \mid x_i = 0, i \in \{1, \dots, m\}\}\tag{3.2}$$

Remark 2.2. [55]

The function $F_i(x)$ denotes the rate at which new infections appear in compartment i , while $V_i^+(x)$ represents the inflow rate of individuals into compartment i for any other cause.

The functions $V_i^-(x)$ denote the transfer rate (outflow) of individuals outside compartment i . For biological reasons, the functions $F_i(x)$, $V_i^+(x)$, $V_i^-(x)$ are assumed to satisfy the hypotheses (1) to (5) as follows:

1. If $x \geq 0$, then $F_i(x)$, $V_i^+(x)$, $V_i^-(x) \geq 0$ for all $i \in \{1, \dots, n\}$.
2. If $x_i = 0$, then $V_i^-(x) = 0$ in particular, if $x \in \omega$ then $V_i^-(x) = 0$ for all $i \in \{1, \dots, m\}$.
3. If $i > m$, then $F_i(x) = 0$.
4. If $x \in \omega$, then $F_i(x) = V_i^+(x) = 0$ for $i \in \{1, \dots, m\}$.
5. If $F_i(x) = 0$, the system admits a locally asymptotically stable equilibrium point x such that $D(V_i^+ - V_i^-)(x)$ has strictly negative eigenvalues.

The functions $F_i(x)$, $V_i^+(x)$, $V_i^-(x)$ are assumed to be at least twice differentiable.

We introduce Lemma and the following theorem for utility, based on the reference below [55].

Definition 2.3. A state x_0 of the system is considered disease-free if no infected individuals are present. This equilibrium, where the disease cannot persist, is known as the Disease-Free Equilibrium (DFE).

Lemma 2.4. [55] If x_0 is a disease-free equilibrium (DFE) of (3.1) and $f_i(x)$ satisfies the hypotheses (1)-(5), then the derivatives $DF(x_0)$ and $DV(x_0)$ are partitioned as

$$DF(x_0) = \begin{pmatrix} \mathbf{F} & 0 \\ 0 & 0 \end{pmatrix}, \quad DV(x_0) = \begin{pmatrix} \mathbf{V} & 0 \\ J_3 & J_4 \end{pmatrix}, \quad (3.3)$$

where F and V are the $m \times m$ matrices defined by

$$\mathbf{F} = \left[\frac{\partial F^i}{\partial x_j}(x_0) \right] \quad \text{and} \quad \mathbf{V} = \left[\frac{\partial V^i}{\partial x_j}(x_0) \right] \quad \text{with } 1 \leq i, j \leq m. \quad (3.4)$$

Moreover, $\mathbf{F} \geq 0$, $\mathbf{V}$ is a nonsingular M -matrix, and every eigenvalue of J_4 has a positive real part.

Definition 2.5. [56] The basic reproduction number $\mathcal{R}_0$ is given by the spectral radius of the next-generation matrix $K = -\mathbf{F}\mathbf{V}^{-1}$. That is,

$$\mathcal{R}_0 = \rho(-\mathbf{F}\mathbf{V}^{-1}). \quad (3.5)$$

where

- $\mathbf{F}$ is the matrix of new infection rates,
- $\mathbf{V}$ is the matrix of transition rates between infected compartments.

Theorem 2.6. [55] The disease-free equilibrium x_0 of system (3.1) (under assumptions (1)-(5)) is locally asymptotically stable when $(\mathcal{R}_0 < 1)$ and unstable when $(\mathcal{R}_0 > 1)$.

2.2 Lyapunov stability

In stability analysis, the Lyapunov method is a standard tool for proving the global stability of both disease-free and endemic equilibrium points. The process consists of finding a function V , which is generally complicated. The general form of the Lyapunov function is defined as follows:

$$V = \sum_{i=1}^n a_i (x_i - \bar{x}_i \log x_i). \quad (3.6)$$

Theorem 2.7. [57] Let $x_0 = 0$ be an equilibrium point for (3.1) and $D \subset \mathbb{R}^n$ be a domain containing $x_0 = 0$. Let $W : D \rightarrow \mathbb{R}$ be a continuously differentiable function such that

$$V(0) = 0 \quad \text{and} \quad V(x_0) > 0 \text{ in } D \setminus \{0\} \quad (3.7)$$

$$\dot{V}(x_0) \leq 0 \text{ in } D \quad (3.8)$$

Then, $x_0 = 0$ is stable. Moreover, if

$$\dot{V}(x_0) < 0 \text{ in } D \setminus \{0\} \quad (3.9)$$

Then $x_0 = 0$ is asymptotically stable.

Theorem 2.8. (*The principle of invariance of Salle*)[58] Consider an autonomous system $\dot{x} = f(x)$, where $f : D \rightarrow \mathbb{R}^n$ is locally Lipschitz (or continuous) and $D \subset \mathbb{R}^n$ is a compact positively invariant set. Let $V : D \rightarrow \mathbb{R}$ be a continuously differentiable **Lyapunov function** such that:

1. $V(x) \geq 0$ for all $x \in D$,
2. $\dot{V}(x) \leq 0$ for all $x \in D$ (where $\dot{V}(x) = \nabla V \cdot f(x)$).

Define the set $E = \{x \in D \mid \dot{V}(x) = 0\}$, and let M be the largest invariant subset of E . Then, every solution starting in D approaches M as $t \rightarrow \infty$.

Corollary 2.9. [59] Assume the hypotheses of the previous theorem hold. If the set L is the singleton $\{\bar{x}\}$, where $\bar{x} \in D$, then $\bar{x}$ is globally asymptotically stable.

3 Stability analysis of the SEIR ODE model

Dynamic systems generally seek to maintain an equilibrium state, and we study their ability to return to it after a disturbance (asymptotic stability). In epidemiology, models often have a disease-free equilibrium. The stability of this equilibrium is determined by the basic reproduction number $\mathcal{R}_0$: if $\mathcal{R}_0 < 1$, the equilibrium is stable; if $\mathcal{R}_0 > 1$, it is unstable. $\mathcal{R}_0$ represents the average number of secondary cases generated by an infectious individual in a completely susceptible population. The Kermack and McKendrick model is a classic example, but it does not account for demographic factors.

3.1 Problem statement

SEIR models are also important as conceptual models (similar to prey-predator models and competitive species in ecology). In this section, we study the stability of the following system:

$$\begin{cases} \dot{S}(t) &= \mu N(t) - \beta S(t)I(t) - \mu S(t) \\ \dot{E}(t) &= \beta S(t)I(t) - (\kappa + \mu)E(t) \\ \dot{I}(t) &= \kappa E(t) - (\gamma + \mu)I(t) \\ \dot{R}(t) &= \gamma I(t) - \mu R(t) \end{cases} \quad (3.10)$$

3.2 Positivity of solutions

Theorem 3.1. Let $\Omega = \{(S(t), E(t), I(t), R(t)) \in \mathbb{R}^4 \mid S(t) + E(t) + I(t) + R(t) \leq N(t), S(0) > 0, E(0) > 0, I(0) > 0, R(0) > 0\}$, then the solutions $\{S(t), E(t), I(t), R(t)\}$ of the system (3.10) are positive for $t \geq 0$.

Proof. From the first equation of system (3.10), we have:

$$\begin{aligned} \frac{dS}{dt} &= \mu N(t) - \beta S(t)I(t) - \mu S(t) \\ \implies \frac{dS}{dt} &\geq -\mu S(t). \end{aligned} \quad (3.11)$$

Integrating Equation (3.11), we get

$$S(t) \geq S_0 e^{-\mu t} \geq 0. \quad (3.12)$$

Hence, $S(t)$ remains non-negative for all $t > 0$. Similarly, using the remaining equations of system (3.10), we obtain $E(t) \geq 0$, $I(t) \geq 0$, and $R(t) \geq 0$. □

Remark 3.2. Birth and death rates are typically assumed to be equal.

3.3 Determination of equilibrium points

In the case of epidemiological models, it is essential to determine the equilibrium points. For every initial condition, the system trajectories (the state variables $\{S(t), E(t), I(t), R(t), t > 0\}$) will asymptotically approach one of these equilibrium points as $t \rightarrow +\infty$. An equilibrium point is asymptotically stable if the nearby trajectories converge as time progresses.

The equilibrium points are obtained by solving the system of equations derived from setting all time derivatives to zero:

$$\begin{cases} \dot{S}(t) = 0 \\ \dot{E}(t) = 0 \\ \dot{I}(t) = 0 \\ \dot{R}(t) = 0 \end{cases} \quad (3.13)$$

$$\dot{N}(t) = \dot{S}(t) + \dot{E}(t) + \dot{I}(t) + \dot{R}(t) = 0 \quad (3.14)$$

So, the size of the population is constant, we can then put the equation of recovery. The reduced system is the following.

With a constant total population, the equation of the recovered compartment can be integrated into the system. The reduced system, therefore, becomes:

$$\begin{cases} \dot{S}(t) = \mu N(t) - \beta S(t)I(t) - \mu S(t) \\ \dot{E}(t) = \beta S(t)I(t) - (\kappa + \mu)E(t) \\ \dot{I}(t) = \kappa E(t) - (\gamma + \mu)I(t) \end{cases} \quad (3.15)$$

Thus,

$$\dot{I} = 0 \implies E = \frac{\gamma + \mu}{\kappa} I. \quad (3.16)$$

By replacing (3.16) in $\dot{E}(t)$ we obtain

$$\begin{aligned}
& \beta SI - \frac{(\kappa + \mu)(\gamma + \mu)}{\kappa} I = 0 \\
\Rightarrow & I \left(\beta S - \frac{(\kappa + \mu)(\gamma + \mu)}{\kappa} \right) = 0 \\
\Rightarrow & \begin{cases} I = 0 \\ \vee \\ S = \frac{(\kappa + \mu)(\gamma + \mu)}{\beta \kappa} \end{cases}
\end{aligned} \tag{3.17}$$

The free-disease equilibrium point X_0 of system (3.10) is obtained by setting all the derivatives to zero with $I = 0$, that yields to:

$$X_0 = (N, 0, 0). \tag{3.18}$$

3.4 Calculation of the Basic Reproduction Number

$\mathcal{R}_0$ is computed using the next-generation matrix concept of Van den Driessche and Watmough [55]. Let $X = (E, I)^T$. System (3.10) can be written as

$$\dot{x}_i = F_i(x) - V_i(x), \tag{3.19}$$

where

$$F_i(x) = (\beta SI, 0)^T \quad \text{and} \quad V_i(x) = ((\mu + \kappa)E, -\kappa E + (\mu + \gamma)I)^T. \tag{3.20}$$

Their corresponding Jacobian matrices at the disease-free equilibrium are

$$\mathbf{F} = DF_i(x) = \begin{pmatrix} 0 & \beta S_0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{V} = DV_i(x) = \begin{pmatrix} \mu + \kappa & 0 \\ -\kappa & \mu + \gamma \end{pmatrix}. \tag{3.21}$$

where D is the derivative with respect to (E, I) , $\mathbf{F}$ is non-negative, and $\mathbf{V}$ is non-singular.

We write the Jacobian matrices F at the point $(S_0, 0, 0)$ without disease:

$$\mathbf{F}(S_0, 0, 0) = \begin{pmatrix} 0 & \beta N \\ 0 & 0 \end{pmatrix} \tag{3.22}$$

$$\begin{aligned}
\mathbf{V}(S_0, 0, 0) &= \begin{pmatrix} \mu + \kappa & 0 \\ -\kappa & \mu + \gamma \end{pmatrix} \\
\Rightarrow \det(\mathbf{V}(S_0, 0, 0)) &= (\mu + \kappa)(\gamma + \mu) \geq 0 \\
\Rightarrow \mathbf{V}^{-1} &= \frac{1}{\det(\mathbf{V}(S_0, 0, 0))} (\text{com } \mathbf{V}(S_0, 0, 0))^T \\
\Rightarrow \mathbf{V}^{-1} &= \begin{pmatrix} \frac{1}{\mu + \kappa} & 0 \\ \frac{\kappa}{(\mu + \kappa)(\gamma + \mu)} & \frac{1}{\gamma + \mu} \end{pmatrix}
\end{aligned} \tag{3.23}$$

To obtain the next-generation matrix:

$$\begin{aligned}
\mathbf{FV}^{-1} &= \begin{pmatrix} \frac{\kappa\beta N}{(\mu + \kappa)(\gamma + \mu)} & \frac{\beta N}{\gamma + \mu} \\ 0 & 0 \end{pmatrix} \\
\det(\mathbf{FV}^{-1} - \lambda I_d) &= 0
\end{aligned} \tag{3.24}$$

$$\begin{aligned}
&\Leftrightarrow \begin{vmatrix} \frac{\kappa\beta N}{(\mu + \kappa)(\gamma + \mu)} - \lambda & \frac{\beta N}{\gamma + \mu} \\ 0 & -\lambda \end{vmatrix} = 0 \\
&\Leftrightarrow \lambda \left(\lambda - \frac{\kappa\beta N}{(\mu + \kappa)(\gamma + \mu)} \right) = 0 \\
&\Leftrightarrow \begin{cases} \lambda = 0 \\ \vee \\ \lambda = \frac{\kappa\beta N}{(\mu + \kappa)(\gamma + \mu)} \end{cases}
\end{aligned} \tag{3.25}$$

Consequently:

$$\begin{aligned}
\mathcal{R}_0 &= \max \left\{ 0, \frac{\kappa\beta N}{(\mu + \kappa)(\gamma + \mu)} \right\} \\
\Rightarrow \mathcal{R}_0 &= \frac{\kappa\beta N}{(\mu + \kappa)(\gamma + \mu)}
\end{aligned} \tag{3.26}$$

3.5 Global Asymptotic Stability of the Disease-Free Equilibrium

Dynamic systems are widely used in biological modeling, particularly in epidemiology. In many epidemiological models, the disease-free equilibrium is stable when a threshold quantity, the introductory reproduction rate, is less than 1. When the same quantity exceeds 1, the disease-free equilibrium loses its stability in favor of an endemic equilibrium.

The overall stability of the endemic equilibrium when $\mathcal{R}_0 > 1$ in SEIR models has

long been conjectural. Li and Muldowney proved this in 1995 [12]. The evidence is based on the competitive nature of the system.

Theorem 3.3. *If $\mathcal{R}_0 \leq 1$ then the DFE is globally asymptotically stable on Ω .*

Proof. We consider the Lyapunov function

$$V(E, I) = E + \frac{\gamma + \mu}{\gamma} I. \quad (3.27)$$

The derivative of the Lyapunov function is given by:

$$\begin{aligned} \dot{V}(E, I) &= \dot{E} + \frac{\kappa + \mu}{\kappa} \dot{I}, \\ &= \beta SI - (\kappa + \mu)E + \frac{\kappa + \mu}{\kappa} (\kappa E - (\gamma + \mu)I), \\ &= \beta SI - \frac{(\kappa + \mu)(\gamma + \mu)}{\kappa} I, \end{aligned} \quad (3.28)$$

by replacing $\beta = \frac{\mathcal{R}_0}{\kappa N} (\kappa + \mu)(\gamma + \mu)$ we obtain

$$\dot{V}(E, I) = \frac{(\kappa + \mu)(\gamma + \mu)}{\kappa} \left[\frac{\mathcal{R}_0 S}{N} - 1 \right] I \leq 0. \quad (3.29)$$

Thus, $\dot{V}(E, I) < 0$ if $\mathcal{R}_0 < 1$ and $\dot{V}(E, I) = 0$ if $I(t) = 0$ and $\mathcal{R}_0 = 1$. Therefore, the largest invariant set contained in this set

$$L = \left\{ (S, E, I) \in \Omega / \dot{V}(E, I) = 0 \right\}$$

which is reduced to DFE. Hence, by LaSalle's Invariance Principle [59], it follows that the disease-free equilibrium point is Globally Asymptotically Stable (GAS) in Ω whenever $\mathcal{R}_0 < 1$.

□

3.6 Global Asymptotic Stability of the Disease Endemic Equilibrium

In this part we consider the system (3.15).

Then, by setting all the derivatives to zero with $I \neq 0$ we obtain $S = \frac{N}{\mathcal{R}_0}$.

By substituting S into the first equation of (3.15) we obtain $I = \frac{\mu}{\beta} (\mathcal{R}_0 - 1)$.

and, by substituting I in (3.16) we obtain $E = \frac{\mu(\kappa + \mu)}{\kappa\beta} (\mathcal{R}_0 - 1)$.

Finally, the disease endemic equilibrium point X_1 of the system (3.10) yields to:

$$X_1 = \left(\frac{N}{\mathcal{R}_0}, \frac{\mu(\gamma + \mu)}{\kappa\beta} (\mathcal{R}_0 - 1), \frac{\mu}{\beta} (\mathcal{R}_0 - 1) \right). \quad (3.30)$$

When $\mathcal{R}_0 = 1$, this equilibrium coincides with DFE; thus, there exists a unique equilibrium within the simplex if and only if $\mathcal{R}_0 > 1$.

Theorem 3.4. *If $\mathcal{R}_0 > 1$, then the endemic point (S, E, I) is globally asymptotically stable.*

Proof. We introduce the Lyapunov candidate function as follows [9]:

$$V(S, E, I) = (S - S_1 \ln S) + (E - E_1 \ln E) + (I - I_1 \ln I) \quad (3.31)$$

which satisfies the imposed conditions:

1.

$$V(S, E, I) \geq 0 \quad (3.32)$$

2.

$$\begin{aligned} \frac{dV}{dt} &= \left(\frac{S - S_1}{S} \right) \frac{dS}{dt} + \left(\frac{E - E_1}{E} \right) \frac{dE}{dt} + \left(\frac{I - I_1}{I} \right) \frac{dI}{dt} \\ &= \left(\frac{S - S_1}{S} \right) (\mu N - \beta SI - \mu S) + \left(\frac{E - E_1}{E} \right) (\beta SI - (\kappa + \mu)E) \\ &\quad + \left(\frac{I - I_1}{I} \right) (\kappa E - (\gamma + \mu)I). \end{aligned} \quad (3.33)$$

Where:

$$\begin{aligned} P &= \mu N + \mu S_1 + \beta S_1 I + E_1(\kappa + \mu) \\ Q &= \mu E + \mu S + I(\gamma + \mu) + \frac{S_1}{S} \mu N + \frac{S_1}{S} \beta SI. \end{aligned} \quad (3.34)$$

We conclude that, if $P < Q$, then $\frac{dV}{dt} \leq 0$ if $S = S_1$, $E = E_1$ and $I = I_1$. Thus, the largest invariant set is given by:

$$L = \{(S, E, I) \in \Omega \mid \frac{dV}{dt} = 0\}$$

It is reduced to the endemic equilibrium according to the principle of invariance at the point (S_1, E_1, I_1) , given $P < Q$. □

4 Stability Analysis of the SEIR Conformable Model

4.1 Problem Statement

The SEIR compartment model is a classical epidemiology approach for modeling infectious disease dynamics. In fractional form, the integer time derivatives are replaced with non-integer differential operators, which give extra flexibility to model long-memory effects or anomalous dynamics. We examine the stability of the fractional-order SEIR model by utilizing the conformable fractional derivative of order $0 < \alpha \leq 1$, as is given by:

$$\begin{cases} \mathbf{T}^\alpha S(t) &= \mu^\alpha N - \beta^\alpha SI - \mu^\alpha S, \\ \mathbf{T}^\alpha E(t) &= \beta^\alpha SI - (\kappa^\alpha + \mu^\alpha)E, \\ \mathbf{T}^\alpha I(t) &= \kappa^\alpha E - (\gamma^\alpha + \mu^\alpha)I, \\ \mathbf{T}^\alpha R(t) &= \gamma^\alpha I - \mu^\alpha R \end{cases} \quad (3.35)$$

4.2 Positivity of solutions

Theorem 4.1. *Let $\Omega = \{(S, E, I, R) \in \mathbb{R}^4 \mid S + E + I + R \leq N, S(0) > 0, E(0) > 0, I(0) > 0, R(0) > 0\}$, then the solutions $\{S(t), E(t), I(t), R(t)\}$ of the system (3.35) are positive for $t \geq 0$.*

