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الجمهوريـــــــة الجزائريـــــة الديمقراطيـــــة الشعبيـــــــة
People's Democratic Republic of Algeria
وزارة التعـلـيـم العـالـــي والبـحـــث الـعـلـمـــي
Ministry of Higher Education and Scientific Research
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Department of Mathematics and Informatics

THESIS

IN VIEW OF OBTAINING THE DIPLOMA OF A DOCTORATE

Speciality : Operational Research and Decision Support

Presented by

BOUKHOBZA Meriem

Titled

VARIABLE ORDER FRACTIONAL DISCRET SYSTEMS AND APPLICATIONS

The Jury is composed of:

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Academic year: 2024/ 2025

The PhD dissertation of **LaTeX**.

In the name of Allah the merciful

I dedicate this work to my father, my husband, my daughters, and my siblings for their encouragement and patience. May God protect them, and to all my friends.

Meriem. B

Contents

Acknowledgements

First and foremost, I thank Almighty God for granting me the strength, determination, and guidance to carry out this research study, despite the challenges I faced.

My supervisor, Prof. Amar DEBBOUCHE, He deserves special recognition for his unwavering support of my Ph.D. studies and related research, as well as his patience, enthusiasm, and extensive expertise. His guidance was invaluable throughout the research and writing of this thesis.

I would also like to extend my heartfelt thanks to my co-supervisor.Prof. Zoubir Dahmani.

I would like to express my sincere gratitude to my committee members for their guidance, support, and valuable feedback throughout this journey.: Prof. Lakehal BELARBI , and Prof. Abdelkader SNOUCI , and Prof. Mohamed HOUAS , and Prof Houari FETTOUCH . I am grateful to my committee members for their support, insightful comments, and thought-provoking questions, which greatly contributed to the quality and depth of my work..

Abstract

The aim of this work is to develop recent methods for the stability of variable order fractional system in discret -time .In particular, during the project of this doctorate thesis we present the theory of variable order fractional calculus and discret-time calculus to prove the questions of existence and uniqueness results, stability and others properties with applications to different biological systems.

keywords: Variable-order fractional derivative;fractional discrete calculus; Optimal control; stability; fixed point techniques;numerical simulation.

Résumé

L'objectif de ce travail est de développer des méthodes récentes pour la stabilité des systemes fractionnaires d'ordre variable en temps discret .
En particulier, au cours du projet de cette thèse de doctorat, nous présentons la théorie du calcul fractionnaire d'ordre variable et du calcul en temps discret pour prouver la questions d'existence et d'unicités de résultats ,de stabilité et autres propriétés avec applications à différents systémes biologiques.

Mots clés: Dérivée fractionnaire d'ordre variable; calcul fractionnaire a temps discret ; Contrôle optimal; stabilité; techniques de point fixe; simulation numérique.

الملخص

الهدف من هذا العمل هو تطوير طرق حديثة لدراسة استقرارية الأنظمة الكسرية ذات الترتيب المتغير في الزمن المتقطع على وجه الخصوص، خلال مشروع هذه الأطروحة سنعرض نظريات الحسابات الكسرية ذات الترتيب المتغير والحسابات في الزمن المتقطع لإثبات قضايا وجود ووحدانية الحل والاستقرارية وخصائص أخرى مع تطبيقات على أنظمة بيولوجية مختلفة.

The List Of Works

1. M. Boukhobza,A. Debbouche,L. Shangerganesh, AND D.F.M. Torres, Modeling the dynamics of the Hepatitis B virus via a variable-order discrete system.
https://doi.org/10.1016/j.chaos.2024.114987

2. Boukhobza, M.; Debbouche, A.; Shangerganesh, L.; Nieto, J.J. The Stability of Solutions of the Variable-Order Fractional Optimal Control Model for the COVID-19 Epidemic in Discrete Time. Mathematics 2024, 12, 1236.
https://doi.org/10.3390/ math12081236

3. M. Boukhobza, A. Debbouche, L. Shangerganesh, S.V. Kashtanova, Stability and sensitivity analysis for a variable-order discrete mathematical model to evaluate the impact of diabetes and its resulting complications, Journal of Mathematical Sciences (2024), accepted.

Introduction

Fractional calculus traces its origins back to the 17th century, where pioneering mathematicians such as Leibniz and L'Hôpital laid the foundational concepts. As the field evolved through the 18th and 19th centuries, significant strides were made with the formulation of the Riemann-Liouville and Caputo definitions. The 20th century saw the emergence of practical applications, particularly in control theory and signal processing. Recent advancements, driven by computational tools, have spurred increasing research interest, solidifying fractional calculus as a pivotal area of study in applied mathematics, see the references [44, 11]

Variable order fractional systems represent a fascinating evolution in fractional calculus, where the order of the derivative can vary over time or space, offering greater flexibility and modeling accuracy. Historically, this concept emerged as researchers sought to address limitations in constant order systems. Its importance is underscored by its growing application in fields such as control theory and signal processing, where traditional models fall short. Unlike constant order systems, variable order systems can adapt to changing dynamics, providing a more robust framework.
Variable order fractional operators are a versatile and significant extension of traditional constant order fractional operators, offering a dynamic approach to mathematical modeling. Unlike their constant counterparts, variable order operators adapt their order based on the system's state or external parameters, providing a more nuanced representation of complex dynamic systems. Formulated using a combination of integral and differential calculus, these operators have evolved through significant historical contributions. Their advantages in describing real-world phenomena, such as viscoelastic materials and anomalous diffusion, are well-documented. However, computational methods for their implementation face challenges in numerical approximation and stability,See [108, 89].

Discrete-time fractional calculus is a field that extends the principles of traditional calculus to discrete-time systems, allowing for the analysis and modeling of processes with memory and hereditary properties. Originating from the foundational work of mathematicians like Grünwald and Letnikov, this area has evolved through significant contributions that have established its mathematical formulations and fundamental theorems. It involves the study of difference equations and discrete-time fractional derivatives, offering a unique perspective compared to continuous-time fractional calculus. The development of numerical methods and algorithms has facilitated its application in signal processing and control theory. However, challenges in discretization and computational complexity persist, driving ongoing research and recent advancements. Future prospects in this area are promising, with potential applications and exploration in various scientific and engineering domains See [87, 51].
Variable order fractional systems in discrete time hold immense promise for revolutionizing various fields. By exploring untapped industries, these systems can introduce unprecedented levels of precision and efficiency. Existing technological applications stand to benefit significantly, with notable enhancements in performance and reliability. Integration with emerging technologies such as AI and IoT can lead to smarter, more adaptive solutions. In biomedical engineering, the potential for enhanced precision is particularly compelling. Advances in materials science and nanotechnology could yield groundbreaking results. Environmental monitoring and sustainability initiatives may also see innovative applications. Additionally, the expansion of smart grid and renewable energy management, improved algorithm design in computational fields, financial modeling, and enhanced control systems in aerospace and automotive industries

highlight the broad scope and transformative potential of variable order fractional systems see [59, 55] and the references therein.
The investigation into variable order fractional systems in discrete time opens several intriguing research avenues. Identifying optimal numerical methods is crucial for accurately modeling these systems, while exploring the impact of variable order on system stability and control provides insights into their practical applications. Developing efficient algorithms is essential for real-time applications, and understanding the physical interpretation of variable order parameters can deepen our theoretical knowledge. Assessing the robustness of models under varying conditions and integrating these systems with machine learning techniques are key to advancing their utility. Additionally, understanding the long-term behavior and evaluating the scalability of these models for large-scale systems are critical for their broader adoption. Interdisciplinary applications in fields such as biology, finance, and engineering highlight the versatility of variable order fractional systems. Addressing computational challenges and improving simulation accuracy remain pivotal for future advancements see [56, 120] and the references therein. This thesis is divided into four chapters.

In the first one, We collect some concepts and results of fractional calculs that are frequently used in this thesis, which is fractional calculus. It also contains definitions and basic results for discrete-time systems, and it contains results for the case of variable order and optimal control.
In the second one, In this work, we want to prove the existence and uniqueness results of the model,using fixed-point theories . Then, we prove that the model exhibits bounded and positive solutions . Also, we explore the local stability of the proposed model by determining the basic reproduction number. Our work improves numerical simulations to illustrate the richness of our results .

The results of this chapter are represented in part by Manuscript entitled:

⋆ M. Boukhobza,A. Debbouche,L. Shangerganesh, AND D.F.M. Torres, Modeling the dynamics of the Hepatitis B virus via a variable-order discrete system.
https://doi.org/10.1016/j.chaos.2024.114987

The third chapter, this work demonstrates the existence and uniqueness conditions of solutions and presents stability theorems for equilibrium points using fixed-point theories.Then,We cover optimal control analysis .Finally,used numerical scheme and presents numerical simulations and results .
The results of this chapter are represented in part by Manuscript entitled:

⋆ Boukhobza, M.; Debbouche,A.; Shangerganesh L.; Nieto, J.J. The Stability of Solutions of the Variable-Order Fractional Optimal Control Model for the COVID-19 Epidemic in Discrete Time.Mathematics 2024, 12, 1236.
https://doi.org/10.3390/math12081236

The fourth chapter, in this work, we want introduces discrete variable-order calculus and formulates an improved diabetes mathematical model .Then,we want to investigates a variable order model of diabetes in discrete-time, exploring its complications, the boundedness,existence and uniqueness of coupled solutions, and also treats the stability property .Then describes the numerical scheme utilized, and we present numerical simulations and their corresponding results.
The results of this chapter are represented in part by Manuscript entitled:

⋆ M. Boukhobza, A. Debbouche, L. Shangerganesh, S.V. Kashtanova, Stability and sensitivity analysis for a variable-order discrete mathematical model to evaluate the impact of diabetes and its resulting complications, Journal of Mathematical Sciences (2024), accepted.

Lastly, we conclude the thesis with a summary of the main findings and a discussion section that provides suggestions for future research in this area.

Chapter 1

Preliminary Background

The goal of this chapter is to synthesize ideas and results related to various aspects of fractional functional analysis. we discuss the basic concepts and results related to the variable fractional order and the discret -time system . In this thesis, basic definitions and results are presented,We specify this in the references: [19, 74, 124, 89].

1.1 Basic concepts of fractional calculus

Fractional calculus traces its origins back to the 17th century, where pioneering mathematicians such as Leibniz and L'Hôpital laid the foundational concepts. As the field evolved through the 18th and 19th centuries, significant strides were made with the formulation of the Riemann-Liouville and Caputo definitions. The 20th century saw the emergence of practical applications, particularly in control theory and signal processing. Recent advancements, driven by computational tools, have spurred increasing research interest, solidifying fractional calculus as a pivotal area of study in applied mathematics,For further details, refer to the listed sources[92, 61].

1.1.1 Special functions

This section covers the set of functions that will be utilized in fractional theory.

Gamma function

Definition 1.1.1

The Gamma function, denoted by $\Gamma(z)$ *is a generalization of the factorial function* $n!$*, i.e.,*

$$\Gamma(n) = (n-1)! \quad \forall n \in \mathbb{N}.$$

For complex arguments with positive real part it is defined as

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \qquad \mathcal{R}e(z) > 0.$$

This function has the following essential results:

Proposition 1.1.1

For a complex argument z with positive real part $\mathcal{R}e(z) > 0$. So we have the following result:

$$\Gamma(z+1) = z\Gamma(z).$$

Some of the most important values are

$$\begin{aligned}\Gamma(1) = \Gamma(2) &= 1,\\ \Gamma\left(\frac{1}{2}\right) &= \sqrt{\pi},\\ \Gamma\left(n+\frac{1}{2}\right) &= \frac{\sqrt{\pi}(2n-1)!}{2^n},\ \forall n \in \mathbb{N}.\end{aligned}$$

Beta function

Definition 1.1.2

The Beta function is defined by the integral

$$B(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}\mathrm{d}t, \quad \mathcal{R}e(z) > 0, \quad \mathcal{R}e(w) > 0.$$

The Beta function is used sometimes for convenience to replace a combination of Gamma function. This relation between the Gamma function and Beta function is given by (see [**?**])

$$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}.$$

It should also be mentioned that the Beta function is symmetric, i.e.

$$B(z,w) = B(w,z).$$

The Mittag-Leffler function

While the Gamma function is a generalization of the factorial function, the Mittag-Leffler function is a generalization of the exponential function

$$\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k+1)}.$$

First introduced as a one parameter function by the series [92]

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}, \quad z, \alpha \in \mathbb{C},\ \mathcal{R}e(\alpha) > 0.$$

Later, the two parameter generalization is introduced by *Agarwal*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \quad z, \alpha, \beta \in \mathbb{C},\ \mathcal{R}e(\alpha) > 0,\ \mathcal{R}e(\beta) > 0,$$

which is of great importance for the fractional calculus. It is called two parameter function of Mittag-Leffler type. Some of its interesting values are [92]

$$\begin{aligned} E_{1,1}(z) &= e^z, \\ E_{2,1}(z^2) &= \cosh(z), \\ E_{2,2}(z^2) &= \frac{\sinh(z)}{z}, \\ E_{\alpha,2}(z) &= E_\alpha(z), \\ E_{\frac{1}{2},1}(z) &= e^{z^2}\operatorname{erfc}(-z). \end{aligned}$$

This function has the following essential results:

Proposition 1.1.2

For a complex argument z with $\mathcal{R}e(z) > 0$,we have the following result:

$$\begin{aligned} E_{\alpha,\beta}(z) &= zE_{\alpha,\alpha+\beta}(z) + \frac{1}{\Gamma(\beta)}, \\ \frac{\mathrm{d}}{\mathrm{d}z}E_{\alpha,\beta}(z) &= \frac{1}{\alpha z}\Big[E_{\alpha,\beta-1}(z) + (\beta-1)E_{\alpha,\beta}(z)\Big]. \end{aligned}$$

We'll need to build estimates in order to illustrate the uniqueness of each solution in the subsequent sections. We'll use the following two outcomes to do so:

Lemma 1.1.1

For positive integers m, λ and α, we have

$$\begin{aligned} \frac{\mathrm{d}^n}{\mathrm{d}z^n}E_{\alpha,1}(-\lambda z^\alpha) &= -\lambda z^{\alpha-n}E_{\alpha,\alpha-n+1}(-\lambda z^\alpha), \quad z > 0, \\ \frac{\mathrm{d}}{\mathrm{d}z}(zE_{\alpha,2}(-\lambda z^\alpha)) &= E_{\alpha,1}(-\lambda z^\alpha), \quad z > 0. \end{aligned}$$

As well as

Theorem 1.1.1

Let $0 < \alpha < 2$, β is an arbitrary real , and we assume that μ is such that

$$\frac{\pi\alpha}{2} < \mu < \min\{\pi, \pi\alpha\}.$$

Then there exists a constant $C = C(\alpha, \beta, \mu) > 0$ such that

$$|E_{\alpha,\beta}(z)| \leq \frac{C}{1+|z|}, \quad \mu \leq |\arg(z)| \leq \pi.$$

The definition of the generalized Mittag-Leffler function is now given.

Definition 1.1.3

Let $\alpha, \beta, \rho \in \mathbb{C}$ such as $\mathcal{R}e(\alpha) > 0$ and $\mathcal{R}e(\beta) > 0$. The generalized Mittag-Leffler function is thus defined as follows:

$$\varsigma^{\rho}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\rho)_n z^n}{\Gamma(\alpha n + \beta) n!}, \quad \forall z \in \mathbb{C},$$

where

$$(\rho)_n = \rho(\rho + 1)...(\rho + n - 1).$$

Remark 1.1.1

Note that when $\rho = 1$ we have

$$\varsigma^{\rho}_{\alpha,\beta}(z) = E_{\alpha,\beta}(z).$$

We'll need the following Lemma in the sequel:

Lemma 1.1.2

Let $\alpha, \beta, \rho \in \mathbb{C}$ such as $\mathcal{R}e(\alpha) > 0$ and $\mathcal{R}e(\beta) > 0$. Then, we have

$$\frac{\mathrm{d}^n}{\mathrm{d}z^n}\varsigma^{\rho}_{\alpha,\beta}(z) = (\rho)_n \varsigma^{\rho+n}_{\alpha,\beta+\alpha n}(z), \quad z \in \mathbb{C}, \ n \in \mathbb{N},$$
$$\alpha\rho\varsigma^{\rho+1}_{\alpha,\beta}(z) = (1 + \alpha\rho - \beta)\varsigma^{\rho}_{\alpha,\beta}(z) + \varsigma^{\rho}_{\alpha,\beta-1}(z), \quad z \in \mathbb{C}.$$

We utilize the Laplace transform to solve our fractional differential equations, just as we did with integer differential equations. As a result, we provide the following definition:

Definition 1.1.4

Let $f : \mathbb{R}^+ \to \mathbb{R}$. The Laplace transform of function is defined by:

$$(\mathcal{L}f)(s) = \mathcal{L}[f(t)](s) = \hat{f}(s) := \int_0^{\infty} \exp(-st) f(t) \mathrm{d}t, \quad s > 0.$$

On occasion, we will run across transforms of the form,

$$H(s) = F(s)G(s),$$

that can't be dealt with easily using partial fractions. We would like a way to take the inverse transform of such a transform. We can use a convolution integral to do this.

Definition 1.1.5

If $f(t)$ and $g(t)$ are piecewise continuous function on $[0, +\infty]$ then the convolution integral of $f(t)$ and $g(t)$ is,

$$(f \star g)(t) = \int_0^t f(t - s) g(s) \mathrm{d}s.$$

A nice property of convolution integrals is

$$(f \star g)(t) = (g \star f)(t).$$

Or,

$$\int_0^t f(t - s) g(s) \mathrm{d}s = \int_0^t f(s) g(t - s) \mathrm{d}s.$$

The following fact will allow us to take the inverse transforms of a product of transforms.

$$\mathcal{L}\{f \star g\}(t) = F(s)G(s), \quad \mathcal{L}^{-1}F(s)G(s) = \{f \star g\}(t).$$

Lemma 1.1.3

Let $\alpha, \beta, \rho \in \mathbb{C}$ such as $\mathcal{R}e(\alpha) > 0$, $\mathcal{R}e(\rho) > 0$ and $\mathcal{R}e(\beta) > 0$. Then, we have

$$\mathcal{L}^{-1}\left[\frac{s^{\rho-1}}{s^{\alpha} + as^{\beta} + b}; z\right] = t^{\alpha-\rho}\sum_{k=0}^{\infty}(-a)^k z^{k(\alpha-\beta)}\varsigma^{k+1}_{\alpha,\alpha+(\alpha-\beta)k-\rho+1}(-bz^{\alpha}),$$

where $|\frac{as^{\beta}}{s^{\alpha}+b}| < 1$. We also assume that the preceding equality's series is convergent.

1.1.2 Riemann-Liouville fractional integral

Calculations of integrals and derivatives of arbitrary real or complex order are referred to as "fractional calculations." In this thesis, we are only concerned with Riemann-Liouville and Caputo derivatives.

Definition 1.1.6 (*See [61]*)

Cauchy's formula for repeated integration is given by

$$I^n f(t) := \int_a^t \int_a^{\tau_1} \cdots \int_a^{\tau_{n-1}} f(\tau)\mathrm{d}\tau \cdots \mathrm{d}\tau_2 \mathrm{d}\tau_1 = \frac{1}{(n-1)!}\int_a^t f(\tau)(t-\tau)^{n-1}\mathrm{d}\tau, \ \forall n \in \mathbb{N}_0, a, t \in \mathbb{R}, t > 0.$$

If n is substituted by a positive real number α and $(n-1)!$ by its generalization $\Gamma(\alpha)$ a formula for fractional integration is obtained.

Definition 1.1.7

The fractional operator

$$I^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} f(s)\mathrm{d}s, \quad t > a, \quad \alpha > 0.$$

is referred to as Riemann-Liouville fractional integral of order α.

Proposition 1.1.3

- *By convention*

$$I^0 f(t) := f(t), \text{ i.e., } I^0 := I \text{ is the identity operator.}$$

- *The linearity*

$$I^\alpha(\lambda f(t) + g(t)) = \lambda I^\alpha f(t) + I^\alpha g(t), \ \alpha \in \mathbb{R}_+, \lambda \in \mathbb{C}.$$

- *If $f(t)$ is continuous for $t \geq 0$ the following equalities hold*

$$lim_{\alpha \to 0} I^\alpha f(t) = f(t),$$
$$I^\alpha(I^\beta f(t)) = I^\beta(I^\alpha f(t)) = I^{\alpha+\beta} f(t) \quad \alpha, \beta \in \mathbb{R}_+, \lambda \in \mathbb{C}.$$

Definition 1.1.8

The Laplace transform of Riemann-Liouville fractional integral is defined by:

$$\begin{aligned}\mathcal{L}\Big[I^\alpha f(x)\Big] &= \frac{1}{\Gamma(\alpha)} \mathcal{L}(x^{\alpha-1} \star f(x)) \\ &= \frac{1}{s^\alpha} \mathcal{L}\Big[f(x)\Big].\end{aligned}$$

1.1.3 Riemann-Liouville fractional derivative operator

Definition 1.1.9

Let f be a real function, the Riemann-Liouville fractional derivative or the Riemann-Liouville fractional differential operator of order α is defined by

$$\begin{aligned}D^\alpha_{RL} f(t) &= \frac{\mathrm{d}^n}{\mathrm{d}t^n}(I^{n-\alpha} f(t)) \\ &= \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_0^t (t-s)^{n-\alpha-1} f(s)\, \mathrm{d}s, \quad t > 0, \quad \alpha \in (n-1, n), \ n \in \mathbb{N}.\end{aligned}$$

In the following lemma, we give some relations between the Riemann-Liouville fractional derivative and the Riemann-Liouville fractional integral.

Lemma 1.1.4

Let $u \in \mathbb{C}^n([0,T]),\ \alpha \in (n-1,n),\ n \in \mathbb{N}$ and $v \in \mathbb{C}^1([0,T])$.

- *The Riemann-Liouville fractional differential operator D^α_{RL} is the left inverse operator of the fractional integral I^α, i.e.,*

$$D^\alpha_{RL} I^\alpha = I,$$

By convention it is defined

$$D^0_{RL} v(t) := v(t),\ \textit{i.e., } D^0_{RL} := I \textit{ is the identity operator.}$$

-

$$D^\alpha_{RL} v(t) = \frac{\mathrm{d}}{\mathrm{d}t} I^{1-\alpha} v(t), \quad n = 1,$$
$$D^\alpha_{RL} v(t) = \frac{\mathrm{d}^2}{\mathrm{d}t^2} I^{2-\alpha} v(t), \quad n = 2,$$
$$I^\alpha D^\alpha_{RL} u(t) = u(t) - \frac{t^{\alpha-1}}{\Gamma(\alpha)} (I^{\alpha-1} u)(0).$$

Remark 1.1.2

As we can see from the previous definition, the Riemann-Liouville fractional derivative of a constant is non-zero, unlike the integer order derivative of a constant C. To be more specific, the Riemann-Liouville fractional derivative of order $0 < \alpha < 1$ of a constant C is given by

$$D^\alpha_{RL} I^\alpha C = \frac{C t^{-\alpha}}{\Gamma(1-\alpha)}.$$

Definition 1.1.10

The Laplace transform of the Riemann-Liouville fractional derivative is defined by:

$$\begin{aligned}
\mathcal{L}\Big[D^\alpha_{RL} f(t)\Big] &= \mathcal{L}\left[\frac{\mathrm{d}^n}{\mathrm{d}t^n}(I^{n-\alpha} f(t))\right] \\
&= s^\alpha \mathcal{L}(f(t)) - \sum_{k=0}^{n-1} s^k D^{\alpha-k-1} f(0) \\
&= s^\alpha F(s) - \sum_{k=0}^{n-1} s^k D^{\alpha-k-1} f(0).
\end{aligned}$$

Remark 1.1.3

- *The Laplace transform of $f^{(n)}$ is defined as follows:*

$$\mathcal{L}\Big[f^{(n)}(t)\Big] = s^n\mathcal{L}\Big[f(t)\Big] - \sum_{k=0}^{n-1} s^k f^{n-k-1}(0).$$

- $$D^{\alpha}_{RL}f(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}(I^{n-\alpha}f(t)) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}(D^{(\alpha-n)}f(t)).$$

As a result, we've arrived at the following two theorems:

Theorem 1.1.2

Let $0 < \alpha < 1$. The derivative Riemann-Liouville fractional equation of order α is then transformed by the Laplace transform:

$$\mathcal{L}\Big[D^{\alpha}_{RL}f(t)\Big] = s^{\alpha}F(s) - \lim_{t\to 0} I^{1-\alpha}f(t).$$

Theorem 1.1.3

Assume that $1 < \alpha < 2$. The derivative Riemann-Liouville fractional equation of order α is then transformed by the Laplace transform:

$$\mathcal{L}\Big[D^{\alpha}_{RL}f(t)\Big] = s^{\alpha}F(s) - s\lim_{t\to 0} I^{2-\alpha}f(t) - \lim_{t\to 0}\frac{\mathrm{d}}{\mathrm{d}t}I^{2-\alpha}f(t).$$

In the formulation of the Laplace transforms, we can see the terms $\lim\limits_{t\to 0} I^{1-\alpha}f(t)$, $\lim\limits_{t\to 0} I^{2-\alpha}f(t)$ and $\lim\limits_{t\to 0}\dfrac{\mathrm{d}}{\mathrm{d}t}I^{2-\alpha}f(t)$. Contrary, in integer order derivatives, where we can see the initial values of the functions f and f'.

1.1.4 The left and right Caputo fractional derivatives

The concepts of left and right Caputo fractional derivatives will be discussed here.

Definition 1.1.11

If $f(t)$ is defined in $\mathcal{C}^n[a,\infty)$, then the left Caputo fractional derivative or left Caputo fractional differential operator of order α is defined as

$$\begin{aligned}\mathcal{D}^{\alpha}_{C}f(t) &= I^{n-\alpha}\left(\frac{\mathrm{d}^n}{\mathrm{d}t^n}f(t)\right)\\ &= \frac{1}{\Gamma(n-\alpha)}\int_0^t f^{(n)}(s)(t-s)^{n-\alpha-1}ds, \quad t>0, \quad \alpha\in(n-1,n),\ n\in\mathbb{N}.\end{aligned}$$

A constant's Caputo derivative is equal to zero.

