Kragujevac Journal of Mathematics
Volume 38(1) (2014), Pages 147–161.

RELATION BETWEEN SMALL FUNCTIONS WITH DIFFERENTIAL POLYNOMIALS GENERATED BY MEROMORPHIC SOLUTIONS OF HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS

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Abstract. This paper is devoted to studying the growth and oscillation of higher order differential polynomial with meromorphic coefficients generated by meromorphic solutions of the linear differential equation

$$f^{(k)} + A(z) f = 0 \ \ (k \geq 2),$$

where A is a meromorphic function in the complex plane.

1. Introduction and main results

In this paper, we shall assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of meromorphic functions see [10, 17]. For the definition of the iterated order of a meromorphic function, we use the same definition as in [11] , ([2], p. 317), ([12], p. 129). For all $r \in \mathbb{R}$, we define $\exp_1 r := e^r$ and $\exp_{p+1} r := \exp\left(\exp_p r\right), p \in \mathbb{N}$. We also define for all r sufficiently large $\log_1 r := \log r$ and $\log_{p+1} r := \log\left(\log_p r\right), p \in \mathbb{N}$. Moreover, we denote by $\exp_0 r := r$, $\log_0 r := r$, $\log_{-1} r := \exp_1 r$ and $\exp_{-1} r := \log_1 r$.

Definition 1.1. [11, 12] Let f be a meromorphic function. Then the iterated $p-$order $\rho_p(f)$ of f is defined as

$$\rho_p(f) = \limsup_{r \to +\infty} \frac{\log_p T(r, f)}{\log r} \quad (p \geq 1 \ \text{ is an integer}),$$

Key words and phrases. Linear differential equations, Differential polynomials, Meromorphic solutions, Iterated order, Iterated exponent of convergence of the sequence of distinct zeros.

2010 *Mathematics Subject Classification.* Primary: 34M10. Secondary: 30D35.

Received: March 21, 2013

Revised: May 6, 2014.

where $T(r, f)$ is the Nevanlinna characteristic function of f. For $p = 1$, this notation is called order and for $p = 2$ hyper-order, see [10, 17].

Definition 1.2. [11] The finiteness degree of the order of a meromorphic function f is defined as

$$i(f) = \begin{cases} 0, & \text{for } f \text{ rational,} \\ \min\{j \in \mathbb{N} : \ \rho_j(f) < +\infty\}, & \text{for } f \text{ transcendental for which} \\ & \text{some } j \in \mathbb{N} \text{ with } \rho_j(f) < +\infty \text{ exists,} \\ +\infty, & \text{for } f \text{ with } \rho_j(f) = +\infty \text{ for all } j \in \mathbb{N}. \end{cases}$$

Definition 1.3. [4, 6] The type of a meromorphic function f of iterated $p-$order ρ $(0 < \rho < \infty)$ is defined as

$$\tau_p(f) = \limsup_{r \to +\infty} \frac{\log_{p-1} T(r, f)}{r^{\rho}} \quad (p \geq 1 \ \text{ is an integer}).$$

Definition 1.4. [11] Let f be a meromorphic function. Then the iterated exponent of convergence of the sequence of zeros of $f(z)$ is defined as

$$\lambda_p(f) = \limsup_{r \to +\infty} \frac{\log_p N\left(r, \frac{1}{f}\right)}{\log r} \quad (p \geq 1 \ \text{ is an integer}),$$

where $N\left(r, \frac{1}{f}\right)$ is the counting function of zeros of $f(z)$ in $\{z : |z| \leq r\}$. For $p = 1$, this notation is called exponent of convergence of the sequence of zeros and for $p = 2$ hyper-exponent of convergence of the sequence of zeros, see [9]. Similarly, the iterated exponent of convergence of the sequence of distinct zeros of $f(z)$ is defined as

$$\overline{\lambda}_p(f) = \limsup_{r \to +\infty} \frac{\log_p \overline{N}\left(r, \frac{1}{f}\right)}{\log r} \quad (p \geq 1 \ \text{ is an integer}),$$

where $\overline{N}\left(r, \frac{1}{f}\right)$ is the counting function of distinct zeros of $f(z)$ in $\{z : |z| \leq r\}$. For $p = 1$, this notation is called exponent of convergence of the sequence of distinct zeros and for $p = 2$ hyper-exponent of convergence of the sequence of distinct zeros, see [9].

Consider for $k \geq 2$ the complex linear differential equation

$$f^{(k)} + A(z) f = 0 \tag{1.1}$$

and the differential polynomial

$$g_f = d_k f^{(k)} + d_{k-1} f^{(k-1)} + \cdots + d_1 f' + d_0 f, \tag{1.2}$$

where A and d_j $(j = 0, 1, \ldots, k)$ are meromorphic functions in the complex plane.

In [9], Chen studied the fixed points and hyper-order of solutions of second order linear differential equations with entire coefficients and obtained the following results.

Theorem 1.1. [9] *For all non-trivial solutions f of*

$$f'' + A(z) f = 0, \tag{1.3}$$

the following hold:

(i) *If A is a polynomial with* $\deg A = n \geq 1$, *then we have*

$$\lambda(f - z) = \rho(f) = \frac{n+2}{2}.$$

(ii) *If A is transcendental and $\rho(A) < \infty$, then we have*

$$\lambda(f - z) = \rho(f) = \infty$$

and

$$\lambda_2(f - z) = \rho_2(f) = \rho(A).$$

After him, in [16] Wang, Yi and Cai generalized the precedent theorem for the differential polynomial g_f with constant coefficients as follows.

Theorem 1.2. [16] *For all non-trivial solutions f of* (1.3) *the following hold:*

(i) *If A is a polynomial with* $\deg A = n \geq 1$, *then we have*

$$\lambda(g_f - z) = \rho(f) = \frac{n+2}{2}.$$

(ii) *If A is transcendental and $\rho(A) < \infty$, then we have*

$$\lambda(g_f - z) = \rho(f) = \infty$$

and

$$\lambda_2(g_f - z) = \rho_2(f) = \rho(A).$$

Theorem 1.1 has been generalized from entire to meromorphic solutions for higher order differential equations by the author as follows, see [3].