Proof. From the first equation of system (3.35), we have:

$$\mathbf{T}^\alpha(S + E + I + R)(t) = \mu^\alpha N - \mu^\alpha(S + E + I + R)(t)$$

which implies that

$$\mathbf{T}^\alpha N(t) = \mu^\alpha N - \mu^\alpha N(t)$$

and therefore

$$\mathbf{T}^\alpha N(t) = 0$$

Applying Sumudu transform, we obtain

$$\frac{1}{v} [\mathcal{S}_\alpha[N(t)](v) - N(0)] = 0$$

Thus,

$$\mathcal{S}_\alpha[N(t)](v) = N(0)$$

As a result,

$$N(t) = N(0) \geq 0$$

So, the functions S, E, I, and R are all non-negative. □

Remark 4.2. *Usually, the death rate and birth rate are assumed to be equal.*

4.3 Determination of Equilibrium points

The equilibrium points are obtained by solving the system of equations derived from setting all time derivatives to zero:

$$\begin{cases} \mathbf{T}^\alpha S(t) = 0 \\ \mathbf{T}^\alpha E(t) = 0 \\ \mathbf{T}^\alpha I(t) = 0 \\ \mathbf{T}^\alpha R(t) = 0 \end{cases} \quad (3.36)$$

$$\mathbf{T}^\alpha N(t) = \mathbf{T}^\alpha S(t) + \mathbf{T}^\alpha E(t) + \mathbf{T}^\alpha I(t) + \mathbf{T}^\alpha R(t) = 0 \quad (3.37)$$

So, the size of the population is constant, we can then put the equation of recovery. The reduced system is:

Since the population size remains constant, we can incorporate the equation for recovered individuals. The reduced system, therefore, becomes:

$$\begin{cases} \mathbf{T}^\alpha S(t) = \mu^\alpha N - \beta^\alpha SI - \mu^\alpha S, \\ \mathbf{T}^\alpha E(t) = \beta^\alpha SI - (\kappa^\alpha + \mu^\alpha)E, \\ \mathbf{T}^\alpha I(t) = \kappa^\alpha E - (\gamma^\alpha + \mu^\alpha)I \end{cases} \quad (3.38)$$

Thus,

$$\mathbf{T}^\alpha I(t) = 0 \implies E = \frac{\gamma^\alpha + \mu^\alpha}{\kappa^\alpha} I. \quad (3.39)$$

By replacing (3.39) in the second equation ($\mathbf{T}^\alpha E(t)$) of (3.38), we obtain

$$\begin{aligned} \beta^\alpha SI - \frac{(\kappa^\alpha + \mu^\alpha)(\gamma^\alpha + \mu^\alpha)}{\kappa^\alpha} I &= 0 \\ \implies I \left(\beta^\alpha S - \frac{(\kappa^\alpha + \mu^\alpha)(\gamma^\alpha + \mu^\alpha)}{\kappa^\alpha} \right) &= 0 \\ \implies \begin{cases} I = 0 \\ \vee \\ S = \frac{(\kappa^\alpha + \mu^\alpha)(\gamma^\alpha + \mu^\alpha)}{\beta^\alpha \kappa^\alpha} \end{cases} & \end{aligned} \quad (3.40)$$

The free-disease equilibrium point X_0 of system (3.35) is obtained by setting all the derivatives to zero with $I = 0$, that yields to:

$$\tilde{X}_0 = X_0 = (N, 0, 0). \quad (3.41)$$

4.4 Calculation of the Basic Reproduction Number

$\mathcal{R}_0$ is computed using the next-generation matrix concept of Van den Driessche and Watmough [55]. Let $X = (E, I)^T$, system (3.35) can be written as

$$\dot{x}_i = \mathcal{F}_i(x) - \mathcal{V}_i(x), \quad (3.42)$$

where

$$\mathcal{F}_i(x) = (\beta^\alpha SI, 0)^T \quad \text{and} \quad \mathcal{V}_i(x) = ((\mu^\alpha + \kappa^\alpha)E, -\kappa^\alpha E + (\mu^\alpha + \gamma^\alpha)I)^T. \quad (3.43)$$

Their corresponding Jacobian matrices at the disease-free equilibrium are

$$\mathbf{F} = D\mathcal{F}_i(x) = \begin{pmatrix} 0 & \beta^\alpha S_0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{V} = D\mathcal{V}_i(x) = \begin{pmatrix} \mu^\alpha + \kappa^\alpha & 0 \\ -\kappa^\alpha & \mu^\alpha + \gamma^\alpha \end{pmatrix}. \quad (3.44)$$

where D is the derivative with respect to (E, I) , $\mathbf{F}$ is non-negative, and $\mathbf{V}$ is non-singular.

We write the Jacobian matrices F at the point without disease $X_0 = (S_0, 0, 0) = (N, 0, 0)$ which is defined in (3.41):

$$\mathbf{F}(N, 0, 0) = \begin{pmatrix} 0 & \beta^\alpha N \\ 0 & 0 \end{pmatrix} \quad (3.45)$$

$$\mathbf{V}(N, 0, 0) = \begin{pmatrix} \mu^\alpha + \kappa^\alpha & 0 \\ -\kappa^\alpha & \mu^\alpha + \gamma^\alpha \end{pmatrix}$$

$$\Rightarrow \det(\mathbf{V}(N, 0, 0)) = (\mu^\alpha + \kappa^\alpha)(\gamma^\alpha + \mu^\alpha) > 0$$

$$\Rightarrow \mathbf{V}^{-1} = \frac{1}{\det(\mathbf{V}(N, 0, 0))} (\text{com } \mathbf{V}(N, 0, 0))^T \quad (3.46)$$

$$\Rightarrow \mathbf{V}^{-1} = \begin{pmatrix} \frac{1}{\mu^\alpha + \kappa^\alpha} & 0 \\ \frac{\kappa^\alpha}{(\mu^\alpha + \kappa^\alpha)(\gamma^\alpha + \mu^\alpha)} & \frac{1}{\gamma^\alpha + \mu^\alpha} \end{pmatrix}$$

To obtain the next-generation matrix:

$$\mathbf{FV}^{-1} = \begin{pmatrix} \frac{\kappa^\alpha \beta^\alpha N}{(\mu^\alpha + \kappa^\alpha)(\gamma^\alpha + \mu^\alpha)} & \frac{\beta^\alpha N}{\gamma^\alpha + \mu^\alpha} \\ 0 & 0 \end{pmatrix} \quad (3.47)$$

$$\det(\mathbf{FV}^{-1} - \lambda I_d) = 0$$

$$\Leftrightarrow \begin{vmatrix} \frac{\kappa^\alpha \beta^\alpha N}{(\mu^\alpha + \kappa^\alpha)(\gamma^\alpha + \mu^\alpha)} - \lambda & \frac{\beta^\alpha N}{\gamma^\alpha + \mu^\alpha} \\ 0 & -\lambda \end{vmatrix} = 0 \quad (3.48)$$

$$\Leftrightarrow \lambda \left(\lambda - \frac{\kappa^\alpha \beta^\alpha N}{(\mu^\alpha + \kappa^\alpha)(\gamma^\alpha + \mu^\alpha)} \right) = 0$$

$$\Leftrightarrow \begin{cases} \lambda = 0 \\ \lambda = \frac{\kappa^\alpha \beta^\alpha N}{(\mu^\alpha + \kappa^\alpha)(\gamma^\alpha + \mu^\alpha)} \end{cases}$$

The basic reproduction number $\mathcal{R}_0$ is given by:

$\mathcal{R}_0 = \rho(-\mathbf{FV}^{-1})$ the spectral radius of the next-generation matrix.

Consequently,

$$\mathcal{R}_0 = \max \left\{ 0, \frac{\kappa^\alpha \beta^\alpha N}{(\mu^\alpha + \kappa^\alpha)(\gamma^\alpha + \mu^\alpha)} \right\} \quad (3.49)$$

$$\text{thus } \mathcal{R}_0 = \frac{\kappa^\alpha \beta^\alpha N}{(\mu^\alpha + \kappa^\alpha)(\gamma^\alpha + \mu^\alpha)}$$

The basic reproduction number is used to measure the transmission potential of a disease. It is the number of secondary cases or new infections produced by a single infectious individual in a completely susceptible population [55, 56, 60]. An epidemic is expected to increase exponentially if $\mathcal{R}_0 > 1$ and to end if $\mathcal{R}_0 < 1$ [55].

4.5 Global stability of the DFE

The disease-free equilibrium is stable when a threshold quantity, the introductory reproduction rate, is less than 1. When the same quantity exceeds 1, the disease-free equilibrium loses its stability in favor of an endemic equilibrium.

Theorem 4.3. *If $\mathcal{R}_0 \leq 1$ then the DFE is globally asymptotically stable on Ω .*

Proof. We consider the Lyapunov function [12]

$$\mathcal{V}(E, I) = E + \frac{\gamma^\alpha + \mu^\alpha}{\gamma^\alpha} I. \quad (3.50)$$

The Conformal derivative of the Lyapunov function, denoted by $\mathbf{T}^\alpha \mathcal{V}(E, I)$ is given by:

$$\begin{aligned} \mathbf{T}^\alpha \mathcal{V}(E, I) &= \mathbf{T}^\alpha E(t) + \frac{\kappa^\alpha + \mu^\alpha}{\kappa^\alpha} \mathbf{T}^\alpha I(t), \\ &= \beta^\alpha SI - (\kappa^\alpha + \mu^\alpha)E + \frac{\kappa^\alpha + \mu^\alpha}{\kappa^\alpha} (\kappa^\alpha E - (\gamma^\alpha + \mu^\alpha)I), \\ &= \beta^\alpha SI - \frac{(\kappa^\alpha + \mu^\alpha)(\gamma^\alpha + \mu^\alpha)}{\kappa^\alpha} I, \end{aligned} \quad (3.51)$$

by replacing $\beta^\alpha = \frac{\mathcal{R}_0}{\kappa^\alpha N} (\kappa^\alpha + \mu^\alpha)(\gamma^\alpha + \mu^\alpha)$ then

$$\mathbf{T}^\alpha \mathcal{V}(E, I) = \frac{(\kappa^\alpha + \mu^\alpha)(\gamma^\alpha + \mu^\alpha)}{\kappa^\alpha} \left[\frac{\mathcal{R}_0 S}{N} - 1 \right] I \leq 0. \quad (3.52)$$

Thus, $\mathbf{T}^\alpha \mathcal{V}(E, I) < 0$ if $\mathcal{R}_0 < 1$ and $\mathbf{T}^\alpha \mathcal{V}(E, I) = 0$ if $I(t) = 0$ and $\mathcal{R}_0 = 1$. Therefore, the largest invariant set contained in this set

$$L = \{(S, E, I) \in \Omega / \mathbf{T}^\alpha \mathcal{V}(E, I) = 0\}$$

which is reduced to DFE. Hence, by LaSalle's Invariance Principle [59], it follows that the disease-free equilibrium point is Globally Asymptotically Stable in Ω whenever $\mathcal{R}_0 < 1$.

□

4.6 Global stability of the DEE

In this part we consider the system (3.38).

Then, by setting all the derivatives to zero with $I \neq 0$ we obtain $S = \frac{N}{\mathcal{R}_0}$.

By substituting S into the first equation of (3.38), we obtain $I = \frac{\mu^\alpha}{\beta^\alpha} (\mathcal{R}_0 - 1)$,

and I in (3.39), gives $E = \frac{\mu^\alpha(\kappa^\alpha + \mu^\alpha)}{\kappa^\alpha \beta^\alpha} (\mathcal{R}_0 - 1)$.

Finally, the disease endemic equilibrium point $\tilde{X}_1$ of the system (3.10) yields to,

$$\tilde{X}_1 = \left(\frac{N}{\mathcal{R}_0}, \frac{\mu^\alpha(\gamma^\alpha + \mu^\alpha)}{\kappa^\alpha \beta^\alpha} (\mathcal{R}_0 - 1), \frac{\mu^\alpha}{\beta^\alpha} (\mathcal{R}_0 - 1) \right). \quad (3.53)$$

When $\mathcal{R}_0 = 1$, this equilibrium coincides with DFE; thus, there exists a unique equilibrium within the simplex if and only if $\mathcal{R}_0 > 1$.

Theorem 4.4. *If $\mathcal{R}_0 > 1$, then the endemic point (S, E, I) is globally asymptotically stable.*

Proof. We introduce the Lyapunov candidate function as follows [9],

$$V(S, E, I) = (S - \tilde{S}_1 \ln S) + (E - \tilde{E}_1 \ln E) + (I - \tilde{I}_1 \ln I) \quad (3.54)$$

which satisfies the imposed conditions:

1.

$$V(S, E, I) \geq 0 \quad (3.55)$$

2.

$$\begin{aligned} \mathbf{T}^\alpha \mathcal{V}(E, I) &= \left(\frac{S - \tilde{S}_1}{S} \right) \frac{dS}{dt} + \left(\frac{E - \tilde{E}_1}{E} \right) \frac{dE}{dt} + \left(\frac{I - \tilde{I}_1}{I} \right) \frac{dI}{dt} \\ &= \left(\frac{S - \tilde{S}_1}{S} \right) (\mu^\alpha N - \beta^\alpha SI - \mu^\alpha S) + \left(\frac{E - \tilde{E}_1}{E} \right) (\beta^\alpha SI - (\kappa^\alpha + \mu^\alpha)E) \\ &\quad + \left(\frac{I - \tilde{I}_1}{I} \right) (\kappa^\alpha E - (\gamma^\alpha + \mu^\alpha)I). \end{aligned} \quad (3.56)$$

Let:

$$\begin{aligned} P &= \mu^\alpha N + \mu^\alpha \tilde{S}_1 + \beta^\alpha \tilde{S}_1 I + \tilde{E}_1 (\kappa^\alpha + \mu^\alpha) \\ Q &= \mu^\alpha E + \mu^\alpha S + I(\gamma^\alpha + \mu^\alpha) + \frac{\tilde{S}_1}{S} \mu^\alpha N + \frac{\tilde{S}_1}{S} \beta^\alpha SI. \end{aligned} \quad (3.57)$$

We conclude that, if $P < Q$, then $\mathbf{T}^\alpha \mathcal{V}(E, I) \leq 0$ if $S = \tilde{S}_1$, $E = \tilde{E}_1$ and $I = \tilde{I}_1$. Thus, the largest invariant set is given by:

$$L = \{(S, E, I) \in \Omega \mid \mathbf{T}^\alpha \mathcal{V}(E, I) = 0\}$$

It is reduced to the endemic equilibrium according to the principle of invariance at the point $(\tilde{S}_1, \tilde{E}_1, \tilde{I}_1)$, given $P < Q$.

□

5 Conclusion

In this chapter, we explored the SEIR model's global stability in a classical mathematical setting, under first-order ODEs, whose dynamics are governed by classical derivatives. Key contributions are the deduction of (DFE) and (DEE) equilibria along with the basic reproduction number $\mathcal{R}_0$, and stability analysis through Lyapunov functions confirms convergence under threshold conditions ($\mathcal{R}_0 < 1$ for DFE, $\mathcal{R}_0 > 1$ for DEE).

The fractional approach, by showing the presence of memory-dependent dynamics that influence stability and convergence, improves the simulation tools for complex epidemics, as well as the investigation of generalized operators.

Positivity and Stability Analysis of Conformable Fractional COVID-19 Models

1 Introduction

This chapter develops a fractional SEIHRDV model with vaccination to better understand COVID-19 dynamics. The model stability is studied using the Sumudu transform and conformable derivatives, showing local and global asymptotic stability. The new approach involving Lyapunov function construction and $\mathcal{R}_t$ calculation is checked using numerical simulation and MATLAB plots. The study demonstrates the use of fractional models and the Sumudu transform in predicting and controlling infectious disease spread.

2 Model formulation

Several models have been developed in the literature to depict and study the dynamics of this disease. Rabih Ghostine's model provides a better explanation and understanding of the emergence and spread of COVID-19. Several factors contribute to its significance: Accuracy and Predictive Power, Involvement of Different Variables, Adjustability, Transdisciplinarity, Openness and Accessibility, and Influence in Reality. This work focuses on an extended and reformulated version of Ghostine's model [20], expressed in a fractional form using a conformable derivative, with a subsequent investigation of its global stability.

In our model, the human population denoted by N is composed of seven compartments (subpopulations) according to the status of the disease. The number of susceptible individuals $S(t)$, infected individuals in the incubation period $E(t)$, infectious individuals $I(t)$, hospitalized individuals $H(t)$, recovered $R(t)$, and vaccinated cases $V(t)$. The fractional-order SEIHRDV model, under the conformable fractional derivative of order $0 < \alpha \leq 1$, is given by

$$\begin{aligned}
 \mathbf{T}^\alpha S(t) &= \Lambda^\alpha - \beta^\alpha SI - \eta^\alpha S - \mu^\alpha S, \\
 \mathbf{T}^\alpha E(t) &= \beta^\alpha SI + (\sigma\beta)^\alpha VI - (\gamma^\alpha + \mu^\alpha)E, \\
 \mathbf{T}^\alpha I(t) &= \gamma^\alpha E - (\delta^\alpha + \mu^\alpha)I, \\
 \mathbf{T}^\alpha H(t) &= \delta^\alpha I - ((1 - \kappa^\alpha)\lambda^\alpha + (\kappa\rho)^\alpha + \mu^\alpha)H, \\
 \mathbf{T}^\alpha R(t) &= (1 - \kappa^\alpha)\lambda^\alpha H - \mu^\alpha R, \\
 \mathbf{T}^\alpha D(t) &= (\kappa\rho)^\alpha H, \\
 \mathbf{T}^\alpha V(t) &= \eta^\alpha S - (\sigma\beta)^\alpha VI - \mu^\alpha V.
 \end{aligned} \tag{4.1}$$

With the following conditions

$$\begin{aligned}
 S(0) = S_0 \geq 0, \quad E(0) = E_0 \geq 0, \quad I(0) = I_0 \geq 0, \quad H(0) = H_0 \geq 0, \quad R(0) = R_0 \geq 0, \\
 D(0) = D_0 \geq 0, \quad V(0) = V_0 \geq 0
 \end{aligned} \tag{4.2}$$

and

$$N(t) = S(t) + E(t) + I(t) + H(t) + R(t) + D(t) + V(t) \tag{4.3}$$

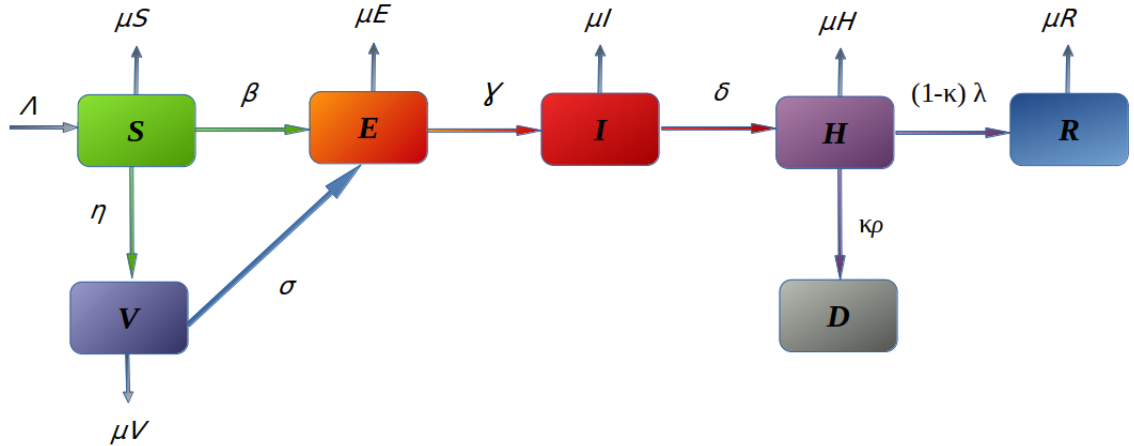


Fig. 4.1: Diagram of a compartmental model for COVID-19 transmission.