Definition 1.1.12

The right Caputo fractional derivative or the right Caputo fractional differential operator of order α is defined by

$$\mathcal{D}_C^{\alpha} f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^T f^{(n)}(s)(s-t)^{n-\alpha-1} ds, \quad 0 < t < T, \quad \alpha \in (n-1, n), \ n \in \mathbb{N}.$$

The adjoint operator of the right fractional derivative is represented by the left fractional derivative. In the following lemma, we give some relations between the Riemann-Liouville fractional derivative and the Caputo fractional integral:

Lemma 1.1.5

Let $u \in \mathbb{C}^n([0,T]),\ \alpha \in (n-1,n),\ n \in \mathbb{N}$ and $v \in \mathbb{C}^1([0,T])$.

$$\mathcal{D}_C^{\alpha} I^{\alpha} v(t) = v(t);$$
$$I^{\alpha} \mathcal{D}_C^{\alpha} u(t) = u(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} u^{(k)}(0);$$
$$I^{\alpha} \mathcal{D}_C^{\alpha} u(t) = u(t) - \frac{t^{\alpha-1}}{\Gamma(\alpha)} (I^{1-\alpha-} u)(0), \quad n = 1;$$
$$I^{\alpha} \mathcal{D}_C^{\alpha} u(t) = u(t) - u(0), \quad n = 1.$$

Lemma 1.1.6

Let $(n-1) < \alpha < n,\ n \in \mathbb{N},\ \alpha \in \mathbb{R}$ and $f(t)$ be such that $\mathcal{D}_C^{\alpha} f(t)$ exists. Then

$$\mathcal{D}_C^{\alpha} f(t) = I^{n-\alpha} D^n f(t) = I^{n-\alpha} \frac{\mathrm{d}^n}{\mathrm{d}t^n} f(t).$$

This implies that the Caputo fractional differential operator is equivalent to an $(n-\alpha)$-fold integration following an n-th order differentiation.

Proposition 1.1.4

In general, the two operators, Riemann-Liouville and Caputo, do not coincide, i.e.,

$$D_{RL}^{\alpha} f(t) \neq \mathcal{D}_C^{\alpha} f(t).$$

Lemma 1.1.7

- *Let* $(n-1) < \alpha < n,\ n, m \in \mathbb{N},\ \alpha \in \mathbb{R}$ *and the functions* $f(t)$ *and* $g(t)$ *be such that both* $\mathcal{D}_C^\alpha f(t)$ *and* $\mathcal{D}_C^\alpha g(t)$ *exist. Then the Caputo fractional derivative is a linear operator, i.e.,*

$$\mathcal{D}_C^\alpha((\lambda f(t) + g(t)) = \lambda \mathcal{D}_C^\alpha f(t) + \mathcal{D}_C^\alpha g(t),\ \alpha \in \mathbb{R}_+, \lambda \in \mathbb{C}.$$

- *The Riemann-Liouville fractional differential operator satisfies*

$$D_{RL}^\alpha(\lambda f(t) + g(t)) = \lambda D_{RL}^\alpha f(t) + D_{RL}^\alpha g(t),\ \alpha \in \mathbb{R}_+, \lambda \in \mathbb{C}.$$

- *Let* $(n-1) < \alpha < n,\ n, m \in \mathbb{N},\ \alpha \in \mathbb{R}$ *and the functions* $f(t)$ *is such that* $\mathcal{D}_C^\alpha f(t)$ *exists. Then in general*

$$\mathcal{D}_C^\alpha D^m f(t) = \mathcal{D}_C^{\alpha+m} f(t) \neq D^m \mathcal{D}_C^\alpha f(t).$$

- *Suppose that* $(n-1) < \alpha < n,\ 0 < \beta = \alpha - (n-1) < 1,\ n \in \mathbb{N},\ \alpha, \beta \in \mathbb{R}$ *and the function*$f(t)$ *is such that both* $\mathcal{D}_C^\alpha f(t)$ *exists. Then*

$$\mathcal{D}_C^\alpha f(t) = \mathcal{D}_C^\beta D^{n-1} f(t).$$

Proof: We refer the reader to[61]. ■

Definition 1.1.13

The Laplace transform of Caputo's fractional derivative is defined by:

$$\begin{aligned}
\mathcal{L}\left[\mathcal{D}_C^\alpha f(t)\right] &= \mathcal{L}\Big[I^{n-\alpha}(\frac{\mathrm{d}^n}{\mathrm{d}t^n} f(t))\Big] \\
&= s^{\alpha-n}\mathcal{L}\left[\frac{\mathrm{d}^n}{\mathrm{d}t^n} f(t)\right] \\
&= s^\alpha \mathcal{L}(f(t)) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0) \\
&= s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0).
\end{aligned}$$

1.2 discrete-time fractional-order calculus

Discrete-time fractional calculus is a field that extends the principles of traditional calculus to discrete-time systems, allowing for the analysis and modeling of processes with memory and hereditary properties. Originating from the foundational work of mathematicians like Grünwald and Letnikov, this area has evolved through significant contributions that have established its mathematical formulations and fundamental theorems. It involves the study of difference equations and discrete-time fractional derivatives, offering a unique perspective compared to

continuous-time fractional calculus. The development of numerical methods and algorithms has facilitated its application in signal processing and control theory. However, challenges in discretization and computational complexity persist, driving ongoing research and recent advancements. Future prospects in this area are promising, with potential applications and exploration in various scientific and engineering domains. In this section, we introduce the fundamental concepts of discrete-time fractional calculus that will be utilized throughout this work.

For a natural number n, the fractional polynomial is defined by,

$$t^{(n)} = \prod_{j=0}^{n-1}(t-j) = \frac{\Gamma(t+1)}{\Gamma(t+1-n)}$$

where Γ denotes the special gamma function and the product is zero when $t+1-j=0$ for some j. More generally, for arbitrary α, define

$$t^{(\alpha)} = \frac{\Gamma(t+1)}{\Gamma(t+1-\alpha)}$$

where the convention is that division at the pole yields zero. Given that the forward and backward difference operators are defined by

$$\Delta f(t) = f(t+1) - f(t), \quad \nabla f(t) = f(t) - f(t-1)$$

respectively, we define iteratively the operators $\Delta^m = \Delta(\Delta^{m-1})$ and $\nabla^m = \nabla(\nabla^{m-1})$, where m is a natural number.

Here are some of the properties of the above factorial function.

Lemma 1.2.1

[18]

Assume the following factorial functions are well defined.

1. $\Delta t^{(\alpha)} = \alpha t^{(\alpha-1)}$.
2. $(t-\mu)t^{(\mu)} = t^{(\mu+1)}$, *where* $\mu \in \mathbb{R}$.
3. $\mu^{(\mu)} = \Gamma(\mu+1)$.
4. *If* $t \le r$, *then* $t^{(\alpha)} \le r^{(\alpha)}$ *for any* $\alpha > r$.
5. *If* $0 < \alpha < 1$, *then* $t^{(\alpha\nu)} \ge \left(t^{(\nu)}\right)^{\alpha}$.
6. $t^{(\alpha+\beta)} = (t-\beta)^{(\alpha)}t^{(\beta)}$.

Also, for our purposes, we list down the following two properties, which can be easily proved:

$$\nabla_s (s-t)^{(\alpha-1)} = (\alpha-1)(\rho(s)-t)^{(\alpha-2)}, \tag{1.1}$$

$$\nabla_t (\rho(s)-t)^{(\alpha-1)} = -(\alpha-1)(\rho(s)-t)^{(\alpha-2)}. \tag{1.2}$$

For two real numbers a and b, we write $N_a = \{a, a+1, a+2, \ldots\}$ and $N_b = \{b, b-1, b-2, \ldots\}$.

If $\alpha > 0$ and $\sigma(s) = s + 1$, then the α-th fractional sum of f is defined (as done in [**?**] and used in [18, 6]) by

$$\Delta^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{-t-\alpha} (t - \sigma(s))^{(\alpha-1)} f(s).$$

Note that $\Delta^{-\alpha}$ maps functions defined on N_a to functions defined on $N_{a+\alpha}$. Also note that

(i) $u(t) = \Delta^{-n} f(t)$, $n \in \mathbb{N}$, satisfies the initial value problem

$$\Delta^n u(t) = f(t), \quad t \in N_a, \quad u(a + j - 1) = 0, \quad j = 1, 2, \ldots, n.$$

(ii) the Cauchy function $\dfrac{(t - \sigma(s))^{(n-1)}}{(n-1)!}$ vanishes at $s = t - (n-1), \ldots, t - 1$.

If $\alpha > 0$ and $\rho(s) = s - 1$, then we define the α-th (right) fractional sum of f by

$$\nabla^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=t+\alpha}^{b} (\rho(s) - t)^{(\alpha-1)} f(s).$$

Note that $\nabla^{-\alpha}$ maps functions defined on bN to functions defined on $b - \alpha N$. Also note that

(i) $u(t) = \nabla^{-n} f(t)$, $n \in \mathbb{N}$, satisfies the initial value problem

$$\nabla^n u(t) = (-1)^n f(t), \quad t \in bN, \quad u(b - j + 1) = 0, \quad j = 1, 2, \ldots, n.$$

(ii) the Cauchy function $\dfrac{(\rho(s) - t)^{(n-1)}}{(n-1)!}$ vanishes at $s = t + 1, t + 2, \ldots, t + (n-1)$.

As is done in usual fractional calculus, the Riemann left and right fractional differences are to be, respectively, defined by

$$\Delta^{\alpha} f(t) = \Delta^n \Delta^{-(n-\alpha)} f(s) \quad \text{and} \quad \nabla^{\alpha} f(t) = (-1)^n \nabla^n \nabla^{-(n-\alpha)} f(s)$$

where $n = \lfloor \alpha \rfloor + 1$. It is clear that the fractional left difference operator Δ^{α} maps functions defined on N_a to functions defined on $N_{a+n-\alpha}$, while the fractional right difference operator ∇^{α} maps functions defined on bN to functions defined on $b - (n - \alpha)N$.
Throughout this article, for simplicity we write Δ^{α} and ∇^{α} in place of Δ_a^{α} and ∇_b^{α}, respectively, where $\alpha \in \mathbb{R}$. Otherwise, we point to the endpoints up to which we take the fractional sum or difference. However, one has to note that if $\alpha = n \in \mathbb{N}$, then

$$\Delta_a^n f(t) = \Delta^n f(t) \quad \text{and} \quad \nabla_b^n f(t) = (-1)^n \nabla^n f(t).$$

The ν-th left fractional sum behaves well in composition. in [18] states...

Lemma 1.2.2

[19] Let f be a real-valued function, and let $\mu, \nu > 0$. Then, for all t such that $t = a + \mu + \nu \mod (1)$, we have

$$\Delta^{-\nu} \left[\Delta^{-\mu} f(t) \right] = \Delta^{-(\mu+\nu)} f(t) = \Delta^{-\mu} \left[\Delta^{\nu} f(t) \right].$$

Lemma 1.2.3

[19] For any $\nu \in \mathbb{R}$, we have

$$\Delta^{-\nu+1} f(t) = \Delta\Delta^{-\nu} f(t) - \frac{(t-a)^{(\nu-1)}}{\Gamma(\nu)} f(a),$$

where f is defined on N_a.

Lemma 1.2.4

[19]For any $\alpha \in \mathbb{R}$ and any positive integer p, the following equality holds:

$$\Delta^{-\alpha}\Delta^{p} f(t) = \Delta^{p}\Delta^{-\alpha} f(t) - \sum_{k=0}^{p-1} \frac{(t-a)^{(\alpha-p+k)}}{\Gamma(\alpha+k-p+1)} \Delta^{k} f(a),$$

where f is defined on N_a.

Lemma 1.2.5

[19] For any $\alpha \in \mathbb{R}$ and any positive integer p, the following equality holds:

$$\Delta^{-\alpha}\Delta^{p} f(t) = \Delta^{p}\Delta^{-\alpha} f(t) - \sum_{k=0}^{p-1} \frac{(t-a)^{(\alpha-p+k)}}{\Gamma(\alpha+k-p+1)} \Delta^{k} f(a) \tag{34}$$

where f is defined on N_a.

Lemma 1.2.6

([19]). For any $\alpha > 0$, the following equality holds:

$$\nabla^{-\alpha}\nabla_{b} f(t) = \nabla_{b}\nabla^{-\alpha} f(t) - \frac{(b-t)^{(\alpha-1)}}{\Gamma(\alpha)} f(b) \tag{35}$$

where f is defined on bN.

Remark 1.2.1

([19]). Let $\alpha > 0$ and $n = \lfloor \alpha \rfloor + 1$. Then, we can have

$$\nabla_{b}\nabla^{\alpha} f(t) = \nabla_{b}\nabla_{b}^{n}\left(\nabla^{-(n-\alpha)} f(t)\right) = \nabla_{b}^{n}\left(\nabla_{b}\nabla^{-(n-\alpha)} f(t)\right)$$

or

$$\nabla_{b}\nabla^{\alpha} f(t) = \nabla_{b}^{n}\left[\nabla^{-(n-\alpha)}\nabla_{b} f(t) + \frac{(b-t)^{(n-\alpha-1)}}{\Gamma(n-\alpha)} f(b)\right].$$

Then, using the identity

$$\nabla_{b}^{n}\left(\frac{(b-t)^{(n-\alpha-1)}}{\Gamma(n-\alpha)}\right) = \frac{(b-t)^{(-\alpha-1)}}{\Gamma(-\alpha)}$$

Theorem 1.2.1

([6])
For any real number α and any positive integer p, the following equality holds:

$$\nabla^{-\alpha}\nabla_b^p f(t) = \nabla_b^p \nabla^{-\alpha} f(t) - \sum_{k=0}^{p-1} \frac{(b-t)^{(\alpha-p+k)}}{\Gamma(\alpha+k-p+1)} \nabla_b^k f(b) \tag{39}$$

where f is defined on bN and we remind that $\nabla_b^k f(t) = (-1)^k \nabla^k f(t)$.

Lemma 1.2.7

[6]
Let $\alpha > 0$, $\mu > 0$. Then,

$$\nabla_{b-\mu}^{-\alpha}(b-t)^{(\mu)} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}(b-t)^{(\mu+\alpha)}.$$

Theorem 1.2.2

[6] Let $\alpha > 0$, $\mu > 0$. Then, for all t such that $t \equiv b - (\mu+\alpha)\ (mod\ 1)$, we have

$$\nabla^{-\nu}[\nabla^{-\mu} f(t)] = \nabla^{-(\mu+\nu)} f(t) = \nabla^{-\mu}[\nabla^{-\nu} f(t)]$$

where f is defined on $b\mathbb{N}$.

Proposition 1.2.1

[6] For $\alpha > 0$, and f defined in a suitable domain N_a, we have for $t \in N_{a+n} \subset N_a$

$$\Delta^{\alpha}\Delta^{-\alpha} f(t) = f(t),$$

and

$$\Delta^{-\alpha}\Delta^{\alpha} f(t) = f(t),\ \text{when } \alpha \notin \mathbb{N},$$

$$\Delta^{-\alpha}\Delta^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{(k)}}{k!}\Delta^k f(a),\ \text{when } \alpha = n \in \mathbb{N}.$$

Proposition 1.2.2

[6] For $\alpha > 0$, and f defined in a suitable domain N_a, we have for $t \in b - n\mathbb{N} \subset b\mathbb{N}$

$$\nabla^{\alpha}\nabla^{-\alpha} f(t) = f(t),$$

and

$$\nabla^{-\alpha}\nabla^{\alpha} f(t) = f(t),\ \text{when } \alpha \notin \mathbb{N},$$

$$\nabla^{-\alpha}\nabla^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(b-t)^{(k)}}{k!}\nabla^k f(b),\ \text{when } \alpha = n \in \mathbb{N}.$$

Definition 1.2.1 (*Stochastic basis.*)

Let $\alpha > 0$, $\alpha \notin \mathbb{N}$. Then, the α-order Caputo left fractional and right fractional differences of a function f defined on N_a and $b\mathbb{N}$, respectively, are defined by

$$\Delta_C^{\alpha} f(t), \Delta^{-(n-\alpha)} \Delta^n f(t) = \frac{1}{\Gamma(n-\alpha)} \sum_{s=a}^{t-(n-\alpha)} (t - \sigma(s))^{(n-\alpha-1)} \Delta_s^n f(s)$$

and

$$\nabla_C^{\alpha} f(t), \nabla^{-(n-\alpha)} \nabla^n f(t) = \frac{1}{\Gamma(n-\alpha)} \sum_{s=t+(n-\alpha)}^{b} (\rho(s) - t)^{(n-\alpha-1)} \nabla_b^n f(s)$$

where $n = \lfloor \alpha \rfloor + 1$.
If $\alpha = n \in \mathbb{N}$, then

$$\Delta_C^{\alpha} f(t), \Delta^n f(t)$$

and

$$\nabla_C^{\alpha} f(t), \nabla_b^n f(t).$$

Definition 1.2.2 (*Discrete Mittag-Leffler*)

For $\lambda \in \mathbb{R}$ and $\alpha, \beta, z \in \mathbb{C}$ with $Re(\alpha) > 0$, the discrete (like) Mittag-Leffler functions are defined by

$$E_{(\alpha,\beta)}(\lambda, z) = \sum_{k=0}^{\infty} \frac{\lambda^k \, (z + (k-1)(\alpha-1))^{(k\alpha)} \, (z + k(\alpha-1))^{(\beta-1)}}{\Gamma(\alpha k + \beta)}.$$

For $\beta = 1$, it is written as

$$E_{(\alpha)}(\lambda, z), E_{(\alpha,1)}(\lambda, z) = \sum_{k=0}^{\infty} \frac{\lambda^k \, (z + (k-1)(\alpha-1))^{(k\alpha)}}{\Gamma(\alpha k + 1)}.$$

1.3 Variable order fractional systems

Variable order fractional operators are a versatile and significant extension of traditional constant order fractional operators, offering a dynamic approach to mathematical modeling. Unlike their constant counterparts, variable order operators adapt their order based on the system's state or external parameters, providing a more nuanced representation of complex dynamic systems. Formulated using a combination of integral and differential calculus, these operators have evolved through significant historical contributions. Their advantages in describing real-world phenomena, such as viscoelastic materials and anomalous diffusion, are well-documented. However, computational methods for their implementation face challenges in numerical approximation and stability, an area ripe for future research. As a starting point of this section, let us introduce the notion of Variable order fractional operators by following [74, 124, 89]

Definition 1.3.1

(VO RL FI, type A):

$$ {}_cI_t^{\beta(t)} f(t) = \frac{1}{\Gamma(\beta(t))} \int_c^t (t-\tau)^{\beta(t)-1} f(\tau)\, d\tau, \quad if \beta(t) \in \mathbb{R}^+. $$

Definition 1.3.2

(VO RL FI, type B).[74, 124, 89]

$$ {}_cI_t^{\beta(t)} f(t) = \frac{1}{\Gamma(\beta(\tau))} \int_c^t (t-\tau)^{\beta(\tau)-1} f(\tau)\, d\tau, \quad if \beta(t) \in \mathbb{R}^+. $$

Definition 1.3.3

(VO RL FI, type C)[74, 124, 89]

$$ {}_cI_t^{\beta(t)} f(t) = \frac{1}{\Gamma(\beta(t-\tau))} \int_c^t (t-\tau)^{\beta(t-\tau)-1} f(\tau)\, d\tau, \quad if \beta(t) \in \mathbb{R}^+. $$

Definition 1.3.4

(VO RL FD Definition)[74, 124, 89] The VO RL FD is defined as an integer-order derivative of the RL FI:

$$ {}_cD_t^{\alpha(t)} f(t) = \frac{d^{\alpha(t)}}{dt^{\alpha(t)}} \left({}_cI_t^{\alpha(t)-\alpha(t)} f(t) \right). $$

Nevertheless, the procedure does not seem suitable for implementing a VO FD, since we are simultaneously differentiating both $f(t)$ and $\alpha(t)$.

The Grünwald–Letnikov definition can be modified leading to alternative VO derivatives.

Definition 1.3.5

(VO GL FD, type A).[74, 124, 89]

$$ {}_cD_t^{\alpha(t)} f(t) = \lim_{h \to 0^+} h^{-\alpha(t)} \sum_{k=0}^{\frac{t-c}{h}} (-1)^k \binom{\alpha(t)}{k} f(t-kh), $$

where $c \in \mathbb{R}$ and $h \in \mathbb{R}^+$.

Definition 1.3.6

(VO GL FD, type B).[74, 124, 89]

$$ {}_cD_t^{\alpha(t)} f(t) = \lim_{h \to 0^+} \sum_{k=0}^{\frac{t-c}{h}} h^{-\alpha(kh)} (-1)^k \binom{\alpha(kh)}{k} f(t-kh). $$

Definition 1.3.7

(VO GL FD, type C).

$$ {}_cD_t^{\alpha(t)} f(t) = \lim_{h \to 0^+} \sum_{k=0}^{\frac{t-c}{h}} h^{\alpha(t-kh)} (-1)^k \binom{\alpha(t-kh)}{k} f(t-kh). $$

Definition 1.3.8

(VO GL FD, type D)[74, 124, 89] A type D derivative definition, based upon a recursive relation, was introduced in [67, 68] :

$$ {}_cD_t^{\alpha(t)} f(t) = \lim_{h\to 0^+} \left(\frac{f(t)}{h^{\alpha(t)}} - \sum_{k=1}^{\frac{t-c}{h}} (-1)^k \binom{-\alpha(t)}{k} {}_cD_{t-kh}^{\alpha(t)} f(t) \right). $$

Definition 1.3.9

[58] The VO (Variable Order) Mittag-Leffler function is given by:

$$ E_{\alpha(t),\beta(t)}(pt^{\alpha(t)}) = \sum_{n=0}^{\infty} \frac{p^n t^{n\alpha(t)}}{\Gamma(n\alpha(t)+\beta(t))}. $$

$$ y_k(t) = \int_{-\infty}^{t} x(\tau)(t-\tau)^{\alpha(t)-1} E_{\alpha(t),\alpha(t)} \left(p_k (t-\tau)^{\alpha(t)} \right) d\tau, $$

for $t \geq 0$, or alternatively:

$$ y_k(t) = \int_{0}^{\infty} x(t-\tau)\tau^{\alpha(t)-1} E_{\alpha(t),\alpha(t)} \left(p_k \tau^{\alpha(t)} \right) d\tau. $$

1.4 Optimal control

the discrete-time variable-order fractional linear system.

$$ \begin{cases} \Delta^{\alpha(t)} x(t) & = Ax(t) + Bu(t),\ x(s) = x_0, \\ y(t) & = Cx(t) \end{cases} \tag{1.3} $$

The corresponding discrete fractional optimal control problem with variable order in the Caputo sense is considered as follows:

$$ J(u^*) = \min_{u\in\Omega} J(u), $$

where J is defined by

$$ \mathcal{J}(u,s;x) = \frac{1}{2} \sum_{k=0}^{N-1} [(u(t), Ru(t)) + (x(t), Qx(t))]\mathrm{d}t + \frac{1}{2}(x(T), Sx(T)), $$

and the control space Ω is defined by the set

$$ \Omega = \{\ u \in \mathcal{R}^n / 0 < u_{min} < u < u_{max} < 1 \}. $$

where $R = R^* > 0,\ Q = Q^* = C^*C \geq 0$, and $S = S^* \geq 0$. *A* control $\hat{u(\cdot)}$ minimizing $\mathcal{J}(u,x)$ will be called an optimal control. We shall not be concerned with the question of existence and uniqueness of solutions but with the characterization of the optimal control $\hat{u}(\cdot)$ and the corresponding optimal trajectory $\hat{x}(\cdot)$.

Chapter 2

Modelling the dynamics of the Hepatitis B virus via a variable-order discrete system

The work presented in this Chapter (2) is the dynamics of the hepatitis B virus by integrating variable-order calculus and discrete analysis. Specifically, we utilize the Caputo variable-order difference operator in this study. To establish the existence and uniqueness results of the model, we employ a fixed-point technique. Furthermore, we prove that the model exhibits bounded and positive solutions. Additionally, we explore the local stability of the proposed model by determining the basic reproduction number. Finally, we present several numerical simulations to illustrate the richness of our results. This work is attributed to the [28].

2.1 Introduction

The body experiences a diminished immune response upon exposure to viruses, particularly viral hepatitis, as a defense mechanism to protect the liver. There are various types of hepatitis, including hepatitis A, hepatitis B, hepatitis C, hepatitis D, and hepatitis E. Our paper specifically concentrates on hepatitis B. The hepatitis B virus is recognized as one of the most harmful viruses, significantly contributing to the development of liver cancer [105]. Its rapid transmission is evident, with approximately 80% of hepatitis B cases attributed to infections transmitted from person to person through blood and sexual contact [93, 112]. Horizontal transmission from mother to fetus is also documented [65, 119].
Hepatitis B manifests in two forms: acute and chronic. Many carriers of the virus, especially in chronic cases, may remain asymptomatic. Symptoms, when present, typically emerge between two weeks to six months after infection. Symptoms of acute hepatitis include fatigue, fever, muscle and joint pain, nausea, vomiting, loss of appetite, weight loss, abdominal pain, jaundice, pale stool, dark urine, itchy skin, and an overall feeling of fatigue and malaise. Symptoms of chronic hepatitis may include blood in the stool or vomit and swelling of the lower extremities. Additionally, the skin may become yellow, and the whites of the eyes may also turn yellow. The prevalence of Hepatitis B is widespread globally, as highlighted in [72, 128]. Despite its global reach, there exists an effective vaccination to prevent infection, as emphasized in [78, 107].
Mathematical epidemiology is a specialized field dedicated to exploring the dynamics of disease transmission. The dynamics of hepatitis viruses have been extensively studied through mathematical models, as demonstrated by numerous examples [98, 125]. In a seminal study, Anderson

et al. [16] presented a simple model to examine the influence of carriers on the transmission of hepatitis B. The transmission dynamics and control of the hepatitis B virus were modeled in [144], while a model forecasting a mechanism for eliminating hepatitis B was proposed in [127]. Similar concepts were explored in [79]. Control analysis using an SIR epidemic model was suggested in [143]. Further investigation of epidemic models with control strategies was conducted in [23, 62]. A model studying multiple endemic solutions was developed in [86]. Similarly, the dynamics of hepatitis B were explored in [141]. Recent research, exemplified by works such as [67, 123], has developed epidemiological models to investigate the influences of various parameters on disease transmission and to propose control measures for infection elimination.