Theorem 1.3. [3] *Let $k \geq 2$ and $A(z)$ be a transcendental meromorphic function of finite iterated order $\rho_p(A) = \rho > 0$ such that $\delta(\infty, A) = \liminf_{r\to+\infty} \frac{m(r,A)}{T(r,A)} = \delta > 0$. Suppose, moreover, that either:*

(i) *all poles of f are of uniformly bounded multiplicity or that*

(ii) $\delta(\infty, f) > 0$.

If $\varphi(z) \not\equiv 0$ is a meromorphic function with finite iterated $p-$order $\rho_p(\varphi) < +\infty$, then every meromorphic solution $f(z) \not\equiv 0$ of (1.1) *satisfies*

$$\overline{\lambda}_p(f - \varphi) = \overline{\lambda}_p(f' - \varphi) = \cdots = \overline{\lambda}_p\left(f^{(k)} - \varphi\right) = \rho_p(f) = +\infty,$$

$$\overline{\lambda}_{p+1}(f - \varphi) = \overline{\lambda}_{p+1}(f' - \varphi) = \cdots = \overline{\lambda}_{p+1}\left(f^{(k)} - \varphi\right) = \rho_{p+1}(f) = \rho.$$

Let $\mathcal{L}(\mathbf{G})$ denote a differential subfield of the field $\mathcal{M}(\mathbf{G})$ of meromorphic functions in a domain $\mathbf{G} \subset \mathbb{C}$. If $\mathbf{G} = \mathbb{C}$, we simply denote $\mathcal{L}$ instead of $\mathcal{L}(\mathbb{C})$. Special case of such differential subfield

$$\mathcal{L}_{p+1,\rho} = \{g \text{ meromorphic: } \rho_{p+1}(g) < \rho\},$$

where ρ is a positive constant. In [13], Laine and Rieppo have investigated the fixed points and iterated order of the second order differential equation (1.3) and have obtained the following result.

Theorem 1.4. [13] *Let $A(z)$ be a transcendental meromorphic function of finite iterated order $\rho_p(A) = \rho > 0$ such that $\delta(\infty, A) = \delta > 0$, and let f be a transcendental meromorphic solution of equation* (1.3)*. Suppose, moreover, that either:*

(i) *all poles of f are of uniformly bounded multiplicity or that*
(ii) $\delta(\infty, f) > 0$.

Then $\rho_{p+1}(f) = \rho_p(A) = \rho$. Moreover, let

$$P[f] = P\left(f, f', \dots, f^{(m)}\right) = \sum_{j=0}^{m} p_j f^{(j)} \tag{1.4}$$

be a linear differential polynomial with coefficients $p_j \in \mathcal{L}_{p+1,\rho}$, assuming that at least one of the coefficients p_j does vanish identically. Then for the fixed points of $P[f]$, we have $\overline{\lambda}_{p+1}(P[f] - z) = \rho$, provided that neither $P[f]$ nor $P[f]-z$ vanishes identically.

Remark 1.1. ([13], p. 904) In Theorem 1.4, in order to study $P[f]$, the authors consider $m \leq 1$. Indeed, if $m \geq 2$, we obtain, by repeated differentiation of (1.3), that $f^{(k)} = q_{k,0} f + q_{k,1} f'$, $q_{k,0}$, $q_{k,1} \in \mathcal{L}_{p+1,\rho}$ for $k = 2, \dots, m$. Substitution into (1.4) yields the required reduction.

The present article may be understood as an extension and improvement of the recent article of the authors [14] from usual order to iterated order. The main purpose of this paper is to study the growth and oscillation of the differential polynomial (1.2) generated by meromorphic solutions of equation (1.1). The method used in the proofs of our theorems is simple and quite different from the method used in the paper of Laine and Rieppo [13]. For some related papers in the unit disc see [7, 8, 15]. Before we state our results, we define the sequence of functions $\alpha_{i,j}$ $(j = 0, \dots, k-1)$ by

$$\alpha_{i,j} = \begin{cases} \alpha'_{i,j-1} + \alpha_{i-1,j-1}, & \text{for all } i = 1, \dots, k-1, \\ \alpha'_{0,j-1} - A\alpha_{k-1,j-1}, & \text{for } i = 0 \end{cases}$$

and

$$\alpha_{i,0} = \begin{cases} d_i, & \text{for all } i = 1, \dots, k-1, \\ d_0 - d_k A, & \text{for } i = 0. \end{cases}$$

We define also

$$h = \begin{vmatrix} \alpha_{0,0} & \alpha_{1,0} & \cdot & \cdot & \alpha_{k-1,0} \\ \alpha_{0,1} & \alpha_{1,1} & \cdot & \cdot & \alpha_{k-1,1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_{0,k-1} & \alpha_{1,k-1} & \cdot & \cdot & \alpha_{k-1,k-1} \end{vmatrix}$$

and

$$\psi(z) = C_0\varphi + C_1\varphi' + \cdots + C_{k-1}\varphi^{(k-1)},$$

where C_j $(j = 0, \ldots, k-1)$ are finite iterated $p-$order meromorphic functions depending on $\alpha_{i,j}$ and $\varphi \not\equiv 0$ is a meromorphic function with $\rho_p(\varphi) < \infty$.