Where the model parameters are defined in Table 4.1

Tab. 4.1: Epidemiological Parameters of the Model

Parameter	Interpretation
Λ	New births
β	Transmission rate divided by N
γ^{-1}	Incubation period
δ^{-1}	Infection time
η	Vaccination rate (rate of people who are vaccinated)
σ	Vaccine inefficacy
λ^{-1}	Recovery time
κ	Case fatality rate
ρ^{-1}	Time until death
μ	Natural birth and death rate

3 Non-negativity and boundness of solutions

This section aims to analyze and demonstrate the positivity and boundness of the system's solutions, such as introducing a new innovative integral technique named Sumudu transform to solve differential systems.

Proposition 3.1. *The region $\Omega = \{(S(t), E(t), I(t), H(t), R(t), D(t), V(t)) \in \mathbb{R}^7 : 0 < N(t) \leq \left(\frac{\Lambda}{\mu}\right)^\alpha\}$ is non-negative invariant for the model (4.1) for $t \geq 0$.*

Proof. We have.

$$\mathbf{T}^\alpha(S + E + I + H + R + D + V)(t) = \Lambda^\alpha - \mu^\alpha(S + E + I + H + R + D + V)(t).$$

$$\implies \mathbf{T}^\alpha N(t) = \Lambda^\alpha - \mu^\alpha N(t)$$

$$\implies \mathbf{T}^\alpha N(t) + \mu^\alpha N(t) = \Lambda^\alpha.$$

Taking Sumudu transform, we have

$$\frac{1}{v} [\mathcal{S}_\alpha[N(t)](v) - N(0)] + \mu^\alpha \mathcal{S}_\alpha[N(t)](v) = \Lambda^\alpha \quad (4.4)$$

$$\implies \mathcal{S}_\alpha[N(t)](v) + v\mu^\alpha \mathcal{S}_\alpha[N(t)](v) = N(0) + v\Lambda^\alpha$$

$$\implies \mathcal{S}_\alpha[N(t)](v) = \frac{N(0) + v\Lambda^\alpha}{1 + v\mu^\alpha}$$

$$\implies \mathcal{S}_\alpha[N(t)](v) = \frac{N(0)}{1 + v\mu^\alpha} - \frac{\Lambda^\alpha}{\mu^\alpha(1 + v\mu^\alpha)} + \frac{\Lambda^\alpha}{\mu^\alpha}$$

Applying inverse Sumudu transform, we get

$$N(t) = \frac{\Lambda^\alpha}{\mu^\alpha} + \left(N(0) - \frac{\Lambda^\alpha}{\mu^\alpha} \right) e^{\frac{-\mu^\alpha t^\alpha}{\alpha}}. \quad (4.5)$$

Thus

$$\lim_{t \rightarrow \infty} \text{Sup} N(t) \leq \left(\frac{\Lambda}{\mu} \right)^\alpha. \quad (4.6)$$

As a result, the functions S, E, I, H, R, D and V are all non-negative. $\square$

4 Equilibrium points and basic reproduction number of the model

The primary objective of this section is to determine two essential equilibrium states: the Disease-Free Equilibrium and the Epidemic Equilibrium. Additionally, we aim to determine the Basic Reproduction Number $\mathcal{R}_0$.

Based on [55], Basic Reproduction Number $\mathcal{R}_0$ of the model (4.1) is given by

$$\mathcal{R}_0 = \frac{(\beta\gamma\Lambda)^\alpha \epsilon_5}{\mu^\alpha \epsilon_1 \epsilon_2 \epsilon_3}. \quad (4.7)$$

the Disease-Free Equilibrium point (DFE) is given by

$$X^0 = \left(\frac{\Lambda^\alpha}{\epsilon_1}, 0, 0, 0, 0, \frac{(\eta\Lambda)^\alpha}{\mu^\alpha \epsilon_1} \right), \quad (4.8)$$

and the Endemic Equilibrium point (DEE) is given by

$$\left\{ \begin{array}{l} S^1 = \frac{\Lambda^\alpha}{\beta^\alpha I^1 + \epsilon_1}, \\ E^1 = \frac{\epsilon_3}{\gamma^\alpha} I^1, \\ I^1 = \frac{\sqrt{M_1} + M_2}{M_3}, \\ H^1 = \frac{\delta^\alpha}{\epsilon_4} I^1, \\ R^1 = \frac{(1 - \kappa^\alpha)(\lambda\delta)^\alpha}{\mu^\alpha \epsilon_4} I^1, \\ V^1 = \frac{(\eta\Lambda)^\alpha}{(\beta^\alpha I^1 + \epsilon_1)((\sigma\beta)^\alpha I^1 + \mu^\alpha)}, \end{array} \right. \quad (4.9)$$

where

$$\begin{aligned}
\epsilon_1 &= \mu^\alpha + \eta^\alpha, \\
\epsilon_2 &= \mu^\alpha + \gamma^\alpha, \\
\epsilon_3 &= \mu^\alpha + \delta^\alpha, \\
\epsilon_4 &= \mu^\alpha + \lambda^\alpha(1 - \kappa^\alpha) + (\kappa\rho)^\alpha, \\
\epsilon_5 &= \mu^\alpha + (\eta\sigma)^\alpha, \\
M_1 &= (\epsilon_2\epsilon_3\mu^\alpha)^2 + (\Lambda\beta\gamma\sigma)^{2\alpha} + (\epsilon_1\epsilon_2\epsilon_3\sigma^\alpha)^2 - 2(\sigma\mu)^\alpha\epsilon_1\epsilon_2^2\epsilon_3^2 + 4\mathcal{R}_0(\sigma\mu)^\alpha\epsilon_1\epsilon_2^2\epsilon_3^2 - 2(\Lambda\beta\sigma\gamma\mu)^\alpha\epsilon_2\epsilon_3 \\
&\quad - 2(\beta\sigma^2\gamma\Lambda)^\alpha\epsilon_1\epsilon_2\epsilon_3, \\
M_2 &= (\beta\sigma\gamma\Lambda)^\alpha - \sigma^\alpha\epsilon_1\epsilon_2\epsilon_3 - \mu^\alpha\epsilon_2\epsilon_3, \\
M_3 &= 2(\beta\sigma)^\alpha\epsilon_2\epsilon_3
\end{aligned} \tag{4.10}$$

5 Stability analysis

In this section, our objective is to investigate the Local Asymptotic Stability (LAS) and to study also the Global Asymptotic Stability (GAS) of the Equilibrium points (DFE) X^0 and (DEE) X^1 for the model (4.1).

5.1 Local stability

By the use of the basic reproduction number $\mathcal{R}_0$ [55], we can present the theorem that establishes the local stability of the Disease-Free Equilibrium (DFE), which is defined as follows:

Theorem 5.1. *The disease-free equilibrium X^0 is locally asymptotically stable if $\mathcal{R}_0 < 1$ and unstable if $\mathcal{R}_0 > 1$.*

Proof. The Jacobian matrix (J) of the model (4.1) at (S, E, I, H, R, V) is given by

$$J = \begin{pmatrix} -\beta^\alpha I - \epsilon_1 & 0 & -\beta^\alpha S & 0 & 0 & 0 \\ \beta^\alpha I & -\epsilon_2 & \beta^\alpha S + (\sigma\beta)^\alpha V & 0 & 0 & (\sigma\beta)^\alpha I \\ 0 & \gamma^\alpha & -\epsilon_3 & 0 & 0 & 0 \\ 0 & 0 & \delta^\alpha & -\epsilon_4 & 0 & 0 \\ 0 & 0 & 0 & (1 - \kappa^\alpha)\lambda^\alpha & -\mu^\alpha & 0 \\ \eta^\alpha & 0 & -(\sigma\beta)^\alpha V & 0 & 0 & -(\sigma\beta)^\alpha I - \mu^\alpha \end{pmatrix}, \tag{4.11}$$

The Jacobian matrix (J) at the DFE point X^0 becomes

$$J(X^0) = \begin{pmatrix} -\epsilon_1 & 0 & -\frac{\mathcal{R}_0 \mu^\alpha \epsilon_2 \epsilon_3}{\epsilon_5 \gamma^\alpha} & 0 & 0 & 0 \\ 0 & -\epsilon_2 & \frac{\epsilon_2 \epsilon_3 \mathcal{R}_0}{\gamma^\alpha} & 0 & 0 & 0 \\ 0 & \gamma^\alpha & -\epsilon_3 & 0 & 0 & 0 \\ 0 & 0 & \delta^\alpha & -\epsilon_4 & 0 & 0 \\ 0 & 0 & 0 & (1 - \kappa^\alpha) \lambda^\alpha & -\mu^\alpha & 0 \\ \eta^\alpha & 0 & -\frac{\mathcal{R}_0 (\sigma \eta)^\alpha \epsilon_2 \epsilon_3}{\gamma^\alpha \epsilon_5} & 0 & 0 & -\mu^\alpha \end{pmatrix}, \quad (4.12)$$

The roots of the characteristic equation are defined as follow:

$$\begin{aligned} s_1 &= -\epsilon_1 < 0, \\ s_2 &= -\epsilon_4 < 0, \\ s_3 &= -\mu^\alpha < 0, \\ s_4 &= -\mu^\alpha < 0, \\ s_5 &= -\frac{1}{2} \left(\epsilon_2 + \epsilon_3 + \sqrt{(\epsilon_2 - \epsilon_3)^2 + 4\mathcal{R}_0 \epsilon_2 \epsilon_3} \right) < 0, \\ s_6 &= -\frac{1}{2} \left(\epsilon_2 + \epsilon_3 - \sqrt{(\epsilon_2 - \epsilon_3)^2 + 4\mathcal{R}_0 \epsilon_2 \epsilon_3} \right) < 0 \quad \text{iff } \mathcal{R}_0 < 1, \end{aligned} \quad (4.13)$$

The disease-free equilibrium X^0 is locally asymptotically stable (LAS) or unstable according to $\mathcal{R}_0 < 1$ or $\mathcal{R}_0 > 1$. □

5.2 Global stability

The theorems that establish the global stability of the Equilibrium points (DFE) and (DEE) are defined as follows:

Theorem 5.2. *The disease-free equilibrium X^0 is global asymptotically stable in $\Omega = \{(S, E, I, H, R, D, V) \in \mathbb{R}_+^7 / S + E + I + H + R + D + V \leq N, \quad S(0) > 0, E(0) > 0, I(0) > 0, H(0) > 0, R(0) > 0, D(0) > 0, V(0) > 0\}$ if $\mathcal{R}_0 \leq 1$, otherwise unstable.*

Proof. Following [12], let consider the Lyapunov function

$$\mathcal{V}(E, I) = E + \frac{\gamma^\alpha + \mu^\alpha}{\gamma^\alpha} I. \quad (4.14)$$

The Conformal derivative of the Lyapunov function, denoted by $\mathbf{T}^\alpha \mathcal{V}(E, I)$ is given by:

$$\begin{aligned} \mathbf{T}^\alpha \mathcal{V}(E, I) &= \mathbf{T}^\alpha E(t) + \mathbf{T}^\alpha I(t) \left(\frac{\epsilon_2}{\gamma^\alpha} \right), \\ &= \beta^\alpha S I + (\sigma \beta)^\alpha V I - \epsilon_2 E + \frac{\epsilon_2}{\gamma^\alpha} (\gamma^\alpha E - \epsilon_3 I), \\ &= \beta^\alpha (S + \sigma^\alpha V) - \frac{\epsilon_2 \epsilon_3}{\gamma^\alpha}, \\ &= \frac{\epsilon_2 \epsilon_3}{\gamma^\alpha} \left[\frac{\epsilon_1 \mu^\alpha \mathcal{R}_0 (S + \sigma^\alpha V)}{\Lambda^\alpha \epsilon_5} - 1 \right] I, \\ &\leq \frac{\epsilon_2 \epsilon_3}{\gamma^\alpha} (\mathcal{R}_0 - 1) I. \end{aligned} \quad (4.15)$$

Thus, $\mathbf{T}^\alpha \mathcal{V}(E, I) < 0$ if $\mathcal{R}_0 < 1$ and $\mathbf{T}^\alpha \mathcal{V}(E, I) = 0$ if $I(t) = 0$ and $\mathcal{R}_0 = 1$.

The largest invariant set within $L = \{(S, E, I, H, R, D, V) \in \Omega \mid \mathbf{T}^\alpha \mathcal{V}(E, I) = 0\}$ is reduced to the Disease-Free Equilibrium (DFE).

Thus, by LaSalle's Invariance Principle [59], the disease-free equilibrium is Globally Asymptotically Stable (GAS) in Ω for $\mathcal{R}_0 < 1$.

□

Theorem 5.3. *If $\mathcal{R}_0 > 1$ then the endemic equilibrium of the model (4.1) given by X^1 is globally asymptotically stable in Ω .*

Proof. The non-linear Lyapunov function of the Goh-Volterra form [9] is as follows:

$$\begin{aligned} \mathcal{V}(S, E, I, H, V) &= (S - S^1 \ln(S)) + (E - E^1 \ln(E)) + \frac{\beta^\alpha S^1 I^1}{\gamma^\alpha E^1} (I - I^1 \ln(I)) \\ &\quad + \frac{\beta^\alpha S^1}{\delta^\alpha} (H - H^1 \ln(H)) + (V - V^1 \ln(V)), \end{aligned} \quad (4.16)$$

the Conformal derivative of the Lyapunov function, $\mathbf{T}^\alpha \mathcal{V}(S, E, I, H, V)$, is expressed as:

$$\begin{aligned} \mathbf{T}^\alpha \mathcal{V}(S, E, I, H, V) = & (1 - \frac{S^1}{S}) \mathbf{T}^\alpha S(t) + (1 - \frac{E^1}{E}) \mathbf{T}^\alpha E(t) + \frac{\beta^\alpha S^1 I^1}{\gamma^\alpha E^1} (1 - \frac{I^1}{I}) \mathbf{T}^\alpha I(t) \\ & + \frac{\beta^\alpha S^1}{\delta^\alpha} (1 - \frac{H^1}{H}) \mathbf{T}^\alpha H(t) + (1 - \frac{V^1}{V}) \mathbf{T}^\alpha V(t), \end{aligned} \quad (4.17)$$

furthermore, we have

$$\begin{aligned} \mathbf{T}^\alpha S(t) & \leq \Lambda^\alpha - \beta^\alpha SI \\ \mathbf{T}^\alpha E(t) & \leq \beta^\alpha SI + (\sigma\beta)^\alpha VI, \\ \mathbf{T}^\alpha I(t) & \leq \gamma^\alpha E, \\ \mathbf{T}^\alpha H(t) & \leq \delta^\alpha I, \\ \mathbf{T}^\alpha V(t) & \leq \eta^\alpha S - (\sigma\beta)^\alpha VI. \end{aligned} \quad (4.18)$$

therefore,

$$\begin{aligned} \mathbf{T}^\alpha \mathcal{V}(S, E, I, H, V) & < \Lambda^\alpha (1 - \frac{S^1}{S}) + \eta^\alpha S^1 (\frac{S}{S^1} - \frac{V^1 S}{VS^1}) + (\sigma\beta)^\alpha V^1 I^1 (\frac{I}{I^1} - \frac{E^1 VI}{EV^1 I^1}) \\ & + \beta^\alpha S^1 I^1 \left(\frac{E}{E^1} + 2 \frac{I}{I^1} - \frac{E^1 SI}{ES^1 I^1} - \frac{I^1 E}{IE^1} - \frac{H^1 I}{HI^1} \right) < 0, \end{aligned} \quad (4.19)$$

such as the following inequalities hold

$$\begin{aligned} 1 - \frac{S^1}{S} & \leq 0, \\ \frac{S}{S^1} - \frac{V^1 S}{VS^1} & \leq 0, \\ \frac{I}{I^1} - \frac{E^1 VI}{EV^1 I^1} & \leq 0, \end{aligned} \quad (4.20)$$

and

$$\frac{E}{E^1} + 2 \frac{I}{I^1} - \frac{E^1 SI}{ES^1 I^1} - \frac{I^1 E}{IE^1} - \frac{H^1 I}{HI^1} \leq 0.$$

Thus $\mathbf{T}^\alpha \mathcal{V}(t) \leq 0$ for $\mathcal{R}_0 > 1$.

The point X^1 is globally asymptotically stable (GAS) if $\mathcal{R}_0 > 1$. □

6 Numerical results and discussion

In this section, we present a numerical simulation of the SEIHRDV model with a conformable fractional derivative for COVID-19 transmission in Algeria. The parameters are determined based on real-time COVID-19 case data in Algeria from the World Health Organization (WHO) and the Ministry of Health of Algeria. The primary objective of this simulation is to assess the impact of the effectiveness of the vaccination campaign on disease transmission. This study uses the Runge-Kutta 4th order (RK4) method in MATLAB. Baseline values and initial conditions of our new model are given in Tables 4.2 and 4.3.

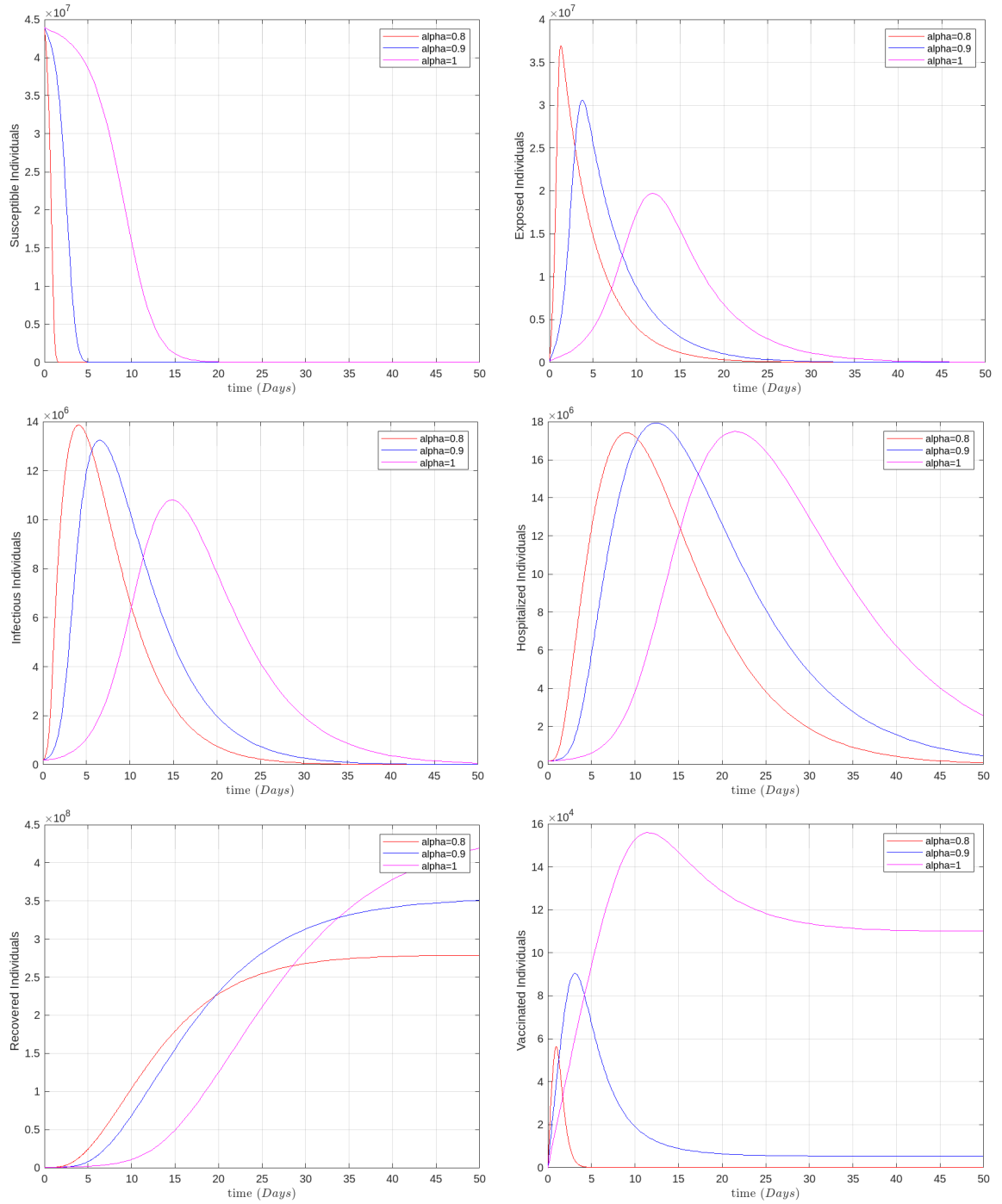
Tab. 4.2: Initial model parameters .