In these investigations, differential equations with integer orders were initially employed. However, it became evident over time that these models were insufficient for comprehending complex biological systems. Consequently, there has been a shift towards mathematical models with fractional orders, a trend that has recently gained prominence. The utilization of fractional derivatives and fractional integrals has found numerous applications in applied sciences and engineering. Many classical models have demonstrated limited accuracy in predicting the future dynamics of a system. In contrast, models incorporating fractional orders have proven more effective in capturing and retaining missing information [15, 76]. It is worth noting that classical derivatives may not adequately capture the dynamics between two distinct points [103, 25].

Recent efforts have been directed towards enhancing discrete fractional calculus, as demonstrated in [13]. These advancements underscore the growing importance of discrete fractional calculus, as evidenced by works such as [25, 63, 18]. Variable-order calculus is recognized as a natural extension of classical calculus [33], with foundational work in this area dating back to 1993 by Samko and his co-authors [104]. Subsequently, variable-order problems have found applications in fields such as photoelasticity [114], and the stability and convergence of novel explicit finite-difference approaches for variable-order nonlinear fractional diffusion equations have been studied [73]. Various numerical schemes for variable-order problems have been developed, as referenced in [34] and [106]. Moreover, existence theories for variable-order problems have been established [96, 131]. Given that variable-order operators extend classical ordinary and fractional orders, their utilization provides sophisticated tools for studying the dynamical systems of infectious diseases [33].

Building upon the aforementioned motivation and work, here we introduce a mathematical model for the dynamics of the hepatitis B virus, employing variable-order calculus. The model encompasses populations of susceptible ($S(t)$), acute ($A(t)$), immune ($M(t)$), chronic ($C(t)$), recovered ($R(t)$), and vaccinated ($V(t)$), while incorporating discrete variable-order time differences. It is expressed as follows:

$$\begin{cases} \Delta^{\alpha(t)}S(t) &= \delta d(1-rC) + wV - (\mu_1 + \beta M + n\beta C + k)S, \\ \Delta^{\alpha(t)}A(t) &= (\beta M + n\beta C)S - (\mu_1 + \gamma)A, \\ \Delta^{\alpha(t)}M(t) &= \gamma A + (\delta dr - \mu_1 - \mu_2 - a - m)M, \\ \Delta^{\alpha(t)}C(t) &= aM - eC - \mu_1 C, \\ \Delta^{\alpha(t)}R(t) &= mM + eC - \mu_1 R, \\ \Delta^{\alpha(t)}V(t) &= \delta(1-d) + kS - \mu_1 V - wV, \end{cases} \tag{2.1}$$

where the initial conditions are given as $S(0) = S_0$, $A(0) = A_0$, $M(0) = M_0$, $C(0) = C_0$, $R(0) = R_0$ and $V(0) = V_0$, and where we use the variable-order $\alpha(t) \in (0, 1)$. The unknown variables $(S(t), A(t), M(t), C(t), R(t), V(t))$ are in $\mathbb{R}^6_+$. The total population is given by $B(t) = S(t)+A(t)+M(t)+C(t)+R(t)+V(t)$. All the parameters $(\delta, \mu_1, \mu_2, \beta, \gamma, a, e, k, d, r, w, n, m)$ are

Table 2.1: Parameter values for our model (2.1), borrowed from [133].

Parameters		ParameterValue
δ	Birth Rate	0.0121
μ_1	Natural mortality rate	0.000034857
μ_2	Hepatitis-B related mortality rate	0.1019
β	Transmission coefficient of the disease	0.00014334
γ	Transition rate from Latent population to Acute population	0.1989
a	Transition rate of individuals with Acute infection to carrier-class	0.3387
e	Recovery rate of individuals in the carrier class	0.0741
k	Vaccination rate	0.8569
d	Rate of births without successful vaccination	0.00043102
r	Infected rate of mothers with HB Acute virus	0.0137
w	The rate of decrease in immunity with the effect of vaccine	0.9472
n	Reduced transmission rate compared to Acute	0.7534
m	Recovery rate of individuals with Acute infection	0.0277

positive real numbers and their values are provided in Table 2.1. Here, the delta variable-order difference of model (2.1) is given in the sense of Caputo, where $\alpha(t) \in (0, 1)$.
The central concept of the presented model revolves around the integration of discrete variable-order time difference, representing a novel class of fractional calculus with wide-ranging real-world applications. The proposed model explores the impact of different phases of infected individuals and various transmission routes on the dynamics of a hepatitis B virus model utilizing discrete time-variable-order differences. This model provides significant advantages in comprehending the transmission dynamics of hepatitis B virus within the human population.
The paper is structured as follows. In Section 2.2 we recall necessary notions and results from the literature needed in the sequel. In Section 2.3, we delve into the conditions for the existence and uniqueness of solutions as well as their boundedness and positivity. Section 2.4 establishes the stability of the equilibrium points while in Section 2.5 we present some numerical results. We end the paper with Section 2.6 of conclusion.

2.2 Mathematical preliminaries

Here, we introduce some definitions and notations from the papers [29, 58]. We denote $\mathbb{N}_a$ and $\mathbb{N}_a^T$ as $\mathbb{N}_a = \{a, a+1, a+2, \ldots\}$, $\mathbb{N}_a^T = \{a, a+1, a+2, \ldots, T\}$.
Let $\alpha(t) > 0$ and $\sigma(s) = s + 1$. For $u(t)$ defined on $\mathbb{N}_a$, the delta variable-order sum of order $\alpha(t)$ is defined by

$$\Delta_a^{-\alpha(t)} u(t) = \frac{1}{\Gamma(\alpha(t))} \sum_{s=a}^{t-\alpha(t)} (t - \sigma(s))^{(\alpha(t)-1)} u(s), \tag{2.2}$$

where $t^{(\alpha(t))}$ is the discrete factorial functional given by $t^{(\alpha(t))} = \dfrac{\Gamma(t+1)}{\Gamma(t-\alpha(t)+1)}$.

Definition 2.2.1

For $u(t)$ defined on $\mathbb{N}_a$, $\alpha(t) > 0, \alpha \notin \mathbb{N}$, the delta Caputo variable-order difference is defined by

$$ {}^{C}\Delta_a^{\alpha(t)}u(t) = \Delta_a^{-(m-\alpha(t))}\Delta^m u(t) = \frac{1}{\Gamma(m-\alpha(t))}\sum_{s=a}^{t-(m-\alpha(t))}(t-\sigma(s))^{(m-\alpha(t)-1)}\Delta^m u(s), \quad (2.3) $$

where $t \in \mathbb{N}_{a+m-\alpha(t)}$, $m = [\alpha(t)] + 1$. Note that the forward difference operator is defined by $\Delta u(t) = u(t+1) - u(t)$.

Theorem 2.2.1 (*See [50, 91]*)

Consider the following variable-order discrete system:

$$ \Delta^{\alpha(t)}x = f(x), \quad x(0) = x_0 \quad (2.4) $$

with $x \in \mathbb{R}^n$ and $\underline{\alpha} = \inf\alpha(t)$, $\overline{\alpha} = \sup\alpha(t)$, $0 < \underline{\alpha} < \alpha(t) < \overline{\alpha} < 1$. The equilibrium points of the system (2.4) are solutions to the equation $f(x) = 0$. An equilibrium is locally asymptotically stable if all the eigenvalues λ_i $(i = 1, 2, 3, \ldots, n)$ of the Jacobian matrix $J = \Delta f$ evaluated at the equilibrium satisfy

$$ |\arg(\lambda_i)| < \frac{\pi}{2}\underline{\alpha}. \quad (2.5) $$

On the other hand, if $|\arg(\lambda_i)| > \frac{\pi}{2}\overline{\alpha}$, then the equilibrium point is unstable.

Theorem 2.2.2 (*See [13]*)

Let $s \in \mathbb{N}_{a+1}$. The following holds:

$$ \sum_{k=a+1}^{s}(s-k+1)^{\alpha(s)-1} = \frac{(s-a)^{\alpha(s)}}{\alpha(s)}. \quad (2.6) $$

2.3 Wellposedness of the discrete time variable-order model

In this section, we prove the existence and uniqueness of solutions for the given system (2.1) (Theorem 2.3.1). Moreover, we show that the solution is positive and bounded (Theorem 2.3.2).

2.3.1 Existence and uniqueness of solution

Using the properties of discrete variable-order calculus, we establish the existence and uniqueness of a solution to system (2.1). Now, system (2.1) can be rewritten as,

$$ \begin{cases} S(t) &= S(0) + \Delta^{-\alpha(t)}(\delta d(1-rC) + wV - (\mu_1 + \beta M + n\beta C + k)S), \\ A(t) &= A(0) + \Delta^{-\alpha(t)}(\beta M + n\beta C)S - (\mu_1 + \gamma)A, \\ M(t) &= M(0) + \Delta^{-\alpha(t)}(\gamma A + (\delta dr - \mu_1 - \mu_2 - a - m)M), \\ C(t) &= C(0) + \Delta^{-\alpha(t)}(aM - eC - \mu_1 C), \\ R(t) &= R(0) + \Delta^{-\alpha(t)}(mM + eC - \mu_1 R), \\ V(t) &= V(0) + \Delta^{-\alpha(t)}(\delta(1-d) + kS - \mu_1 V - wV). \end{cases} \quad (2.7) $$

Using the definitions of variable-order calculus, it is easy to obtain

$$\begin{aligned}
S(t) &= S(0)+\frac{1}{\Gamma(\alpha(t))}\sum_{s=0}^{t-\alpha(t)}(t-\sigma(s))^{(\alpha(t)-1)}(\delta d(1-rC)+wV-(\mu_1+\beta M+n\beta C+k)S),\\
A(t) &= A(0)+\frac{1}{\Gamma(\alpha(t))}\sum_{s=0}^{t-\alpha(t)}(t-\sigma(s))^{(\alpha(t)-1)}(\beta M+n\beta C)S-(\mu_1+\gamma)A,\\
M(t) &= M(0)+\frac{1}{\Gamma(\alpha(t))}\sum_{s=0}^{t-\alpha(t)}(t-\sigma(s))^{(\alpha(t)-1)}(\gamma A+(\delta dr-\mu_1-\mu_2-a-m)M),\\
C(t) &= C(0)+\frac{1}{\Gamma(\alpha(t))}\sum_{s=0}^{t-\alpha(t)}(t-\sigma(s))^{(\alpha(t)-1)}(aM-eC-\mu_1 C),\\
R(t) &= R(0)+\frac{1}{\Gamma(\alpha(t))}\sum_{s=0}^{t-\alpha(t)}(t-\sigma(s))^{(\alpha(t)-1)}(mM+eC-\mu_1 R),\\
V(t) &= V(0)+\frac{1}{\Gamma(\alpha(t))}\sum_{s=0}^{t-\alpha(t)}(t-\sigma(s))^{(\alpha(t)-1)}(\delta(1-dC)+kS-\mu_1 V-wV).
\end{aligned} \tag{2.8}$$

We define the kernels L_1 ,L_2, L_3, L_4, L_5 and L_6 from the RHS of (2.1):

$$\begin{cases}
L_1(t,S(t)) &= \delta d(1-rC)+wV-(\mu_1+\beta M+n\beta C+k)S,\\
L_2(t,A(t)) &= (\beta M+n\beta C)S-(\mu_1+\gamma)A,\\
L_3(t,M(t)) &= \gamma A+(\delta dr-\mu_1-\mu_2-a-m)M,\\
L_4(t,C(t)) &= aM-eC-\mu_1 C,\\
L_5(t,R(t)) &= mM+eC-\mu_1 R,\\
L_6(t,V(t)) &= \delta(1-dC)+kS-\mu_1 V-wV.
\end{cases} \tag{2.9}$$

Lemma 2.3.1

The kernels L_1, L_2, L_3, L_4, L_5 and L_6 satisfy a Lipschitz condition.

Proof: Suppose $S(t)$ and $S^*(t)$ to be the two unknowns of the first equation of (2.9). Then we have

$$\|L_1(t,S(t))-L_1(t,S^*(t))\| = \|(\mu_1+\beta M+n\beta C+k)(S(t)-S^*(t))\|.$$

Assume that $m_1=\|\mu_1+\beta M+n\beta C+k\|$. Then the above can be rewritten as

$$\|L_1(t,S(t))-L_1(t,S^*(t))\| < m_1\|S(t)-S^*(t)\|.$$

Using the similar procedure for L_2, L_3, L_4, L_5 and L_6, we get

$$\begin{aligned}
\|L_2(t,A(t))-L_2(t,A^*(t))\| &< m_2\|A(t)-A^*(t)\|,\\
\|L_3(t,M(t))-L_3(t,M^*(t))\| &< m_3\|M(t)-M^*(t)\|,\\
\|L_4(t,C(t))-L_4(t,C^*(t))\| &< m_4\|C(t)-C^*(t)\|,\\
\|L_5(t,R(t))-L_5(t,R^*(t))\| &< m_5\|R(t)-R^*(t)\|,\\
\|L_6(t,V(t))-L_6(t,V^*(t))\| &< m_6\|V(t)-V^*(t)\|,
\end{aligned}$$

where m_1, m_2, m_3, m_4, m_5 and m_6 are the Lipschitz constants. ■

Now using (2.9) in (2.8), we get

$$\begin{cases} S(t) &= S(0)+\dfrac{1}{\Gamma(\alpha(t))}\displaystyle\sum_{s=0}^{t-\alpha(t)}(t-\sigma(s))^{(\alpha(t)-1)}L_1(s,S(s)),\\ A(t) &= A(0)+\dfrac{1}{\Gamma(\alpha(t))}\displaystyle\sum_{s=0}^{t-\alpha(t)}(t-\sigma(s))^{(\alpha(t)-1)}L_2(s,A(s)),\\ M(t) &= M(0)+\dfrac{1}{\Gamma(\alpha(t))}\displaystyle\sum_{s=0}^{t-\alpha(t)}(t-\sigma(s))^{(\alpha(t)-1)}L_3(s,M(s)),\\ C(t) &= C(0)+\dfrac{1}{\Gamma(\alpha(t))}\displaystyle\sum_{s=0}^{t-\alpha(t)}(t-\sigma(s))^{(\alpha(t)-1)}L_4(s,C(s)),\\ R(t) &= R(0)+\dfrac{1}{\Gamma(\alpha(t))}\displaystyle\sum_{s=0}^{t-\alpha(t)}(t-\sigma(s))^{(\alpha(t)-1)}L_5(s,R(s)),\\ V(t) &= V(0)+\dfrac{1}{\Gamma(\alpha(t))}\displaystyle\sum_{s=0}^{t-\alpha(t)}(t-\sigma(s))^{(\alpha(t)-1)}L_6(s,V(s)). \end{cases} \tag{2.10}$$

Then, using the recursive formula, we get

$$\begin{cases} S_n(t) &= S(0)+\dfrac{1}{\Gamma(\alpha(t))}\displaystyle\sum_{s=0}^{t-\alpha(t)}(t-\sigma(s))^{(\alpha(t)-1)}L_1(s,S_{n-1}(s)),\\ A_n(t) &= A(0)+\dfrac{1}{\Gamma(\alpha(t))}\displaystyle\sum_{s=0}^{t-\alpha(t)}(t-\sigma(s))^{(\alpha(t)-1)}L_2(s,A_{n-1}(s)),\\ M_n(t) &= M(0)+\dfrac{1}{\Gamma(\alpha(t))}\displaystyle\sum_{s=0}^{t-\alpha(t)}(t-\sigma(s))^{(\alpha(t)-1)}L_3(s,M_{n-1}(s)),\\ C_n(t) &= C(0)+\dfrac{1}{\Gamma(\alpha(t))}\displaystyle\sum_{s=0}^{t-\alpha(t)}(t-\sigma(s))^{(\alpha(t)-1)}L_4(s,C_{n-1}(s)),\\ R_n(t) &= R(0)+\dfrac{1}{\Gamma(\alpha(t))}\displaystyle\sum_{s=0}^{t-\alpha(t)}(t-\sigma(s))^{(\alpha(t)-1)}L_5(s,R_{n-1}(s)),\\ V_n(t) &= V(0)+\dfrac{1}{\Gamma(\alpha(t))}\displaystyle\sum_{s=0}^{t-\alpha(t)}(t-\sigma(s))^{(\alpha(t)-1)}L_6(s,V_{n-1}(s)). \end{cases} \tag{2.11}$$

Subtract the (2.11) with previous successive term and denote the resulting expressions as follows:

$$\Phi_{S,n}(t)=S_n(t)-S_{n-1}(t),\Phi_{A,n}(t)=A_n(t)-A_{n-1}(t),\Phi_{M,n}(t)=M_n(t)-M_{n-1}(t),$$

$$\Phi_{R,n}(t)=R_n(t)-R_{n-1}(t),\Phi_{V,n}(t)=V_n(t)-V_{n-1}(t).$$

Then, we have

$$\begin{aligned} \Phi_{S,n}(t) &= S_n(t)-S_{n-1}(t)=\frac{1}{\Gamma(\alpha(t))}\sum_{s=0}^{t-\alpha(t)}(t-\sigma(s))^{(\alpha(t)-1)}(L_1(s,S_{n-1}(s))-L_1(s,S_{n-2}(s))),\\ \Phi_{A,n}(t) &= A_n(t)-A_{n-1}(t)=\frac{1}{\Gamma(\alpha(t))}\sum_{s=0}^{t-\alpha(t)}(t-\sigma(s))^{(\alpha(t)-1)}(L_2(s,A_{n-1}(s))-L_2(s,A_{n-2}(s))),\\ \Phi_{M,n}(t) &= M_n(t)-M_{n-1}(t)=\frac{1}{\Gamma(\alpha(t))}\sum_{s=0}^{t-\alpha(t)}(t-\sigma(s))^{(\alpha(t)-1)}(L_3(s,M_{n-1}(s))-L_3(s,M_{n-2}(s))), \end{aligned}$$

$$\begin{aligned}
\Phi_{C,n}(t) &= C_n(t) - C_{n-1}(t) = \frac{1}{\Gamma(\alpha(t))} \sum_{s=0}^{t-\alpha(t)} (t-\sigma(s))^{(\alpha(t)-1)} (L_4(s, C_{n-1}(s)) - L_4(s, C_{n-2}(s))),\\
\Phi_{R,n}(t) &= R_n(t) - R_{n-1}(t) = \frac{1}{\Gamma(\alpha(t))} \sum_{s=0}^{t-\alpha(t)} (t-\sigma(s))^{(\alpha(t)-1)} (L_5(s, R_{n-1}(s)) - L_5(s, R_{n-2}(s))),\\
\Phi_{V,n}(t) &= V_n(t) - V_{n-1}(t) = \frac{1}{\Gamma(\alpha(t))} \sum_{s=0}^{t-\alpha(t)} (t-\sigma(s))^{(\alpha(t)-1)} (L_6(s, V_{n-1}(s)) - L_6(s, V_{n-2}(s))).
\end{aligned} \tag{2.12}$$

In the above,

$$\begin{cases}
S_n(t) = \sum_{j=0}^{n} \Phi_{S,j}(t), \quad A_n(t) = \sum_{j=0}^{n} \Phi_{A,j}(t),\\
M_n(t) = \sum_{j=0}^{n} \Phi_{M,j}(t), \quad C_n(t) = \sum_{j=0}^{n} \Phi_{C,j}(t),\\
R_n(t) = \sum_{j=0}^{n} \Phi_{R,j}(t), \quad V_n(t) = \sum_{j=0}^{n} \Phi_{V,j}(t).
\end{cases} \tag{2.13}$$

Suppose

$$\begin{cases}
\Phi_{S,n-1}(t) = S_{n-1}(t) - S_{n-2}(t), \quad \Phi_{A,n-1}(t) = A_{n-1}(t) - A_{n-2}(t),\\
\Phi_{M,n-1}(t) = M_{n-1}(t) - M_{n-2}(t), \quad \Phi_{C,n-1}(t) = C_{n-1}(t) - C_{n-2}(t),\\
\Phi_{R,n-1}(t) = R_{n-1}(t) - R_{n-2}(t), \quad \Phi_{V,n-1}(t) = V_{n-1}(t) - V_{n-2}(t).
\end{cases} \tag{2.14}$$

Thus, we obtain

$$\begin{cases}
\|\Phi_{S,n}(t)\| < \frac{m_1}{\Gamma(\alpha(t))} \sum_{s=0}^{t-\alpha(t)} (t-\sigma(s))^{(\alpha(t)-1)} \|\Phi_{S,n-1}(s)\|,\\
\|\Phi_{A,n}(t)\| < \frac{m_2}{\Gamma(\alpha(t))} \sum_{s=0}^{t-\alpha(t)} (t-\sigma(s))^{(\alpha(t)-1)} \|\Phi_{A,n-1}(s)\|,\\
\|\Phi_{M,n}(t)\| < \frac{m_3}{\Gamma(\alpha(t))} \sum_{s=0}^{t-\alpha(t)} (t-\sigma(s))^{(\alpha(t)-1)} \|\Phi_{M,n-1}(s)\|,\\
\|\Phi_{C,n}(t)\| < \frac{m_4}{\Gamma(\alpha(t))} \sum_{s=0}^{t-\alpha(t)} (t-\sigma(s))^{(\alpha(t)-1)} \|\Phi_{C,n-1}(s)\|,\\
\|\Phi_{R,n}(t)\| < \frac{m_5}{\Gamma(\alpha(t))} \sum_{s=0}^{t-\alpha(t)} (t-\sigma(s))^{(\alpha(t)-1)} \|\Phi_{R,n-1}(s)\|,\\
\|\Phi_{V,n}(t)\| < \frac{m_6}{\Gamma(\alpha(t))} \sum_{s=0}^{t-\alpha(t)} (t-\sigma(s))^{(\alpha(t)-1)} \|\Phi_{V,n-1}(s)\|.
\end{cases} \tag{2.15}$$

Theorem 2.3.1

The solution of (2.1) exists for $t \in [0, \mathrm{T}]$, if $\frac{km_i}{\Gamma(\alpha(t))} < 1$, $i = 1, \ldots, 6$. Furthermore, the solution is unique if $\|\chi(t)\| \left(1 - \frac{km_i}{\Gamma(\alpha(t))}\right) > 0$ holds for $i = 1, \ldots, 6$.

Proof: Using the Lemma 2.3.1 and (2.15), we get

$$\begin{cases} \|\Phi_{S,n}(t)\| & < & \|S_0(t)\|(\frac{km_1}{\Gamma(\alpha(t))})^n, \\ \|\Phi_{A,n}(t)\| & < & \|A_0(t)\|(\frac{km_2}{\Gamma(\alpha(t))})^n, \\ \|\Phi_{M,n}(t)\| & < & \|M_0(t)\|(\frac{km_3}{\Gamma(\alpha(t))})^n, \\ \|\Phi_{C,n}(t)\| & < & \|C_0(t)\|(\frac{km_4}{\Gamma(\alpha(t))})^n, \\ \|\Phi_{R,n}(t)\| & < & \|R_0(t)\|(\frac{km_5}{\Gamma(\alpha(t))})^n, \\ \|\Phi_{V,n}(t)\| & < & \|V_0(t)\|(\frac{km_6}{\Gamma(\alpha(t))})^n. \end{cases} \tag{2.16}$$

As $n \to \infty$, we have $\|\Phi_{\cdot,n}(t)\| \longrightarrow 0$. This shows the existence of solutions to (2.1).