Theorem 1.5. *Let $A(z)$ be a meromorphic function of finite iterated $p-$order. Let $d_j(z)$ $(j = 0, 1, \ldots, k)$ be finite iterated $p-$order meromorphic functions that are not all vanishing identically such that $h \not\equiv 0$. If $f(z)$ is an infinite iterated $p-$order meromorphic solution of* (1.1) *with $\rho_{p+1}(f) = \rho$, then the differential polynomial* (1.2) *satisfies*

$$\rho_p(g_f) = \rho_p(f) = \infty$$

and

$$\rho_{p+1}(g_f) = \rho_{p+1}(f) = \rho.$$

Furthermore, if f is a finite iterated $p-$order meromorphic solution of (1.1) *such that*

$$\rho_p(f) > \max\{\rho_p(A), \rho_p(d_j) \quad (j = 0, 1, \ldots, k)\}, \tag{1.5}$$

then

$$\rho_p(g_f) = \rho_p(f).$$

Remark 1.2. In Theorem 1.5, if we do not have the condition $h \not\equiv 0$, then the conclusions of Theorem 1.5 cannot hold. For example, if we take $d_k = 1, d_0 = A$ and $d_j \equiv 0$ $(j = 1, \ldots, k-1)$, then $h \equiv 0$. It follows that $g_f \equiv 0$ and $\rho_p(g_f) = 0$. So, if $f(z)$ is an infinite iterated $p-$order meromorphic solution of (1.1), then $\rho_p(g_f) = 0 < \rho_p(f) = \infty$, and if f is a finite iterated $p-$order meromorphic solution of (1.1) such that (1.5) holds, then $\rho_p(g_f) = 0 < \rho_p(f)$.

Corollary 1.1. *Let $A(z)$ be a transcendental entire function of finite iterated order $\rho_p(A) = \rho > 0$, and let $d_j(z)$ $(j = 0, 1, \ldots, k)$ be finite iterated $p-$ order entire functions that are not all vanishing identically such that $h \not\equiv 0$. If $f \not\equiv 0$ is a solution of* (1.1), *then the differential polynomial* (1.2) *satisfies*

$$\rho_p(g_f) = \rho_p(f) = \infty$$

and

$$\rho_{p+1}(g_f) = \rho_{p+1}(f) = \rho_p(A) = \rho.$$

Corollary 1.2. *Let $k \geq 2$ and $A(z)$ be a transcendental meromorphic function of finite iterated order $\rho_p(A) = \rho > 0$ such that $\delta(\infty, A) = \delta > 0$, and let $f \not\equiv 0$ be a meromorphic solution of equation* (1.1). *Suppose, moreover, that either:*

(i) *all poles of* f *are of uniformly bounded multiplicity or that*
(ii) $\delta(\infty, f) > 0$.

Let $d_j(z)$ $(j = 0, 1, \ldots, k)$ *be finite iterated* $p-$*order meromorphic functions that are not all vanishing identically such that* $h \not\equiv 0$. *Then the differential polynomial* (1.2) *satisfies* $\rho_p(g_f) = \rho_p(f) = \infty$ *and* $\rho_{p+1}(g_f) = \rho_{p+1}(f) = \rho_p(A)$.

Theorem 1.6. *Under the hypotheses of Theorem 1.5, let* $\varphi(z) \not\equiv 0$ *be a meromorphic function with finite iterated* $p-$*order such that* $\psi(z)$ *is not a solution of* (1.1). *If* $f(z)$ *is an infinite iterated* $p-$*order meromorphic solution of* (1.1) *with* $\rho_{p+1}(f) = \rho$, *then the differential polynomial* (1.2) *satisfies*

$$\overline{\lambda}_p(g_f - \varphi) = \lambda_p(g_f - \varphi) = \rho_p(f) = \infty$$

and

$$\overline{\lambda}_{p+1}(g_f - \varphi) = \lambda_{p+1}(g_f - \varphi) = \rho_{p+1}(f) = \rho.$$

Furthermore, if f *is a finite iterated* $p-$*order meromorphic solution of* (1.1) *such that*

$$\rho_p(f) > \max\{\rho_p(A), \rho_p(\varphi), \rho_p(d_j) \quad (j = 0, 1, lcdots, k)\}, \tag{1.6}$$

then

$$\overline{\lambda}_p(g_f - \varphi) = \lambda_p(g_f - \varphi) = \rho_p(f).$$

Corollary 1.3. *Under the hypotheses of Corollary 1.1, let* $\varphi(z) \not\equiv 0$ *be an entire function with finite iterated* $p-$*order such that* $\psi(z) \not\equiv 0$. *Then the differential polynomial* (1.2) *satisfies*

$$\overline{\lambda}_p(g_f - \varphi) = \lambda_p(g_f - \varphi) = \rho_p(f) = \infty$$

and

$$\overline{\lambda}_{p+1}(g_f - \varphi) = \lambda_{p+1}(g_f - \varphi) = \rho_{p+1}(f) = \rho_p(A).$$

Corollary 1.4. *Under the hypotheses of Corollary 1.2, let* $\varphi(z) \not\equiv 0$ *be a meromorphic function with finite iterated* $p-$*order such that* $\psi(z) \not\equiv 0$. *Then the differential polynomial* (1.2) *satisfies*

$$\overline{\lambda}_p(g_f - \varphi) = \lambda_p(g_f - \varphi) = \rho_p(f) = \infty$$

and

$$\overline{\lambda}_{p+1}(g_f - \varphi) = \lambda_{p+1}(g_f - \varphi) = \rho_{p+1}(f) = \rho_p(A).$$

In the following we give two applications of the above results without the additional conditions $h \not\equiv 0$ and ψ is not a solution of (1.1).