Parameter	Interpretation	Initial Value	Reference
μ	Natural birth and death rate	3×10^{-5} person/day	Estimated
Λ	New births	1350 person/day	[61]
γ^{-1}	Incubation period	5.5 days	[62]
δ^{-1}	Infection time	3.8 days	[62]
λ^{-1}	Recovery time	10 days	[62]
ρ^{-1}	Time until death	15 days	[62]
η	Vaccination rate (rate of people who are vaccinated)	4×10^{-4} day $^{-1}$	Estimated
σ	Vaccine inefficacy	0.05 day $^{-1}$	[63]
β	Transmission rate divided by N	5.7×10^{-8} day $^{-1}$	Estimated
κ	Case fatality rate	0.028	[64]

Tab. 4.3: The initial values for the model variables.

Variable	N	I_0	E_0	H_0	R_0	D_0	V_0	S_0
Initial Value	44,600,000	202122	150000	180000	138362	5739	0	44457156
Reference	[65]	[64]	Estimated	Estimated	[64]	[64]	[64]	Estimated

The graphs show the pattern of disease spread and impact of parameter α . Susceptibles decrease over time as more and more people become exposed and infected, and smaller values of α mean fewer susceptibles. The exposed peak is higher and decreases faster for lower values of α , suggesting faster development to the infectious state. Infectious individuals peak and then decline, where lower α values lead to an earlier and higher peak, indicating a quicker and worse spreading. Hospitalized individuals peak and then decline, where lower α values lead to a later and lower peak, indicating a lighter burden on the healthcare system. The recovered individuals increase, with the lowest recovery rates at $\alpha = 1$. Vaccinated individuals rise rapidly at first and then plateau, with lower values of α leading to faster vaccination rates but reaching this plateau sooner.

Fig. 4.2: Impact of α on System Dynamics

7 Conclusion

In this study, we analyze fractional order derivatives utilizing the Conformable derivatives with an order of $0 < \alpha \leq 1$ within the SEIHRDV model with vaccination. Using COVID-19 case data from Algeria at the beginning of vaccine implementation (September 2021). To prove the positivity of the system's solutions, we introduce the Sumudu transform (an integral technique for solving differential systems) as a new innovative approach. After that, we establish the local stability of the equilibrium points by calculating the Jacobian matrix and proving that its eigenvalues are negative. We construct a Lyapunov function to demonstrate the equilibrium points' global stability. Our study shows that a lower value of α is associated with a more effective response to the disease. However, it is essential to note that α is not the only factor determining a response's effectiveness. Other factors, such as the vaccine's effectiveness and the availability of healthcare resources, also play a role.

Introduction to Optimal Control Theory

1 Introduction

Optimal control theory considers the problem of controlling a dynamic system to achieve a cost function's minimization (or maximization) subject to constraints. Direct and indirect solution methods may be employed to obtain optimal trajectories. Widely utilized in engineering, economics, and robotics, the theory allows one to optimize performance (energy, time, cost) under various conditions. Variational methods and numerical algorithms are crucial for practical solutions.

2 Optimal Control Problem Statement

Let $t_0, t_1 \in \mathbb{R}$ with $t_0 < t_1$, and $f : [t_0, t_1] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$, a function where $x^0 = (x_1(0), x_2(0), \dots, x_n(0))^T \in \mathbb{R}^n$.

We consider the following system of ordinary differential equations (ODEs): for a function $u : [t_0, t_1] \rightarrow \mathbb{R}^m$ with $u(t) = (u_1(t), u_2(t), \dots, u_m(t))^T$ (with certain properties that will be specified later) find a function $x : [t_0, t_1] \rightarrow \mathbb{R}^n$ such that $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$ satisfies the Cauchy problem

$$\begin{cases} x'(t) &= f(t, x(t), u(t)), \quad t \in [t_0, t_1] \\ x(t_0) &= x^0 \end{cases} \quad (5.1)$$

- System (5.1) is called a controlled system.

- $x(t)$ is called the state of system (5.1).
- $u(t)$ is called the control of system (5.1).

We introduce a function S depending on x and u , called the cost function, which takes the form

$$S(x, u) = \int_{t_0}^{t_1} L(t, x(t), u(t)) dt + \varphi(x(t_1)) \quad (5.2)$$

where

$$L : [t_0, t_1] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R} \quad \text{and} \quad \varphi : \mathbb{R}^n \rightarrow \mathbb{R} \quad (5.3)$$

are given functions

The optimal control problem consists in finding, among all pairs of functions $(x, u) \in \mathbb{R}^\times \times \mathbb{R}^\geq$ satisfying system (5.1) with $u \in [0, 1]$ and $x(t_1) \in C_1$, those that minimize the function S given by (5.2).

So, the formulation of an optimal control problem is as follows:

$$(P) = \begin{cases} \min S(x, u) = \int_{t_0}^{t_1} L(t, x(t), u(t)) dt + \varphi(x(t_1)). \\ \dot{x}(t) = f(t, x(t), u(t)), \quad t \in [t_0, t_1] \\ x(0) = x^0, \\ x(t_1) = x^1, \\ u \in [0, 1] \end{cases} \quad (5.4)$$

3 Numerical Methods in Optimal Control

Optimal control problems are solved mainly by two kinds of methods: direct and indirect methods. Both have strengths and domains of preferred application.

3.1 Direct Method

Direct methods reformulate the original problem as an optimization problem in a standard form by discretizing the control and state variables and then resolving the discretized problem. Among the various approaches, we consider the Nonsmooth Newton method as a means of solving the optimal control problem in this chapter. This method is well-suited to handling complex issues with nonlinearities or discontinuities. These robust and versatile techniques are widely used in modern industrial processes where ease of implementation matters.

Nonsmooth Newton Method

This method [41, 42] is used to solve discrete optimal control problems with inequality constraints or nonsmooth functions (entries). We employ a general one-step process for differential equations of infinite-dimensional problems. By discretizing the differential equations, we transform the problem into a finite set of equations that can be solved numerically. We then use the Fischer Burmeister function to transform

the necessary first-order conditions for discrete problems into equivalent nonlinear and nonsmooth equations. The general form of the Optimal control problem is defined as follows:

$$(P_g) = \begin{cases} \min & \int_0^T \tilde{f}(x(t), u(t)) dt. \\ \dot{x}(t) & = f(x(t), u(t)), \quad t \in [t_0, t_1] \\ \psi(x(0), x(T)) & = 0, \\ c(x(t), u(t)) & \leq 0, \\ s(x(t)) & \leq 0 \end{cases} \quad (5.5)$$

Without loss of generality, the discussion is restricted to autonomous problems on the fixed time interval $[0, 1]$. The functions

$$\tilde{f} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \rightarrow \mathbb{R}, \quad f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \rightarrow \mathbb{R}^{n_x}, \quad \psi : \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_\psi},$$

$$c : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \rightarrow \mathbb{R}^{n_c}, \quad \text{and} \quad s : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_s}$$

are assumed to be at least twice continuously differentiable with respect to all arguments. Furthermore, we assume that the mixed control-state constraint c satisfies the rank condition $\text{rank}(c'_u(t)) = n_c$ for almost every $t \in [0, 1]$.

The Discretized Optimal Control Problem (DOCP) is defined as follows

$$(P_d) = \begin{cases} \min & \sum_{i=0}^{\tilde{N}-1} \Delta t \tilde{f}(x_i, u_i) \\ x_{i+1} - x_i - \Delta t f(x_i, u_i, \Delta t) & = 0, & i = 0, 1, \dots, \tilde{N} - 1, \\ \psi(x_0, x_{\tilde{N}}) & = 0, \\ c(x_i, u_i) & \leq 0 & i = 0, 1, \dots, \tilde{N}, \\ s(x_i) & \leq 0 & i = 0, 1, \dots, \tilde{N}. \end{cases} \quad (5.6)$$

We define the variable

$$y := (x_0, x_1, \dots, x_{\tilde{N}}, u_0, \dots, u_{\tilde{N}})^\top \in \mathbb{R}^{(n_x+n_u)(\tilde{N}+1)}, \quad (5.7)$$

and the functions

$$g(y) := \begin{pmatrix} c(x_0, u_0) \\ \vdots \\ c(x_{\tilde{N}}, u_{\tilde{N}}) \\ s(x_0) \\ \vdots \\ s(x_{\tilde{N}}) \end{pmatrix} \in \mathbb{R}^{(n_c+n_s)(\tilde{N}+1)}, \quad (5.8)$$

$$\mathbf{h}(y) := \begin{pmatrix} x_1 - x_0 - \Delta t \Phi(x_0, u_0, \Delta t) \\ \vdots \\ x_{\tilde{N}} - x_{\tilde{N}-1} - \Delta t \Phi(x_{\tilde{N}-1}, u_{\tilde{N}-1}, \Delta t) \\ \psi(x_0, x_{\tilde{N}}) \end{pmatrix} \in \mathbb{R}^{\tilde{N}n_x + n_\psi}. \quad (5.9)$$

The Lagrangian of the Discretized Optimal Control Problem (DOCP) is given by

$$\begin{aligned} \mathbf{L}_d &:= \sum_{i=0}^{\tilde{N}-1} \Delta t \tilde{f}(x_i, u_i) + \sum_{i=0}^{\tilde{N}-1} \lambda_i [x_{i+1} - x_i - \Delta t f(x_i, u_i, \Delta t)] + \sum_{i=0}^{\tilde{N}} \eta_i c(x_i, u_i) \\ &\quad + \sum_{i=0}^{\tilde{N}} \tau_i s(x_i) + \sum_{k=0}^{n_\psi} \delta_k \psi_k(x_0, x_{\tilde{N}}) \\ &= \mathbf{B}(x_{\tilde{N}}, u_{\tilde{N}}, \lambda_{\tilde{N}}, \tau_{\tilde{N}}, \delta) + \sum_{i=0}^{\tilde{N}-1} \mathbf{G}(x_i, u_i, \lambda_i, \tau_i). \end{aligned} \quad (5.10)$$

with multipliers $\lambda = (\lambda_0, \dots, \lambda_{\tilde{N}-1})^\top \in \mathbb{R}^{\tilde{N}}$, $\eta = (\eta_0, \dots, \eta_{\tilde{N}})^\top \in \mathbb{R}^{(\tilde{N}+1)}$, $\tau = (\tau_0, \dots, \tau_{\tilde{N}})^\top \in \mathbb{R}^{(\tilde{N}+1)}$, and $\delta = (\delta_1, \dots, \delta_{n_\psi})^\top \in \mathbb{R}^{n_\psi}$.

The first order necessary **KKT** conditions reads as

$$\begin{aligned} \nabla_x \mathbf{L}_d(x, u, \lambda, \tau, \eta, \delta) &= 0, & (1) \\ \nabla_u \mathbf{L}_d(x, u, \lambda, \tau, \eta, \delta) &= 0, & (2) \\ \mathbf{h}(x, u) &= 0, & (3) \\ c(x_i, u_i) \leq 0, \quad \eta_i \geq 0, \quad \eta_i c(x_i, u_i) &= 0, \quad i = 0, \dots, \tilde{N}, & (4) \\ s(x_i) \leq 0, \quad \tau_i \geq 0, \quad \tau_i s(x_i) &= 0, \quad i = 0, \dots, \tilde{N}. & (5) \end{aligned} \quad (5.11)$$

$$\nabla_x \mathbf{L}_d := \begin{pmatrix} \nabla_{x_0} \mathbf{G}(x_0, u_0, \lambda_0, \tau_0, \eta_0) \\ \vdots \\ \nabla_{x_{\tilde{N}-1}} \mathbf{G}(x_{\tilde{N}-1}, u_{\tilde{N}-1}, \lambda_{\tilde{N}-1}, \tau_{\tilde{N}-1}, \eta_{\tilde{N}-1}) \\ \nabla_{x_{\tilde{N}}} \mathbf{B}(x_{\tilde{N}}, u_{\tilde{N}}, \lambda_{\tilde{N}}, \tau_{\tilde{N}}, \eta_{\tilde{N}}, \delta) \end{pmatrix} \in \mathbb{R}^{n_x(\tilde{N}+1)} \quad (5.12)$$

$$\nabla_u \mathbf{L}_d := \begin{pmatrix} \nabla_{u_0} \mathbf{G}(x_0, u_0, \lambda_0, \tau_0, \eta_0) \\ \vdots \\ \nabla_{u_{\tilde{N}-1}} \mathbf{G}(x_{\tilde{N}-1}, u_{\tilde{N}-1}, \lambda_{\tilde{N}-1}, \tau_{\tilde{N}-1}, \eta_{\tilde{N}-1}) \\ \nabla_{u_{\tilde{N}}} \mathbf{B}(x_{\tilde{N}}, u_{\tilde{N}}, \lambda_{\tilde{N}}, \tau_{\tilde{N}}, \eta_{\tilde{N}}, \delta) \end{pmatrix} \in \mathbb{R}^{n_u(\tilde{N}+1)} \quad (5.13)$$

If the inequality constraints $g(y) \leq 0$ were not present in NLP, the classical Newton's method could be applied to the remaining nonlinear equations (1-3) of 5.11 in order to solve them numerically. This approach is just the well-known

Lagrange-Newton method. In the presence of the inequality constraints we are focused with a coupled system of equalities (1-3) of 5.11 and complementarity conditions (4-5) of 5.11 and the numerical solution becomes more intricate. The sequential quadratic programming (SQP) method and the trust region method are based on quadratic approximations of NLP and they require to solve a quadratic program within each iteration. Other approaches work more or less directly with the optimality conditions (1-5) of 5.11. For example, the most basic interior point method applies Newton's method to a perturbed KKT system with an additional strategy for letting the perturbation tend to zero. We will concentrate on a closely related method which was suggested by Fischer [66]. The idea is to apply a so-called NCP function to the complementarity system (4) and transform the KKT system into an equivalent system of equations. This can be achieved by application of the Fischer–Burmeister function $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by

$$\varphi(a, b) := \sqrt{a^2 + b^2} - a - b.$$

The Fischer–Burmeister function has the nice property that

$$\varphi(a, b) = 0 \quad \Leftrightarrow \quad a \geq 0, \quad b \geq 0, \quad ab = 0. \quad (6)$$

Notice, that there exist other NCP functions with this property, e.g. $\min\{a, b\}$. However, we prefer the Fischer–Burmeister function as it has additional properties which are particularly useful in view of the globalization strategy below. More precisely, we will exploit the fact that the squared function $\varphi(a, b)^2$ is continuously differentiable. Notice, that other NCP functions may not have this property, e.g. $\min\{a, b\}^2$ is not differentiable.

Exploitation of (6) immediately allows us to restate the KKT conditions equivalently as the nonlinear equation F, which is given by

$$F(z) := \begin{pmatrix} \nabla_x L_d(x, u, \lambda, \sigma, \eta, \tau) \\ \nabla_u L_d(x, u, \lambda, \sigma, \eta, \tau) \\ \mathbf{h}(x, u) \\ \varphi(-c(x_0, u_0), \eta_0) \\ \vdots \\ \varphi(-c(x_{\tilde{N}}, u_{\tilde{N}}), \eta_{\tilde{N}}) \\ \varphi(-s(x_0), \tau_0) \\ \vdots \\ \varphi(-s(x_{\tilde{N}}), \tau_{\tilde{N}}) \end{pmatrix} = 0, \quad z = (x, u, \lambda, \delta, \eta, \tau)^\top, \quad (5.14)$$

For brevity we use the notations

$$\varphi(-s(x_i), \tau_i) := \begin{pmatrix} \varphi(-s_1(x_i), \tau_{i,1}) \\ \vdots \\ \varphi(-s_{n_s}(x_i), \tau_{i,n_s}) \end{pmatrix}, \quad i = 0, \dots, \tilde{N}, \quad (5.15)$$

$$\varphi(-c(x_i, u_i), \eta_i) := \begin{pmatrix} \varphi(-c_1(x_i, u_i), \eta_{i,1}) \\ \vdots \\ \varphi(-c_{n_c}(x_i, u_i), \eta_{i,n_c}) \end{pmatrix}, \quad i = 0, \dots, \tilde{N}, \quad (5.16)$$

For every $z \in D_F := \{z = (x, u, \lambda, \delta, \eta, \tau)^\top \mid |g_i(y)| + |c_i| + |s_i| > 0, i = 1, \dots, m\}$, F is continiously differentiable in a neighborhood of z and it holds $\partial_B F(z) = \{F'(z)\}$

with

$$\begin{aligned} F'(z) &= \begin{pmatrix} \nabla_{xx}^2 \mathbf{L}_d & \nabla_{xu}^2 \mathbf{L}_d & \nabla_{x\lambda}^2 \mathbf{L}_d & \nabla_{x\eta}^2 \mathbf{L}_d & \nabla_{x\tau}^2 \mathbf{L}_d \\ \nabla_{ux}^2 \mathbf{L}_d & \nabla_{uu}^2 \mathbf{L}_d & \nabla_{u\lambda}^2 \mathbf{L}_d & \nabla_{u\eta}^2 \mathbf{L}_d & \nabla_{u\tau}^2 \mathbf{L}_d \\ \mathbf{h}'_x & \mathbf{h}'_u & \mathbf{h}'_\lambda & \mathbf{h}'_\eta & \mathbf{h}'_\tau \\ \varphi'_x & \varphi'_u & \varphi'_\lambda & \varphi'_\eta & \varphi'_\tau \end{pmatrix} \\ &= \begin{pmatrix} \nabla_{xx}^2 \mathbf{L}_d & \nabla_{xu}^2 \mathbf{L}_d & (\mathbf{h}'_x)^\top & \mathbf{C}_x^\top & \mathbf{S}^\top \\ \nabla_{ux}^2 \mathbf{L}_d & \nabla_{uu}^2 \mathbf{L}_d & (\mathbf{h}'_u)^\top & \mathbf{C}_u^\top & 0 \\ \mathbf{h}'_x & \mathbf{h}'_u & 0 & 0 & 0 \\ -R_c \mathbf{C}_x & -R_c \mathbf{C}_u & 0 & T_c & 0 \\ -R_s \mathbf{S} & 0 & 0 & 0 & T_s \end{pmatrix} \end{aligned} \quad (5.17)$$

$\nabla_{xx}^2 \mathbf{L}_d$ is define as fallow

$$\nabla_{xx}^2 \mathbf{L}_d = \begin{pmatrix} \nabla_{x_0 x_0}^2 \mathbf{G} & \mathbf{0} & \nabla_{x_0 x_{\tilde{N}}}^2 \mathbf{B} \\ & \ddots & \vdots \\ \mathbf{0} & \nabla_{x_{\tilde{N}-1} x_{\tilde{N}-1}}^2 \mathbf{G} & \vdots \\ \nabla_{x_{\tilde{N}} x_0}^2 \mathbf{B} & \dots & \dots & \nabla_{x_{\tilde{N}} x_{\tilde{N}}}^2 \mathbf{B} \end{pmatrix} \in \mathbb{R}^{n_x(\tilde{N}+1) \times n_x(\tilde{N}+1)} \quad (5.18)$$