Suppose there are two solutions for the system (2.1) ($S(t)$, $A(t)$, $M(t)$, $C(t)$, $R(t)$, $V(t)$) and ($S(t)$, $A(t)$, $M(t)$, $C(t)$, $R(t)$, $V(t)$). Then,

$$\begin{cases} S(t) - S_2(t) & = & \frac{1}{\Gamma(\alpha(t))} \sum_{s=0}^{t-\alpha(t)} (t-\sigma(s))^{(\alpha(t)-1)}(L_1(s,S(s)) - L_1(s,S_2(s))), \\ A(t) - A_2(t) & = & \frac{1}{\Gamma(\alpha(t))} \sum_{s=0}^{t-\alpha(t)} (t-\sigma(s))^{(\alpha(t)-1)}(L_2(s,A(s)) - L_2(s,A_2(s))), \\ M(t) - M_2(t) & = & \frac{1}{\Gamma(\alpha(t))} \sum_{s=0}^{t-\alpha(t)} (t-\sigma(s))^{(\alpha(t)-1)}(L_3(s,M_(s)) - L_3(s,M_2(s))), \\ C(t) - C_2(t) & = & \frac{1}{\Gamma(\alpha(t))} \sum_{s=0}^{t-\alpha(t)} (t-\sigma(s))^{(\alpha(t)-1)}(L_4(s,C(s)) - L_4(s,C_2(s))), \\ R(t) - R_2(t) & = & \frac{1}{\Gamma(\alpha(t))} \sum_{s=0}^{t-\alpha(t)} (t-\sigma(s))^{(\alpha(t)-1)}(L_5(s,R(s)) - L_5(s,R_2(s))), \\ V(t) - V_2(t) & = & \frac{1}{\Gamma(\alpha(t))} \sum_{s=0}^{t-\alpha(t)} (t-\sigma(s))^{(\alpha(t)-1)}(L_6(s,V(s)) - L_6(s,V_2(s))). \end{cases} \tag{2.17}$$

From the above, we get

$$\begin{cases} \|S(t) - S_2(t)\| & = & \|\frac{1}{\Gamma(\alpha(t))} \sum_{s=0}^{t-\alpha(t)} (t-\sigma(s))^{(\alpha(t)-1)}(L_1(s,S(s)) - L_1(s,S_2(s)))\|, \\ \|A(t) - A_2(t)\| & = & \|\frac{1}{\Gamma(\alpha(t))} \sum_{s=0}^{t-\alpha(t)} (t-\sigma(s))^{(\alpha(t)-1)}(L_2(s,A(s)) - L_2(s,A_2(s)))\|, \\ \|M(t) - M_2(t)\| & = & \|\frac{1}{\Gamma(\alpha(t))} \sum_{s=0}^{t-\alpha(t)} (t-\sigma(s))^{(\alpha(t)-1)}(L_3(s,M_(s)) - L_3(s,M_2(s)))\|, \\ \|C(t) - C_2(t)\| & = & \|\frac{1}{\Gamma(\alpha(t))} \sum_{s=0}^{t-\alpha(t)} (t-\sigma(s))^{(\alpha(t)-1)}(L_4(s,C(s)) - L_4(s,C_2(s)))\|, \\ \|R(t) - R_2(t)\| & = & \|\frac{1}{\Gamma(\alpha(t))} \sum_{s=0}^{t-\alpha(t)} (t-\sigma(s))^{(\alpha(t)-1)}(L_5(s,R(s)) - L_5(s,R_2(s)))\|, \\ \|V(t) - V_2(t)\| & = & \|\frac{1}{\Gamma(\alpha(t))} \sum_{s=0}^{t-\alpha(t)} (t-\sigma(s))^{(\alpha(t)-1)}(L_6(s,V(s)) - L_6(s,V_2(s)))\|. \end{cases} \tag{2.18}$$

The above leads to,

$$\begin{cases} \|S(t)-S_2(t)\| &= \frac{1}{\Gamma(\alpha(t))}\sum_{s=0}^{t-\alpha(t)}(t-\sigma(s))^{(\alpha(t)-1)}\|(L_1(s,S(s))-L_1(s,S_2(s)))\|, \\ \|A(t)-A_2(t)\| &= \frac{1}{\Gamma(\alpha(t))}\sum_{s=0}^{t-\alpha(t)}(t-\sigma(s))^{(\alpha(t)-1)}\|(L_2(s,A(s))-L_2(s,A_2(s)))\|, \\ \|M(t)-M_2(t)\| &= \frac{1}{\Gamma(\alpha(t))}\sum_{s=0}^{t-\alpha(t)}(t-\sigma(s))^{(\alpha(t)-1)}\|(L_3(s,M_(s))-L_3(s,M_2(s)))\|, \\ \|C(t)-C_2(t)\| &= \frac{1}{\Gamma(\alpha(t))}\sum_{s=0}^{t-\alpha(t)}(t-\sigma(s))^{(\alpha(t)-1)}\|(L_4(s,C(s))-L_4(s,C_2(s)))\|, \\ \|R(t)-R_2(t)\| &= \frac{1}{\Gamma(\alpha(t))}\sum_{s=0}^{t-\alpha(t)}(t-\sigma(s))^{(\alpha(t)-1)}\|(L_5(s,R(s))-L_5(s,R_2(s)))\|, \\ \|V(t)-V_2(t)\| &= \frac{1}{\Gamma(\alpha(t))}\sum_{s=0}^{t-\alpha(t)}(t-\sigma(s))^{(\alpha(t)-1)}\|(L_6(s,V(s))-L_6(s,V_2(s)))\|. \end{cases}$$

Using Lemma 2.3.1, we have

$$\begin{cases} \|S(t)-S_2(t)\| &< \frac{m_1}{\Gamma(\alpha(t))}\sum_{s=0}^{t-\alpha(t)}(t-\sigma(s))^{(\alpha(t)-1)}\|S(t)-S_2(t)\|, \\ \|A(t)-A_2(t)\| &< \frac{m_2}{\Gamma(\alpha(t))}\sum_{s=0}^{t-\alpha(t)}(t-\sigma(s))^{(\alpha(t)-1)}\|A(t)-A_2(t)\|, \\ \|M(t)-M_2(t)\| &< \frac{m_3}{\Gamma(\alpha(t))}\sum_{s=0}^{t-\alpha(t)}(t-\sigma(s))^{(\alpha(t)-1)}\|M(t)-M_2(t)\|, \\ \|C(t)-C_2(t)\| &< \frac{m_4}{\Gamma(\alpha(t))}\sum_{s=0}^{t-\alpha(t)}(t-\sigma(s))^{(\alpha(t)-1)}\|C(t)-C_2(t)\|, \\ \|R(t)-R_2(t)\| &< \frac{m_5}{\Gamma(\alpha(t))}\sum_{s=0}^{t-\alpha(t)}(t-\sigma(s))^{(\alpha(t)-1)}\|R(t)-R_2(t)\|, \\ \|V(t)-V_2(t)\| &< \frac{m_6}{\Gamma(\alpha(t))}\sum_{s=0}^{t-\alpha(t)}(t-\sigma(s))^{(\alpha(t)-1)}\|V(t)-V_2(t)\|. \end{cases} \tag{2.19}$$

As a consequence, we get

$$\begin{cases} \|S(t)-S_2(t)\|(1-\frac{km_1}{\Gamma(\alpha(t))}) &< 0, \\ \|A(t)-A_2(t)\|(1-\frac{km_2}{\Gamma(\alpha(t))}) &< 0, \\ \|M(t)-M_2(t)\|(1-\frac{km_3}{\Gamma(\alpha(t))}) &< 0, \\ \|C(t)-C_2(t)\|(1-\frac{km_4}{\Gamma(\alpha(t))}) &< 0, \\ \|R(t)-R_2(t)\|(1-\frac{km_5}{\Gamma(\alpha(t))}) &< 0, \\ \|V(t)-V_2(t)\|(1-\frac{km_6}{\Gamma(\alpha(t))}) &< 0. \end{cases} \tag{2.20}$$

The result is proved. ■

2.3.2 Non-negativity and boundedness of the solutions

Now we address the positivity and boundedness of the solutions of model (2.1). The main result here establishes that $S(t)$, $A(t)$, $M(t)$, $C(t)$, $R(t)$, and $V(t)$ are all positive and bounded.

Remark 2.3.1

As $(1-d)$ represents the efficacy of the successful vaccine and δ is a non-negative parameter, then $\delta(1-d)$ is always non-negative.

Theorem 2.3.2

The solutions $S(t)$, $A(t)$, $M(t)$, $C(t)$, $R(t)$ and $V(t)$ of (2.1) are in $\mathbb{R}_+^6$ for all t.

Proof: Suppose a general discrete time variable-order model (2.1) is given as

$$\begin{cases} \Delta^{\alpha(t)} S(t)|_{S=0} &= \delta d(1-rC) + wV \geq 0, \\ \Delta^{\alpha(t)} A(t)|_{A=0} &= (\beta M + n\beta C)S \geq 0, \\ \Delta^{\alpha(t)} M(t)|_{M=0} &= \gamma A \geq 0, \\ \Delta^{\alpha(t)} C(t)|_{C=0} &= aM \geq 0, \\ \Delta^{\alpha(t)} R(t)|_{R=0} &= mM + eC \geq 0, \\ \Delta^{\alpha(t)} V(t)|_{V=0} &= \delta(1-d) + kS \geq 0. \end{cases} \tag{2.21}$$

Using Remark 2.3.1, we get that $S(t), A(t), M(t), C(t), R(t)$ and $V(t)$ are positive. ■

Theorem 2.3.3

Let $B(t)$ be the total population of (2.1). Then,

$$B(t) = (S(t), A(t), M(t), C(t), R(t), V(t)) \in \mathbb{R}_+^6$$

and

$$0 < S(t) + A(t) + M(t) + C(t) + R(t) + V(t) < \frac{\delta}{\mu_1}.$$

Proof: Here $B(t)$ represents the total population and it is given as $B = S + A + M + C + R + V$. Then the discrete Caputo variable-order difference of the total population is:

$$\Delta^{\alpha(t)} B(t) = \delta - \mu_1 B - \mu_2 M.$$

Then, we have

$$\Delta^{\alpha(t)} B(t) < \delta - \mu_1 B.$$

Using the Laplace transform, it is easy to obtain

$$s^{\alpha(s)} \tilde{B}(s) - s^{\alpha(s)-1} \tilde{B}(0) < \frac{\delta}{s} - \mu_1 \tilde{B}(s).$$

Then

$$\tilde{B}(s) < \frac{s^{\alpha(s)-1}}{(s^{\alpha(s)} + \mu_1)} \tilde{B}(0) + \frac{\delta s^{-1}}{(s^{\alpha(s)} + \mu_1)}.$$

Further, using the inverse Laplace transform, we have

$$B(s) < \frac{\delta}{\mu_1} (\mu_1 t^{\alpha(t)} E_{\alpha(t),\alpha(t+1)}(-\mu_1 t^{\alpha(t)})) + E_{\alpha(t),1}(-\mu_1 t^{\alpha(t)}) < \frac{\delta}{\mu_1}.$$

This completes the proof. ■

2.4 Stability analysis of the variable-order hepatitis-B virus model

The epidemiological model (2.1) is analyzed for equilibrium points: the disease-free state denoted by B_1 and endemic states denoted by B_2. These equilibrium points can be obtained by setting

$$\Delta^{\alpha(t)}S(t) = \Delta^{\alpha(t)}A(t) = \Delta^{\alpha(t)}M(t) = \Delta^{\alpha(t)}C(t) = \Delta^{\alpha(t)}R(t) = \Delta^{\alpha(t)}V(t) = 0.$$

Using the above conditions in (2.1), we get

$$\begin{cases} \delta d(1-rC) + wV - (\mu_1 + \beta M + n\beta C + k)S &= 0, \\ (\beta M + n\beta C)S - (\mu_1 + \gamma)A &= 0, \\ \gamma A + (\delta dr - \mu_1 - \mu_2 - a - m)M &= 0, \\ aM - eC - \mu_1 C &= 0, \\ mM + eC - \mu_1 R &= 0, \\ \delta(1-d) + kS - \mu_1 V - wV &= 0. \end{cases} \tag{2.22}$$

We obtain the disease free equilibrium point as given below:

$$B_1 = \left(\frac{\delta\left(d\mu_1 + w\right)}{\mu_1\left(\mu_1 + k + w\right)}, 0, 0, 0, 0, \frac{\delta\left(\mu_1 + k - \mu_1 d\right)}{\mu_1\left(\mu_1 + k + w\right)} \right).$$

Further, a unique endemic equilibrium is given as $B_2 = (S_2, A_2, M_2, C_2, R_2, V_2)$, where

$$\begin{aligned} S_2 &= \frac{\left(-\delta dr + \mu_1 + \mu_2 + a + m\right)\left(\mu_1 + \gamma\right)\left(e + \mu_1\right)}{\gamma\beta\left(e + \mu_1 + na\right)}; \\ V_2 &= \frac{\delta - \delta d + kS_2}{\mu_1 + w}; \\ C_2 &= \frac{(\delta d + wV_2 - (\mu_1 + k)S_2)a}{ar\delta d + S_2\beta(e + \mu_1 + na)}; \\ A_2 &= \frac{(\delta dr - \mu_1 - \mu_2 - a - m)(e + \mu_1)}{a\gamma} C_2; \\ M_2 &= \frac{e + \mu_1}{a} C_2; \\ R_2 &= \frac{m(e + \mu_1) + ea}{a\mu_1} C_2. \end{aligned} \tag{2.23}$$

To derive the expression for the reproduction number $\mathcal{R}_0$ of the system (2.1), we employ the method outlined in [133]. The necessary matrices are obtained as follows:

$$\mathcal{F} = \begin{pmatrix} 0 \\ \left(\beta M + n\beta C\right) S \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathcal{V} = \begin{pmatrix} -\delta d(1 - rC) + wV + Q_1 S \\ \left(\mu_1 + \gamma\right) A \\ -\gamma A - \delta dr M + Q_2 M \\ -aM + eC + \mu_1 C \\ -mM - eC + \mu_1 R \\ V\left(\mu_1 + w\right) - kS - C\left(1 - d\right) \end{pmatrix},$$

where $Q_1 = \mu_1 + \beta M + n\beta C + k$ and $Q_2 = \mu_1 + \mu_2 + a + m$. Now, for linearization, the Jacobian matrix of the upper matrix at the disease-free state is given by

$$\mathbf{F} = \begin{pmatrix} 0 & \beta S_0 & n\beta S_0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{V} = \begin{pmatrix} \mu_1 + \gamma & 0 & 0 & 0 \\ -\gamma & \mu_1 + \mu_2 + a + m - \delta dr & 0 & 0 \\ 0 & -a & e + \mu_1 & 0 \\ 0 & -m & -e & \mu_1 \end{pmatrix}.$$

So the spectral of the matrix FV^{-1} is the expression of and $\mathcal{R}_0$ written as follows:

$$\mathcal{R}_0 = \frac{\delta\beta\gamma\left(\mu_1 + e + na\right)\left(w + d\mu_1\right)}{\mu_1\left(\mu_1 + \gamma\right)\left(\mu_1 + e\right)\left(w + k + \mu_1\right)\left(\mu_1 + \mu_2 + a + m - r\delta d\right)}.$$

We now investigate the local stability.

Theorem 2.4.1

The disease-free equilibrium B_1 of the proposed discrete variable-order hepatitis B model (2.1) is locally asymptotically stable if $\mathcal{R}_0 < 1$.

Proof: The Jacobian of model at B_1 given by

$$\mathcal{J}\left(B_1\right) = \begin{pmatrix} -\left(\mu_1 + k\right) & 0 & -\beta S_0 & -rd\delta - n\beta S_0 & 0 & w \\ 0 & -\left(\mu_1 + \gamma\right) & \beta S_0 & n\beta S_0 & 0 & 0 \\ 0 & \gamma & \delta dr - \left(\mu_1 + \mu_2 + a + m\right) & 0 & 0 & 0 \\ 0 & 0 & a & -\left(e + \mu_1\right) & 0 & 0 \\ 0 & 0 & m & e & -\mu_1 & 0 \\ k & 0 & 0 & 0 & 0 & -\left(\mu_1 + w\right) \end{pmatrix}.$$

The characteristic equation is

$$(\lambda + \mu_1)(\lambda^2 + a_1\lambda + a_2)(\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3) = 0,$$

where

$$a_1 = 2\mu_1 + k + w; \quad a_2 = (\mu_1 + k)(\mu_1 + w) + kw; \quad b_1 = a + e + \gamma + m + 3\mu_1 - \mu_2 - \delta dr;$$

$$b_2 = -(\gamma\beta S_0 + (\mu_1 + \gamma)(\delta dr - Q_2) + (\delta dr - Q_2)(e + \mu_1) - (\mu_1 + \gamma)(e + \mu_1));$$

$$b_3 = -\gamma\beta S_0(na + e + \mu_1) + (\mu_1 + \gamma)(\delta dr - Q_2)(e + \mu_1),$$

here $Q_2 = (\mu_1 + \mu_2 + a + m)$. By the Routh–Hurwitz criterion, $a_1 > 0$, $a_2 > 0$, $b_1 > 0$, $b_3 > 0$, $b_1 b_2 > b_3$, and if $\mathcal{R}_0 < 1$, then B_1 is asymptotically stable. ■

2.5 Numerical results

This section provides a discussion on Caputo variable-order hepatitis B virus model (2.1). We utilize the numerical values presented in Table 2.1 for our analysis. The considered hepatitis B virus model with a variable-order in discrete time is solved numerically utilizing the methodology outlined in [23] and [49].

The initial conditions for the system (2.1) are set as follows: $S(0) = 0.8565$, $A(0) = 0.01223$, $M(0) = 0.10675$, $C(0) = 0.00489$, $R(0) = 0.00734$, and $V(0) = 0.01224$, as in [133]. We consider different values for the variable-order, denoted as $\alpha(t)$:

$$\alpha_1 = \begin{pmatrix} 0.95 & 0.9 & 0.85 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0.93 & 0.84 & 0.88 \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} 0.78 & 0.8 & 0.82 \end{pmatrix}.$$

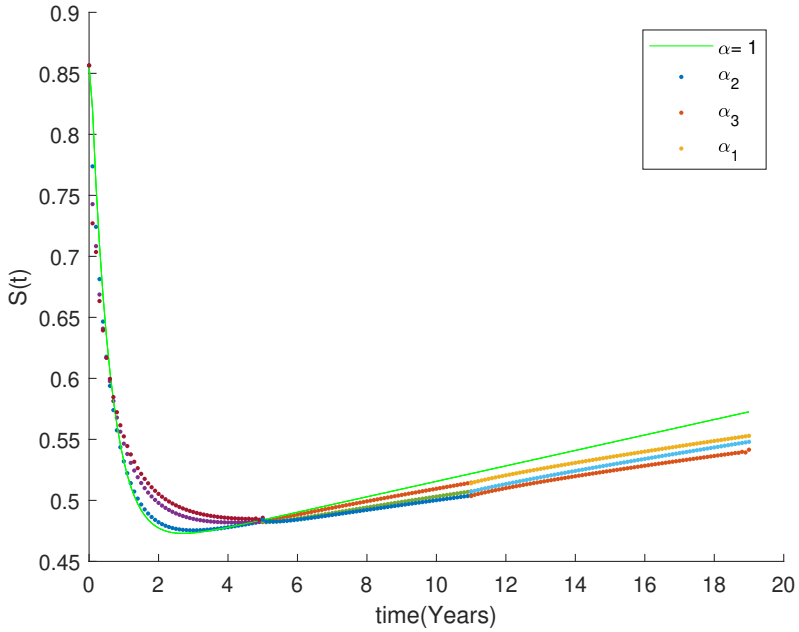


Figure 2.1: The graph illustrates the dynamics of the susceptible population $(S(t))$ with respect to the variable-order parameter $\alpha(k)$.

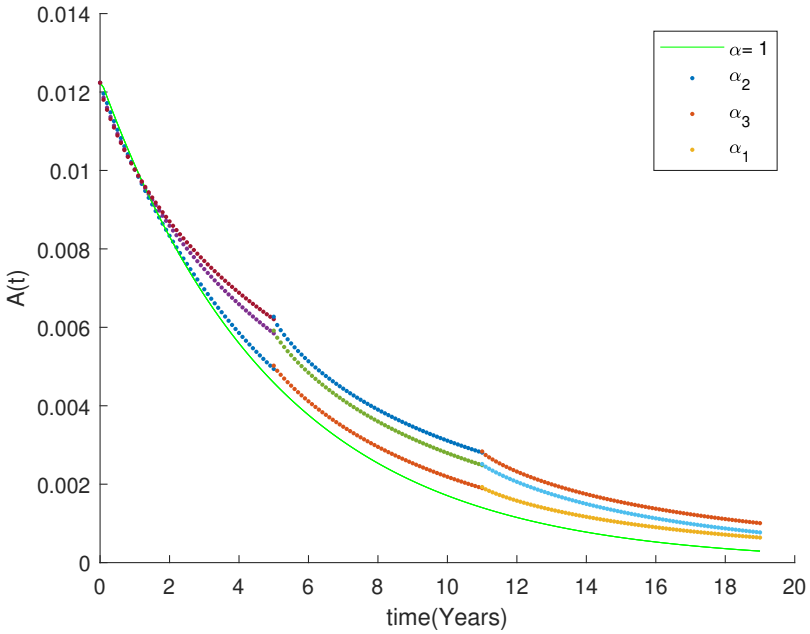


Figure 2.2: The graph illustrates the dynamics of the acute individual class $(A(t))$ with respect to the variable-order parameter $\alpha(k)$.

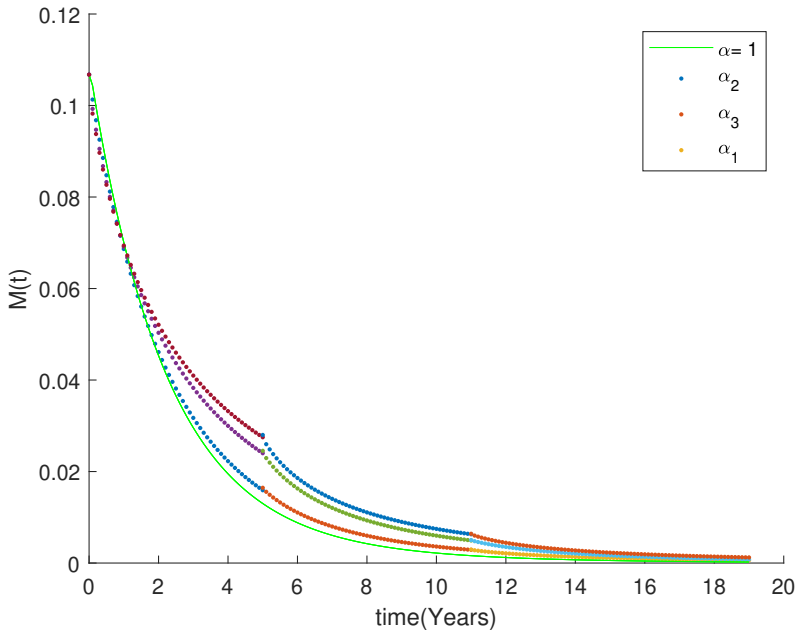


Figure 2.3: The graph illustrates the dynamics of the immune individuals class ($M(t)$) with respect to the variable-order parameter $\alpha(k)$.

The results are presented in Figs. 2.1–2.6, showcasing the dynamic behaviors of the susceptible, latent, acute, carrier, recovered, and vaccinated population groups.
Fig. 2.1 illustrates that as the value of α increases, the number of susceptible individuals also increases, indicating a proportional relationship. Conversely, Figs. 2.2 and 2.3 show an inverse proportionality between the variable-order and the number of acute and immune individuals, respectively. Notably, variable and integer order cases tend to converge and stabilize over time. In Fig. 2.4, for variable-order values, the peak in carrier individuals is observed in the year seven, whereas for $\alpha = 1$, the peak occurs in earlier years. Fig. 2.5 shows an increase in the value of α after two years. Fig. 2.6 reveals the dynamics of the vaccinated population, indicating a direct relationship between the variable-order parameter $\alpha(k)$ and the vaccinated population. The computational results demonstrate that, by incorporating discrete time and variable-orders, the hepatitis B virus model is capable of capturing richer and more complex dynamics. Moreover, it offers valuable insights and enhances understanding of the model. Various morphological changes in the unknown variables, due to the influence of discrete time and the variable-order, are visible and comparableas illustrated in Fig. 2.1–2.6.

2.6 Conclusion

This Chapter (2) we have established the existence and uniqueness of a solution for a novel discrete variable-order model of the hepatitis-B virus. Initially, we ensured the positivity and boundedness of model's solutions. To further investigate the local stability of the proposed system, we computed the basic reproduction number $\mathcal{R}_0$. If $\mathcal{R}_0 < 1$, we have proven that the disease-free equilibrium B_1 is locally asymptotically stable. To illustrate the dynamics of the model under various $\alpha(t)$ values, we have used Matlab to generate dynamic graphs for the proposed discrete variable-order hepatitis-B virus model. Our results show that, due to the incorporation of discrete time and variable-orders, the new model is able to cover more rich

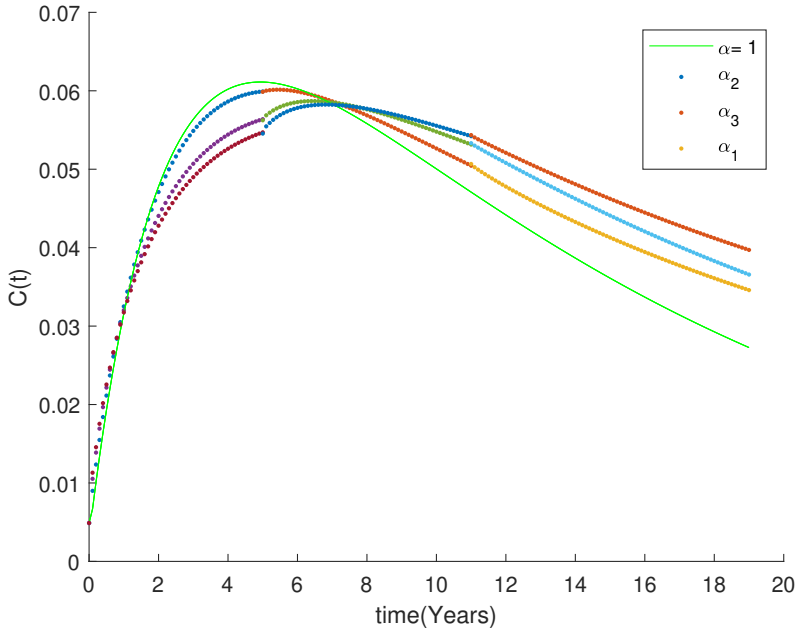


Figure 2.4: The graph illustrates the dynamics of the carrier (chronic) individual class ($C(t)$) with respect to the variable-order parameter $\alpha(k)$.

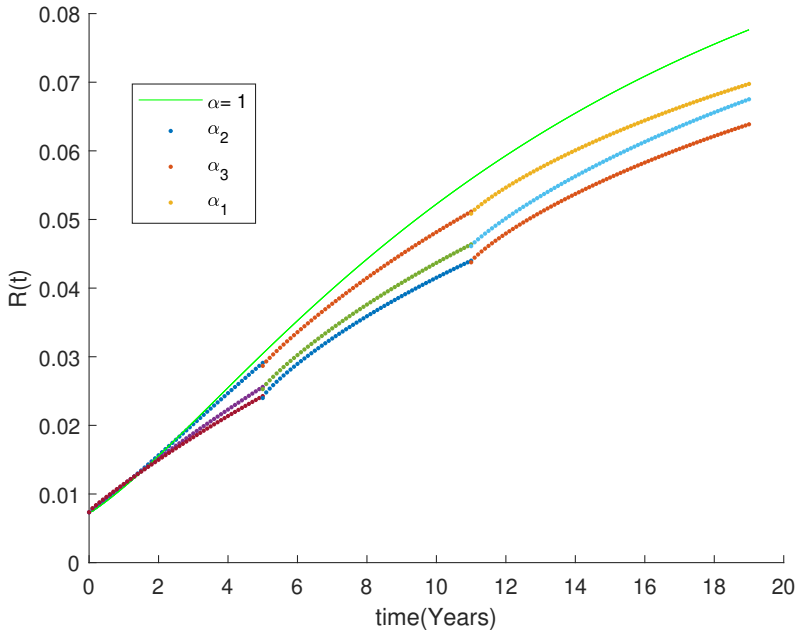


Figure 2.5: The graph illustrates the dynamics of the recovered individuals class ($R(t)$) with respect to the variable-order parameter $\alpha(k)$.