Theorem 1.7. *Let* $A(z)$ *be an entire function of finite iterated* $p-$*order satisfying* $0 < \rho_p(A) < \infty$ *and* $0 < \tau_p(A) < \infty$, *and let* $d_j(z)$ $(j = 0, 1, 2, 3)$ *be finite iterated* $p-$*order entire functions that are not all vanishing identically such that*

$$\max\{\rho_p(d_j) \quad (j = 0, 1, 2, 3)\} < \rho_p(A).$$

If f *is a nontrivial solution of the equation*

$$f''' + A(z) f = 0, \tag{1.7}$$

then the differential polynomial

$$g_f = d_3 f^{(3)} + d_2 f'' + d_1 f' + d_0 f \tag{1.8}$$

satisfies

$$\rho_p(g_f) = \rho_p(f) = \infty$$

and

$$\rho_{p+1}(g_f) = \rho_{p+1}(f) = \rho_p(A).$$

Theorem 1.8. *Under the hypotheses of Theorem 1.7, let $\varphi(z) \not\equiv 0$ be an entire function with finite iterated $p-$order. If f is a nontrivial solution of (1.7), then the differential polynomial $g_f = d_3 f^{(3)} + d_2 f'' + d_1 f' + d_0 f$ such that $d_3 \not\equiv 0$ satisfies*

$$\overline{\lambda}_p(g_f - \varphi) = \lambda_p(g_f - \varphi) = \rho_p(f) = \infty$$

and

$$\overline{\lambda}_{p+1}(g_f - \varphi) = \lambda_{p+1}(g_f - \varphi) = \rho_{p+1}(f) = \rho_p(A).$$

2. Auxiliary lemmas

Here, we give a special case of the result due to T. B. Cao, Z. X. Chen, X. M. Zheng and J. Tu in [5].

Lemma 2.1. [3] *Let $p \geq 1$ be an integer and let $A_0, A_1, \ldots, A_{k-1}, F \not\equiv 0$ be finite iterated $p-$order meromorphic functions. If f is a meromorphic solution with $\rho_p(f) = +\infty$ and $\rho_{p+1}(f) = \rho < +\infty$ of the differential equation*

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = F, \tag{2.1}$$

then $\overline{\lambda}_p(f) = \lambda_p(f) = \rho_p(f) = \infty$ and $\overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \rho_{p+1}(f) = \rho$.

Lemma 2.2. [5] *Let $p \geq 1$ be an integer and let $A_0, A_1, \ldots, A_{k-1}, F \not\equiv 0$ be meromorphic functions. If f is a meromorphic solution of equation (2.1) such that*

(i) $\max\{i(F), i(A_j)\ \ (j = 0, \ldots, k-1)\} < i(f) = p$ *or that*
(ii) $\max\{\rho_p(F), \rho_p(A_j)\ \ (j = 0, \ldots, k-1)\} < \rho_p(f) < +\infty$,

then $i_{\overline{\lambda}}(f) = i_\lambda(f) = i(f) = p$ and $\overline{\lambda}_p(f) = \lambda_p(f) = \rho_p(f)$.

The following lemma is a corollary of Theorem 2.3 in [11].

Lemma 2.3. *Assume A is an entire function with $i(A) = p$, and assume $1 \leq p < +\infty$. Then, for all non-trivial solutions f of (1.1), we have*

$$\rho_p(f) = \infty \text{ and } \rho_{p+1}(f) = \rho_p(A).$$

Lemma 2.4. [3] *Let $k \geq 2$ and $A(z)$ be a transcendental meromorphic function of finite iterated order $\rho_p(A) = \rho > 0$ such that $\delta(\infty, A) = \delta > 0$, and let $f \not\equiv 0$ be a meromorphic solution of equation (1.1). Suppose, moreover, that either:*

(i) *all poles of f are of uniformly bounded multiplicity or that*
(ii) $\delta(\infty, f) > 0$.

Then $\rho_p(f) = +\infty$ *and* $\rho_{p+1}(f) = \rho_p(A) = \rho$.

Remark 2.1. For $k = 2$, Lemma 2.4 was obtained by Laine and Rieppo in [13].

Lemma 2.5. [4] *Let* f, g *be meromorphic functions with iterated* $p-$*order* $0 < \rho_p(f)$, $\rho_p(g) < \infty$ *and iterated* $p-$*type* $0 < \tau_p(f), \tau_p(g) < \infty$ $(1 \le p < \infty)$. *Then the following statements hold:*

(i) *If* $\rho_p(g) < \rho_p(f)$, *then*

$$\tau_p(f+g) = \tau_p(fg) = \tau_p(f).$$

(ii) *If* $\rho_p(f) = \rho_p(g)$ *and* $\tau_p(g) \ne \tau_p(f)$, *then*

$$\rho_p(f+g) = \rho_p(fg) = \rho_p(f).$$

Lemma 2.6. [11] *Let* f *be a meromorphic function for which* $i(f) = p \ge 1$ *and* $\rho_p(f) = \rho$, *and let* $k \ge 1$ *be an integer. Then for any* $\varepsilon > 0$,

$$m\left(r, \frac{f^{(k)}}{f}\right) = O\left(\exp_{p-2}\left\{r^{\rho+\varepsilon}\right\}\right),$$

outside of a possible exceptional set E_1 *of finite linear measure.*

Lemma 2.7. [1] *Let* $g : [0, +\infty) \to \mathbb{R}$ *and* $h : [0, +\infty) \to \mathbb{R}$ *be monotone non-decreasing functions such that* $g(r) \le h(r)$ *outside of an exceptional set* E_2 *of finite linear measure. Then for any* $\lambda > 1$, *there exists* $r_0 > 0$ *such that* $g(r) \le h(\lambda r)$ *for all* $r > r_0$.

Lemma 2.8. [14] *Assume* $f \not\equiv 0$ *is a solution of equation* (1.1). *Then the differential polynomial* g_f *defined in* (1.2) *satisfies the system of equations*

$$\begin{cases} g_f & = \alpha_{0,0} f + \alpha_{1,0} f' + \cdots + \alpha_{k-1,0} f^{(k-1)}, \\ g_f' & = \alpha_{0,1} f + \alpha_{1,1} f' + \cdots + \alpha_{k-1,1} f^{(k-1)}, \\ g_f'' & = \alpha_{0,2} f + \alpha_{1,2} f' + \cdots + \alpha_{k-1,2} f^{(k-1)}, \\ \cdots & \\ g_f^{(k-1)} & = \alpha_{0,k-1} f + \alpha_{1,k-1} f' + \cdots + \alpha_{k-1,k-1} f^{(k-1)}, \end{cases}$$

where

$$\alpha_{i,j} = \begin{cases} \alpha'_{i,j-1} + \alpha_{i-1,j-1}, & \text{for all } i = 1, \ldots, k-1, \\ \alpha'_{0,j-1} - A\alpha_{k-1,j-1}, & \text{for } i = 0 \end{cases}$$

and

$$\alpha_{i,0} = \begin{cases} d_i, & \text{for all } i = 1, \ldots, k-1, \\ d_0 - d_k A, & \text{for } i = 0. \end{cases}$$