$\nabla_{xu}^2 \mathbf{L}_d$ and $\nabla_{ux}^2 \mathbf{L}_d$ are defined as fallow

$$\nabla_{xu}^2 \mathbf{L}_d = (\nabla_{ux}^2 \mathbf{L}_d)^\top = \begin{pmatrix} \nabla_{x_0 u_0}^2 \mathbf{G} & \mathbf{0} & \nabla_{x_0 u_{\tilde{N}}}^2 \mathbf{B} \\ & \ddots & \vdots \\ \mathbf{0} & \nabla_{x_{\tilde{N}-1} u_{\tilde{N}-1}}^2 \mathbf{G} & \vdots \\ \nabla_{x_{\tilde{N}} u_0}^2 \mathbf{B} & \dots & \dots & \nabla_{x_{\tilde{N}} u_{\tilde{N}}}^2 \mathbf{B} \end{pmatrix} \in \mathbb{R}^{n_x(\tilde{N}+1) \times n_u(\tilde{N}+1)} \quad (5.19)$$

$\nabla_{uu}^2 \mathbf{L}_d$ is defined as follow

$$\nabla_{uu}^2 \mathbf{L}_d = \begin{pmatrix} \nabla_{u_0 u_0}^2 \mathbf{G} & \mathbf{0} & \nabla_{u_0 u_{\tilde{N}}}^2 \mathbf{B} \\ & \ddots & \vdots \\ \mathbf{0} & & \nabla_{u_{\tilde{N}-1} u_{\tilde{N}-1}}^2 \mathbf{G} & \vdots \\ \nabla_{u_{\tilde{N}} u_0}^2 \mathbf{B} & \dots & \dots & \nabla_{u_{\tilde{N}} u_{\tilde{N}}}^2 \mathbf{B} \end{pmatrix} \in \mathbb{R}^{n_u(\tilde{N}+1) \times n_u(\tilde{N}+1)} \quad (5.20)$$

$\mathbf{h}'_x$ and $\mathbf{h}'_u$ are defined as follow

$$\mathbf{h}'_X = \begin{pmatrix} -M_0 & I_d & \mathbf{0} & \mathbf{0} \\ & \ddots & \ddots & \\ \mathbf{0} & & -M_{\tilde{N}-1} & I_d \\ \frac{\psi'_x(0)}{\psi'_x(1)} & & & \end{pmatrix} \in \mathbb{R}^{n_x(\tilde{N})+n_\psi \times n_x(\tilde{N}+1)} \quad (5.21)$$

where $M_i = I_d + h f'_x(x_i, u_i, h)$

$$\mathbf{h}'_u = \begin{pmatrix} -h f'_u[t_0] & \mathbf{0} & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & -h f'_u[t_{\tilde{N}-1}] & \mathbf{0} \end{pmatrix} \in \mathbb{R}^{n_x(\tilde{N}) \times n_u(\tilde{N}+1)} \quad (5.22)$$

$\mathbf{C}_x$, $\mathbf{C}_u$, and $\mathbf{S}$ are defined as follow

$$\mathbf{C}_x = \begin{pmatrix} c'_x[t_0] & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & c'_x[t_{\tilde{N}}] \end{pmatrix} \in \mathbb{R}^{n_c(\tilde{N}+1) \times n_x(\tilde{N}+1)} \quad (5.23)$$

$$\mathbf{C}_u = \begin{pmatrix} c'_u[t_0] & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & c'_u[t_{\tilde{N}}] \end{pmatrix} \in \mathbb{R}^{n_c(\tilde{N}+1) \times n_u(\tilde{N}+1)} \quad (5.24)$$

$$\mathbf{S} = \begin{pmatrix} s'(x_0) & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & s'(x_{\tilde{N}}) \end{pmatrix} \in \mathbb{R}^{n_s(\tilde{N}+1) \times n_x(\tilde{N}+1)} \quad (5.25)$$

$\mathbf{R}_c$, $\mathbf{T}_c$, $\mathbf{R}_s$, and $\mathbf{T}_s$ are defined as follow

$$\mathbf{R}_c = \begin{pmatrix} \mathbf{R}_c^0 & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & \mathbf{R}_c^{\tilde{N}} \end{pmatrix} \in \mathbb{R}^{n_c(\tilde{N}+1) \times n_c(\tilde{N}+1)} \quad (5.26)$$

$$\mathbf{T}_c = \begin{pmatrix} \mathbf{T}_c^0 & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & \mathbf{T}_c^{\tilde{N}} \end{pmatrix} \in \mathbb{R}^{n_c(\tilde{N}+1) \times n_c(\tilde{N}+1)} \quad (5.27)$$

$$\mathbf{R}_s = \begin{pmatrix} \mathbf{R}_s^0 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{R}_s^{\tilde{N}} \end{pmatrix} \in \mathbb{R}^{n_s(\tilde{N}+1) \times n_s(\tilde{N}+1)} \quad (5.28)$$

$$\mathbf{T}_s = \begin{pmatrix} \mathbf{T}_s^0 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{T}_s^{\tilde{N}} \end{pmatrix} \in \mathbb{R}^{n_s(\tilde{N}+1) \times n_s(\tilde{N}+1)}, \quad (5.29)$$

Herein, $\mathbf{R}_s^i$, $\mathbf{T}_s^i$, $\mathbf{R}_c^i$, and $\mathbf{T}_c^i$ are diagonal matrices. A specific element of the B-differential F can be computed according to Lemma 2.5 in Section 2 of [41]

$$\begin{pmatrix} (\mathbf{R}_s^i)_{kk} \\ (\mathbf{T}_s^i)_{kk} \end{pmatrix} = \begin{cases} \begin{pmatrix} \frac{-s_k(x_i)}{\sqrt{(s_k(x_i))^2 + \tau_{k,n}^2} - 1} \\ \frac{\tau_{k,i}}{\sqrt{(s_k(x_i))^2 + \tau_{k,i}^2} - 1} \end{pmatrix} & \text{if } (s(x_i), \tau_{k,i}) \neq (0, 0). \\ \begin{pmatrix} -1 \\ 0 \end{pmatrix} & \text{otherwise.} \end{cases} \quad (5.30)$$

$$\begin{pmatrix} \mathbf{R}_c^i \\ \mathbf{T}_c^i \end{pmatrix} = \begin{cases} \begin{pmatrix} \frac{-c_j[t_i]}{\sqrt{c_j[t_i]^2 + \eta_{i,j}^2} - 1} \\ \frac{\eta_{i,j}}{\sqrt{c_j[t_i]^2 + \eta_{i,j}^2} - 1} \end{pmatrix} & \text{if } (c_j[t_i], \eta_{i,j}) \neq (0, 0). \\ \begin{pmatrix} -1 \\ 0 \end{pmatrix} & \text{otherwise.} \end{cases} \quad (5.31)$$

for $i = 1, \dots, \tilde{N}$, $j = 0, \dots, n_c$, and $k = 0, \dots, n_s$.

3.2 Indirect Method

Indirect methods are based on underlying theoretical frameworks like Pontryagin's maximum principle [30, 31]. They comprise the solution of optimality conditions that form equations and are exact but require more mathematical derivation and may be more sensitive to initial conditions.

Pontryagin's Maximum Principle

Theorem 3.1. *We consider the control system in $\mathbb{R}^n$*

$$\dot{x}(t) = f(t, x(t), u(t)), \quad (5.32)$$

where $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is of class C^1 , and where the controls are measurable and bounded functions defined on an interval $[0, t_e(u)] \subset \mathbb{R}^+$ with values in $\Omega \subset \mathbb{R}^m$. Let M_0 and M_1 be two subsets of $\mathbb{R}^n$. We denote by U the set of admissible controls u whose associated trajectories connect an initial point in M_0 to a final point in M_1 in time $t(u) < t_e(u)$.

Moreover, we define the cost of a control u over $[0, t]$ by

$$C(t, u) = \int_0^t f^0(s, x(s), u(s)) ds + g(t, x(t)), \quad (5.33)$$

where $f^0 : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ and $g : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ are C^1 , and $x(\cdot)$ is the trajectory solution of (5.32) associated with the control u .

We consider the following optimal control problem: determine a trajectory connecting M_0 to M_1 that minimizes the cost. The final time may or may not be fixed.

If the control $u \in U$, associated with the trajectory $x(\cdot)$, is optimal over $[0, T]$, then there exists an absolutely continuous function $p(\cdot) : [0, T] \rightarrow \mathbb{R}^n$, called the adjoint vector, and a scalar $p^0 \leq 0$, such that the pair $(p(\cdot), p^0)$ is nontrivial, and such that, for almost every $t \in [0, T]$,

$$\begin{aligned} \dot{x}(t) &= \frac{\partial H}{\partial p}(t, x(t), p(t), p^0 u(t)), \\ \dot{p}(t) &= \frac{\partial H}{\partial x}(t, x(t), p(t), p^0, u(t)), \end{aligned} \quad (5.34)$$

where $H(t, x, p, p^0, u) = \langle p, f(t, x, u) \rangle + p^0 f^0(t, x, u)$ is the Hamiltonian of the system, and we have the maximization condition almost everywhere on $[0, T]$:

$$H(t, x(t), p(t), p^0(t), u(t)) = \max_{v \in \Omega} H(t, x(t), p(t), p^0(t), v). \quad (5.35)$$

If, in addition, the final time to reach the target set M_1 is not fixed, then the final time condition at T is:

$$\max_{v \in \Omega} H(T, x(T), p(T), p^0, v) = -p^0 \frac{\partial g}{\partial t}(T, x(T)). \quad (5.36)$$

If furthermore M_0 and M_1 (or just one of the two sets) are manifolds in $\mathbb{R}^n$ with tangent spaces at $x(0) \in M_0$ and $x(T) \in M_1$, then the adjoint vector can be constructed to satisfy the transversality conditions at both endpoints (or just one of them):

$$p(0) \perp T_{x(0)} M_0 \quad (5.37)$$

and

$$p(T) - p^0 \frac{\partial g}{\partial x}(T, x(T)) \perp T_{x(T)} M_1. \quad (5.38)$$

Remark 3.2. If the control u is continuous at time T , condition (5.36) can be written as:

$$H(T, x(T), p(T), p^0, u(T)) = -p^0 \frac{\partial g}{\partial t}(T, x(T)). \quad (5.39)$$

Remark 3.3. If the manifold M_1 is expressed as

$$M_1 = \{x \in \mathbb{R}^n \mid F_1(x) = \cdots = F_p(x) = 0\}, \quad (5.40)$$

where the F_i are C^1 functions on $\mathbb{R}^n$, then condition (5.39) takes the form:

$$\exists \lambda_1, \dots, \lambda_p \in \mathbb{R} \mid p(T) = \sum_{i=1}^p \lambda_i \nabla F_i(x(T)) + p^0 \frac{\partial g}{\partial x}(T, x(T)). \quad (5.41)$$

Remark 3.4. Under the conditions of the theorem, we also have for almost every $t \in [0, T]$

$$\frac{d}{dt} H(t, x(t), p(t), p^0, u(t)) = \frac{\partial H}{\partial t}(t, x(t), p(t), p^0, u(t)). \quad (5.42)$$

In particular, if the augmented system is autonomous, i.e., if f and f^0 do not depend on t , then H does not depend on t , and we have

$$\forall t \in [0, T] \quad \max_{v \in \Omega} H(x(t), p(t), p^0, v) = \text{constant}.$$

Note that this equality holds everywhere on $[0, T]$ (indeed, this function of t is Lipschitz continuous).

Remark 3.5. The convention $p^0 \leq 0$ leads to the maximum principle. The convention $p^0 \geq 0$ would lead to the minimum principle, i.e., condition (5.35) would become a minimization condition.

Definition 3.6. An extremal of the optimal control problem is a quadruple $(x(\cdot), p^0, u(\cdot))$ that solves equations (5.34) and (5.35). If $p^0 = 0$, the extremal is said to be abnormal, and if $p^0 \neq 0$, the extremal is said to be normal.

Remark 3.7. Conditions (5.37) and (5.38) are called transversality conditions on the adjoint vector. Condition (5.36) is called a transversality condition on the Hamiltonian.

Remark 3.8. The important minimum time problem corresponds to $f^0 = 1$ and $g = 0$, or $f^0 = 0$ and $g(x) = 1$. In both cases, the resulting transversality conditions are the same.

4 Conclusion

The choice between these two paradigms is primarily a function of the type of problem. While direct methods dominate when dealing with complex systems where computation time is an issue, indirect methods remain useful for theoretical problems or when high precision is required. Knowledge of both methods enables one to deal effectively with diverse challenges in dynamic optimization.

Nonsmooth Newton's Method for Discretized SEIR Optimal Control Problem

1 Introduction

This chapter's study utilizes the nonsmooth Newton method (NNM) to overcome the limitations of standard procedures like Pontryagin's maximum principle by proposing a comparative analysis of optimal vaccination schemes for an SEIR model. NNM, with its higher convergence rate and enhanced robustness, deals with optimization problems with challenging constraints, like mixed state-control constraints. This novel methodology assesses vaccine efficacy in real-world situations, taking into account population dynamics and available resources while capitalizing on the role of mathematical modeling in the control of epidemics.

2 Formulation of the model

This section develops an optimal control model, building on the work of Biswas, Paiva, and de Pinho [67], to control the spread of disease through vaccination.

Such as the total population, $N(t)$, at the time, t , is divided into subpopulations: Susceptible, $S(t)$; Exposed, $E(t)$; Infectious individuals, $I(t)$; Recovered, $R(t)$. We assume that all vaccinated susceptible individuals gain immunity. Let $u(t)$ represent the percentage of susceptible individuals vaccinated per unit of time. The system of differential equations related to the schematic diagram in Figure 6.1 is as follows:

$$\begin{aligned}
 \dot{S}(t) &= bN(t) - cS(t)I(t) - dS(t) - u(t)S(t), \\
 \dot{E}(t) &= cS(t)I(t) - (e + d)E(t), \\
 \dot{I}(t) &= eE(t) - (g + a + d)I(t), \\
 \dot{R}(t) &= gI(t) - dR(t) + u(t)S(t), \\
 \dot{N}(t) &= (b - d)N(t) - aI(t).
 \end{aligned} \tag{6.1}$$

such as

$$S(t) + E(t) + I(t) + R(t) = N(t) \tag{6.2}$$

where the model parameters are defined in Table 6.1 .

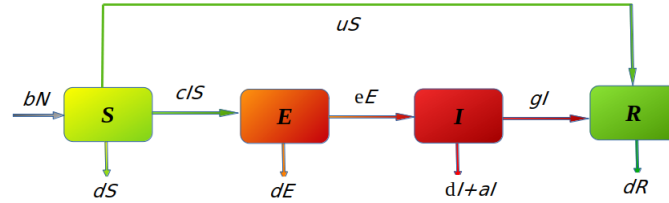


Fig. 6.1: SEIR compartmental model with vaccination.

Tab. 6.1: Description of model parameters.

Parameters	Definition of Parameters
b	Natural birth rate
d	Natural death rate
c	Incidence coefficient
e	Exposed to infectious rate
g	Recovery rate
a	Disease induced death rate

The strategy employed in this study aims to minimize the number of infected individuals within a given time interval $[0, T]$ and determine the vaccination policy u by minimizing the following objective function.

$$\mathcal{J}(x, u) := \int_0^T (AI(t) + u^2(t))dt. \tag{6.3}$$

Where the control effort is modeled utilizing a linear combination of quadratic terms, $u^2(t)$, the constants A represent a measure of the relative cost of the interventions over time $[0, T]$.

Our analysis aims to find an optimal control for vaccination effort u , such as.

$$\mathcal{J}(x^*, u^*) := \min\{\mathcal{J}(x, u), u \in \mathcal{U}\}, \quad (6.4)$$

Where

$$x(t) = [S(t), E(t), I(t), R(t), N(t)]^\top$$

The control effort u is assumed to be bounded and Lebesgue measurable time-dependent functions on the interval $[0, T]$. The control effort set subject to the model (6.1) with initial conditions (6.2) are defined as

$$\mathcal{U} := \{u \mid u \text{ is Lebesgue measurable}, 0 \leq u(t) \leq 1, t \in [0, T]\}, \quad (6.5)$$

We introduce an extra variable (W) denoted by the number of vaccines used to handle such a situation. Assuming that W_M is the total amount of vaccines available during the whole time, the constraint can be represented by

$$\dot{W}(t) = u(t)S(t), \quad W(0) = 0, \quad W(T) = W_M. \quad (6.6)$$

With $W(T) = W_M$, it assumes that all the vaccines should be used. We believe working with $W(T) \leq W_M$ is more realistic.

The optimal control problem studied in [67] is then:

$$(P) = \begin{cases} \min \mathcal{J}(x, u) = \int_0^T (AI(t) + u^2(t)) dt. \\ \text{st.} \\ \begin{aligned} \dot{S}(t) &= bN(t) - cS(t)I(t) - dS(t) - u(t)S(t), & t \in [0, T] \\ \dot{E}(t) &= cS(t)I(t) - (e + d)E(t), & t \in [0, T] \\ \dot{I}(t) &= eE(t) - (g + a + d)I(t), & t \in [0, T] \\ \dot{R}(t) &= gI(t) - dR(t) + u(t)S(t), & t \in [0, T] \\ \dot{N}(t) &= (b - d)N(t) - aI(t), & t \in [0, T] \\ \dot{W}(t) &= u(t)S(t), & t \in [0, T] \\ u(t) &\in [0, 1], \\ W(T) &\leq W_M. \end{aligned} \end{cases} \quad (6.7)$$

2.1 Model without control

In this part, we propose a problem without control ($u(t) = 0$), such as this type of analysis reveals the spontaneous spread of the disease and provides a reference to compare the impact of control strategies. It is essential to assess the need for and effectiveness of measures such as vaccination. The problem without control is defined as follows:

$$(P_{WC}) = \begin{cases} \min \mathcal{J}(x) = \int_0^T AI(t)dt. \\ \text{st.} \\ \dot{S}(t) = bN(t) - cS(t)I(t) - dS(t), \quad t \in [0, T] \\ \dot{E}(t) = cS(t)I(t) - (e + d)E(t), \quad t \in [0, T] \\ \dot{I}(t) = eE(t) - (g + a + d)I(t), \quad t \in [0, T] \\ \dot{N}(t) = (b - d)N(t) - aI(t), \quad t \in [0, T] \end{cases} \quad (6.8)$$

2.2 A Model with an unlimited supply of the vaccine

This part presents the problem of an unlimited supply of vaccines by posing $W(T) \leq +\infty$. It allows exploring the maximum impact of vaccination on the spread of an epidemic. It also serves as a basis for comparison with situations of limited resources. The problem with an unlimited supply of vaccines is defined as follows:

$$(P_{UV}) = \begin{cases} \min \mathcal{J}(x, u) = \int_0^T (AI(t) + u^2(t))dt. \\ \text{st.} \\ \dot{S}(t) = bN(t) - cS(t)I(t) - dS(t) - u(t)S(t), \quad t \in [0, T] \\ \dot{E}(t) = cS(t)I(t) - (e + d)E(t), \quad t \in [0, T] \\ \dot{I}(t) = eE(t) - (g + a + d)I(t), \quad t \in [0, T] \\ \dot{N}(t) = (b - d)N(t) - aI(t), \quad t \in [0, T] \\ \dot{W}(t) = u(t)S(t), \quad t \in [0, T] \\ u(t) \in [0, 1]. \end{cases} \quad (6.9)$$

2.3 A model with a limited supply of vaccines at each instant of time

Here, the overall limit of vaccines $W(T) \leq W_M$ considered in [67] is replaced by a mixed control-state constraint of the form:

$$u(t)S(t) \leq V_0 \quad a.e. \quad t \in [0, T], \quad (6.10)$$

Where V_0 is an upper bound on vaccines available at each instant t , the constraint $W(T) \leq W_M$ in [67] should be satisfied only at the end terminal T , while the mixed constraint (6.10) is to be satisfied at almost every instant of time during the whole vaccination program.