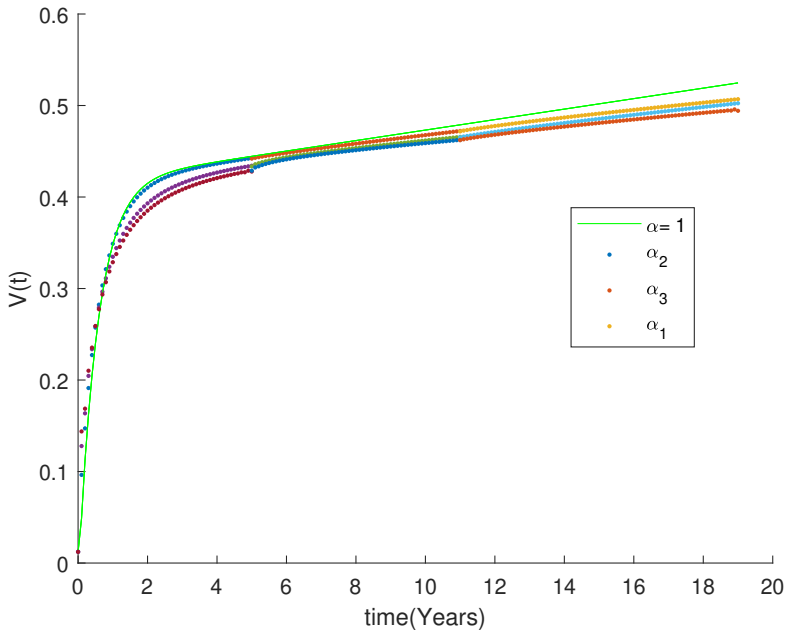


Figure 2.6: The graph illustrates the dynamics of the vaccination class ($V(t)$) with respect to the variable-order parameter $\alpha(k)$.

and more complex dynamics when compared to models available in the literature, providing valuable insights.

Chapter 3

The stability of solution of variable-order fractional optimal control COVID-19 epidemic in discrete time

The work presented in this Chapter (3) introduces a discrete-time fractional variable order over a SEIQR model, incorporated for COVID-19. Initially, we establish the well-possedness of solution. Further, the disease-free and the endemic equilibrium points are determined. Moreover, the local asymptotic stability of the model is analyzed. We develop a novel discrete fractional optimal control problem tailored for COVID-19, utilizing a discrete mathematical model featuring a variable order fractional derivative. Finally, we validate the reliability of these analytical findings through numerical simulations and offer insights from a biological perspective. This work is attributed to the [29].

3.1 Introduction

COVID-19, first identified in Wuhan, the capital of Hubei Province, China, in 2019 [99], is an acute respiratory disease. Since 2002, severe acute respiratory syndrome (SARS-CoV) and Middle East respiratory syndrome (MERS-CoV) have been responsible for outbreaks in humans, despite primarily infecting animals [138]. As per the International Committee on Taxonomy of Viruses (ICTV), coronaviruses are classified within the sub-family Coronavirinae, which is a part of the family Coronaviridae and the order Nidovirales. The sub-family Coronavirinae encompasses four biological groups: α, β, γ, and δ-coronaviruses [129, 130]. Studies indicate that all coronaviruses have their origins in animals [129, 116]. Moreover, recent research findings [37] suggest that although the precise origin of SARS-CoV-2 cannot be definitively determined, the potential for laboratory origin cannot be easily ruled out. α-coronaviruses, like HumanCoV-NL63 and HumanCoV-229E, usually result in mild infections in humans. However, SADS-CoV (Swine acute diarrhea syndrome coronavirus), which utilizes swine as an intermediate carrier, does not induce infectious symptoms in humans. While both HCoV-OC43 and HCoV-HKU1 belong to the β-coronavirus category, they typically pose no serious threat to humans [113]. However, the perception of highly pathogenic coronaviruses changed significantly following the outbreaks of SARS-CoV (Severe Acute Respiratory Syndrome coronavirus) in 2003 and MERS-CoV (Middle East Respiratory Syndrome coronavirus) in 2012 [113].

Mathematical models that depict infectious diseases are pivotal in both theoretical understanding and practical application [32, 137]. Developing and scrutinizing models of this nature aids in comprehending the mechanisms of transmission and disease characteristics. This understanding facilitates the formulation of effective strategies for prediction, prevention, and control, ultimately safeguarding population health. To date, numerous mathematical models for infectious diseases, formulated using differential equations, have been constructed and analyzed to study virus spread [77, 142]. Recently, mathematical models for the COVID-19 epidemic have attracted considerable interest from mathematicians, biologists, epidemiologists, pharmacists, and chemists, producing noteworthy and vital outcomes [142, 139]. Furthermore, these investigations have extended to encompass fractional-order models, as evidenced by studies like [3, 134].
Recent research has extensively explored optimal control strategies for managing COVID-19 and its co-infections [8, 117]. Fractional variable-order optimal control problems (V-FOCPs) have been formulated using various definitions of fractional derivatives, such as Riemann-Liouville and Caputo derivatives, with illustrative examples provided in [24, 111]. Moreover, the discrete-time fractional optimal control model has been investigated in studies like [39, 118]. Additionally, research has delved into optimal control problems for variable fractional systems [53].

3.1.1 Model formulation

We shall present a detailed proposal for the discrete-time variable-order fractional COVID-19 system. Motivated by [40, 132], we introduce the following notation: Let $S(t)$, $E(t)$, $I(t)$, $Q(t)$, and $R(t)$ denote the susceptible, exposed, infected, isolated, and removed populations at time $t \geq 0$, respectively. The total population is represented by $N(t) = S(t)+E(t)+I(t)+Q(t)+R(t)$. All parameters $(\chi, \beta, \nu, \rho, \lambda, \delta, \mu, \gamma, \psi, \phi, \varphi_1, \tau, \varphi_2)$ are positive real numbers which are provided in Table 3.1. Here χ represent the recruitment rate. The proposed model is presented as follows:

$$\begin{aligned}
\Delta^{\alpha(t)} S(t) &= \chi - \beta\nu \frac{SI}{N+\rho I} + \lambda R(t) - (\delta+\mu)S(t), \\
\Delta^{\alpha(t)} E(t) &= \beta\nu \frac{SI}{N+\rho I} - (\gamma+\mu)E(t), \\
\Delta^{\alpha(t)} I(t) &= \gamma E(t) - (\psi+\phi+\varphi_1+\mu)I(t), \\
\Delta^{\alpha(t)} Q(t) &= \psi I(t) - (\tau+\varphi_2+\mu)Q(t), \\
\Delta^{\alpha(t)} R(t) &= \delta S(t) + \phi I(t) + \tau Q(t) - (\lambda+\mu)R(t),
\end{aligned} \tag{3.1}$$

with the initial conditions :

$$S(0) = S_0, E(0) = E_0, I(0) = I_0, Q(0) = Q_0, R(0) = R_0.$$

Here, the delta variable-order fractional difference of model (3.1) is given in sense of Caputo where $\alpha(t) \in (0, 1)$.
This paper is organized as follows. In Section 3.2, we provide definitions of variable-order fractional calculus in discrete-time along with some important auxiliaries related to VOFDD. Section 3.3 discusses the existence and uniqueness conditions of solutions and presents stability theorems for equilibrium points. Optimal control analysis is covered in Section 3.4. Section 3.5 outlines the used numerical scheme. Section 3.6 presents numerical simulations and results. Finally, Section 3.7 concludes our contribution.

Table 3.1: Description of the model parameters

Parameters	Description	Value	Reference
χ	Recruitment rate	*assumed*	$--$
ρ	Saturation factor	*assumed*	$--$
ν	Contact rate	9.0631	[132]
β	The transmission probabilities	0.2761	[132]
δ	Self protection rate	0.0439	[132]
λ	Transmission rate from temporarily removed to susceptible population	0.0028	[132]
γ	Rate of progression from exposed group to the infected group	0.1736	[132]
ψ	Isolation rate	0.516	[132]
φ_1	Death rate in infected group caused by COVID-19	0.018	[132]
φ_2	Death rate in isolated group caused by COVID-19	$3.7559e-5$	[132]
τ	Recovery rate	0.0534	[132]
ϕ	Self recovery rate	$6.1462e-6$	[132]
μ	Natural death rate	$3.2811e-5$	[132]

3.2 Preliminaries

In this context, we introduce certain definitions and notations referenced from the papers [7, 58]. Let $\mathbb{N}_a$ denotes the set $\{a, a+1, a+2, \ldots\}$ and $\mathbb{N}_a^T$ represents the set $\{a, a+1, a+2, \ldots, T\}$.

Definition 3.2.1

Let $\alpha(t) > 0$ and $\sigma(s) = s+1$. For $u(t)$ defined on $\mathbb{N}_a$, the delta variable-order fractional sum of order $\alpha(t)$ is defined by

$$\Delta_a^{-\alpha(t)} u(t) = \frac{1}{\Gamma(\alpha(t))} \sum_{s=a}^{t-\alpha(t)} (t-\sigma(s))^{(\alpha(t)-1)} u(s), \tag{3.2}$$

where $t^{(\alpha(t))}$ is the discrete factorial functional given by $t^{(\alpha(t))} = \dfrac{\Gamma(t+1)}{\Gamma(t-\alpha(t)+1)}$.

Definition 3.2.2

For $u(t)$ defined on defined on $\mathbb{N}_a, \alpha(t) > 0, \alpha \notin \mathbb{N}$, the delta Caputo variable-order fractional difference is defined by

$${}^{C}\Delta_a^{\alpha(t)} u(t) = \Delta_a^{-(m-\alpha(t))} \Delta^m u(t) = \frac{1}{\Gamma(m-\alpha(t))} \sum_{s=a}^{t-(m-\alpha(t))} (t-\sigma(s))^{(m-\alpha(t)-1)} \Delta^m u(s), \tag{3.3}$$

where $t \in \mathbb{N}_{a+m-\alpha(t)}, m = [\alpha(t)] + 1$. Note that, the forward difference operator is defined by $\Delta u(t) = u(t+1) - u(t)$.

Theorem 3.2.1

[13] Let $s \in \mathbb{N}_{a+1}$, then the following hold

$$\sum_{k=a+1}^{s} (s-k+1)^{\alpha(s)-1} = \frac{(s-a)^{\alpha(s)}}{\alpha(s)}. \tag{3.4}$$

Theorem 3.2.2

[50, 91] Consider the following fractional variable-order discrete system

$$\Delta^{\alpha(t)} x = f(x), \quad x(0) = x_0, \tag{3.5}$$

with $x \in \mathbb{R}^n, \underline{\alpha} = \inf \alpha(t)$ and $\overline{\alpha} = \sup \alpha(t)$ where $0 < \underline{\alpha} < \alpha(t) < \overline{\alpha} < 1$. The equilibrium points of the system (3.5) are solutions to the equation $f(x) = 0$.
An equilibrium is locally asymptotically stable if all the eigenvalues λ_i $(i = 1, 2, \ldots, n)$ of the Jacobian matrix $J = \Delta f$ evaluated at the equilibrium satisfy

$$|\arg(\lambda_i)| < \frac{\pi}{2}\underline{\alpha}. \tag{3.6}$$

On the other hand, if $|\arg(\lambda_i)| > \frac{\pi}{2}\overline{\alpha}$, then the equilibrium point is unstable.

Theorem 3.2.3

[10] Consider the polynomial equation

$$p(\lambda) = \lambda^2 + a_1\lambda + a_2.$$

1. *For $n = 1$, the condition for stability is $a_1 > 0$.*
2. *For $n = 2$, the condition for stability either Routh-Hurwitz conditions [122] $(a_1 > 0, a_2 > 0)$ or $a_1 < 0$, $4a_2 > a_1^2$, $\tan^{-1}(4a_2 - a_1^2) > \frac{\pi}{2}\overline{\alpha}$.*

Definition 3.2.3

[48] Given a system of characteristic equations in the form of n-order polynomials as follows

$$f(z) = d_1 z^n + d_1 z^{n-1} + ... + d_{n-1} z + d_n. \tag{3.7}$$

If all the real parts of equation from the root are negative, then

$$\frac{d_1}{d_0} > 0, \frac{d_2}{d_0} > 0, \cdots, \frac{d_n}{d_0} > 0. \tag{3.8}$$

Suppose d_k are real numbers for $k = 0, 1, 2, \cdots, 2n-1$ and d_k are positive numbers. The Hurwitz matrix for equation (3.7) is defined as a square matrix of size $n \times n$ as follows :

$$H_n = \begin{pmatrix} d_1 & d_3 & d_5 & d_7 & \cdots & d_{2n-1} \\ d_0 & d_2 & d_4 & d_6 & \cdots & d_{2n-2} \\ 0 & d_1 & d_3 & d_5 & \cdots & d_{2n-3} \\ 0 & d_0 & d_2 & d_4 & \cdots & d_{2n-4} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & d_n \end{pmatrix}, \tag{3.9}$$

where $d_k = 0$ for $k < 0$ or $k > n$. Therefore, the matrix element index is greater than n, or the negative index must be replaced by zero. The k-level Hurwitz determinant, denoted by $detH_k$; $k = 1, 2, \cdots, n$, formed from the Hurwitz matrix (4.35), is defined as follows:

$$detH_n = \begin{vmatrix} d_1 & d_3 & d_5 & d_7 & \cdots & d_2n-1 \\ d_0 & d_2 & d_4 d_6 & & \cdots & d_2n-2 \\ 0 & d_1 & d_3 & d_5 & \cdots & d_2n-3 \\ 0 & d_0 & d_2 & d_4 & \cdots & d_2n-4 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & d_n \end{vmatrix}$$

Theorem 3.1

[48] The polynomial root (3.7) has a real part of its root is negative if and only if the inequality (3.2.3) is fulfilled and

$$detH_1 > 0;\ detH_2 > 0;\ detH_3 > 0;\ \cdots;\ detH_n > 0. \tag{3.10}$$

Thus, the equilibrium point $\bar{z}$ is stable if and only if $detH_j > 0$ for each $j = 1, 2, \cdots, n$.

3.3 Properties of Solution

3.3.1 Non-negativity and Boundedness of the Solutions

In this subsection, we discuss some properties related to the non negativity and boundedness of the solution of model (3.1). To this aim, we show the following result.

Theorem 3.2

All solutions of model (3.1) are non negative for any initial value $(S(0), E(0), I(0), Q(0), R(0)) \in (0,0,0,0,0) \cup \mathbb{R}^5_+$ *and the feasible region of model (3.1) is defined by* $N = \{(S(t), E(t), I(t), Q(t), R(t)) \in \mathbb{R}^5_+;\ 0 < S(t) + E(t) + I(t) + Q(t) + R(t) < \frac{\chi}{\mu}\}$.

Proof: Suppose a general fractional variable order discrete time model of system (3.1) as

$$\begin{aligned}
\Delta^{\alpha(t)} S(t)|_{S=0} &= \chi + \lambda R(t), \\
\Delta^{\alpha(t)} E(t)|_{E=0} &= \beta\nu \frac{SI}{N + \rho I}, \\
\Delta^{\alpha(t)} I(t)|_{I=0} &= \gamma E(t), \\
\Delta^{\alpha(t)} Q(t)|_{Q=0} &= \psi I(t), \\
\Delta^{\alpha(t)} R(t)|_{R=0} &= \delta S(t) + \phi I(t) + \tau Q(t).
\end{aligned}$$

From the above results, it easy to deduce that the solution $S(t), E(t), I(t), Q(t)$ and $R(t)$ are positive. Next, we have to show that the boundedness of the solution of model (3.1). We have,

$$\Delta^{\alpha(t)} N(t) = \chi - \mu N(t) - \varphi_1 I(t) - \varphi_2 Q(t) < \chi - \mu N(t).$$

According on the fractional order comparison Theorem in [71], we get,

$$N(t) < \frac{\chi}{\mu} + \left(N(0) - \frac{\chi}{\mu}\right) E_{\alpha(t),1}(-\mu t^{\alpha(t)}) < \frac{\chi}{\mu},$$

where $N(0) < \frac{\chi}{\mu}$. ■

3.3.2 Equilibrium Points and Basic Reproduction Number

First to discover equilibria of the model (3.1) where $X = (S, E, I, Q, R)^T$, we set

$$\Delta^{\alpha(t)} X(t) = 0. \tag{3.11}$$

We get the following algebraic system :

$$\begin{aligned}
\chi - \beta\nu \frac{SI}{N + \rho I} + \lambda R(t) - (\delta + \mu) S(t) &= 0, \\
\beta\nu \frac{SI}{N + \rho I} - (\gamma + \mu) E(t) &= 0, \\
\gamma E(t) - (\psi + \phi + \varphi_1 + \mu) I(t) &= 0, \\
\psi I(t) - (\tau + \varphi_2 + \mu) Q(t) &= 0, \\
\delta S(t) + \phi I(t) + \tau Q(t) - (\lambda + \mu) R(t) &= 0.
\end{aligned} \tag{3.12}$$

Using some algebraic calculations, we find two solutions of system (3.12). We have disease-free equilibrium point noted by $P_0 = (S_0, 0, 0, 0, R_0)$, and an a endemic equilibrium point denoted as $P_* = (S_*, E_*, I_*, Q_*, R_*)$, here

$$S_0 = \frac{\chi(\lambda + \mu)}{\mu(\lambda + \delta + \mu)}; \qquad R_0 = \frac{\chi\delta}{\mu(\lambda + \delta + \mu)}.$$

More detailed discussion about an endemic equilibrium point are provided later. Now we identify the basic reproduction number of the model (3.1) denoted by R_0 using the spectral radius of the next generation matrix as in [82].
Let $Y = (E, I, Q)^T$, we have

$$\Delta^{\alpha(t)} Y = \mathcal{F}(Y) - \mathcal{V}(Y)$$

where

$$\mathcal{F} = \begin{pmatrix} \left(\beta\nu \dfrac{SI}{N + \rho I}\right) \\ 0 \\ 0 \end{pmatrix}, \quad \mathcal{V} = \begin{pmatrix} (\gamma + \mu) E \\ -\gamma E + (\psi + \phi + \varphi_1 + \mu) I \\ -\psi I + (\tau + \varphi_2 + \mu) Q \end{pmatrix}.$$

The Jacobi matrix of the above at disease free stateP_0 is given by,

$$\mathbf{F} = \begin{pmatrix} 0 & \dfrac{\beta\nu S_0}{N} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{V} = \begin{pmatrix} \gamma + \mu & 0 & 0 \\ -\gamma & \psi + \phi + \varphi_1 + \mu & 0 \\ 0 & -\psi & \tau + \varphi_2 + \mu \end{pmatrix}.$$

Hence, the spectral radius is denote by ς,

$$\mathcal{R}_0 = \varsigma(FV^{-1}) = \frac{\beta\nu\gamma S_0}{N(\gamma + \mu)(\psi + \phi + \varphi_1 + \mu)}.$$

Theorem 3.3

The proposed variable-order fractional model (3.1) has a unique disease endemic equilibrium point $P_* = (S_*, E_*, I_*, Q_*, R_*)$ if and only if $\mathcal{R}_0 > 1$.

Proof: We are able to obtain the endemic equilibrium $P_* = (S_*, E_*, I_*, Q_*, R_*)$, and it is written as

$$I_* = q_1 E_*; \quad IQ_* = q_2 I_*; \quad R_* = \frac{(\delta + \mu) S_* + (\gamma + \mu) E_* - \chi}{\lambda}; \quad S_* = \frac{\chi(\lambda + \mu) - (B_1 - B_2) E_*}{\mu(\lambda + \delta + \mu)}, \tag{3.13}$$

where

$$q_1 = \frac{\gamma}{\psi + \phi + \varphi_1 + \mu}; \quad q_2 = \frac{\psi}{\tau + \varphi_2 + \mu}; \quad B_1 = (\lambda + \mu)(\gamma + \mu); \quad B_2 = \lambda\phi q_1 + \lambda\tau q_1 q_2. \tag{3.14}$$

We can obtain that

$$B_1 - B_2 = \frac{(\lambda + \mu)(\gamma + \mu)(\psi + \phi + \varphi_1 + \mu)(\tau + \varphi_2 + \mu) - \lambda\phi\gamma(\tau + \varphi_2 + \mu) - \lambda\tau\gamma\psi}{(\psi + \phi + \varphi_1 + \mu)(\tau + \varphi_2 + \mu)} > 0.$$

Further, from the above equations, we have

$$\beta\nu \frac{(\chi(\lambda + \mu) + (B_2 - B_1) x) q_1 E_*}{\mu(\lambda + \delta + \mu)(N + \rho q_1 E_*)} = (\gamma + \mu) E_*.$$

Define

$$g(x) = \beta\nu \frac{(\chi(\lambda + \mu) + (B_2 - B_1) x) q_1 x}{\mu(\lambda + \delta + \mu)(N + \rho q_1 x)} - (\gamma + \mu) x, \tag{3.15}$$

with $g(x) = 0$, at $x \in \left[0, \dfrac{\chi(\lambda + \mu)}{B_1 - B_2}\right]$ where

$$\beta\nu \frac{(\chi(\lambda + \mu) + (B_2 - B_1) x) q_1 x}{\mu(\lambda + \delta + \mu)(N + \rho q_1 x)} - (\gamma + \mu) x = 0,$$

and $g(0) = 0$, $g\left(\frac{\chi(\lambda+\mu)}{B_1-B_2}\right) = \frac{\chi(\lambda+\mu)(\gamma+\mu)}{B_1-B_2} < 0$. Now,

$$g'(x) = \frac{-\beta\nu(B_2-B_1)q_1x}{\mu(\lambda+\delta+\mu)(N+\rho q_1 x)} + q_1\beta\nu\frac{SN}{(N+\rho q_1 x)^2} - (\gamma+\mu),$$
$$g'(0) = q_1\beta\nu\frac{S_*}{N} - (\gamma+\mu) = (\gamma+\mu)(\mathcal{R}_0 - 1).$$

Here

$$\mathcal{R}_0 = \frac{\beta\nu\gamma S_0}{N(\gamma+\mu)(\psi+\phi+\varphi_1+\mu)}.$$

Since $g(x)$ is a continuous differentiable function $x \in \left[0, \frac{\chi(\lambda+\mu)}{B_1-B_2}\right]$. That $g(x) = 0$ has a positive solution E_*. If $g(x) > 0$ ($\mathcal{R}_0 > 1$) is proved. ■

3.3.3 Existence and Uniqueness (E&U) of the Solution

In this subsection, we prove the existence and uniqueness of the solutions of our problem (3.1). The kernels H_1, H_2, H_3, H_3, H_4 and H_5 are defined by

$$\begin{aligned}
H_1(t,S(t)) &= \chi - \beta\nu\frac{SI}{N+\rho I} + \lambda R(t) - (\delta+\mu)S(t),\\
H_2(t,E(t)) &= \beta\nu\frac{SI}{N+\rho I} - (\gamma+\mu)E(t),\\
H_3(t,I(t)) &= \gamma E(t) - (\psi+\phi+\varphi_1+\mu)I(t),\\
H_4(t,Q(t)) &= \psi I(t) - (\tau+\varphi_2+\mu)Q(t),\\
H_5(t,R(t)) &= \delta S(t) + \phi I(t) + \tau Q(t) - (\lambda+\mu)R(t).
\end{aligned}$$

Theorem 3.4

The Kernels H_1, H_2, H_3, H_4 and H_5 have the Lipschitz condition.