3. Proofs of the Theorems and the Corollaries

Proof of Theorem 1.5. Suppose that f is an infinite iterated $p-$order meromorphic solution of (1.1) with $\rho_{p+1}(f) = \rho$. By Lemma 2.8, g_f satisfies the system of equations

$$(3.1)\qquad \begin{cases} g_f & = \alpha_{0,0}f + \alpha_{1,0}f' + \cdots + \alpha_{k-1,0}f^{(k-1)}, \\ g_f' & = \alpha_{0,1}f + \alpha_{1,1}f' + \cdots + \alpha_{k-1,1}f^{(k-1)}, \\ g_f'' & = \alpha_{0,2}f + \alpha_{1,2}f' + \cdots + \alpha_{k-1,2}f^{(k-1)}, \\ & \cdots \\ g_f^{(k-1)} & = \alpha_{0,k-1}f + \alpha_{1,k-1}f' + \cdots + \alpha_{k-1,k-1}f^{(k-1)}, \end{cases}$$

where

$$(3.2)\qquad \alpha_{i,j} = \begin{cases} \alpha'_{i,j-1} + \alpha_{i-1,j-1}, & \text{for all } i = 1, \ldots, k-1, \\ \alpha'_{0,j-1} - A\alpha_{k-1,j-1}, & \text{for } i = 0 \end{cases}$$

and

$$(3.3)\qquad \alpha_{i,0} = \begin{cases} d_i, & \text{for all } i = 1, \ldots, k-1, \\ d_0 - d_k A, & \text{for } i = 0. \end{cases}$$

By Cramer's rule, since $h \not\equiv 0$ we have

$$(3.4)\qquad f = \frac{\begin{vmatrix} g_f & \alpha_{1,0} & . & . & \alpha_{k-1,0} \\ g_f' & \alpha_{1,1} & . & . & \alpha_{k-1,1} \\ . & . & . & . & . \\ . & . & . & . & . \\ g_f^{(k-1)} & \alpha_{1,k-1} & . & . & \alpha_{k-1,k-1} \end{vmatrix}}{h}.$$

Then

$$(3.5)\qquad f = C_0 g_f + C_1 g_f' + \cdots + C_{k-1} g_f^{(k-1)},$$

where C_j are finite iterated $p-$order meromorphic functions depending on $\alpha_{i,j}$, where $\alpha_{i,j}$ are defined in (3.2).

If $\rho_p(g_f) < +\infty$, then by (3.5) we obtain $\rho_p(f) < +\infty$, and this is a contradiction. Hence $\rho_p(g_f) = \rho_p(f) = +\infty$.

Now, we prove that $\rho_{p+1}(g_f) = \rho_{p+1}(f) = \rho$. By (1.2), we get $\rho_{p+1}(g_f) \leq \rho_{p+1}(f)$ and by (3.5) we have $\rho_{p+1}(f) \leq \rho_{p+1}(g_f)$. This yield $\rho_{p+1}(g_f) = \rho_{p+1}(f) = \rho$.

Furthermore, if f is a finite iterated $p-$order meromorphic solution of equation (1.1) such that

$$(3.6)\qquad \rho_p(f) > \max\{\rho_p(A), \rho_p(d_j) \quad (j = 0, 1, \ldots, k)\},$$

then

$$(3.7)\qquad \rho_p(f) > \max\{\rho_p(\alpha_{i,j}) : i = 0, \ldots, k-1, j = 0, \ldots, k-1\}.$$

By (1.2) and (3.6) we have $\rho_p(g_f) \leq \rho_p(f)$. Now, we prove $\rho_p(g_f) = \rho_p(f)$. If $\rho_p(g_f) < \rho_p(f)$, then by (3.5) and (3.7) we get

$$\rho_p(f) \leq \max\{\rho_p(C_j) \quad (j = 0, \ldots, k-1), \rho_p(g_f)\} < \rho_p(f)$$

and this is a contradiction. Hence $\rho_p(g_f) = \rho_p(f)$. □

Remark 3.1. From (3.5), it follows that the condition $h \not\equiv 0$ is equivalent to the condition $g_f, g_f', g_f'', \ldots, g_f^{(k-1)}$ are linearly independent over the field of meromorphic functions of finite iterated $p-$order.

Proof of Corollary 1.1. Suppose that $f \not\equiv 0$ is a solution of (1.1). Since A is an entire function with $i(A) = p$, then by Lemma 2.3, we have $\rho_p(f) = \infty$ and $\rho_{p+1}(f) = \rho_p(A)$. Thus, by Theorem 1.5 we obtain $\rho_p(g_f) = \rho_p(f) = \infty$ and $\rho_{p+1}(g_f) = \rho_{p+1}(f) = \rho_p(A)$. □

Proof of Corollary 1.2. Suppose that $f \not\equiv 0$ is a meromorphic solution of (1.1) such that

(i) all poles of f are uniformly bounded multiplicity or that
(ii) $\delta(\infty, f) > 0$.