The problem with a limited supply of vaccines at each instant of time of vaccine is defined as follows:

$$(P_{mixed}) = \left\{ \begin{array}{ll} \min & \mathcal{J}(x, u) = \int_0^T (AI(t) + u^2(t))dt. \\ \text{st.} & \\ & \dot{S}(t) = bN(t) - cS(t)I(t) - dS(t) - u(t)S(t), \quad t \in [0, T] \\ & \dot{E}(t) = cS(t)I(t) - (e + d)E(t), \quad t \in [0, T] \\ & \dot{I}(t) = eE(t) - (g + a + d)I(t), \quad t \in [0, T] \\ & \dot{N}(t) = (b - d)N(t) - aI(t), \quad t \in [0, T] \\ & \dot{W}(t) = u(t)S(t), \quad t \in [0, T] \\ & u(t) \in [0, 1], \\ & u(t)S(t) \leq V_0. \end{array} \right. \quad (6.11)$$

3 Numerical Analysis of the Model

This approach utilizes a nonsmooth Newton method [41] to solve discrete optimal control problems. We employ a general one-step process for differential equations of infinite dimensional problems. By discretizing the differential equations, we transform the problem into a finite set of equations that can be solved using numerical techniques. We then use the Fischer Burmeister function to transform the necessary first-order conditions for discrete problems into equivalent nonlinear and nonsmooth equations.

3.1 Model without control

The general form of the NLP (P_{WC}) is defined as follows:

$$(P_{WC}) = \left\{ \begin{array}{ll} \min & \int_0^T AI(t)dt. \\ & \dot{x}(t) = f(t, x(t)), \quad t \in [0, T] \end{array} \right. \quad (6.12)$$

To use the nonsmooth Newton method, we need to discretize the problem, such as using the direct discretization method for a problem based on a discretization of the infinite-dimensional problem. The resulting discretized problem will be a finite-dimensional nonlinear program, like in NLP, with the structure of the objective function.

By using the Rectangle Method to discretize the integral, we get

$$\min \mathcal{J}_d(x_0, x_{\tilde{N}}) = \sum_{n=0}^{\tilde{N}-1} \Delta t AI_n \quad (6.13)$$

By using the Euler discretization method, where its general form is written as follows:

$$x_{n+1} = x_n + \Delta t f(t_n, x_n), \quad \text{for } n = 0, 1, \dots, \tilde{N} - 1 \quad (6.14)$$

where h is the time step size.

Most often, the Discretized Problem is defined as follows:

$$(P_d) = \begin{cases} \min & \mathcal{J}_d(x_0, x_{\tilde{N}}) \\ x_{n+1} & = x_n + \Delta t f(t_n, x_n), \quad n = 0, 1, \dots, \tilde{N} - 1. \end{cases} \quad (6.15)$$

we suppose the optimization functions are defined as:

$$\mathbf{h}(x_n) := \begin{pmatrix} x_1 - x_0 - \Delta t f(t_0, x_0) \\ \vdots \\ x_{\tilde{N}} - x_{\tilde{N}-1} - \Delta t f(t_{\tilde{N}-1}, x_{\tilde{N}-1}) \end{pmatrix} \in \mathbb{R}^{4\tilde{N}} \quad (6.16)$$

The Lagrangian of the Discretized Problem is given by

$$\mathbf{L}_d := \sum_{n=0}^{\tilde{N}-1} \Delta t A I_n + \sum_{n=0}^{\tilde{N}-1} [\lambda_{S,n}, \lambda_{E,n}, \lambda_{I,n}, \lambda_{N,n}] [x_{n+1} - x_n - \Delta t f(t_n, x_n)]. \quad (6.17)$$

with multipliers $\lambda_j = (\lambda_{j,0}, \dots, \lambda_{j,\tilde{N}-1})^\top \in \mathbb{R}^{4 \times \tilde{N}}$, $j = \{S, E, I, N\}$.

The first order necessary **KKT** conditions reads as

$$\begin{aligned} \nabla_x \mathbf{L}_d &= 0, \\ \mathbf{h}(x_n) &= 0, \end{aligned} \quad (6.18)$$

$$\nabla_{x_n} \mathbf{L}_d := \begin{pmatrix} \Delta t [(\lambda_{S,n} - \lambda_{E,n})cI_n + \lambda_{S,n}d] - \lambda_{S,n} \\ -\lambda_{E,n} + \Delta t [\lambda_{E,n}d + e(\lambda_{E,n} - \lambda_{I,n})] \\ \Delta t A - \lambda_{I,n} + \Delta t [(\lambda_{S,n} - \lambda_{E,n})cS_n + \lambda_{I,n}(d + g) + (\lambda_{I,n} + \lambda_{N,n})a] \\ -\lambda_{N,n} + \Delta t [\lambda_{N,n}d - (\lambda_{S,n} + \lambda_{N,n})b] \end{pmatrix} \quad (6.19)$$

$$F(z) := \begin{pmatrix} \nabla_x \mathbf{L}_d(x_n, \lambda_{j,n}) \\ \mathbf{h}(x_n) \end{pmatrix} = 0, \quad z = (x_n, \lambda_{j,n})^\top, \quad (6.20)$$

For every $z \in D_F$, F is continuously differentiable in a neighborhood of z , and it holds $\partial_B F(z) := \{V \mid V = \lim_{z_i \rightarrow z} F'(z_i)\}$

with

$$\partial_B F(z) = \begin{pmatrix} \nabla_{xx}^2 \mathbf{L}_d & \nabla_{x\lambda_j}^2 \mathbf{L}_d \\ \mathbf{h}'_x & \mathbf{h}'_{\lambda_j} \end{pmatrix} = \begin{pmatrix} \nabla_{xx}^2 \mathbf{L}_d & (\mathbf{h}'_x)^\top \\ \mathbf{h}'_x & 0 \end{pmatrix} \in \mathbb{R}^{8\tilde{N} \times 8\tilde{N}} \quad (6.21)$$

$\nabla_{xx}^2 \mathbf{L}_d$ is defined as follows:

$$\nabla_{xx}^2 \mathbf{L}_d = \begin{pmatrix} \nabla_{x_0 x_0}^2 \mathbf{L}_d & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \nabla_{x_{\tilde{N}-1} x_{\tilde{N}-1}}^2 \mathbf{L}_d \end{pmatrix} \in \mathbb{R}^{4\tilde{N} \times 4\tilde{N}} \quad (6.22)$$

where

$$\nabla_{x_n x_n}^2 \mathbf{L}_d = \begin{pmatrix} 0 & 0 & \Delta t(\lambda_{S,n} - \lambda_{E,n})c & 0 \\ 0 & 0 & 0 & 0 \\ \Delta t(\lambda_{S,n} - \lambda_{E,n})c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (6.23)$$

$\mathbf{h}'_x$ is defined as follows

$$\mathbf{h}'_x = \begin{pmatrix} -M_0 & I_d & & \mathbf{0} \\ & \ddots & \ddots & \\ \mathbf{0} & & -M_{\tilde{N}-1} & I_d \end{pmatrix} \in \mathbb{R}^{4\tilde{N} \times 4\tilde{N}} \quad (6.24)$$

where $M_n = I_d + \Delta t f'_x(t_n, x_n)$

$$f'_x = \begin{pmatrix} -d - cI_n & 0 & -cS_n & b \\ cI_n & -e - d & cS_n & 0 \\ 0 & e & -(g + d + a) & 0 \\ 0 & 0 & -a & b - d \end{pmatrix}, \quad (6.25)$$

3.2 Model with unlimited supply of vaccine

The general forme of the NLP (P_{UV}) is defined as follows.

$$(P_{UV}) = \begin{cases} \min \mathcal{J}(x, u) = \int_0^T (AI(t) + u^2(t)) dt. \\ \dot{x}(t) = f(t, x(t), u(t)), & t \in [0, T] \\ -u(t) \leq 0, \\ u(t) - 1 \leq 0. \end{cases} \quad (6.26)$$

where,

$$y(t) = [S(t) \ E(t) \ I(t) \ N(t) \ u(t)]^\top \quad (6.27)$$

By using the Rectangle Method to discretize the integral, we get

$$\min \mathcal{J}_d(y_0, y_{\tilde{N}}) = \sum_{n=0}^{\tilde{N}-1} \Delta t [AI_n + u_n^2] \quad (6.28)$$

By using the Euler discretization method, where its general form is written as follows:

$$x_{n+1} = x_n + \Delta t f(t_n, x_n, u_n), \quad \text{for } n = 0, 1, \dots, \tilde{N} - 1 \quad (6.29)$$

where h is the time step size.

Most often, the Discretized Optimal Control Problem (DOCP) is defined as follows:

$$(P_d) = \begin{cases} \min = \mathcal{J}_d(y_0, y_{\tilde{N}}) \\ x_{n+1} = x_n + \Delta t f(t_n, x_n, u_n), & n = 0, 1, \dots, \tilde{N} - 1, \\ -u_n \leq 0 & n = 0, 1, \dots, \tilde{N}, \\ u_n - 1 \leq 0 & n = 0, 1, \dots, \tilde{N}. \end{cases} \quad (6.30)$$

we suppose the optimization functions are defined as:

$$\mathbf{h}(y_n) := \begin{pmatrix} x_1 - x_0 - \Delta t f(t_0, x_n, u_n) \\ \vdots \\ x_{\tilde{N}} - x_{\tilde{N}-1} - \Delta t f(t_{\tilde{N}-1}, x_{\tilde{N}-1}, u_{\tilde{N}-1}) \end{pmatrix} \in \mathbb{R}^{4\tilde{N}} \quad (6.31)$$

$$\mathbf{w}(u_n) := \begin{pmatrix} -u_n \\ u_n - 1 \end{pmatrix} \in \mathbb{R}^{2(\tilde{N}+1)} \quad \text{for } n = 0, 1, \dots, \tilde{N}. \quad (6.32)$$

The Lagrangian of the Discretized Optimal Control Problem (DOCP) is given by:

$$\begin{aligned} \mathbf{L}_d &:= \mathcal{J}_d(y_n, y_{\tilde{N}}) + \sum_{n=0}^{\tilde{N}-1} [\lambda_{S,n}, \lambda_{E,n}, \lambda_{I,n}, \lambda_{N,n}] [x_{n+1} - x_n - \Delta t f(t_n, x_n, u_n)] \\ &\quad - \sum_{n=0}^{\tilde{N}} \tau_{1,n} u_n + \sum_{n=0}^{\tilde{N}} \tau_{2,n} (u_n - 1). \\ &= \mathbf{B}(x_{\tilde{N}}, u_{\tilde{N}}, \lambda_{j,\tilde{N}}, \tau_{k,\tilde{N}}) + \sum_{n=0}^{\tilde{N}-1} \mathbf{G}(x_n, u_n, \lambda_{j,n}, \tau_{k,n}). \end{aligned} \quad (6.33)$$

with multipliers

$$\lambda_j = (\lambda_{j,0}, \dots, \lambda_{j,\tilde{N}-1})^\top \in \mathbb{R}^{4 \times \tilde{N}}, j = \{S, E, I, N\}, \tau_k = (\tau_{k,0}, \dots, \tau_{k,\tilde{N}})^\top \in \mathbb{R}^{2(\tilde{N}+1)}, k = 1, 2.$$

where

$$\mathbf{B}(x_{\tilde{N}}, u_{\tilde{N}}, \lambda_{j,\tilde{N}}, \tau_{k,\tilde{N}}) = -\tau_{1,\tilde{N}} u_{\tilde{N}} + \tau_{2,\tilde{N}} (u_{\tilde{N}} - 1). \quad (6.34)$$

and,

$$\begin{aligned} \mathbf{G}(x_n, u_n, \lambda_{j,n}, \tau_{k,n}) &= \Delta t [A I_n + u_n^2] - \tau_{1,n} u_n + \tau_{2,n} (u_n - 1) \\ &\quad + [\lambda_{S,n}, \lambda_{E,n}, \lambda_{I,n}, \lambda_{N,n}] [x_{n+1} - x_n - \Delta t f(t_n, x_n, u_n)]. \end{aligned} \quad (6.35)$$

The first order necessary **KKT** conditions reads as

$$\begin{aligned} \nabla_x \mathbf{L}_d &= 0, \\ \nabla_u \mathbf{L}_d &= 0, \quad , \\ \mathbf{h}(x, u) &= 0, \end{aligned} \quad (6.36)$$

$$\mathbf{w}_k(u_n) \leq 0, \quad \tau_{k,n} \geq 0, \quad \tau_{k,n} \mathbf{w}_k(u_n) = 0, \quad k = 1, 2 \quad n = 0, 1, \dots, \tilde{N}$$

$$\nabla_x \mathbf{L}_d := \begin{pmatrix} \nabla_{x_0} \mathbf{G}(x_0, u_0, \lambda_{j,0}, \tau_{k,0}) \\ \vdots \\ \nabla_{x_{\tilde{N}-1}} \mathbf{G}(x_{\tilde{N}-1}, u_{\tilde{N}-1}, \lambda_{j,\tilde{N}-1}, \tau_{k,\tilde{N}-1}) \\ \nabla_{x_{\tilde{N}}} \mathbf{B}(x_{\tilde{N}}, u_{\tilde{N}}, \lambda_{j,\tilde{N}}, \tau_{k,\tilde{N}}) \end{pmatrix} \in \mathbb{R}^{4(\tilde{N}+1)} \quad (6.37)$$

$$\nabla_u \mathbf{L}_d := \begin{pmatrix} \nabla_{u_0} \mathbf{G}(x_0, u_0, \lambda_{j,0}, \tau_{k,0}) \\ \vdots \\ \nabla_{u_{\tilde{N}-1}} \mathbf{G}(x_{\tilde{N}-1}, u_{\tilde{N}-1}, \lambda_{j,\tilde{N}-1}, \tau_{k,\tilde{N}-1}) \\ \nabla_{u_{\tilde{N}}} \mathbf{B}(x_{\tilde{N}}, u_{\tilde{N}}, \lambda_{j,\tilde{N}}, \tau_{k,\tilde{N}}) \end{pmatrix} \in \mathbb{R}^{\tilde{N}+1} \quad (6.38)$$

where

$$\nabla_{x_n} \mathbf{G} := \begin{pmatrix} \Delta t [(\lambda_{S,n} - \lambda_{E,n})cI_n + \lambda_{S,n}d + \lambda_{S,n}u_n] - \lambda_{S,n} \\ -\lambda_{E,n} + \Delta t [\lambda_{E,n}d + e(\lambda_{E,n} - \lambda_{I,n})] \\ \Delta t A - \lambda_{I,n} + \Delta t [(\lambda_{S,n} - \lambda_{E,n})cS_n + \lambda_{I,n}(d + g) + (\lambda_{I,n} + \lambda_{N,n})a] \\ -\lambda_{N,n} + \Delta t [\lambda_{N,n}d - (\lambda_{S,n} + \lambda_{N,n})b] \end{pmatrix} \quad (6.39)$$

$$\nabla_{u_n} \mathbf{G} := 2\Delta t u_n + \lambda_{S,n}\Delta t S_n - \tau_1 + \tau_2, \quad \nabla_{x_{\tilde{N}}} \mathbf{B} := \mathbf{0}_{4,1}, \quad \text{and} \quad \nabla_{u_{\tilde{N}}} \mathbf{B} := \tau_{2,\tilde{N}} - \tau_{1,\tilde{N}} \quad (6.40)$$

The application of the Fischer-Burmeister function leads to the nonsmooth equation.

$$F(z) := \begin{pmatrix} \nabla_x \mathbf{L}_d(x_n, u_n, \lambda_{j,n}, \tau_{k,n}) \\ \nabla_u \mathbf{L}_d(x_n, u_n, \lambda_{j,n}, \tau_{k,n}) \\ \mathbf{h}(x_n, u_n) \\ \varphi(u_n, \tau_{1,0}) \\ \varphi(1 - u_n, \tau_{2,0}) \\ \vdots \\ \varphi(u_{\tilde{N}}, \tau_{1,\tilde{N}}) \\ \varphi(1 - u_{\tilde{N}}, \tau_{2,\tilde{N}}) \end{pmatrix} = 0 \quad , \quad z = (x_n, u_n, \lambda_{j,n}, \tau_{k,n})^\top, \quad (6.41)$$

where

$$\begin{aligned} \varphi(u_n, \tau_{1,n}) &= \sqrt{(u_n)^2 + (\tau_{1,n})^2} - (u_n + \tau_{1,n}) \\ \varphi(1 - u_n, \tau_{2,n}) &= \sqrt{(1 - u_n)^2 + (\tau_{2,n})^2} - (1 - u_n + \tau_{2,n}) \end{aligned} \quad (6.42)$$

For every $z \in D_F := \{z = (x, u, \lambda_j, \tau_1, \tau_2)^\top \mid |w_k(u)| + |\tau_k| > 0, k = 1, 2\}$, F is continuously differentiable in a neighborhood of z and it holds $\partial_B F(z) := \{V \mid V = \lim_{z_i \rightarrow z} F'(z_i)\}$

with

$$\partial_B F(z) = \begin{pmatrix} \nabla_{xx}^2 \mathbf{L}_d & \nabla_{xu}^2 \mathbf{L}_d & \nabla_{x\lambda_j}^2 \mathbf{L}_d & \nabla_{x\tau_k}^2 \mathbf{L}_d \\ \nabla_{ux}^2 \mathbf{L}_d & \nabla_{uu}^2 \mathbf{L}_d & \nabla_{u\lambda_j}^2 \mathbf{L}_d & \nabla_{u\tau_k}^2 \mathbf{L}_d \\ \mathbf{h}'_x & \mathbf{h}'_u & \mathbf{h}'_{\lambda_j} & \mathbf{h}'_{\tau_k} \\ \varphi'_x & \varphi'_u & \varphi'_{\lambda_j} & \varphi'_{\tau_k} \end{pmatrix} = \begin{pmatrix} \nabla_{xx}^2 \mathbf{L}_d & \nabla_{xu}^2 \mathbf{L}_d & (\mathbf{h}'_x)^\top & 0 \\ \nabla_{ux}^2 \mathbf{L}_d & \nabla_{uu}^2 \mathbf{L}_d & (\mathbf{h}'_u)^\top & \mathbf{W}_u^\top \\ \mathbf{h}'_x & \mathbf{h}'_u & 0 & 0 \\ 0 & -\mathbf{R}_w \mathbf{W}_u & 0 & \mathbf{T}_w \end{pmatrix} \quad (6.43)$$

$\nabla_{xx}^2 \mathbf{L}_d$ is defined as follows:

$$\nabla_{xx}^2 = \begin{pmatrix} \nabla_{x_0 x_0}^2 \mathbf{G} & & \mathbf{0} & \nabla_{x_0 x_{\tilde{N}}}^2 \mathbf{G} \\ & \ddots & & \vdots \\ \mathbf{0} & & \nabla_{x_{\tilde{N}-1} x_{\tilde{N}-1}}^2 \mathbf{G} & \nabla_{x_{\tilde{N}-1} x_{\tilde{N}}}^2 \mathbf{G} \\ \nabla_{x_{\tilde{N}} x_0}^2 \mathbf{B} & \dots & \dots & \nabla_{x_{\tilde{N}} x_{\tilde{N}}}^2 \mathbf{B} \end{pmatrix} \in \mathbb{R}^{4(\tilde{N}+1) \times 4(\tilde{N}+1)} \quad (6.44)$$

where

$$\nabla_{x_n x_n}^2 \mathbf{G} = \nabla_{x_n x_n}^2 \mathbf{L}_d \quad \text{as defined in (6.23), and} \quad (6.45)$$

$$\nabla_{x_{\tilde{N}}x_n}^2 \mathbf{B} = \nabla_{x_nx_{\tilde{N}}}^2 \mathbf{G} = \nabla_{x_{\tilde{N}}x_{\tilde{N}}}^2 \mathbf{B} = \mathbf{0}_{4,4} \quad (6.46)$$