Proof: Depending on the fractional discrete variable-order calculus properties, a solution of (3.1) is given by

$$\begin{aligned}
S(t) &= S(0) + \Delta^{-\alpha(t)}(\chi - \beta\nu\frac{SI}{N+\rho I} + \lambda R(t) - (\delta+\mu)S(t)),\\
E(t) &= E(0) + \Delta^{-\alpha(t)}(\beta\nu\frac{SI}{N+\rho I} - (\gamma+\mu)E(t)),\\
I(t) &= I(0) + \Delta^{-\alpha(t)}(\gamma E(t) - (\psi+\phi+\varphi_1+\mu)I(t)),\\
Q(t) &= Q(0) + \Delta^{-\alpha(t)}(\psi I(t) - (\tau+\varphi_2+\mu)Q(t)),\\
R(t) &= R(0) + \Delta^{-\alpha(t)}(\delta S(t) + \phi I(t) + \tau Q(t) - (\lambda+\mu)R(t)).
\end{aligned} \tag{3.16}$$

By definition

$$\begin{aligned}
S(t) &= S(0) + \frac{1}{\Gamma(\alpha(t))}\sum_{s=0}^{t-\alpha(t)}(t-\sigma(s))^{(\alpha(t)-1)}(\chi - \beta\nu\frac{SI}{N+\rho I} + \lambda R(t) - (\delta+\mu)S(t)),\\
E(t) &= E(0) + \frac{1}{\Gamma(\alpha(t))}\sum_{s=0}^{t-\alpha(t)}(t-\sigma(s))^{(\alpha(t)-1)}(\beta\nu\frac{SI}{N+\rho I} - (\gamma+\mu)E(t)),
\end{aligned}$$

$$
\begin{aligned}
I(t) &= I(0) + \frac{1}{\Gamma(\alpha(t))} \sum_{s=0}^{t-\alpha(t)} (t-\sigma(s))^{(\alpha(t)-1)}(\gamma E(t) - (\psi+\phi+\varphi_1+\mu)I(t)), \\
Q(t) &= Q(0) + \frac{1}{\Gamma(\alpha(t))} \sum_{s=0}^{t-\alpha(t)} (t-\sigma(s))^{(\alpha(t)-1)}(\psi I(t) - (\tau+\varphi_2+\mu)Q(t)), \\
R(t) &= R(0) + \frac{1}{\Gamma(\alpha(t))} \sum_{s=0}^{t-\alpha(t)} (t-\sigma(s))^{(\alpha(t)-1)}(\delta S(t) + \phi I(t) + \tau Q(t) - (\lambda+\mu)R(t)).
\end{aligned} \tag{3.17}
$$

We consider the two functions $S(t)$ and $S^*(t)$, we have

$$
\|H_1(t,S(t)) - H_1(t,S^*(t))\| = \left\|(\beta\nu\frac{I}{N+\rho I} + (\delta+\mu))(S(t)-S^*(t))\right\|. \tag{3.18}
$$

Suppose that

$$
p_1 = \|\beta\nu + (\delta+\mu)\|.
$$

If $\dfrac{I}{N+\rho I} < 1$, we get

$$
\|H_1(t,S(t)) - H_1(t,S^*(t))\| < p_1\|S(t)-S^*(t)\|.
$$

We use similar arguments for other functions, we obtain

$$
\begin{aligned}
\|H_2(t,E(t)) - H_2(t,E^*(t))\| &< p_2\|E(t)-E^*(t)\|, \\
\|H_3(t,I(t)) - H_3(t,I^*(t))\| &< p_3\|I(t)-I^*(t)\|, \\
\|H_4(t,Q(t)) - H_4(t,Q^*(t))\| &< p_4\|Q(t)-Q^*(t)\|, \\
\|H_5(t,R(t)) - H_5(t,R^*(t))\| &< p_5\|R(t)-R^*(t)\|.
\end{aligned} \tag{3.19}
$$

The respective Lipschitz constants to the functions H_1, H_2, H_3, H_4 and H_5 are p_1, p_2, p_3, p_4 and p_5, . Therefore, equations in (3.17) become

$$
\begin{aligned}
S(t) &= S(0) + \frac{1}{\Gamma(\alpha(t))} \sum_{s=0}^{t-\alpha(t)} (t-\sigma(s))^{(\alpha(t)-1)} H_1(s,S(s)), \\
E(t) &= E(0) + \frac{1}{\Gamma(\alpha(t))} \sum_{s=0}^{t-\alpha(t)} (t-\sigma(s))^{(\alpha(t)-1)} H_2(s,A(s)), \\
I(t) &= I(0) + \frac{1}{\Gamma(\alpha(t))} \sum_{s=0}^{t-\alpha(t)} (t-\sigma(s))^{(\alpha(t)-1)} H_3(s,M(s)), \\
Q(t) &= Q(0) + \frac{1}{\Gamma(\alpha(t))} \sum_{s=0}^{t-\alpha(t)} (t-\sigma(s))^{(\alpha(t)-1)} H_4(s,C(s)), \\
R(t) &= R(0) + \frac{1}{\Gamma(\alpha(t))} \sum_{s=0}^{t-\alpha(t)} (t-\sigma(s))^{(\alpha(t)-1)} H_5(s,R(s)).
\end{aligned} \tag{3.20}
$$

The recursive formula is presented

$$
\begin{aligned}
S_n(t) &= S(0) + \frac{1}{\Gamma(\alpha(t))} \sum_{s=0}^{t-\alpha(t)} (t-\sigma(s))^{(\alpha(t)-1)} H_1(s,S_{n-1}(s)), \\
E_n(t) &= E(0) + \frac{1}{\Gamma(\alpha(t))} \sum_{s=0}^{t-\alpha(t)} (t-\sigma(s))^{(\alpha(t)-1)} H_2(s,E_{n-1}(s)), \\
I_n(t) &= I(0) + \frac{1}{\Gamma(\alpha(t))} \sum_{s=0}^{t-\alpha(t)} (t-\sigma(s))^{(\alpha(t)-1)} H_3(s,I_{n-1}(s)),
\end{aligned} \tag{3.21}
$$

$$
\begin{aligned}
Q_n(t) &= Q(0) + \frac{1}{\Gamma(\alpha(t))} \sum_{s=0}^{t-\alpha(t)} (t-\sigma(s))^{(\alpha(t)-1)} H_4(s, Q_{n-1}(s)),\\
R_n(t) &= R(0) + \frac{1}{\Gamma(\alpha(t))} \sum_{s=0}^{t-\alpha(t)} (t-\sigma(s))^{(\alpha(t)-1)} H_5(s, R_{n-1}(s)),
\end{aligned}
$$

the initial conditions are given by $S_0(t) = S(0)$, $E_0(t) = E(0)$, $I_0(t) = I(0)$, $Q_0(t) = Q(0)$ and $R_0(t) = R(0)$. Then, we take the expressions for difference of successive terms

$$\vartheta_{S,n}(t) = S_n(t) - S_{n-1}(t) = \frac{1}{\Gamma(\alpha(t))} \sum_{s=0}^{t-\alpha(t)} (t-\sigma(s))^{(\alpha(t)-1)} (H_1(s, S_{n-1}(s)) - H_1(s, S_{n-2}(s))),$$

$$\vartheta_{E,n}(t) = E_n(t) - E_{n-1}(t) = \frac{1}{\Gamma(\alpha(t))} \sum_{s=0}^{t-\alpha(t)} (t-\sigma(s))^{(\alpha(t)-1)} (H_2(s, E_{n-1}(s)) - H_2(s, E_{n-2}(s))),$$

$$\vartheta_{I,n}(t) = I_n(t) - I_{n-1}(t) = \frac{1}{\Gamma(\alpha(t))} \sum_{s=0}^{t-\alpha(t)} (t-\sigma(s))^{(\alpha(t)-1)} (H_3(s, I_{n-1}(s)) - H_3(s, I_{n-2}(s))),$$

$$\vartheta_{Q,n}(t) = Q_n(t) - Q_{n-1}(t) = \frac{1}{\Gamma(\alpha(t))} \sum_{s=0}^{t-\alpha(t)} (t-\sigma(s))^{(\alpha(t)-1)} (H_4(s, Q_{n-1}(s)) - H_4(s, Q_{n-2}(s))),$$

$$\Phi_{R,n}(t) = R_n(t) - R_{n-1}(t) = \frac{1}{\Gamma(\alpha(t))} \sum_{s=0}^{t-\alpha(t)} (t-\sigma(s))^{(\alpha(t)-1)} (H_5(s, R_{n-1}(s)) - H_5(s, R_{n-2}(s))),$$

where

$$S_n(t) = \sum_{j=0}^{n} \vartheta_{S,j}(t);\ E_n(t) = \sum_{j=0}^{n} \vartheta_{E,j}(t);\ I_n(t) = \sum_{j=0}^{n} \vartheta_{I,j}(t);$$
$$Q_n(t) = \sum_{j=0}^{n} \vartheta_{Q,j}(t);\ R_n(t) = \sum_{j=0}^{n} \vartheta_{R,j}(t).$$

Considering

$$
\begin{aligned}
\vartheta_{S,n-1}(t) &= S_{n-1}(t) - S_{n-2}(t),\\
\vartheta_{E,n-1}(t) &= E_{n-1}(t) - E_{n-2}(t),\\
\vartheta_{I,n-1}(t) &= I_{n-1}(t) - I_{n-2}(t),\\
\vartheta_{Q,n-1}(t) &= Q_{n-1}(t) - Q_{n-2}(t),\\
\vartheta_{R,n-1}(t) &= R_{n-1}(t) - R_{n-2}(t).
\end{aligned}
\tag{3.22}
$$

We obtain the following

$$
\begin{aligned}
\|\vartheta_{S,n}(t)\| &< \frac{p_1}{\Gamma(\alpha(t))} \sum_{s=0}^{t-\alpha(t)} (t-\sigma(s))^{(\alpha(t)-1)} \|\vartheta_{S,n-1}(s)\|,\\
\|\vartheta_{E,n}(t)\| &< \frac{p_2}{\Gamma(\alpha(t))} \sum_{s=0}^{t-\alpha(t)} (t-\sigma(s))^{(\alpha(t)-1)} \|\vartheta_{E,n-1}(s)\|,\\
\|\vartheta_{I,n}(t)\| &< \frac{p_3}{\Gamma(\alpha(t))} \sum_{s=0}^{t-\alpha(t)} (t-\sigma(s))^{(\alpha(t)-1)} \|\vartheta_{I,n-1}(s)\|,\\
\|\vartheta_{Q,n}(t)\| &< \frac{p_4}{\Gamma(\alpha(t))} \sum_{s=0}^{t-\alpha(t)} (t-\sigma(s))^{(\alpha(t)-1)} \|\vartheta_{Q,n-1}(s)\|,\\
\|\vartheta_{R,n}(t)\| &< \frac{p_5}{\Gamma(\alpha(t))} \sum_{s=0}^{t-\alpha(t)} (t-\sigma(s))^{(\alpha(t)-1)} \|\vartheta_{R,n-1}(s)\|.
\end{aligned}
\tag{3.23}
$$

■

Theorem 3.5

The solution of model (3.1) exists for $t \in [0, \mathrm{T}]$ provided

$$\frac{kp_i}{\Gamma(\alpha(t))} < 1, i = 1, \ldots, 5.$$

Proof: Here, the function $S(t)$, $E(t)$, $I(t)$, $Q(t)$ and $R(t)$ are bounded, and the kernels H_1, H_2, H_3, H_4 and H_5 satisfy the Lipschitz condition. By using the recursive principle, the inequalities (3.23) involve

$$\begin{aligned}
\|\vartheta_{S,n}(t)\| &< \|S_0(t)\|(\frac{kp_1}{\Gamma(\alpha(t))})^n,\\
\|\vartheta_{E,n}(t)\| &< \|E_0(t)\|(\frac{kp_2}{\Gamma(\alpha(t))})^n,\\
\|\vartheta_{I,n}(t)\| &< \|I_0(t)\|(\frac{kp_3}{\Gamma(\alpha(t))})^n,\\
\|\vartheta_{Q,n}(t)\| &< \|Q_0(t)\|(\frac{kp_4}{\Gamma(\alpha(t))})^n,\\
\|\vartheta_{R,n}(t)\| &< \|R_0(t)\|(\frac{kp_5}{\Gamma(\alpha(t))})^n.
\end{aligned} \tag{3.24}$$

Applying a limit, as n approaches to ∞, we get $\|\vartheta_{.,n}(t)\| \longrightarrow 0$. Hence, we have the existence of the solutions of equations (3.1). ■

Theorem 3.6

The solution of (3.1) is unique if $\|\aleph(t)\|\left(1 - \frac{kp}{\Gamma(\alpha(t))}\right) > 0$ *holds true.*

Proof: Assume that there exists another solution to model (3.1) and it is given as ($S_2(t)$, $E_2(t)$, $I_2(t)$, $Q_2(t)$, $R_2(t)$) where

$$\aleph(t) = \begin{pmatrix} S(t) - S_2(t) \\ E(t) - E_2(t) \\ I(t) - I_2(t) \\ Q(t) - Q_2(t) \\ R(t) - R_2(t) \end{pmatrix}; \quad p = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{pmatrix}.$$

$$\begin{aligned}
S(t) - S_2(t) &= \frac{1}{\Gamma(\alpha(t))} \sum_{s=0}^{t-\alpha(t)} (t - \sigma(s))^{(\alpha(t)-1)}(L_1(s, S(s)) - H_1(s, S_2(s))),\\
E(t) - E_2(t) &= \frac{1}{\Gamma(\alpha(t))} \sum_{s=0}^{t-\alpha(t)} (t - \sigma(s))^{(\alpha(t)-1)}(L_2(s, E(s)) - H_2(s, E_2(s))),\\
I(t) - I_2(t) &= \frac{1}{\Gamma(\alpha(t))} \sum_{s=0}^{t-\alpha(t)} (t - \sigma(s))^{(\alpha(t)-1)}(L_3(s, I_(s)) - H_3(s, I_2(s))),\\
Q(t) - Q_2(t) &= \frac{1}{\Gamma(\alpha(t))} \sum_{s=0}^{t-\alpha(t)} (t - \sigma(s))^{(\alpha(t)-1)}(L_4(s, Q(s)) - H_4(s, Q_2(s))),\\
R(t) - R_2(t) &= \frac{1}{\Gamma(\alpha(t))} \sum_{s=0}^{t-\alpha(t)} (t - \sigma(s))^{(\alpha(t)-1)}(L_5(s, R(s)) - H_5(s, R_2(s))).
\end{aligned} \tag{3.25}$$

Applying the norm on (3.25), we obtain

$$\begin{aligned}
\|S(t)-S_2(t)\| &= \|\frac{1}{\Gamma(\alpha(t))}\sum_{s=0}^{t-\alpha(t)}(t-\sigma(s))^{(\alpha(t)-1)}(H_1(s,S(s))-H_1(s,S_2(s)))\|,\\
\|E(t)-E_2(t)\| &= \|\frac{1}{\Gamma(\alpha(t))}\sum_{s=0}^{t-\alpha(t)}(t-\sigma(s))^{(\alpha(t)-1)}(H_2(s,E(s))-H_2(s,E_2(s)))\|,\\
\|I(t)-I_2(t)\| &= \|\frac{1}{\Gamma(\alpha(t))}\sum_{s=0}^{t-\alpha(t)}(t-\sigma(s))^{(\alpha(t)-1)}(H_3(s,I_(s))-H_3(s,I_2(s)))\|,\\
\|Q(t)-Q_2(t)\| &= \|\frac{1}{\Gamma(\alpha(t))}\sum_{s=0}^{t-\alpha(t)}(t-\sigma(s))^{(\alpha(t)-1)}(H_4(s,Q(s))-H_4(s,Q_2(s)))\|,\\
\|R(t)-R_2(t)\| &= \|\frac{1}{\Gamma(\alpha(t))}\sum_{s=0}^{t-\alpha(t)}(t-\sigma(s))^{(\alpha(t)-1)}(H_5(s,R(s))-H_5(s,R_2(s)))\|.
\end{aligned}$$

So

$$\begin{aligned}
\|S(t)-S_2(t)\| &= \frac{1}{\Gamma(\alpha(t))}\sum_{s=0}^{t-\alpha(t)}(t-\sigma(s))^{(\alpha(t)-1)}\|(H_1(s,S(s))-H_1(s,S_2(s)))\|,\\
\|E(t)-E_2(t)\| &= \frac{1}{\Gamma(\alpha(t))}\sum_{s=0}^{t-\alpha(t)}(t-\sigma(s))^{(\alpha(t)-1)}\|(H_2(s,E(s))-H_2(s,E_2(s)))\|,\\
\|I(t)-I_2(t)\| &= \frac{1}{\Gamma(\alpha(t))}\sum_{s=0}^{t-\alpha(t)}(t-\sigma(s))^{(\alpha(t)-1)}\|(H_3(s,I_(s))-H_3(s,I_2(s)))\|,\\
\|Q(t)-Q_2(t)\| &= \frac{1}{\Gamma(\alpha(t))}\sum_{s=0}^{t-\alpha(t)}(t-\sigma(s))^{(\alpha(t)-1)}\|(H_4(s,Q(s))-H_4(s,Q_2(s)))\|,\\
\|R(t)-R_2(t)\| &= \frac{1}{\Gamma(\alpha(t))}\sum_{s=0}^{t-\alpha(t)}(t-\sigma(s))^{(\alpha(t)-1)}\|(H_5(s,R(s))-H_5(s,R_2(s)))\|.
\end{aligned} \tag{3.26}$$

Using the lipschitz condition

$$\begin{aligned}
\|S(t)-S_2(t)\| &< \frac{p_1}{\Gamma(\alpha(t))}\sum_{s=0}^{t-\alpha(t)}(t-\sigma(s))^{(\alpha(t)-1)}\|S(t)-S_2(t)\|,\\
\|E(t)-E_2(t)\| &< \frac{p_2}{\Gamma(\alpha(t))}\sum_{s=0}^{t-\alpha(t)}(t-\sigma(s))^{(\alpha(t)-1)}\|E(t)-E_2(t)\|,\\
\|I(t)-I_2(t)\| &< \frac{p_3}{\Gamma(\alpha(t))}\sum_{s=0}^{t-\alpha(t)}(t-\sigma(s))^{(\alpha(t)-1)}\|I(t)-I_2(t)\|,\\
\|Q(t)-Q_2(t)\| &< \frac{p_4}{\Gamma(\alpha(t))}\sum_{s=0}^{t-\alpha(t)}(t-\sigma(s))^{(\alpha(t)-1)}\|Q(t)-Q_2(t)\|,\\
\|R(t)-R_2(t)\| &< \frac{p_5}{\Gamma(\alpha(t))}\sum_{s=0}^{t-\alpha(t)}(t-\sigma(s))^{(\alpha(t)-1)}\|R(t)-R_2(t)\|.
\end{aligned} \tag{3.27}$$

In consequence

$$\|S(t)-S_2(t)\|(1-\frac{kp_1}{\Gamma(\alpha(t))}) < 0,$$

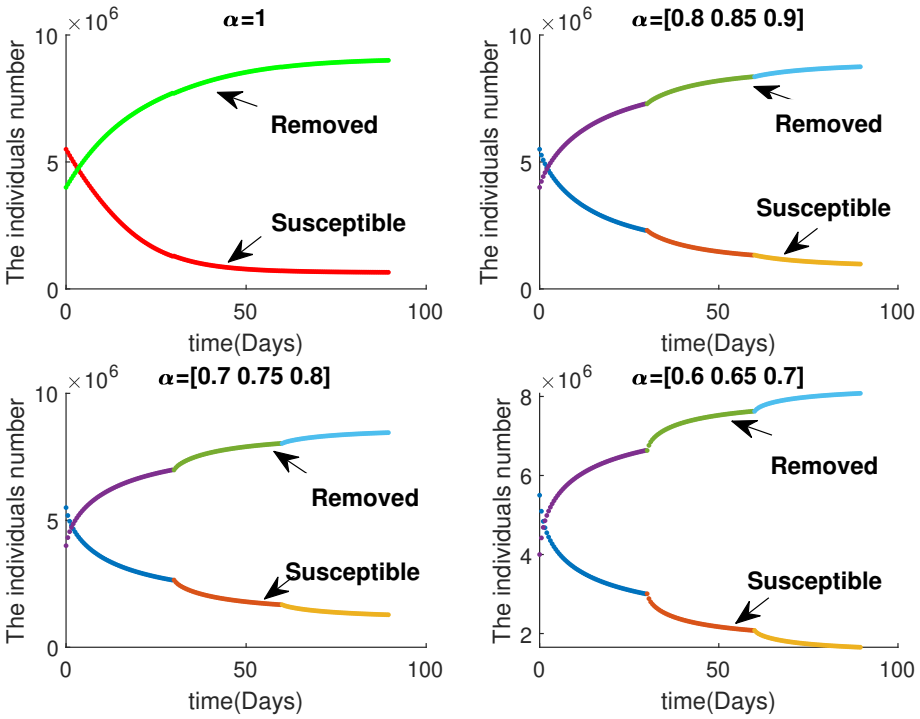


Figure 3.1: The behavior of the susceptible and removed population from some variable order α.

$$\begin{aligned} \|E(t) - E_2(t)\|(1 - \frac{kp_2}{\Gamma(\alpha(t))}) &< 0, \\ \|I(t) - I_2(t)\|(1 - \frac{kp_3}{\Gamma(\alpha(t))}) &< 0, \\ \|Q(t) - Q_2(t)\|(1 - \frac{kp_4}{\Gamma(\alpha(t))}) &< 0, \\ \|R(t) - R_2(t)\|(1 - \frac{kp_5}{\Gamma(\alpha(t))}) &< 0. \end{aligned} \tag{3.28}$$

Contradiction, hence the result. ■

3.3.4 The Stability of Equilibrium Points

In this subsection, we study the local asymptotic stability of the equilibrium points.

Theorem 3.7

The disease-free equilibrium $P_0 = (S_0, 0, 0, 0, R_0)$ of the suggested discrete fractional variable-order model is locally asymptotically stable if $\mathcal{R}_0 < 1$, and is unstable if $\mathcal{R}_0 > 1$.

Proof: The Jacobian matrix of model (3.1) estimated at P_0 given by

$$\mathcal{J}(P_0) = \begin{pmatrix} -(\delta+\mu) & 0 & -\beta\nu\dfrac{S_0}{N} & 0 & \lambda \\ 0 & -(\mu+\gamma) & \beta\nu\dfrac{S_0}{N} & 0 & 0 \\ 0 & \gamma & -(\psi+\varphi_1+\phi+\mu) & 0 & 0 \\ 0 & 0 & \psi & -(\tau+\varphi_2+\mu) & 0 \\ \delta & 0 & \phi & \tau & -(\lambda+\mu) \end{pmatrix}.$$

The characteristic polynomial of $\mathcal{J}$ is represented by

$$|\mathcal{J}(P_0) - \lambda\widehat{\mathcal{I}}| = (\lambda+\mu)(\lambda+\tau+\varphi_1+\mu)(\lambda+\lambda+\delta+\mu)L\left(\lambda^{\alpha(t)}\right) = 0,$$

where

$$L(\lambda) = (\lambda)^2 + a_1\lambda + a_2.$$

Here

$$a_1 = \gamma+\psi+\varphi+\phi+2\mu; \quad a_2 = (1-\mathcal{R}_0)(\gamma+\mu)(\psi+\varphi+\phi+\mu).$$

There are five eigenvalues; $\lambda_1 < 0$; $\lambda_2 < 0$; $\lambda_3 < 0$ and λ_4, λ_5 are the solution of $L(\lambda)$.

If $\mathcal{R}_0 < 1$ then $a_1 > 0$ and $a_2 > 0$, $L(\lambda)$ has two real roots negative, then P_0 is locally asymptotically stable. Likewise condition $\mathcal{R}_0 > 1$ that $\lambda_4\lambda_5 < 0$ holds that the equilibrium P_0 unstable and so the proof of theorem. ■

Theorem 3.8

The endemic equilibrium P_* is locally asymptotically stable if $R_0 > 1$.

Proof: The Jacobian matrix of model (3.1) at P_* is

$$J(P_*) = \begin{bmatrix} -b_1 & 0 & -\beta\nu l_2 & 0 & \lambda \\ \beta\nu l_1 & -b_2 & \beta\nu l_2 & 0 & 0 \\ 0 & \gamma & -b_3 & 0 & 0 \\ 0 & 0 & \psi & -b_4 & 0 \\ \delta & 0 & \phi & \tau & -b_5 \end{bmatrix},$$

where

$$b_1 = \beta\nu l_1 + \delta + \mu, \quad b_2 = \gamma+\mu, \quad b_3 = \psi+\phi+\varphi_1+\mu,$$
$$b_4 = \tau+\varphi_2+\mu, \quad b_5 = \lambda+\mu, \quad l_1 = \frac{I_*}{N+\rho I_*}, \quad l_2 = \frac{S_* N}{(N+\rho I_*)^2}.$$

The characteristic equation is

$$|\lambda E - J(P_*)| = \lambda^5 + \zeta_1\lambda^4 + \zeta_2\lambda^3 + \zeta_3\lambda^2 + \zeta_4\lambda + \zeta_5,$$

where

$$\begin{aligned}
\zeta_1 &= \sum_{i=1}^{5} a_i, \\
\zeta_2 &= a_1(\xi_1 - a_1) + a_2(\xi_1 - a_1 - a_2) + a_3(a_4+a_5) + a_4a_5 - \rho^a\theta^a - \beta c^a\epsilon^a h_2, \\
\zeta_3 &= a_2[a_4a_5 + a_1(a_4+a_5)] + a_3[a_4a_5 + a_2(a_1+a_4+a_5) + a_1(a_4+a_5)] + a_1a_4a_5 \\
&\quad + \epsilon^\alpha\beta^2c^{2\alpha}h_1h_2 - \epsilon^\alpha\beta c^\alpha h_2(a_1+a_4+a_5) - \theta^\alpha\rho^\alpha(a_2+a_3+a_4), \\
\zeta_4 &= a_3[a_2(a_4a_5 + a_1(a_4+a_5)) + a_1a_4a_5] + a_1a_2a_4a_5 + (a_4+a_5)\epsilon^\alpha\beta^2c^{2x}h_1h_2 \\
&\quad - (a_4a_5 + a_1a_5 + a_1a_4)\epsilon^\alpha\beta c^\alpha h_2 - (a_2a_4 + a_3a_4 + a_2a_3)\theta^\alpha\rho^\alpha + \epsilon^a\theta^\alpha\beta c^a(\rho^\alpha h_2 - \gamma^a h_1), \\
\zeta_5 &= a_1a_2a_3a_4a_5 + a_4a_5\epsilon^\alpha\beta^2c^{2x}h_1h_2 - a_1a_4a_5\epsilon^\alpha\beta c^\alpha h_2 - a_2a_3a_4\theta^\alpha\rho^\alpha \\
&\quad + a_4\epsilon^\alpha\theta^\alpha\beta c^\alpha(\rho^\alpha h_2 - \gamma^\alpha h_1) - \epsilon^\alpha\delta^\alpha\theta^\alpha\eta^\alpha\beta c^\alpha h_1.
\end{aligned}$$

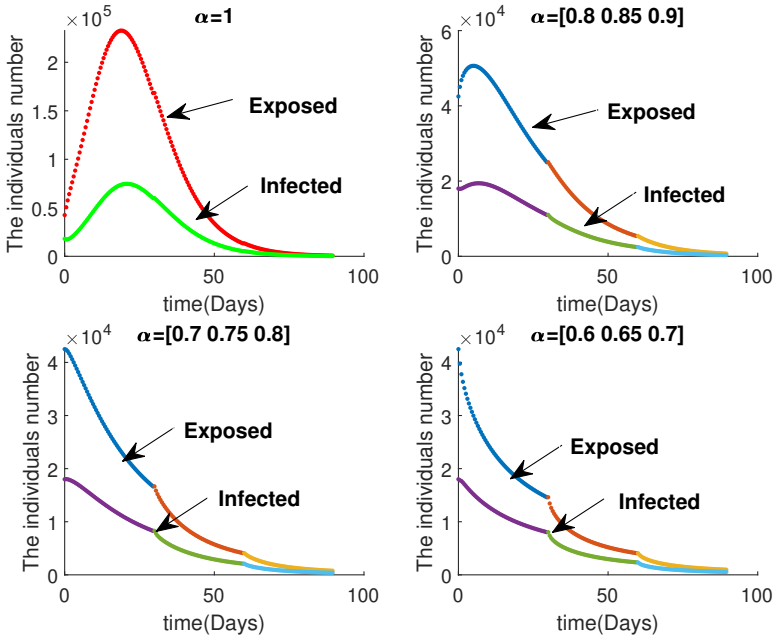


Figure 3.2: The behavior of the exposed and infected population from some variable order α.