Then by Lemma 2.4, we have $\rho_p(f) = \infty$ and $\rho_{p+1}(f) = \rho_p(A)$. Now, by using Theorem 1.5, we obtain $\rho_p(g_f) = \rho_p(f) = \infty$ and $\rho_{p+1}(g_f) = \rho_{p+1}(f) = \rho_p(A)$. □

Proof of Theorem 1.6. Suppose that f is an infinite iterated $p-$order meromorphic solution of equation (1.1) with $\rho_{p+1}(f) = \rho$. Set $w(z) = g_f - \varphi$. Since $\rho_p(\varphi) < \infty$, then by Theorem 1.5 we have $\rho_p(w) = \rho_p(g_f) = \infty$ and $\rho_{p+1}(w) = \rho_{p+1}(g_f) = \rho$. To prove $\overline{\lambda}_p(g_f - \varphi) = \lambda_p(g_f - \varphi) = \infty$ and $\overline{\lambda}_{p+1}(g_f - \varphi) = \lambda_{p+1}(g_f - \varphi) = \rho$ we need to prove $\overline{\lambda}_p(w) = \lambda_p(w) = \infty$ and $\overline{\lambda}_{p+1}(w) = \lambda_{p+1}(w) = \rho$. By $g_f = w + \varphi$ and (3.5), we get

$$f = C_0 w + C_1 w' + \cdots + C_{k-1} w^{(k-1)} + \psi(z), \tag{3.8}$$

where

$$\psi(z) = C_0\varphi + C_1\varphi' + \cdots + C_{k-1}\varphi^{(k-1)}.$$

Substituting (3.8) into (1.1), we obtain

$$C_{k-1} w^{(2k-1)} + \sum_{i=0}^{2k-2} \phi_i w^{(i)} = -\left(\psi^{(k)} + A(z)\psi\right) = H,$$

where ϕ_i $(i = 0, \ldots, 2k-2)$ are meromorphic functions with finite iterated $p-$order. Since $\psi(z)$ is not a solution of (1.1), it follows that $H \not\equiv 0$. Then by Lemma 2.1, we obtain $\overline{\lambda}_p(w) = \lambda_p(w) = \infty$ and $\overline{\lambda}_{p+1}(w) = \lambda_{p+1}(w) = \rho$, i. e., $\overline{\lambda}_p(g_f - \varphi) = \lambda_p(g_f - \varphi) = \infty$ and $\overline{\lambda}_{p+1}(g_f - \varphi) = \lambda_{p+1}(g_f - \varphi) = \rho$.

Suppose that f is a finite iterated $p-$order meromorphic solution of equation (1.1) such that (1.6) holds. Set $w(z) = g_f - \varphi$. Since $\rho_p(\varphi) < \rho_p(f)$, then by Theorem 1.5 we have $\rho_p(w) = \rho_p(g_f) = \rho_p(f)$. To prove $\overline{\lambda}_p(g_f - \varphi) = \lambda_p(g_f - \varphi) = \rho_p(f)$ we need to prove $\overline{\lambda}_p(w) = \lambda_p(w) = \rho_p(f)$. Using the same reasoning as above, we get

$$C_{k-1} w^{(2k-1)} + \sum_{i=0}^{2k-2} \phi_i w^{(i)} = -\left(\psi^{(k)} + A(z)\psi\right) = F,$$

where C_{k-1}, ϕ_i $(i = 0, \ldots, 2k-2)$ are meromorphic functions with finite iterated $p-$order $\rho_p(C_{k-1}) < \rho_p(f) = \rho_p(w)$, $\rho_p(\phi_i) < \rho_p(f) = \rho_p(w)$ $(i = 0, \ldots, 2k-2)$ and

$$\psi(z) = C_0\varphi + C_1\varphi' + \cdots + C_{k-1}\varphi^{(k-1)}, \ \rho_p(F) < \rho_p(w).$$

Since $\psi(z)$ is not a solution of (1.1), it follows that $F \not\equiv 0$. Then by Lemma 2.2, we obtain $\overline{\lambda}_p(w) = \lambda_p(w) = \rho_p(f)$, i. e., $\overline{\lambda}_p(g_f - \varphi) = \lambda_p(g_f - \varphi) = \rho_p(f)$. □

Proof of Corollary 1.3. Suppose that $f \not\equiv 0$ is a solution of (1.1). Then by Lemma 2.3, we have $\rho_p(f) = \infty$ and $\rho_{p+1}(f) = \rho_p(A)$. Since $\psi \not\equiv 0$ and $\rho_p(\psi) < \infty$, then ψ cannot be a solution of equation (1.1). Thus, by Theorem 1.6 we obtain

$$\overline{\lambda}_p(g_f - \varphi) = \lambda_p(g_f - \varphi) = \rho_p(f) = \infty$$

and

$$\overline{\lambda}_{p+1}(g_f - \varphi) = \lambda_{p+1}(g_f - \varphi) = \rho_{p+1}(f) = \rho_p(A).$$

□

Proof of Corollary 1.4. Suppose that $f \not\equiv 0$ is a meromorphic solution of (1.1). Then by Lemma 2.4, we have $\rho_p(f) = \infty$ and $\rho_{p+1}(f) = \rho_p(A)$. Since $\psi \not\equiv 0$ and $\rho_p(\psi) < \infty$, then ψ cannot be a solution of equation (1.1). Now, by using Theorem 1.6, we obtain

$$\overline{\lambda}_p(g_f - \varphi) = \lambda_p(g_f - \varphi) = \rho_p(f) = \infty$$

and

$$\overline{\lambda}_{p+1}(g_f - \varphi) = \lambda_{p+1}(g_f - \varphi) = \rho_{p+1}(f) = \rho_p(A).$$

□

Proof of Theorem 1.7. Suppose that f is a nontrivial solution of (1.7). Then by Lemma 2.3, we have

$$\rho_p(f) = \infty, \ \rho_{p+1}(f) = \rho_p(A).$$

We have by Lemma 2.8

$$\left\{ \begin{array}{ll} g_f & = \alpha_{0,0}f + \alpha_{1,0}f' + \alpha_{2,0}f'', \\ g_f' & = \alpha_{0,1}f + \alpha_{1,1}f' + \alpha_{2,1}f'', \\ g_f'' & = \alpha_{0,2}f + \alpha_{1,2}f' + \alpha_{2,2}f''. \end{array} \right. \tag{3.9}$$

By (3.3) we obtain

$$\alpha_{i,0} = \left\{ \begin{array}{ll} d_i, & \text{for all } i = 1, 2, \\ d_0 - d_3A, & \text{for } i = 0. \end{array} \right. \tag{3.10}$$