$\nabla_{xu}^2 \mathbf{L}_d$ and $\nabla_{ux}^2 \mathbf{L}_d$ are defined as follows

$$\nabla_{xu}^2 \mathbf{L}_d = (\nabla_{ux}^2 \mathbf{L}_d)^\top = \begin{pmatrix} \nabla_{x_0u_0}^2 \mathbf{G} & \mathbf{0} & \nabla_{x_0u_{\tilde{N}}}^2 \mathbf{G} \\ & \ddots & \vdots \\ \mathbf{0} & \nabla_{x_{\tilde{N}-1}u_{\tilde{N}-1}}^2 \mathbf{G} & \nabla_{x_{\tilde{N}-1}u_{\tilde{N}}}^2 \mathbf{G} \\ \nabla_{x_{\tilde{N}}u_0}^2 \mathbf{B} & \dots & \dots & \nabla_{x_{\tilde{N}}u_{\tilde{N}}}^2 \mathbf{B} \end{pmatrix} \in \mathbb{R}^{4(\tilde{N}+1) \times (\tilde{N}+1)} \quad (6.47)$$

where

$$\nabla_{x_nu_n}^2 \mathbf{G} = \begin{pmatrix} \Delta t \lambda_{S,n} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \nabla_{x_{\tilde{N}}u_n}^2 \mathbf{B} = \nabla_{x_nu_{\tilde{N}}}^2 \mathbf{G} = \nabla_{x_{\tilde{N}}u_{\tilde{N}}}^2 \mathbf{B} = \mathbf{0}_{4,1} \quad (6.48)$$

$\nabla_{uu}^2 \mathbf{L}_d$ is defined as follows:

$$\nabla_{uu}^2 \mathbf{L}_d = \begin{pmatrix} \nabla_{u_0u_0}^2 \mathbf{G} & \mathbf{0} & \nabla_{u_0u_{\tilde{N}}}^2 \mathbf{G} \\ & \ddots & \vdots \\ \mathbf{0} & \nabla_{u_{\tilde{N}-1}u_{\tilde{N}-1}}^2 \mathbf{G} & \nabla_{u_{\tilde{N}-1}u_{\tilde{N}}}^2 \mathbf{G} \\ \nabla_{u_{\tilde{N}}u_0}^2 \mathbf{B} & \dots & \dots & \nabla_{u_{\tilde{N}}u_{\tilde{N}}}^2 \mathbf{B} \end{pmatrix} \in \mathbb{R}^{(\tilde{N}+1) \times (\tilde{N}+1)} \quad (6.49)$$

where

$$\nabla_{u_nu_n}^2 \mathbf{G} = 2\Delta t, \quad \text{and} \quad \nabla_{u_{\tilde{N}}u_n}^2 \mathbf{B} = \nabla_{u_nu_{\tilde{N}}}^2 \mathbf{G} = \nabla_{u_{\tilde{N}}u_{\tilde{N}}}^2 \mathbf{B} = \mathbf{0} \quad (6.50)$$

$\mathbf{h}'_x$ and $\mathbf{h}'_u$ are defined as follows:

$$\mathbf{h}'_x = \begin{pmatrix} -M_0 & I_d & \mathbf{0} & \mathbf{0} \\ & \ddots & \ddots & \\ \mathbf{0} & & -M_{\tilde{N}-1} & I_d \end{pmatrix} \in \mathbb{R}^{4(\tilde{N}) \times 4(\tilde{N}+1)} \quad (6.51)$$

where $M_n = I_d + \Delta t f'_x(t_n, x_n, u_n)$

$$f'_x = \begin{pmatrix} -d - cI_n - u_n & 0 & -cS_n & b \\ cI_n & -e - d & cS_n & 0 \\ 0 & e & -(g + d + a) & 0 \\ 0 & 0 & -a & b - d \end{pmatrix}, \quad (6.52)$$

$$\mathbf{h}'_u = \begin{pmatrix} -\Delta t f'_u[t_0] & 0 & 0 \\ \ddots & & \\ 0 & -\Delta t f'_u[t_{N-1}] & 0 \end{pmatrix} \in \mathbb{R}^{4(\tilde{N}) \times \tilde{N}+1}, \quad f'_u = \begin{pmatrix} -S_n \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (6.53)$$

$\mathbf{W}_u$ is defined as follows:

$$\mathbf{W}_u = \text{diag}(\mathbf{w}'(u_0), \dots, \mathbf{w}'(u_{\tilde{N}})) \in \mathbb{R}^{2(\tilde{N}+1) \times 2(\tilde{N}+1)}, \quad \mathbf{w}'(u_n) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \quad (6.54)$$

$\mathbf{R}_w$, and $\mathbf{T}_w$ are defined as follows:

$$\begin{aligned} \mathbf{R}_w &= \text{diag}(\mathbf{R}_w^0, \dots, \mathbf{R}_w^{\tilde{N}}) \in \mathbb{R}^{2(\tilde{N}+1) \times 2(\tilde{N}+1)}, \\ \mathbf{T}_w &= \text{diag}(\mathbf{T}_w^0, \dots, \mathbf{T}_w^{\tilde{N}}) \in \mathbb{R}^{2(\tilde{N}+1) \times 2(\tilde{N}+1)}. \end{aligned} \quad (6.55)$$

Herein, $\mathbf{R}_w^n$ and $\mathbf{T}_w^n$ are diagonal matrices. A specific element of the B-differential F can be computed according to Lemma 2.5 in Section 2 of [41]

$$\begin{pmatrix} (\mathbf{R}_w^n)_{kk} \\ (\mathbf{T}_w^n)_{kk} \end{pmatrix} = \begin{cases} \begin{pmatrix} \left(\frac{-\mathbf{w}_k(u_{i,n})}{\sqrt{(\mathbf{w}_k(u_{i,n}))^2 + \tau_{k,n}^2}} - 1 \right) \\ \frac{\tau_{k,n}}{\sqrt{(\mathbf{w}_k(u_{i,n}))^2 + \tau_{k,n}^2}} - 1 \end{pmatrix} & \text{if } (\mathbf{w}(u_n), \tau_{k,n}) \neq (0, 0). \\ \begin{pmatrix} -1 \\ 0 \end{pmatrix} & \text{otherwise.} \end{cases} \quad (6.56)$$

3.3 Model with limited supply of vaccines at each instant of time

The general forme of the NLP (P_{mixed}) is defined as follows.

$$(P_{mixed}) = \begin{cases} \min \mathcal{J}(y) = \int_0^T (AI(t) + u^2(t)) dt. \\ \dot{x}(t) = f(t, x(t), u(t)), & t \in [0, T] \\ -u(t) \leq 0, \\ u(t) - 1 \leq 0, \\ u(t)S(t) - V_0 \leq 0. \end{cases} \quad (6.57)$$

By using the Rectangle Method to discretize the integral, and the Euler discretization method we obtain the Discretized Optimal Control Problem (DOCP) defined as follows

$$(P_d) = \begin{cases} \min = \mathcal{J}_d(y_0, y_{\tilde{N}}) \\ x_{n+1} = x_n + \Delta t f(t_n, x_n, u_n), & n = 0, 1, \dots, \tilde{N} - 1, \\ -u_n \leq 0 & n = 0, 1, \dots, \tilde{N}, \\ u_n - 1 \leq 0 & n = 0, 1, \dots, \tilde{N}. \\ u_n S_n - V_0 \leq 0 & n = 0, 1, \dots, \tilde{N}. \end{cases} \quad (6.58)$$

The optimization functions $\mathbf{h}(y_n)$ and $\mathbf{w}(u_n)$ are defined by (6.31) and (6.32). $\mathbf{c}(y_n)$ is defined as:

$$\mathbf{c}(y_n) := \begin{pmatrix} u_0 S_0 - V_0 \\ \vdots \\ u_{\tilde{N}} S_{\tilde{N}} - V_0 \end{pmatrix} \in \mathbb{R}^{(\tilde{N}+1)}. \quad (6.59)$$

The Lagrangian of the Discretized Optimal Control Problem (DOCP) is given by

$$\begin{aligned} \mathbf{L}_d &:= \mathcal{J}_d(y_n, y_{\tilde{N}}) + \sum_{n=0}^{\tilde{N}-1} [\lambda_{S,n}, \lambda_{E,n}, \lambda_{I,n}, \lambda_{N,n}] [x_{n+1} - x_n - \Delta t f(t_n, x_n, u_n)] \\ &\quad - \sum_{n=0}^{\tilde{N}} \tau_{1,n} u_n + \sum_{n=0}^{\tilde{N}} \tau_{2,n} (u_n - 1) + \sum_{n=0}^{\tilde{N}} \eta_n [u_n S_n - V_0]. \\ &= \mathbf{B}(x_{\tilde{N}}, u_{\tilde{N}}, \lambda_{j,\tilde{N}}, \tau_{k,\tilde{N}}, \eta_{\tilde{N}}) + \sum_{n=0}^{\tilde{N}-1} \mathbf{G}(x_n, u_n, \lambda_{j,n}, \tau_{k,n}, \eta_n). \end{aligned} \quad (6.60)$$

with multiplier $\eta = (\eta_0, \dots, \eta_{\tilde{N}})^\top \in \mathbb{R}^{(\tilde{N}+1)}$.

where

$$\mathbf{B}(x_{\tilde{N}}, u_{\tilde{N}}, \lambda_{j,\tilde{N}}, \tau_{k,\tilde{N}}, \eta_{\tilde{N}}) = -\tau_{1,\tilde{N}}u_{\tilde{N}} + \tau_{2,\tilde{N}}(u_{\tilde{N}} - 1) + \eta_{\tilde{N}}[u_{\tilde{N}}S_{\tilde{N}} - V_0]. \quad (6.61)$$

and,

$$\begin{aligned} \mathbf{G}(x_n, u_n, \lambda_{j,n}, \tau_{k,n}, \eta_n) &= \Delta t[AI_n + u_n^2] - \tau_{1,n}u_n + \tau_{2,n}(u_n - 1) + \eta_n(u_nS_n - V_0) \\ &\quad + [\lambda_{S,n}, \lambda_{E,n}, \lambda_{I,n}, \lambda_{N,n}][x_{n+1} - x_n - \Delta t f(t_n, x_n, u_n)]. \end{aligned} \quad (6.62)$$

The first order necessary **KKT** conditions reads as

$$\begin{aligned} \nabla_x \mathbf{L}_d &= 0, \\ \nabla_u \mathbf{L}_d &= 0, \quad , \\ \mathbf{h}(x, u) &= 0, \end{aligned} \quad (6.63)$$

$$\mathbf{w}_k(u_n) \leq 0, \quad \tau_{k,n} \geq 0, \quad \tau_{k,n} \mathbf{w}_k(u_n) = 0, \quad k = 1, 2 \quad n = 0, 1, \dots, \tilde{N}$$

$$\mathbf{c}(y_n) \leq 0, \quad \eta_n \geq 0, \quad \eta_n \mathbf{c}(y_n) = 0, \quad n = 0, 1, \dots, \tilde{N}$$

$\nabla_x \mathbf{L}_d$, $\nabla_u \mathbf{L}_d$, and $\nabla_{x_n} \mathbf{G}$ are defined by (6.37), (6.38), and (6.39).

$$\nabla_{u_n} \mathbf{G} := 2\Delta t u_n + \lambda_{S,n} \Delta t S_n - \tau_1 + \tau_2 + \eta_n S_n, \quad \nabla_{u_{\tilde{N}}} \mathbf{B} := \tau_{2,\tilde{N}} - \tau_{1,\tilde{N}} + \eta_n S_n, \quad (6.64)$$

and

$$\nabla_{x_{\tilde{N}}} \mathbf{B} := [\eta_n S_n, 0, 0, 0]^\top. \quad (6.65)$$

Application of the Fischer-Burmeister function leads to the nonsmooth equation

$$F(z) := \begin{pmatrix} \nabla_x \mathbf{L}_d(x_n, u_n, \lambda_{j,n}, \tau_{k,n}, \eta_n) \\ \nabla_u \mathbf{L}_d(x_n, u_n, \lambda_{j,n}, \tau_{k,n}, \eta_n) \\ \mathbf{h}(x_n, u_n) \\ \varphi(u_n, \tau_{1,0}) \\ \varphi(1 - u_n, \tau_{2,0}) \\ \vdots \\ \varphi(u_{\tilde{N}}, \tau_{1,\tilde{N}}) \\ \varphi(1 - u_{\tilde{N}}, \tau_{2,\tilde{N}}) \\ \varphi(V_0 - u_0 S_0, \eta_0) \\ \vdots \\ \varphi(V_0 - u_{\tilde{N}} S_{\tilde{N}}, \eta_{\tilde{N}}) \end{pmatrix} = 0 \quad , \quad z = (x_n, u_n, \lambda_{j,n}, \tau_{k,n}, \eta_n)^\top, \quad (6.66)$$

where $\varphi(u_n, \tau_{1,n})$, $\varphi(1 - u_n, \tau_{2,n})$ are defined by (6.42), and

$$\varphi(V_0 - u_n S_n, \eta_n) = \sqrt{(V_0 - u_n S_n)^2 + (\eta_n)^2} - (V_n - u_n S_n + \eta_n). \quad (6.67)$$

For every $z \in D_F := \left\{ z = (x, u, \lambda_j, \tau_1, \tau_2, \eta)^\top \mid \begin{array}{l} |w_k(u)| + |\tau_k| > 0, \quad k = 1, 2, \\ |c(y)| + |\eta| > 0. \end{array} \right\}$,

F is continuously differentiable in a neighborhood of z and it holds $\partial_B F(z) := \{V \mid V = \lim_{z_i \rightarrow z} F'(z_i)\}$. with

$$\begin{aligned} \partial_B F(z) &= \begin{pmatrix} \nabla_{xx}^2 \mathbf{L}_d & \nabla_{xu}^2 \mathbf{L}_d & \nabla_{x\lambda_j}^2 \mathbf{L}_d & \nabla_{x\tau_k}^2 \mathbf{L}_d & \nabla_{x\eta}^2 \mathbf{L}_d \\ \nabla_{ux}^2 \mathbf{L}_d & \nabla_{uu}^2 \mathbf{L}_d & \nabla_{u\lambda_j}^2 \mathbf{L}_d & \nabla_{u\tau_k}^2 \mathbf{L}_d & \nabla_{u\eta}^2 \mathbf{L}_d \\ \mathbf{h}'_x & \mathbf{h}'_u & \mathbf{h}'_{\lambda_j} & \mathbf{h}'_{\tau_k} & \mathbf{h}'_\eta \\ \varphi'_x & \varphi'_u & \varphi'_{\lambda_j} & \varphi'_{\tau_k} & \varphi'_\eta \end{pmatrix} \\ &= \begin{pmatrix} \nabla_{xx}^2 \mathbf{L}_d & \nabla_{xu}^2 \mathbf{L}_d & (\mathbf{h}'_x)^\top & 0 & \mathbf{C}_x^\top \\ \nabla_{ux}^2 \mathbf{L}_d & \nabla_{uu}^2 \mathbf{L}_d & (\mathbf{h}'_u)^\top & \mathbf{W}_u^\top & \mathbf{C}_u^\top \\ \mathbf{h}'_x & \mathbf{h}'_u & 0 & 0 & 0 \\ 0 & -\mathbf{R}_w \mathbf{W}_u & 0 & \mathbf{T}_w & 0 \\ -\mathbf{R}_c \mathbf{C}_x & -\mathbf{R}_c \mathbf{C}_u & 0 & 0 & \mathbf{T}_c \end{pmatrix} \end{aligned} \quad (6.68)$$

$\nabla_{xx}^2 \mathbf{L}_d$ is defined by (6.44), where $\nabla_{x_n x_n}^2 \mathbf{G}$, $\nabla_{x_n x_{\tilde{N}}}^2 \mathbf{G}$, $\nabla_{x_{\tilde{N}} x_n}^2 \mathbf{B}$, and $\nabla_{x_{\tilde{N}} x_{\tilde{N}}}^2 \mathbf{B}$ are defined by (6.45) and (6.46).

$\nabla_{xu}^2 \mathbf{L}_d$ and $\nabla_{ux}^2 \mathbf{L}_d$ are defined by (6.47).

where

$$\nabla_{x_n u_n}^2 \mathbf{G} = \begin{pmatrix} \Delta t \lambda_{S,n} + \eta_n \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \nabla_{x_{\tilde{N}} u_n}^2 \mathbf{B} = \nabla_{x_n u_{\tilde{N}}}^2 \mathbf{G} = \mathbf{0}_{4,1}, \quad \text{and} \quad \nabla_{x_{\tilde{N}} u_{\tilde{N}}}^2 \mathbf{B} = \begin{pmatrix} \eta_n \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (6.69)$$

$\nabla_{uu}^2 \mathbf{L}_d$, $\nabla_{u_n u_n}^2 \mathbf{G}$, $\nabla_{u_n u_{\tilde{N}}}^2 \mathbf{G}$, $\nabla_{u_{\tilde{N}} u_n}^2 \mathbf{B}$, and $\nabla_{u_{\tilde{N}} u_{\tilde{N}}}^2 \mathbf{B}$ are defined by (6.49) and (6.50).

All of $\mathbf{h}'_x$, $\mathbf{h}'_u$, f'_x , f'_u , $\mathbf{W}_u$, $\mathbf{R}_w$, and $\mathbf{T}_w$ are defined by (6.51) $\rightarrow$ (6.56).

$\mathbf{C}_x$ and $\mathbf{C}_u$ are defined as follows:

$$\begin{aligned} \mathbf{C}_x &= \text{diag}(\mathbf{c}'(x_0), \dots, \mathbf{c}'(x_{\tilde{N}})) \in \mathbb{R}^{(\tilde{N}+1) \times 4(\tilde{N}+1)}, & \mathbf{c}'(x_n) &= (u_n, 0, 0, 0). \\ \mathbf{C}_u &= \text{diag}(\mathbf{c}'(u_0), \dots, \mathbf{c}'(u_{\tilde{N}})) \in \mathbb{R}^{(\tilde{N}+1) \times (\tilde{N}+1)}, & \mathbf{c}'(u_n) &= S_n. \end{aligned} \quad (6.70)$$

$\mathbf{R}_c$ and $\mathbf{T}_c$ are defined as follows:

$$\begin{aligned} \mathbf{R}_c &= \text{diag}(\mathbf{R}_c^0, \dots, \mathbf{R}_c^{\tilde{N}}) \in \mathbb{R}^{(\tilde{N}+1) \times (\tilde{N}+1)}, \\ \mathbf{T}_c &= \text{diag}(\mathbf{T}_c^0, \dots, \mathbf{T}_c^{\tilde{N}}) \in \mathbb{R}^{(\tilde{N}+1) \times (\tilde{N}+1)}. \end{aligned} \quad (6.71)$$

Herein, $\mathbf{R}_c^n$ and $\mathbf{T}_c^n$ are diagonal matrices. A specific element of the B-differential F can be computed according to Lemma 2.5 in Section 2 of [41]

$$\begin{pmatrix} \mathbf{R}_c^n \\ \mathbf{T}_c^n \end{pmatrix} = \begin{cases} \begin{pmatrix} \frac{V_0 - u_n S_n}{\sqrt{(V_0 - u_n S_n)^2 + \eta_n^2}} - 1 \\ \frac{\eta_n}{\sqrt{(V_0 - u_n S_n)^2 + \eta_n^2}} - 1 \end{pmatrix} & \text{if } (\mathbf{c}(y_n), \eta_n) \neq (0, 0). \\ \begin{pmatrix} -1 \\ 0 \end{pmatrix} & \text{otherwise.} \end{cases} \quad (6.72)$$

4 Numerical Results and Discussion

The numerical resolution of optimal control problems and nonsmooth dynamic systems is crucial in optimization and control. Nonsmooth Newton's method has been employed to solve our problem. This numerical part focuses on a detailed comparison of (NNM and PMP) approaches. This chapter discusses and develops the nonsmooth Newton and the maximum principle discussed in [67], highlighting their advantages, disadvantages, and potential applications in epidemiology. We evaluate their effectiveness in terms of convergence and numerical stability.