If $\zeta_i > 0$ $(i = 2, 3, 4, 5)$ and it is clear that $\zeta_1 > 0$ the sufficient conditions can be derived as follows:

$$\begin{vmatrix} \zeta_1 & 1 \\ \zeta_3 & \zeta_2 \end{vmatrix} > 0, \quad \begin{vmatrix} \zeta_1 & 1 & 0 \\ \zeta_3 & \zeta_2 & \zeta_1 \\ \zeta_5 & \zeta_4 & \zeta_3 \end{vmatrix} > 0, \quad \begin{vmatrix} \zeta_1 & 1 & 0 & 0 \\ \zeta_3 & \zeta_2 & \zeta_1 & 1 \\ \zeta_5 & \zeta_4 & \zeta_3 & \zeta_2 \\ 0 & 0 & \zeta_5 & \zeta_4 \end{vmatrix} > 0. \tag{3.29}$$

for which the equilibrium P_* is locally asymptotically stable. ■

3.4 Optimal Control Problem

A vaccine for the emerging coronavirus (COVID-19) has been developed, aiming to decrease the number of contacts between susceptible individuals and infected individuals to limit infection and mitigate the spread of the virus. This can be realized through various measures, including home quarantine, nucleic acid testing, and restrictions on residents' movement. Mathematically, these measures can be represented by a coefficient denoted as u in this section, indicating the intensity of different control measures. Consequently, the model system of equations (3.1) is modified as follows:

$$\begin{aligned}
\Delta^{\alpha(t)}S(t) &= \chi - \beta\nu^{\alpha(t)}(1-u(t))\frac{SI}{N+\rho I} + \lambda^{\alpha(t)}R(t) - (\delta^{\alpha(t)} + \mu^{\alpha(t)})S(t), \\
\Delta^{\alpha(t)}E(t) &= \beta\nu^{\alpha(t)}(1-u(t))\frac{SI}{N+\rho I} - (\gamma^{\alpha(t)} + \mu^{\alpha(t)})E(t), \\
\Delta^{\alpha(t)}I(t) &= \gamma^{\alpha(t)}E(t) - (\psi^{\alpha(t)} + \phi^{\alpha(t)} + \varphi_1^{\alpha(t)} + \mu^{\alpha(t)})I(t), \\
\Delta^{\alpha(t)}Q(t) &= \psi^{\alpha(t)}I(t) - (\tau^{\alpha(t)} + \varphi_2^{\alpha(t)} + \mu^{\alpha(t)})Q(t), \\
\Delta^{\alpha(t)}R(t) &= \delta^{\alpha(t)}S(t) + \phi^{\alpha(t)}I(t) + \tau^{\alpha(t)}Q(t) - (\lambda^{\alpha(t)} + \mu^{\alpha(t)})R(t).
\end{aligned} \tag{3.30}$$

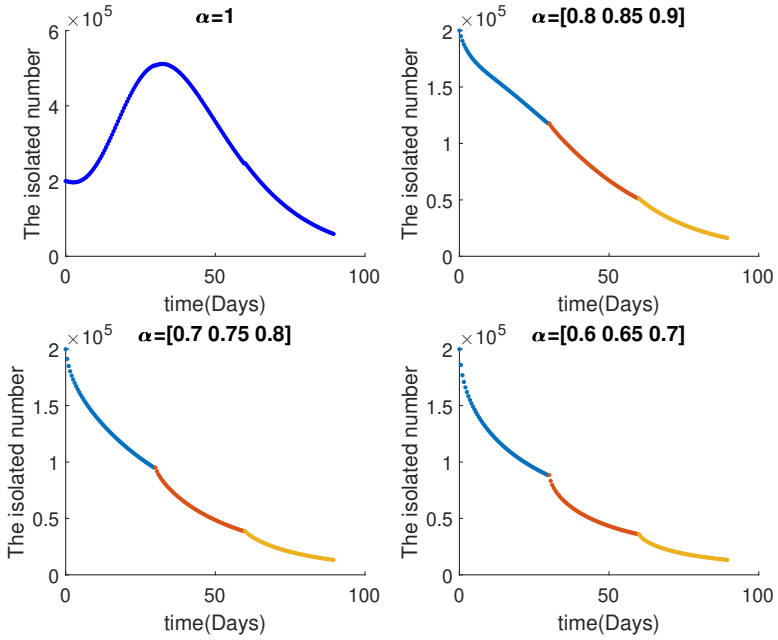


Figure 3.3: The behavior of the isolated population from some variable order α.

The corresponding discrete fractional optimal control problem with variable order in the Caputo sense is considered as follows:

$$J(u^*) = \min_{u \in \Omega} J(u),$$

where J is defined by

$$J(u) = \sum_{k=0}^{N-1} C_1 E(k) + C_2 I(k) + C_3 Q(k) + C_4 u^2(k),$$

and the control space Ω is defined by the set

$$\Omega = \{\, u \in \mathcal{R}^n / 0 < u_{min} < u < u_{max} < 1 \}\,.$$

The coefficients C_1, C_2, C_3 and C_4 represent the positive weight constants of exposed, infected, isolated and control variables.

Theorem 3.9

Let u^* denote the optimal control variable of the discrete fractional optimal control with variable order, and let S^*, E^*, I^*, Q^*, and R^* represent the optimal state solution. Additionally, there exist adjoint variables κ_i, where $i = 1, ..., 5$, satisfying the following equations:

$$\begin{aligned}
\Delta^{\alpha(t)}\kappa_1(t) &= -(\beta\nu^{\alpha(t)}(1-u)\frac{I}{N+\rho I} + \delta^{\alpha(t)} + \mu^{\alpha(t)})\kappa_1(t) + \beta\nu^{\alpha(t)}(1-u)\frac{I}{N+\rho I}\kappa_2(t) + \delta^{\alpha(t)}\kappa_5(t),\\
\Delta^{\alpha(t)}\kappa_2(t) &= C_1 - (\gamma^{\alpha(t)} + \mu^{\alpha(t)})\kappa_2(t) + \gamma^{\alpha(t)}\kappa_2(t),\\
\Delta^{\alpha(t)}\kappa_3(t) &= C_2 - (\beta\nu^{\alpha(t)}(1-u)\frac{I}{(N+\rho I)^2}\kappa_1(t) + (\beta\nu^{\alpha(t)}(1-u)\frac{SN}{(N+\rho I)^2}\kappa_2(t) -\\
&\quad (\psi^{\alpha(t)} + \phi^{\alpha(t)} + \varphi_1^{\alpha(t)} + \mu^{\alpha(t)})\kappa_3(t) + \psi^{\alpha(t)}\kappa_4(t) + \phi^{\alpha(t)}\kappa_5(t),\\
\Delta^{\alpha(t)}\kappa_4(t) &= C_3 - (\tau^{\alpha(t)} + \varphi_2^{\alpha(t)} + \mu^{\alpha(t)})\kappa_4(t) + \tau^{\alpha(t)}\kappa_5,\\
\Delta^{\alpha(t)}\kappa_5(t) &= \lambda^{\alpha(t)}\kappa_1 - (\lambda^{\alpha(t)} + \mu^{\alpha(t)})\kappa_5,
\end{aligned} \tag{3.31}$$

with $\Delta_T^{\alpha(t)}\kappa_i(t) = 0$, $i = 1, \cdots, 5$. In addition, the optimal control $u^*$is characterized by

$$u^* = \min\left(\max\left(\frac{\beta\nu^{\alpha(t)}SI(\kappa_2 - \kappa_1)}{2C_4(N+\rho I)}, 0\right), 1\right). \tag{3.32}$$

Proof: We can determine the discrete optimal control u^* by the application of a discrete version of Pontryagin's maximum principle as in [12, 136] to the discrete Hamiltonian function $\mathcal{M}$ as follows

$$\begin{aligned}
\mathcal{M} = \; & C_1E + C_2I + C_3Q + C_4u^2 + \kappa_1(\chi - \beta\nu^{\alpha(t)}(1-u)\frac{SI}{N+\rho I} + \lambda^{\alpha(t)}R - (\delta^{\alpha(t)} + \mu^{\alpha(t)})S)\\
& + \kappa_2(\beta\nu^{\alpha(t)}(1-u)\frac{SI}{N+\rho I} - (\gamma^{\alpha(t)} + \mu^{\alpha(t)})E) + \kappa_3(\gamma^{\alpha(t)}E - (\psi^{\alpha(t)} + \phi^{\alpha(t)} + \varphi_1^{\alpha(t)} + \mu^{\alpha(t)})I)\\
& + \kappa_4(\psi^{\alpha(t)}I - (\tau^{\alpha(t)} + \varphi_2^{\alpha(t)} + \mu^{\alpha(t)})Q) + \kappa_5(\delta^{\alpha(t)}S + \phi^{\alpha(t)}I + \tau^{\alpha(t)}Q - (\lambda^{\alpha(t)} + \mu^{\alpha(t)})R.
\end{aligned}$$

By using the following formulations

$$\Delta_T^{\alpha(t)}\kappa_1 = \frac{\partial H}{\partial S};\ \Delta_T^{\alpha(t)}\kappa_2 = \frac{\partial H}{\partial E};\ \Delta_T^{\alpha(t)}\kappa_3 = \frac{\partial H}{\partial I};\ \Delta_T^{\alpha(t)}\kappa_4 = \frac{\partial H}{\partial Q};\ \Delta_T^{\alpha(t)}\kappa_5 = \frac{\partial H}{\partial R}.$$

The discrete -time fractional adjoint system with variable order as

$$\begin{aligned}
\Delta^{\alpha(t)}\kappa_1(t) &= -(\beta\nu^{\alpha(t)}(1-u)\frac{I}{N+\rho I} + \delta^{\alpha(t)} + \mu^{\alpha(t)})\kappa_1(t) + \beta\nu^{\alpha(t)}(1-u)\frac{I}{N+\rho I}\kappa_2(t) + \delta^{\alpha(t)}\kappa_5(t),\\
\Delta^{\alpha(t)}\kappa_2(t) &= C_1 - (\gamma^{\alpha(t)} + \mu^{\alpha(t)})\kappa_2(t) + \gamma^{\alpha(t)}\kappa_2(t),\\
\Delta^{\alpha(t)}\kappa_3(t) &= C_2 - (\beta\nu^{\alpha(t)}(1-u)\frac{I}{(N+\rho I)^2}\kappa_1(t) + (\beta\nu^{\alpha(t)}(1-u)\frac{SN}{(N+\rho I)^2}\kappa_2(t)\\
&\quad -(\psi^{\alpha(t)} + \phi^{\alpha(t)} + \varphi_1^{\alpha(t)} + \mu^{\alpha(t)})\kappa_3(t) + \psi^{\alpha(t)}\kappa_4(t) + \phi^{\alpha(t)}\kappa_5(t),\\
\Delta^{\alpha(t)}\kappa_4(t) &= C_3 - (\tau^{\alpha(t)} + \varphi_2^{\alpha(t)} + \mu^{\alpha(t)})\kappa_4(t) + \tau^{\alpha(t)}\kappa_5,\\
\Delta^{\alpha(t)}\kappa_5(t) &= \lambda^{\alpha(t)}\kappa_1 - (\lambda^{\alpha(t)} + \mu^{\alpha(t)})\kappa_5.
\end{aligned} \tag{3.33}$$

The control equation is $\dfrac{\partial H}{\partial u} = 0$, we have the optimal condition

$$\frac{\partial H}{\partial u} = 2C_4u + \beta\nu^{\alpha(t)}\frac{SI}{N+\rho I}\kappa_1 - \beta\nu^{\alpha(t)}\frac{SI}{N+\rho I}\kappa_2 = 0,$$

$$u = \frac{\beta\nu^{\alpha(t)}SI(\kappa_2 - \kappa_1)}{2(N+\rho I)C_4},$$

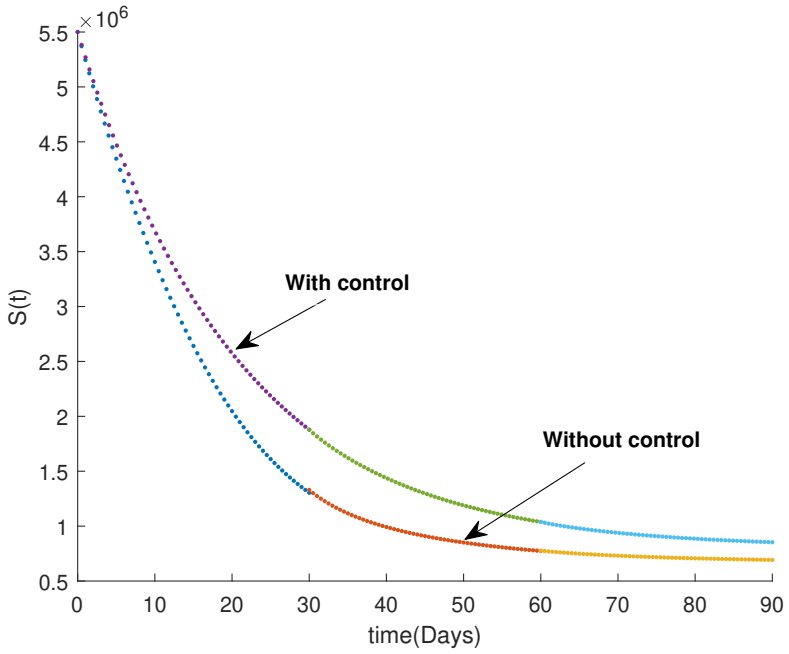


Figure 3.4: Susceptible group with control and without control for $\alpha = [0.99, 0.95, 0.93]$.

for the optimal control u^* we have

$$u^* = \begin{cases} 0 & if \ \dfrac{\partial H}{\partial u} < 0 \\ u & if \ \dfrac{\partial H}{\partial u} = 0 \\ 1 & if \ \dfrac{\partial H}{\partial u} > 0 \end{cases}$$

and

$$u^* = \min\left(\max\left(\frac{\beta\nu^{\alpha(t)}SI(\kappa_2 - \kappa_1)}{2C_4(N + \rho I)}, 0\right), 1\right).$$

■

3.5 Numerical Simulation

3.5.1 Numerical Strategy without Control

In this subsection, we solve the discrete-time fractional variable-order model defined by the system (3.1) using the Adams type prdictor-corrector method proposed in [90].

$$\begin{aligned} \Delta_t^{\alpha(t)} y(t) &= f(t, y(t)), \\ y(0)^b &= y_0^b, 0 < \alpha(t) < 1, 0 < t < \tau, \end{aligned} \tag{3.34}$$

where $b = 0, 1, \ldots, n-1$, and $n = [\alpha(t)]$. Analogous to the fractional order meaning , the above is equivalent to the Volterra equation

$$y(t) = \sum_{b=0}^{n-1} y_0^b \frac{t^b}{b!} + \frac{1}{\Gamma(\alpha(t))} \sum_{s=0}^{t-\alpha(t)} (t - \sigma(s))^{(\alpha(t)-1)} f(s, y(s)), \tag{3.35}$$

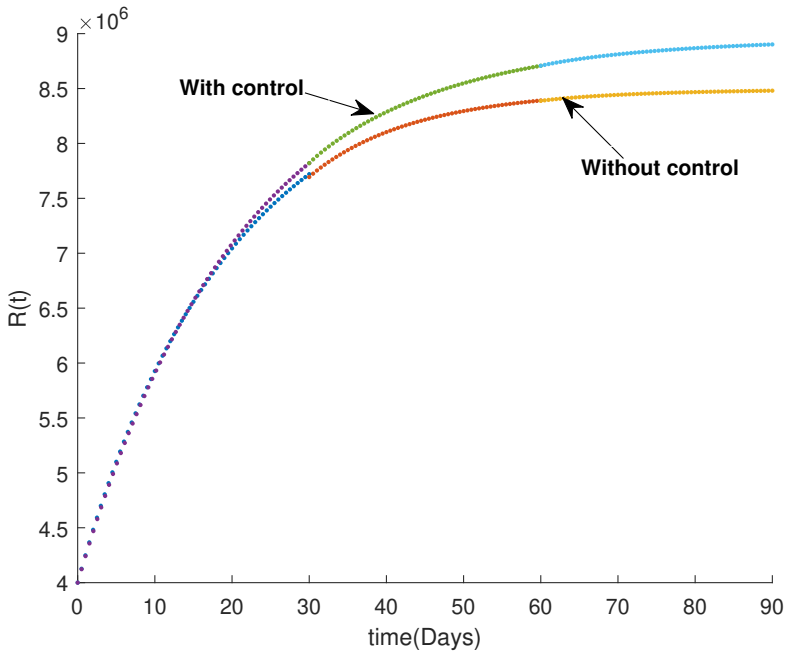


Figure 3.5: Recovered group with control and without control for $\alpha = [0.99, 0.95, 0.93]$.

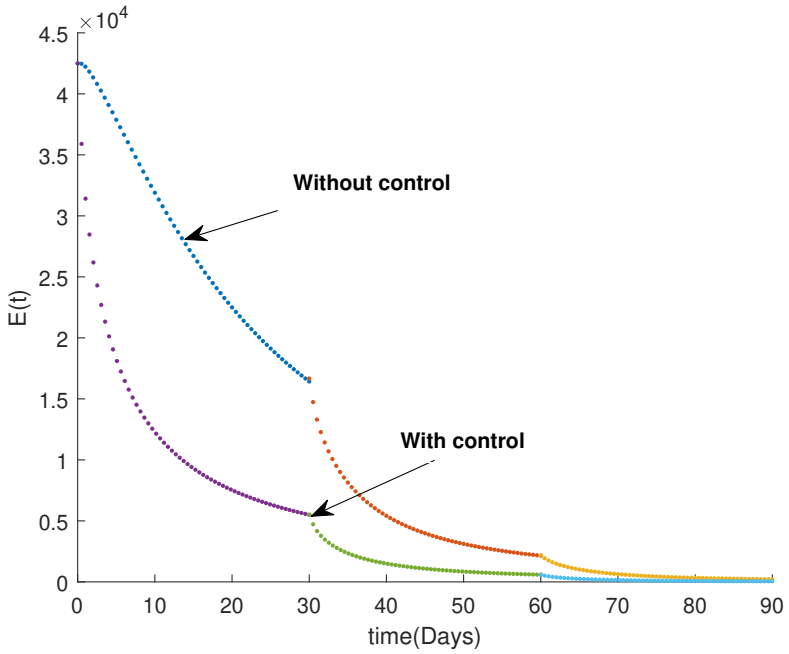


Figure 3.6: Exposed group with control and without control for$\alpha = [0.7, 0.75, 0.8]$.

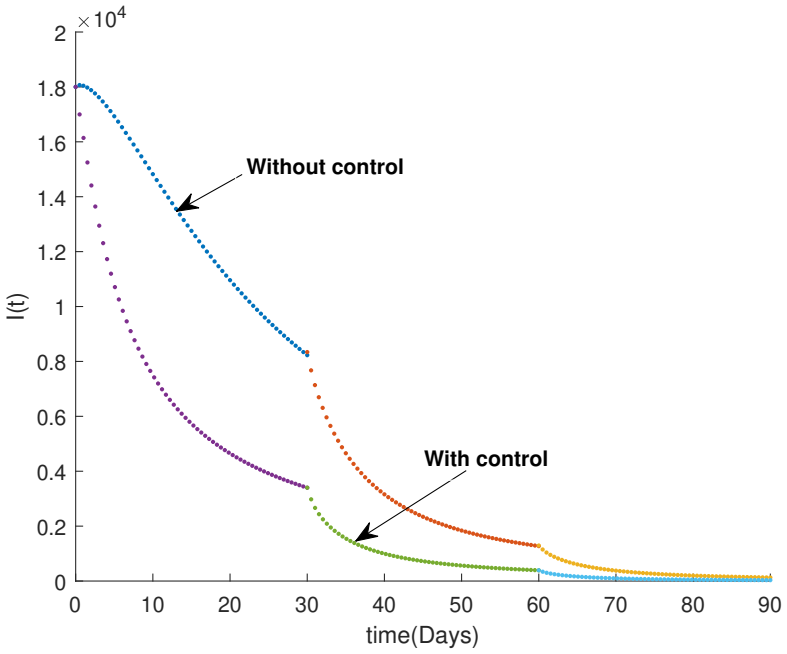


Figure 3.7: Infected group with control and without control for $\alpha = [0.7, 0.75, 0.8]$.

to get the numerical solutions of the suggested model . We take

$$h = \frac{\tau}{N};\ t_z = zh;\ N = \bigcup [N_{k-1}, N_k];\ z = N_{k-1}, \ldots, N_k \in \digamma^+,$$

$\alpha(t) = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ by means of letting $y_z \approx y(t_z)$, the discretization of (3.1):

$$\begin{aligned}
S_{q+1} &= \sum_{z=0}^{[\alpha(k)]} S_0^{(z)} \frac{t_{q+1}^{(z)}}{z!} + \frac{h^{\alpha(k)}}{\Gamma(\alpha(k)+2)} \sum_{z=0}^{q} (p_{z,q+1})(\chi - \beta\nu \frac{S_z I_z}{N + \rho I_z} + \lambda R_z - (\delta + \mu) S_z) \\
&\quad + \frac{h^{\alpha(k)}}{\Gamma(\alpha(k)+2)} \sum_{z=0}^{q} (p_{q+1,q+1})(\chi - \beta\nu \frac{S_{q+1}^{PF} I_{q+1}^{PF}}{N + \rho I_{q+1}^{PF}} + \lambda R_{q+1}^{PF} - (\delta + \mu) S_{q+1}^{PF}), \\
E_{q+1} &= \sum_{z=0}^{[\alpha(k)]} E_0^{(z)} \frac{t_{q+1}^{(z)}}{z!} + \frac{h^{\alpha(k)}}{\Gamma(\alpha(k)+2)} \sum_{z=0}^{q} (p_{z,q+1})(\beta\nu \frac{S_z I_z}{N + \rho I_z} - (\gamma + \mu) E_z) \\
&\quad + \frac{h^{\alpha(k)}}{\Gamma(\alpha(k)+2)} \sum_{z=0}^{q} (p_{q+1,q+1})(\beta\nu \frac{S_{q+1}^{PF} I_{q+1}^{PF}}{N + \rho I_{q+1}^{PF}} - (\gamma + \mu) E_{q+1}^{PF}), \\
I_{q+1} &= \sum_{z=0}^{[\alpha(k)]} I_0^{(z)} \frac{t_{q+1}^{(z)}}{z!} + \frac{h^{\alpha(k)}}{\Gamma(\alpha(k)+2)} \sum_{z=0}^{q} (p_{z,q+1})(\gamma E_z - (\psi + \phi + \varphi_1 + \mu) I_z) \\
&\quad + \frac{h^{\alpha(k)}}{\Gamma(\alpha(k)+2)} \sum_{z=0}^{q} (p_{q+1,q+1})(\gamma E_{q+1}^{PF} (\psi + \phi + \varphi_1 + \mu) I_{q+1}^{PF}, \\
Q_{q+1} &= \sum_{z=0}^{[\alpha(k)]} Q_0^{(z)} \frac{t_{q+1}^{(z)}}{z!} + \frac{h^{\alpha(k)}}{\Gamma(\alpha(k)+2)} \sum_{z=0}^{q} (p_{z,q+1})(\psi I_z - (\tau + \varphi_2 + \mu) Q_z) \\
&\quad + \frac{h^{\alpha(k)}}{\Gamma(\alpha(k)+2)} \sum_{z=0}^{q} (p_{q+1,q+1})(r_1^{\alpha(k)} M_{q+1}^{PF} (\psi I_{q+1}^{PF} - (\tau + \varphi_2 + \mu) Q_{q+1}^{PF}), \\
R_{q+1} &= \sum_{z=0}^{[\alpha(k)]} R_0^{(z)} \frac{t_{q+1}^{(z)}}{z!} + \frac{h^{\alpha(k)}}{\Gamma(\alpha(k)+2)} \sum_{z=0}^{q} (p_{z,q+1})(\delta S_z + \phi I_z + \tau Q_z - (\lambda + \mu) R_z)
\end{aligned}$$

$$+\frac{h^{\alpha(k)}}{\Gamma(\alpha(k)+2)}\sum_{z=0}^{q}(p_{q+1,q+1})(\delta S_{q+1}^{PF}+\phi I_{q+1}^{PF}+\tau Q_{q+1}^{PF}-(\lambda+\mu)R_{q+1}^{PF}), \tag{3.36}$$

where

$$p_{z,q+1}=\begin{cases} q^{\alpha(k)+1}-(q-\alpha(k))(q+1)^{\alpha(k)}, & \text{if } z=0 \\ (q-z+2)^{\alpha(k)+1}+(q-z)^{\alpha(k)+1}-2(q-z+1)^{\alpha(k)+1}, & \text{if } 1<z<q \\ 1, & \text{if } z=q+1\ . \end{cases} \tag{3.37}$$

The proposed prediction formula is calculated as follows:

$$\begin{aligned}
S_{q+1}^{PF} &= \sum_{z=0}^{[\alpha(k)]} S_0^{(z)}\frac{t_{q+1}^{(z)}}{z!}+\frac{h^{\alpha(k)}}{\Gamma(\alpha(k)+1)}\sum_{z=0}^{q}(j_{z,q+1})(\chi-\beta\nu\frac{S_zI_z}{N+\rho I_z}+\lambda R_z-(\delta+\mu)S_z),\\
E_{q+1}^{PF} &= \sum_{z=0}^{[\alpha(k)]} E_0^{(z)}\frac{t_{q+1}^{(z)}}{z!}+\frac{h^{\alpha(k)}}{\Gamma(\alpha(k)+1)}\sum_{z=0}^{q}(j_{z,q+1})(\beta\nu\frac{S_zI_z}{N+\rho I_z}-(\gamma+\mu)E_z),\\
I_{q+1}^{PF} &= \sum_{z=0}^{[\alpha(k)]} I_0^{(z)}\frac{t_{q+1}^{(z)}}{z!}+\frac{h^{\alpha(k)}}{\Gamma(\alpha(k)+1)}\sum_{z=0}^{q}(j_{z,q+1})(\psi+\phi+\varphi_1+\mu)I_z),\\
Q_{q+1}^{PF} &= \sum_{z=0}^{[\alpha(k)]} Q_0^{(z)}\frac{t_{q+1}^{(z)}}{z!}+\frac{h^{\alpha(k)}}{\Gamma(\alpha(k)+1)}\sum_{z=0}^{q}(j_{z,q+1})(\psi I_z-(\tau+\varphi_2+\mu)Q_z),\\
R_{q+1}^{PF} &= \sum_{z=0}^{[\alpha(k)]} R_0^{(z)}\frac{t_{q+1}^{(z)}}{z!}+\frac{h^{\alpha(k)}}{\Gamma(\alpha(k)+1)}\sum_{z=0}^{q}(j_{z,q+1})(\delta S_z+\phi I_z+\tau Q_z-(\lambda+\mu)R_z)
\end{aligned} \tag{3.38}$$

where

$$j_{z,q+1}=(q+1-z)^{\alpha(k)}-(q-z)^{\alpha(k)}.$$

3.5.2 Numerical Strategy with Control

In this subsection to solve the (V-FOCP) in the discrete-time defined, we have

$$Y_{q+1}=\sum_{z=0}^{[\alpha(k)]} Y_0^{(z)}\frac{t_{q+1}^{(z)}}{z!}+\frac{h^{\alpha(k)}}{\Gamma(\alpha(k)+2)}\sum_{z=0}^{q}(p_{z,q+1})g(t_z,Y_z,u_q)+\frac{h^{\alpha(k)}}{\Gamma(\alpha(k)+2)}\sum_{z=0}^{q}(p_{q+1,q+1})g(t_{q+1},Y_{q+1}^{PF},u_{q+1}^{PF}) \tag{3.39}$$

where $Y=(S,E,I,Q,R)^T$. We can rewrite the system of adjoint equations in the compact form with

$$L(t,Y,\kappa,u)=\begin{bmatrix} L_1(t,Y,\kappa,u)\\ L_2(t,Y,\kappa,u)\\ L_3(t,Y,\kappa,u)\\ L_4(t,Y,\kappa,u)\\ L_5(t,Y,\kappa,u)\end{bmatrix}.$$

We get

$$\Delta_t^{\alpha(t)}\kappa(T_f-t)=L(T_f-t,Y(T_f-t),\kappa(T_f-t),u(T_f-t)). \tag{3.40}$$

We are discritized control system as in [132] with following algorithm.