Now, by (3.2) we get

$$\alpha_{i,1} = \left\{ \begin{array}{ll} \alpha_{i,0}' + \alpha_{i-1,0}, & \text{for all } i = 1, 2, \\ \alpha_{0,0}' - A\alpha_{2,0}, & \text{for } i = 0. \end{array} \right.$$

Hence

$$(3.11)\qquad \begin{cases} \alpha_{0,1} &= \alpha'_{0,0} - A\alpha_{2,0} = (d_0 - d_3A)' - Ad_2 = d'_0 - (d_2 + d'_3)\, A - d_3A', \\ \alpha_{1,1} &= \alpha'_{1,0} + \alpha_{0,0} = d_0 + d'_1 - d_3A, \\ \alpha_{2,1} &= \alpha'_{2,0} + \alpha_{1,0} = d_1 + d'_2. \end{cases}$$

Finally, by (3.2) we have

$$\alpha_{i,2} = \begin{cases} \alpha'_{i,1} + \alpha_{i-1,1}, & \text{for all } i = 1, 2, \\ \alpha'_{0,1} - A\alpha_{2,1}, & \text{for } i = 0. \end{cases}$$

So, we obtain

$$(3.12)\qquad \begin{cases} \alpha_{0,2} &= \alpha'_{0,1} - A\alpha_{2,1} = d''_0 - (d_1 + 2d'_2 + d''_3)\, A - (d_2 + 2d'_3)\, A' - d_3A'', \\ \alpha_{1,2} &= \alpha'_{1,1} + \alpha_{0,1} = 2d'_0 + d''_1 - (d_2 + 2d'_3)\, A - 2d_3A', \\ \alpha_{2,2} &= \alpha'_{2,1} + \alpha_{1,1} = d_0 + 2d'_1 + d''_2 - d_3A. \end{cases}$$

Hence

$$(3.13)\qquad \begin{cases} g_f &= (d_0 - d_3A)\, f + d_1f' + d_2f'', \\ g'_f &= (d'_0 - (d_2 + d'_3)\, A - d_3A')\, f + (d_0 + d'_1 - d_3A)\, f' + (d_1 + d'_2)\, f'', \\ g''_f &= (d''_0 - (d_1 + 2d'_2 + d''_3)\, A - (d_2 + 2d'_3)\, A' - d_3A'')\, f \\ & \quad + (2d'_0 + d''_1 - (d_2 + 2d'_3)\, A - 2d_3A')\, f' + (d_0 + 2d'_1 + d''_2 - d_3A)\, f''. \end{cases}$$

First, we suppose that $d_3 \not\equiv 0$. By (3.13), we have

$$h = \begin{vmatrix} H_0 & H_1 & H_2 \\ H_3 & H_4 & H_5 \\ H_6 & H_7 & H_8 \end{vmatrix},$$

where $H_0 = d_0 - d_3A,\ H_1 = d_1,\ H_2 = d_2, H_3 = d'_0 - (d_2 + d'_3)\, A - d_3A',\ H_4 = d_0 + d'_1 - d_3A,\ H_5 = d_1 + d'_2, H_6 = d''_0 - (d_1 + 2d'_2 + d''_3)\, A - (d_2 + 2d'_3)\, A' - d_3A''$, $H_7 = 2d'_0 + d''_1 - (d_2 + 2d'_3)\, A - 2d_3A',\ H_8 = d_0 + 2d'_1 + d''_2 - d_3A$. Then

$$\begin{aligned} h = \big(&3d_0d_1d_2 + 3d_0d_1d'_3 + 3d_0d_2d'_2 - 6d_0d_3d'_1 + 3d_1d_2d'_1 + 3d_1d_3d'_0 \\ &+ d_0d_2d''_3 - 2d_0d_3d''_2 + d_1d_2d''_2 + d_1d_3d''_1 + d_2d_3d''_0 + 2d_0d'_2d'_3 + 2d_1d'_1d'_3 - 4d_2d'_0d'_3 \\ &+ 2d_2d'_1d'_2 + 2d_3d'_0d'_2 - d_1d'_2d''_3 + d_1d'_3d''_2 + d_2d'_1d''_3 - d_2d''_1d'_3 - d_3d'_1d''_2 \\ &+ d_3d'_2d''_1 - d_1^3 - 3d_0^2d_3 - 2d_1(d'_2)^2 - 3d_1^2d'_2 - 2d_3(d'_1)^2 - d_2^2d''_1 - d_1^2d''_3 - 3d_2^2d'_0\big)\, A \\ &+ \big(2d_0d_2d'_3 + 2d_0d_3d'_2 - d_1d_2d'_2 + 2d_1d_3d'_1 - 4d_2d_3d'_0 + d_1d_3d''_2 \\ &- d_2d_3d''_1 - 2d_1d'_2d'_3 + 2d_2d'_1d'_3 + 3d_0d_1d_3 + d_0d_2^2 - d_1^2d_2 + d_2^2d'_1 - 2d_1^2d'_3\big)\, A' \\ &+ (d_2d_3d'_1 + d_0d_2d_3 - d_1d_3d'_2 - d_1^2d_3)A'' + (2d_2d_3d'_3 - 3d_1d_3^2 + 2d_2^2d_3 - 2d_3^2d'_2)AA' \end{aligned}$$

$$
\begin{aligned}
&+\left(d_2^3-3d_1d_2d_3-3d_1d_3d_3'-3d_2d_3d_2'-d_2d_3d_3''-2d_3d_2'd_3'\right.\\
&\left.+3d_0d_3^2+3d_3^2d_1'+2d_2(d_3')^2+3d_2^2d_3'+d_3^2d_2''\right)A^2\\
&-d_3^3A^3+2d_2d_3^2(A')^2-d_2d_3^2AA''-3d_0d_1d_0'-d_0d_1d_1''-d_0d_2d_0''-2d_0d_0'd_2'\\
&+d_1d_0''d_2'+d_2d_0'd_1''-d_2d_1'd_0''+d_0^3+2d_0(d_1')^2+3d_0^2d_1'+2d_2(d_0')^2\\
&+d_1^2d_0''+d_0^2d_2''-2d_1d_0'd_1'+d_0d_1'd_2''-d_0d_2'd_1''-d_1d_0'd_2''.
\end{aligned}
$$