The simulations are performed using MATLAB, an iterative Globalized Nonsmooth Newton algorithm in [41] is developed into a Matlab code to compute the numerical

solution all of (P_{WC}) , (P_{UV}) , and (P_{mixed}) , such as the parameters and the initial values used in the simulation are given in Table 6.2.

Tab. 6.2: Parameters values and constants [67] .

Parameter	b	d	c	e	g	a	A	T	S_0	E_0	I_0	R_0	N_0	W_0
Initial Value	0.525	0.5	0.001	0.5	0.1	0.2	0.1	20	1000	100	50	15	1165	0

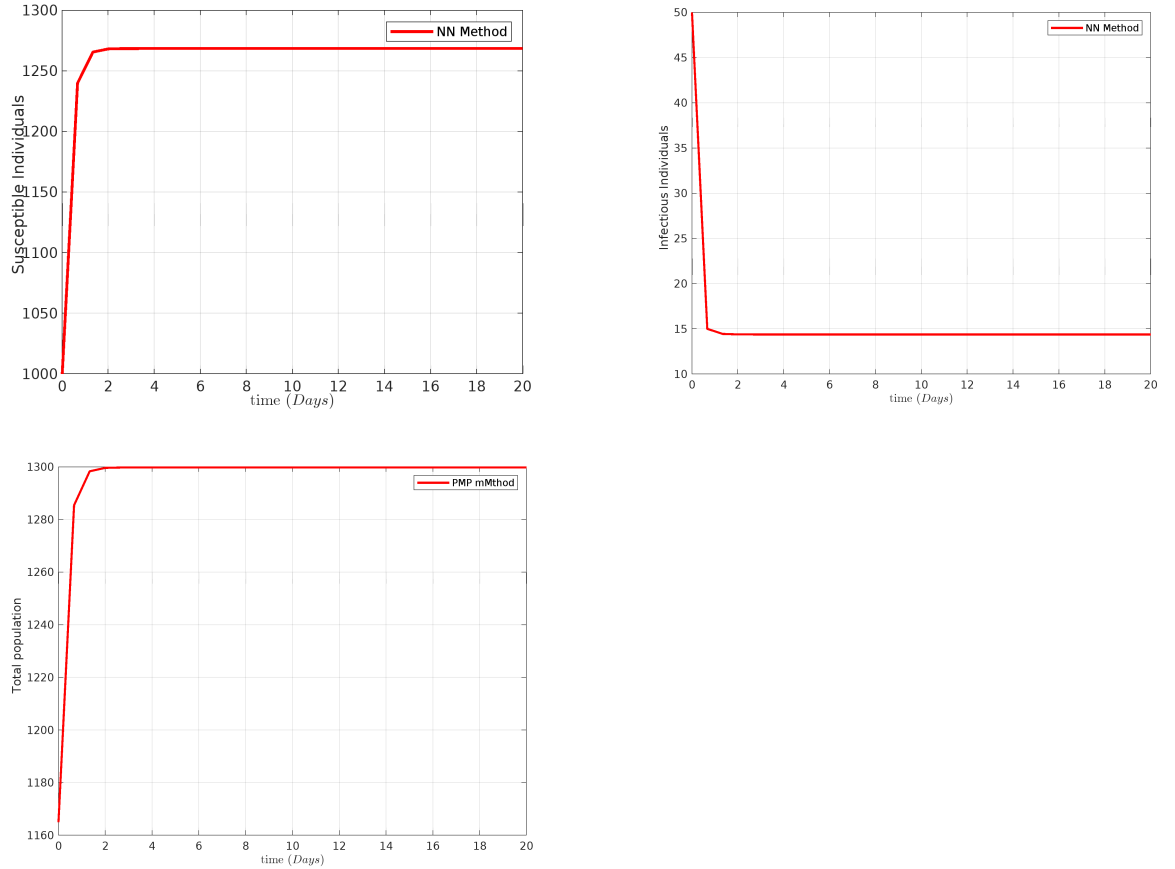


Fig. 6.2: The dynamics of S , I and N when no vaccines are used (without control)

Figure 6.2 shows the transmission of an infectious disease through a population that has not been vaccinated. The infected graph shows the typical epidemic curve with a sudden rise and gradual fall. The absence of vaccination allows the disease to spread widely and infect many of the population.

Figure 6.3 shows the optimal trajectories for an SEIR epidemic model with unbounded vaccination rates.

The plots show how the population of susceptible, infected, and vaccinated people varies over time and the optimal vaccination rate. We can see that the vaccination rate is high initially and then decreases as the infected population decreases.

The number of infected people decreases rapidly and approaches a steady state after a few days. Ultimately, the number of infected people is brought down to zero, which means vaccination has effectively eradicated the epidemic.

Figure 6.4 shows the optimal trajectories of an epidemic model with an instant-

Tab. 6.3: Comparative Analysis of the Methods (Iterations and Cost)

Cases	Iterations	Cost
WC (MP) [67]	NM	46.49
WC (NNM)	≈ 4	31.20
UV (MP) [67]	≈ 31	24.13
UV (NNM)	≈ 8	21.93
MC (MP) [67]	≈ 33	33.68
MC (NNM)	≈ 15	10.26

Tab. 6.4: Initial values of the multipliers

Cases	$\lambda_S(0) \times 10^{-3}$	$\lambda_E(0) \times 10^{-3}$	$\lambda_I(0) \times 10^{-3}$	$\lambda_W(0) \times 10^{-3}$	$\lambda_N(0) \times 10^{-3}$	$\tau_1(0)$	$\tau_2(0)$	$\eta(0)$
WC (MP) [67]	–	–	–	–	–	–	–	–
WC (NNM)	–3.33	–100.6	–204.8	0	–11.05	–	–	–
UV (MP) [67]	–3.33	–100.6	–204.8	0	–11.05	–	–	–
UV (NNM)	–3.33	–100.6	–204.8	0	–11.05	0.01	0.01	–
MC (MP) [67]	–12.61	–151.1	–293.2	0	–45.28	–	–	–
MC (NNM)	–12.61	–151.1	–293.2	0	–45.28	–0.1	–0.1	–10

neous vaccination constraint.

First graph: This graph shows the evolution of the number of susceptible individuals over time. There is a sharp decrease at first, followed by stabilization at around 1600 individuals.

The second: This graph shows the evolution of the number of infected individuals over time. There is a rapid increase at first, followed by a steep decrease and stabilization at five individuals.

Third: This graph illustrates the development of the total population over time. There is steady growth, which stabilizes at about 1700 individuals.

The fourth: This graph illustrates the number of people vaccinated over time. There was an initial rapid growth, followed by a stabilization of 120 individuals.

The fifth: This figure plots the variation of the optimal vaccination rate. It starts with a significant drop and then stabilizes around 0.07.

These findings indicate that an optimal vaccination policy, subject to a limitation on the rate of immediate vaccination, successfully contains the epidemic by reducing the number of infected people to a minimum and the number of vaccinated individuals to a maximum.

The tabular analysis shows that the non-smooth Newton method (NNM) converges faster and has a lower cost when compared to the Pontryagin maximum (PMP) principle for the optimal control problems considered. This is true for the three types of issues: uncontrolled (WC), with unlimited strain on the vaccine (UV), and with mixed strain (MC).

The findings are summarized as follows:

WC (without control): NNM converges to a cost of 31.20, while PMP requires more iterations and converges to 46.49. UV (unlimited strain on the vaccine): NNM converges in 8 iterations to a cost of 21.83, while 31 iterations and 24.13 are required for PMP. MC (mixed constraint): NNM converges in 15 iterations to a cost of 10.26,

while 33 iterations and 33.68 are required for PMP.

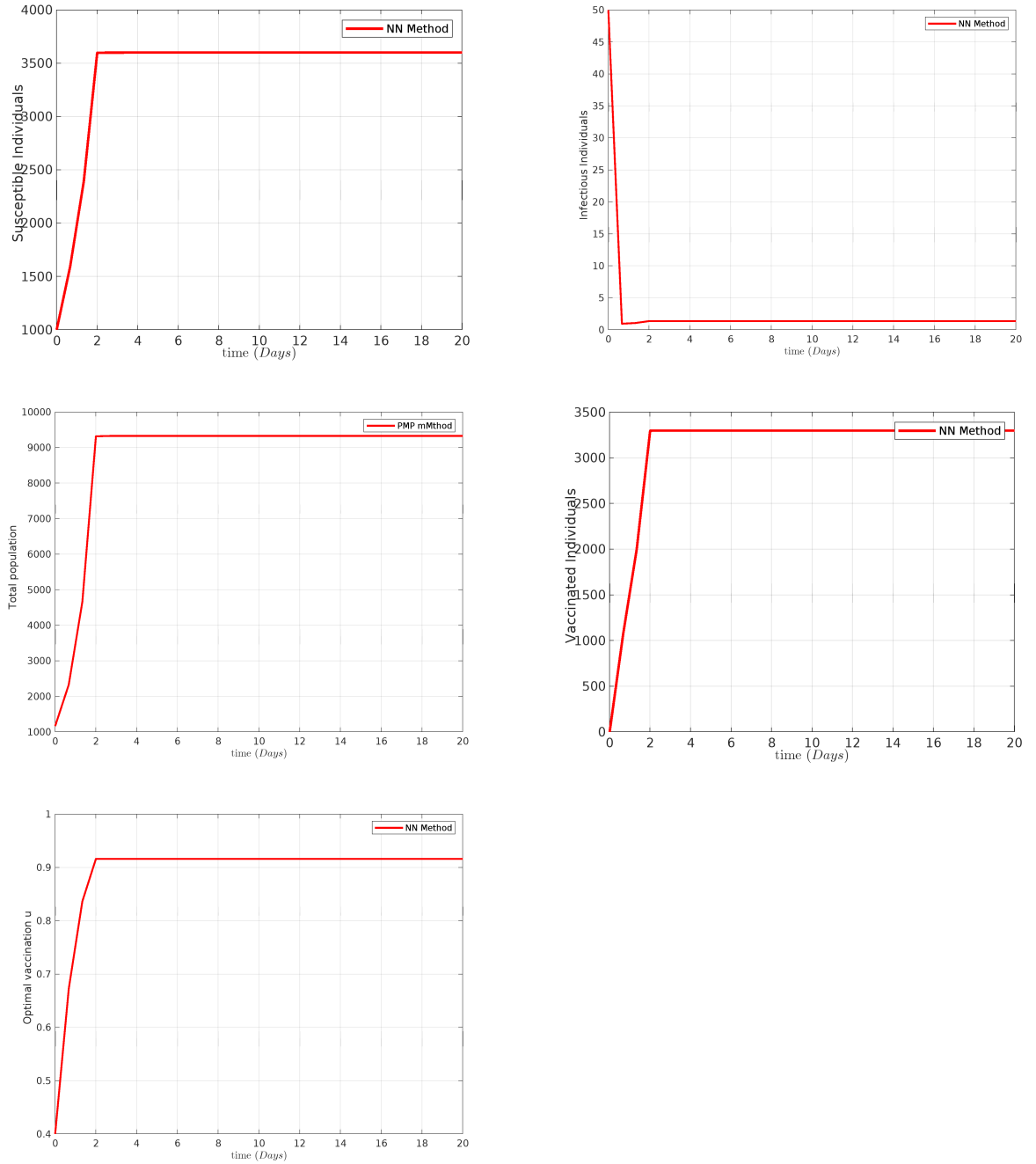


Fig. 6.3: The optimal trajectories and optimal vaccination rate with unlimited vaccines (P_{UV}).

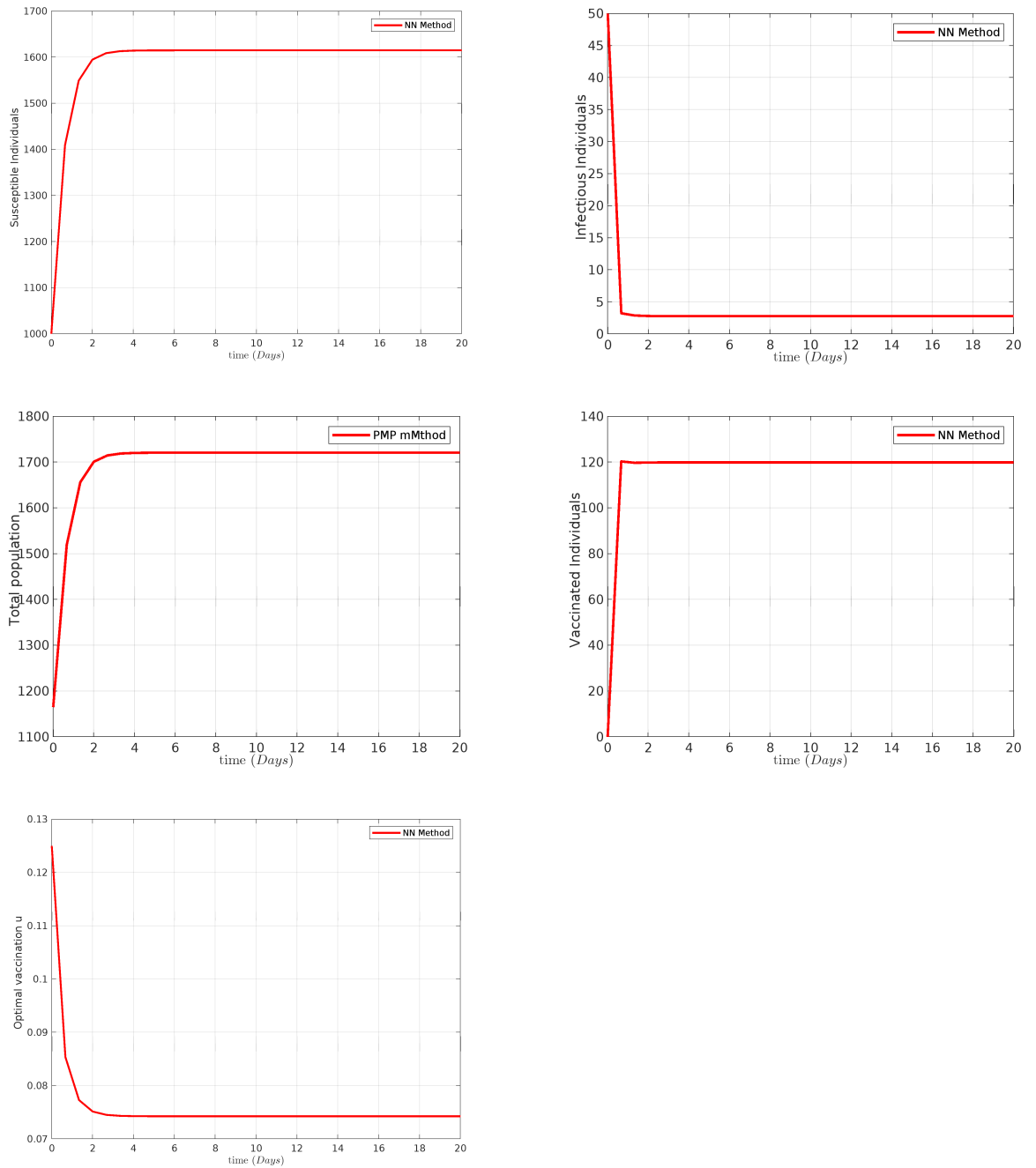


Fig. 6.4: The optimal trajectories and optimal vaccination rate for (P_{mixed})

5 Conclusion

From the comparison of PMP (Pontryagin Maximum Principle) and NNM (Nonsmooth Newton Method), it can be seen that the NNM method is more efficient globally for solving the problem without control, an optimal control system with control constraints, and with a mixed control-state constraint. These observations confirm that NNM can be a more suitable method to solve optimal control problems. The better convergence and lower costs achieved with NNM could be due to its better ability to treat constraints and non-linearity of optimal control problems.

General Conclusion

This thesis develops a new and consistent mathematical toolset for analysis and control in dynamical systems for application in epidemiology. Starting with the classic basics of ordinary differential equations, it incorporates fractional calculus in a systematic way, mainly with the conformable derivative, which is appropriate to model memory effects and non-local behavior crucial to correctly model infectious diseases such as COVID-19.

The work rigorously illustrates, through the SEIR model and its extension to the SEIHRDV model with vaccination, how this fractional setting enriches and modifies the stability analysis of equilibria, thus affecting the convergence rates and threshold conditions linked to the basic reproduction number $\mathcal{R}_0$. One key methodological contribution is in the innovative use of the Sumudu transform to prove the positivity of solutions and to simplify the analysis.

As concerns the optimal control, the thesis has made detailed and comparative evaluations of direct methods and indirect methods. Finally, it ends up with the practical superiority of NNM methods over PMP for constrained problems because their global efficiency is superior, nonlinear effects can be robustly handled, and their convergence is better.

In other words, this thesis makes a dual contribution:

1. **Theoretical and Analytical:** It enhances the toolkit for studying complex dynamical systems by embracing the fractional paradigm and proposing new stability proofs.
2. **Numerical and Applied:** It presents and validates, through the Algeria case study, effective computational strategies (Sumudu transform, NNM method) to simulate epidemics and find optimal control policies that take into account realistic parameters like vaccine efficacy.

Hence, it bridges the mathematical rigor with practical applicability and thus opens up avenues for more realistic modeling of epidemic dynamics and optimization of public health intervention strategies.

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Abstract

This thesis focuses on the application of mathematical models to better understand and control infectious diseases.

First, it uses a classic model (SEIR) to show how a disease moves through a population. The result is that if the reproduction number $\mathcal{R}_0$ is less than 1, the outbreak will die out. In order to model diseases, which have a sort of “memory” of past infections, a more advanced type of math called fractional calculus is used. This new model gives a more realistic view of how diseases like COVID-19 really behave.

The second part of this thesis is concerned with determining optimal methods for halting an outbreak. Two major mathematical approaches to finding these optimal control strategies are compared. The results demonstrate that one approach—the Nonsmooth Newton Method is faster and better at handling real-world, complicated limits on resources than its predecessor, Pontryagin’s Maximum Principle. A new neural network-based approach is also proposed for solving these control problems.

In other words, this work develops better mathematical tools for the simulation of epidemics and offers more efficient means of calculating the optimal public health actions like vaccinations required to control such outbreaks.

Keywords: Compartmental model; SEIR model; Covid-19; Lyapunov stability; Equilibrium points; Fractional calculus; Sumudu transform; Optimal control; Maximum principle method; Nonsmooth Newton’s method.

Résumé

Cette thèse se concentre sur l'application de modèles mathématiques pour mieux comprendre et contrôler les maladies infectieuses.

Dans un premier temps, elle utilise un modèle classique (SEIR) pour montrer comment une maladie se propage au sein d'une population. Le résultat principal est que si le nombre de reproduction de base $\mathcal{R}_0$ est inférieur à 1, l'épidémie s'éteindra d'elle-même. Afin de modéliser des maladies qui présentent une forme de « mémoire » des infections passées, un type de mathématiques plus avancé, le calcul fractionnaire, est employé. Ce nouveau modèle offre une vision plus réaliste du comportement de maladies telles que la COVID-19.

La seconde partie de cette thèse vise à déterminer les méthodes optimales pour arrêter une épidémie. Deux approches mathématiques majeures pour élaborer ces stratégies de contrôle optimal sont comparées. Les résultats démontrent qu'une de ces approches la méthode de Newton non lisse est plus rapide et plus efficace pour prendre en compte les limites pratiques et complexes des ressources disponibles que son prédécesseur, le principe du maximum de Pontryagin. Une nouvelle approche basée sur les réseaux neuronaux est également proposée pour résoudre ces problèmes de contrôle.

En d'autres termes, ce travail développe des outils mathématiques améliorés pour la simulation des épidémies et propose des moyens plus efficaces de calculer les actions de santé publique optimales telles que les campagnes de vaccination nécessaires pour contrôler de telles flambées.

Mots-clés: Modèle compartimental; Modèle SEIR ; COVID-19; Stabilité de Lyapunov; Points d'équilibre; Calcul fractionnaire; Transformée de Sumudu; Contrôle optimal; Principe du maximum; Méthode de Newton non lisse.