By $d_3 \not\equiv 0$, $A \not\equiv 0$, $0 < \rho_p(A) < \infty$, $0 < \tau_p(A) < \infty$ and Lemma 2.5, we have $\rho_p(h) = \rho_p(A)$, hence $h \not\equiv 0$. For the cases (i) $d_3 \equiv 0$, $d_2 \not\equiv 0$; (ii) $d_3 \equiv 0, d_2 \equiv 0$ and $d_1 \not\equiv 0$ by using a similar reasoning as above we get $h \not\equiv 0$. Finally, if $d_3 \equiv 0$, $d_2 \equiv 0$, $d_1 \equiv 0$ and $d_0 \not\equiv 0$, we have $h = d_0^3 \not\equiv 0$. Hence $h \not\equiv 0$. By $h \not\equiv 0$, we obtain

$$
f=\frac{1}{h}\begin{vmatrix} g_f & d_1 & d_2 \\ g_f' & d_0+d_1'-d_3A & d_1+d_2' \\ g_f'' & 2d_0'+d_1''-(d_2+2d_3')A-2d_3A' & d_0+2d_1'+d_2''-d_3A \end{vmatrix},
$$

which we can write

$$
f=\frac{1}{h}\left(D_0g_f+D_1g_f'+D_2g_f''\right), \tag{3.14}
$$

where

$$
\begin{aligned}
D_0 &= (d_1d_2-2d_0d_3+2d_1d_3'+d_2d_2'-3d_3d_1'-d_3d_2''+2d_2'd_3')A\\
&\quad+(2d_1d_3+2d_3d_2')A'+A^2d_3^2+3d_0d_1'-2d_1d_0'+d_0d_2''-d_1d_1''\\
&\quad-2d_0'd_2'+d_1'd_2''-d_2'd_1''+d_0^2+2(d_1')^2,\\
D_1 &= \left(d_1d_3-2d_2d_3'-d_2^2\right)A+d_2d_1''-d_0d_1-2d_1d_1'+2d_2d_0'-d_1d_2'',\\
D_2 &= d_2d_3A+d_1^2-d_2d_1'+d_1d_2'-d_0d_2.
\end{aligned}
$$

If $\rho_p(g_f) < +\infty$, then by (3.14) we obtain $\rho_p(f) < +\infty$, and this is a contradiction. Hence $\rho_p(g_f) = \rho_p(f) = +\infty$.

Now, we prove that $\rho_{p+1}(g_f) = \rho_{p+1}(f) = \rho_p(A)$. By (1.8), we get $\rho_{p+1}(g_f) \leq \rho_{p+1}(f)$ and by (3.14) we have $\rho_{p+1}(f) \leq \rho_{p+1}(g_f)$. This yield $\rho_{p+1}(g_f) = \rho_{p+1}(f) = \rho_p(A)$. □

Proof of Theorem 1.8. Suppose that f is a nontrivial solution of (1.7). By setting $w = g_f - \varphi$ in (3.14), we have

$$
f=\frac{1}{h}\left(D_0w+D_1w'+D_2w''\right)+\psi, \tag{3.15}
$$

where

$$
\psi=\frac{D_2\varphi''+D_1\varphi'+D_0\varphi}{h}. \tag{3.16}
$$

Since $d_3 \not\equiv 0$, then $h \not\equiv 0$. It follows from Theorem 1.7 that g_f is of infinite iterated $p-$order and $\rho_{p+1}(g_f) = \rho_p(A)$. Substituting (3.15) into (1.7), we obtain

$$\frac{D_2}{h} w^{(5)} + \sum_{i=0}^{4} \phi_i w^{(i)} = -\left(\psi^{(3)} + A(z)\psi\right),$$

where ϕ_i $(i = 0, \ldots, 4)$ are meromorphic functions with finite iterated $p-$order. First, we prove that $\psi \not\equiv 0$. Suppose that $\psi \equiv 0$, then by (3.16) we obtain

$$D_2\varphi'' + D_1\varphi' + D_0\varphi = 0 \tag{3.17}$$

and by Lemma 2.5, we have

$$\rho_p(D_0) > \max\{\rho_p(D_1), \rho_p(D_2)\}. \tag{3.18}$$

By (3.17) we can write

$$D_0 = -\left(D_2\frac{\varphi''}{\varphi} + D_1\frac{\varphi'}{\varphi}\right).$$

Since $\rho_p(\varphi) = \beta < \infty$, then by Lemma 2.6 we obtain

$$T(r, D_0) \leq T(r, D_1) + T(r, D_2) + O\left(\exp_{p-2}\left\{r^{\beta+\varepsilon}\right\}\right), \ r \notin E,$$

where E is a set of finite linear measure. Then, by Lemma 2.7 we have

$$\rho_p(D_0) \leq \max\{\rho_p(D_1), \rho_p(D_2)\},$$

which is a contradiction with (3.18). Hence $\psi \not\equiv 0$. It is clear now that $\psi \not\equiv 0$ cannot be a solution of (1.7) because $\rho_p(\psi) < \infty$. Then, by Lemma 2.1 we obtain

$$\overline{\lambda}_p(w) = \lambda_p(w) = \overline{\lambda}_p(g_f - \varphi) = \lambda_p(g_f - \varphi) = \rho_p(f) = \infty$$

and

$$\overline{\lambda}_{p+1}(w) = \lambda_{p+1}(w) = \overline{\lambda}_{p+1}(g_f - \varphi) = \lambda_{p+1}(g_f - \varphi) = \rho_{p+1}(f) = \rho_p(A).$$

□

Acknowledgments: The authors would like to thank the referees for their valuable comments and suggestions to improve the paper.

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