3.5.3 Solution Algorithm of V-FOCP in discrete-time

Step 1: Consider the initial estimation control u and used the initial condition.
Step 2: Find the adjoint variable and the optimal states by solving control problem.
Step 3: Find the control u^* using control function.
Step 4: Take $u_k = \dfrac{u^* + u_k}{2}$ to update the control.
Step 5: Stop the iteration when $\dfrac{\|u_k - u_{k-1}\|}{\|u_k\|}$ otherwise return to Step 2.

3.6 Numerical Results and Discussion

In this section, we use the parameters provided in Table 3.1 to discuss the introduced model (3.1) numerically. Additionally, the proposed model of fractional variable order in discrete time is numerically solved using the method outlined in the previous section. Moreover, the initial value conditions for the system (3.1) are set as follows:
$S(0) = 5.5 \times 10^6$, $E(0) = 4.25 \times 10^4$, $I(0) = 18000$, $Q(0) = 3000$, and $R(0) = 4 \times 10^6$ as in [132]. The following values are assumed for $\alpha(t_k)$

$$\alpha_1 = 1$$

$$\alpha_2 = \begin{pmatrix} 0.8 & 0.85 & 0.9 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 0.7 & 0.75 & 0.8 \end{pmatrix}, \alpha_4 = \begin{pmatrix} 0.6 & 0.65 & 0.7 \end{pmatrix}.$$

The figures from (3.1) to (3.3) depict the influence of different values of the variable fractional parameter α_k on the dynamics of subdivisions for the total population of COVID-19. We obtained interesting results by varying α_k. Figure (3.1) illustrates changes in the susceptible population graph as the value of α varies, with a decrease in the number of days separating the two curves as α decreases. We observe a proportional relationship between the time taken to reach the peak point in the graphs for exposed (E) and infected (I) individuals and the changes in α as depicted in figures (3.2) and (3.3).
Figures (3.4) to (3.8) present simulation results that demonstrate the significance of control variable to control the pandemic. The optimal control measures have a positive impact on reducing the rate of infection and the number of individuals exposed to infection, as depicted in Figure (3.5). Additionally, the number of people recovering increases more than usual. However, the decline in the susceptible population slows down due to the reduction in the number of individuals at risk resulting from the control measures.

3.7 Conclusion

This Chapter (3) we have conducted an analysis of a novel mathematical model for the 'SEIQR' epidemic (COVID-19), incorporating an isolated class. This model is characterized by a discrete-time system of fractional variable order in the Caputo sense. We have examined the non-negativity and boundedness of the solution, as well as determined the reproduction number R_0 by computing the spectral radius of the next-generation matrix. Based on the threshold R_0, we have established the existence and stability of both the disease-free equilibrium and endemic equilibrium points. Moreover, we have applied an optimal control approach to a discrete-time COVID-19 model. A numerical scheme utilizing the Adam's numerical method was employed for the Caputo fractional variable-order system. We tackled the Variable-Order Fractional Optimal Control Problem (VO-FOCP) in discrete time. Numerical simulations have

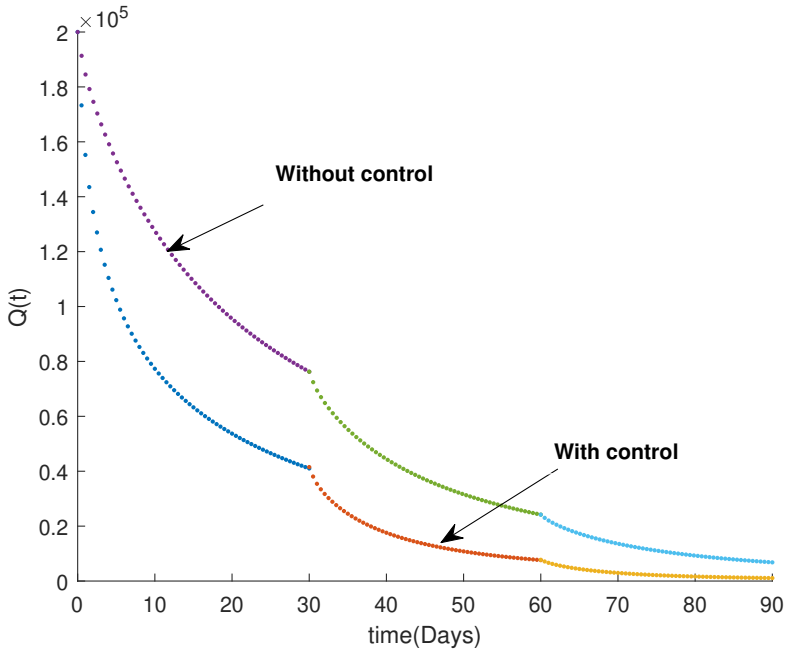


Figure 3.8: Isolated group with control and without control for $\alpha = [0.7, 0.75, 0.8]$.

been conducted to underscore the significance of control measures. It has been observed that upon implementation of control measures, the number of susceptible individuals increases while the number of infected and recovered individuals decreases.

Chapter 4

Stability and sensitivity analysis for a variable-order discrete mathematical model to evaluate the impact of diabetes and its resulting complications

The work presented in this Chapter (4) is a variable-order discrete mathematical model to estimate the impact of diabetes and its resulting complications. Main results show solvability and stability analysis for the proposed system via standard tools such as, variable-order calculus in discrete time. Further, we provide various numerical schemes along with their results to evaluate the proposed model.This work is attributed to the [30].

4.1 Introduction

Diabetes mellitus, a metabolic condition marked by endogenous insulin, results in excessive amounts of glucose in the bloodstream. Based on World Health Organization WHO, Insulin is primarily responsible for the conversion of blood glucose into energy for human activities such as walking. Since diabetes mellitus lacks insulin functionality, glucose levels in the blood remain unchanged [1, 20]. When blood glucose levels are high, hyperglycemia can lead to damage to the eyes, kidneys, and nerves [95]. The goal of insulin-dependent diabetic therapy is often to prevent hyperglycemia by providing continuous exogenous insulin. A low blood glucose level can cause sleepiness, mental impairment, agitation, and unconsciousness if insulin is oversupplied. This condition is known as hypoglycemia in diabetics.Asufficient amount of exogenous insulin must be administered at the right time to treat both hyperglycemia and hypoglycemia in insulin-dependent diabetes. An overview of mathematical diabetes models pertaining to glucose-insulin regulation is provided in [75].
Diabetes can affect anyone regardless of size, age, or gender. It has two main forms, type I and type II diabetes. Diabetes is characterized by anomalies in the insulin-glucose channel in the system due to the loss of glucagon and the subsequent production of glycogen, Diabetes Mellitus is a metabolic condition with persistent hyperglycemia caused by deficiencies in insulin action, insulin secretion, or both. It also causes disruptions in protein, carbohydrate, and lipid metabolism. Type I diabetes does not have any established cause and cannot be cured. Lack of access to insulin can cause several severe problems or endangers the patients life. Symptoms include increased hunger, thirst, urination, weariness, weight loss, blurred vision, and tiredness.

Patients with type II diabetes which are not dependent on insulin, are advised to exercise regularly and follow a specific food plan. Yoga can also be an efficient method for controlling type II diabetes. While type II diabetes symptoms might reflect some type I diabetes, they are usually less apparent or absent. Everywhere, including in children, diabetes is constantly on the rise, with the middle-income countries of the world experiencing the highest rates [36, 60]. According to the latest estimates from the International Diabetes Federation IDF, 537 million adults (20-79 years) are living with diabetes - 1 in 10. This number is predicted to rise to 643 million by 2030 and 783 million by 2045. Over 3 in 4 adults with diabetes live in low- and middle-income countries. Diabetes is responsible for 6.7 million deaths in 2021 - 1 every 5 seconds, it caused at least USD 966 billion dollars in health expenditure a 316% increase over the last 15 years [2].
Nowadays, mathematical modeling is considered as an effective and important tool for describing the cause and transmission dynamics of many common infectious and noninfectious diseases such as the HIV/AIDS [140, 84], COVID-19 [29, 102, 64], Cancer [38, 66], Ebola [45] and others. On the other side, fractional-order systems depend not only on the current state but on all of its historical states and thus have memory properties [69, 92]. Based on the special properties of fractional differentiation and integration, many scholars have applied the concept of fractional calculus to model several nonlinear phenomena in medicine, engineering, physics, and applied sciences, see for instance [26, 85, 126, 14].
Moreover, applied sciences such as pixel points in images, economy time series and sampled signals all hold discrete time or space structures. In addition to the achieved activities by using the standard fractional calculus, the discretization of the fractional derivative or integral readily introduce additional information or numerical errors, and difference equations are much more straightforward [58]. In 1989, fractional sum and differences have been initiated by Miller and Ross [80] where they started from the discrete version of Cauchy integral formula and introduced difference operators with discrete memory effects. Basic properties of fractional differences were given in [57, 5].
Further, variable-order calculus goes back to 1993 by Samko and Ross [104]. Given the fact that it is recognized as a natural extension of classical/fractional calculus with more successful results in many applications, their utilization can provide also sophisticated tools for studying the dynamical systems of diseases. The paper is organized as follows. Section 4.2 introduces discrete variable-order calculus and formulates an improved diabetes mathematical model. Section 4.3 investigates a variable-order model of diabetes in discrete-time, exploring its complications, the boundedness, existence and uniqueness of coupled solutions, and also treats the stability property. Section 4.4 describes the numerical scheme utilized, and in section 4.5, we present numerical simulations and their corresponding results.

4.2 Preliminaries and mathematical model

We start by brief notations and definitions related to discrete variable-order calculus [29, 58, 89, 7, 28]. We denote $\mathbb{N}_a$ and $\mathbb{N}_a^T$ as $\mathbb{N}_a = \{a, a+1, a+2, \ldots\}$, $\mathbb{N}_a^T = \{a, a+1, a+2, \ldots, T\}$.

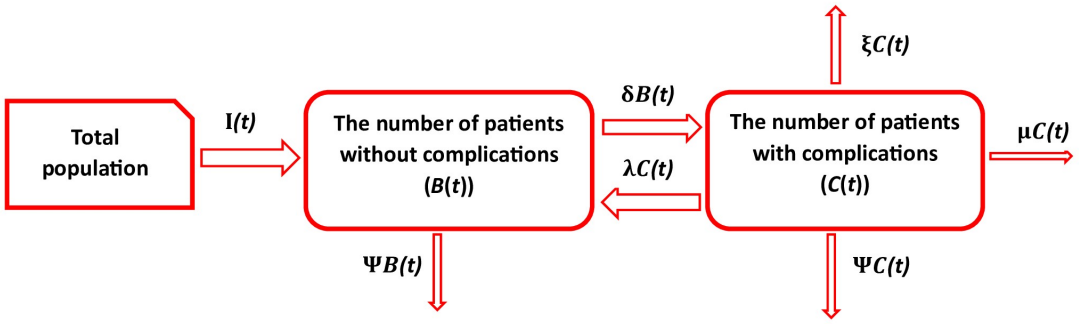


Figure 4.1: Diabetes chart.

Definition 4.2.1

Let $\alpha(t) > 0$ and $\sigma(s) = s + 1$. For $u(t)$ defined on $\mathbb{N}_a$, the delta variable-order sum of order $\alpha(t)$ is defined by

$$\Delta_a^{-\alpha(t)} u(t) = \frac{1}{\Gamma(\alpha(t))} \sum_{s=a}^{t-\alpha(t)} (t - \sigma(s))^{(\alpha(t)-1)} u(s), \tag{4.1}$$

where $t^{(\alpha(t))}$ is the discrete factorial functional given by $t^{(\alpha(t))} = \dfrac{\Gamma(t+1)}{\Gamma(t-\alpha(t)+1)}$.

Definition 4.2.2

For $u(t)$ defined on $\mathbb{N}_a, \alpha(t) > 0, \alpha \notin \mathbb{N}$, the delta Caputo variable-order difference is defined by

$${}^{C}\Delta_a^{\alpha(t)} u(t) = \Delta_a^{-(m-\alpha(t))} \Delta^m u(t) = \frac{1}{\Gamma(m-\alpha(t))} \sum_{s=a}^{t-(m-\alpha(t))} (t - \sigma(s))^{(m-\alpha(t)-1)} \Delta^m u(s), \tag{4.2}$$

where $t \in \mathbb{N}_{a+m-\alpha(t)}$, $m = [\alpha(t)] + 1$. Note that the forward difference operator is defined by $\Delta u(t) = u(t+1) - u(t)$.

Next, we shall formulate a new extended version of mathematical model to evaluate the impact of diabetes and its resulting complications, which was introduced by Boutayeb et al. [31], then improved by Srivastava et al. [115], Ahmad and Kirane [9]. The purpose of this model is to study the diabetic patients both with and without complications. Within this framework, the variables $C(t)$ and $B(t)$ denote patients with and without complications, respectively, at time t. We define the total number of diabetic individuals at time t as $N(t)$, expressed as the sum of patients without complications $B(t)$ and those with complications $C(t)$. To enhance comprehension of this model, we provide a detailed flow chart.
The parameters in Figure 4.1 are given as follow.
$I(t)$: The incidence of diabetes mellitus at time t;
$B(t)$: The number of persons having diabetics without complications at time t;
$C(t)$: The number of persons having diabetics with complications at time t;
$N(t)$: The size of the population of diabetics at time t;
δ: The probability of a person having diabetes and developing complications;
ψ: The natural rate of mortality;

λ: The rate of complications recovered;
ξ: The rate of mortality due to diabetic complications;
μ: The rate of diabetic patients having complication and becoming severely disabled.
We consider $I(t)$ patients of diabetes and diagnosed for some time interval t and without any resulting complications diagnosed. In this chart, $\delta B(t)$ represent the patients who have developed complications, $\lambda C(t)$ represent the patients whose complications are recovered, $\mu C(t)$ represent those patients who become strictly disabled and whose disabilities cannot be recovered, $\xi C(t)$ represent those patients who die from their complications. This can lead to the standard system of ordinary differential equation by [31] as follows

$$\frac{d}{dt}[B(t)] = I(t) - (\delta + \psi)B(t) + \lambda C(t) \tag{4.3}$$

and

$$\frac{d}{dt}[C(t)] = -(\lambda + \mu + \xi + \psi)C(t) + \delta B(t). \tag{4.4}$$

For short, we set

$$\nu = \lambda + \mu + \xi + \psi. \tag{4.5}$$

It is known that individuals with diabetes are learned who to survey, but with some other complication makes more riskier to live, so the model is modifiable to be represented by two unknowns $C(t)$ and $N(t)$. By using the relation

$$N(t) = B(t) + C(t) \tag{4.6}$$

together with (4.5), one can get

$$\begin{aligned} \frac{dC(t)}{dt} &= -(\delta + \nu)C(t) + \delta N(t), \\ \frac{dN(t)}{dt} &= I(t) - (\mu + \xi)C(t) - \psi N(t), \end{aligned} \qquad (t > 0) \tag{4.7}$$

with its initial conditions

$$C(0) = C_0 \quad \text{and} \quad N(0) = N_0. \tag{4.8}$$

Motivated by the above facts, we intend to apply discrete analysis and variable-order calculus on the mathematical model of diabetes and its complications. Below is the model formulation with the mentioned parameters.

$$\begin{aligned} \Delta_t^{\alpha(t)} C(t) &= -(\delta + \nu)C(t) + \delta N(t), & 0 < \alpha(t) \leq 1, \\ \Delta_t^{\beta(t)} N(t) &= I(t) - (\mu + \xi)C(t) - \psi N(t), & 0 < \beta(t) \leq 1, \end{aligned} \qquad (t > 0) \tag{4.9}$$

with the initial condition in (4.8). Here, the delta variable-order differences of model (4.9) are given in the sense of Caputo, where $\alpha(t), \beta(t) \in (0, 1]$.

4.3 Solvability and stability analysis

To prove the main results of this work related to the proposed model, we use the properties of discrete variable-order calculus. Now the model (4.9) can be written as follows:

$$\begin{aligned} C(t) &= C(0) + \Delta^{-\alpha(t)}(-(\delta + \nu)C(t) + \delta N(t)), \\ N(t) &= N(0) + \Delta^{-\beta(t)}(I(t) - (\mu + \xi)C(t) - \psi N(t)). \end{aligned} \tag{4.10}$$

We use the delta variable-order sum, we have

$$\begin{aligned} C(t) &= C_0 + \frac{1}{\Gamma(\alpha(t))} \sum_{s=0}^{t-\alpha(t)} (t-\sigma(s))^{(\alpha(t)-1)}(-(\delta+\nu)C(s)+\delta N(s)), \\ N(t) &= N_0 + \frac{1}{\Gamma(\beta(t))} \sum_{s=0}^{t-\beta(t)} (t-\sigma(s))^{(\beta(t)-1)}(I(s)-(\mu+\xi)C(s)-\psi N(s)). \end{aligned} \tag{4.11}$$

Let us consider the kernels M_1 and M_2 as

$$\begin{aligned} M_1(t, C(t)) &= -(\delta+\nu)C(t)+\delta N(t), \\ M_2(t, N(t)) &= I(t)-(\mu+\xi)C(t)-\psi N(t). \end{aligned} \tag{4.12}$$

Lemma 4.3.1

The kernels M_1 and M_2 satisfy the Lipschitz condition.

Proof: Let $C(t)$ and $\overline{C}(t)$ be any two functions, then

$$\begin{aligned} \|M_1(t, C(t)) - M_1(t, \overline{C}(t))\| &= \| -(\delta+\nu)(C(t)-\overline{C}(t))\| \\ &\leq |-(\delta+\nu)|\|C(t)-\overline{C}(t)\| \\ &\leq c_1\|C(t)-\overline{C}(t)\|. \end{aligned} \tag{4.13}$$

Similarly, for any $N(t)$ and $\overline{N}(t)$, we have

$$\begin{aligned} \|M_2(t, N(t)) - M_2(t, \overline{N}(t))\| &= \| -\psi(N(t)-\overline{N}(t))\| \\ &\leq |-\psi|\|N(t)-\overline{N}(t)\| \\ &\leq c_2\|\|N(t)-\overline{N}(t)\|. \end{aligned} \tag{4.14}$$

This concludes the result by considering that c_1 and c_2 as Lipschitz constants. ■

We make use of the following assumption.

(H) Let $K_{\alpha(t)}(t,s) = (t-\sigma(s))^{(\alpha(t)-1)}/\Gamma(\alpha(t))$, where $t^{(\alpha(t))}$ is the defined discrete factorial functional. We assume that $c_1 \sum_{s=0}^{t-\alpha(t)} K_{\alpha(t)}(t,s) < 1$ and $c_2 \sum_{s=0}^{t-\beta(t)} K_{\beta(t)}(t,s) < 1$.

Theorem 4.3.1

The coupled solutions of discrete variable-order model of diabetic and its complications exists and unique for $t > 0$ provided that assumption (H) is satisfied.

Proof: The system of equations in (4.11) becomes

$$\begin{aligned} C(t) &= C_0 + \frac{1}{\Gamma(\alpha(t))} \sum_{s=0}^{t-\alpha(t)} (t-\sigma(s))^{(\alpha(t)-1)} M_1(s, C(s)), \\ N(t) &= N_0 + \frac{1}{\Gamma(\beta(t))} \sum_{s=0}^{t-\beta(t)} (t-\sigma(s))^{(\beta(t)-1)} M_2(s, N(s)). \end{aligned} \tag{4.15}$$

By using the recursive formula, we have

$$\begin{aligned} C_n(t) &= C_0 + \frac{1}{\Gamma(\alpha(t))} \sum_{s=0}^{t-\alpha(t)} (t-\sigma(s))^{(\alpha(t)-1)} M_1(s, C_{n-1}(s)), \\ N_n(t) &= N_0 + \frac{1}{\Gamma(\beta(t))} \sum_{s=0}^{t-\beta(t)} (t-\sigma(s))^{(\beta(t)-1)} M_2(s, N_{n-1}(s)). \end{aligned} \tag{4.16}$$

Then, the difference between the consecutive terms gives

$$\begin{aligned} \vartheta_{C,n}(t) &= C_n(t) - C_{n-1}(t) \\ &= \frac{1}{\Gamma(\alpha(t))} \sum_{s=0}^{t-\alpha(t)} (t-\sigma(s))^{(\alpha(t)-1)} (M_1(s, C_{n-1}(s)) - M_1(s, C_{n-2}(s))), \\ \vartheta_{N,n}(t) &= N_n(t) - N_{n-1}(t) \\ &= \frac{1}{\Gamma(\beta(t))} \sum_{s=0}^{t-\beta(t)} (t-\sigma(s))^{(\beta(t)-1)} (M_2(s, N_{n-1}(s)) - M_2(s, N_{n-2}(s))), \end{aligned} \tag{4.17}$$

where

$$\begin{aligned} C_n(t) &= \sum_{j=0}^{n} \vartheta_{C,j}(t), \\ N_n(t) &= \sum_{j=0}^{n} \vartheta_{N,j}(t). \end{aligned} \tag{4.18}$$

Setting

$$\begin{aligned} \vartheta_{C,n-1}(t) &= C_{n-1}(t) - C_{n-2}(t), \\ \vartheta_{N,n-1}(t) &= N_{n-1}(t) - N_{n-2}(t). \end{aligned} \tag{4.19}$$

We use Lemma 4.3.1. Since the kernels M_1 and M_2 satisfy the Lipschitz condition, we obtain the following

$$\begin{aligned} \|\vartheta_{C,n}(t)\| &< \frac{c_1}{\Gamma(\alpha(t))} \sum_{s=0}^{t-\alpha(t)} (t-\sigma(s))^{(\alpha(t)-1)} \|\vartheta_{C,n-1}(s)\|, \\ \|\vartheta_{N,n}(t)\| &< \frac{c_2}{\Gamma(\beta(t))} \sum_{s=0}^{t-\beta(t)} (t-\sigma(s))^{(\beta(t)-1)} \|\vartheta_{N,n-1}(s)\|. \end{aligned} \tag{4.20}$$

Moreover, the unknown functions $C(t)$ and $N(t)$ are bounded, by using the recursive principle, the inequalities (4.20) involve

$$\begin{aligned} \|\vartheta_{C,n}(t)\| &< \|C_0(t)\| \left(c_1 \sum_{s=0}^{t-\alpha(t)} K_{\alpha(t)}(t,s) \right)^n, \\ \|\vartheta_{N,n}(t)\| &< \|N_0(t)\| \left(c_2 \sum_{s=0}^{t-\beta(t)} K_{\beta(t)}(t,s) \right)^n. \end{aligned} \tag{4.21}$$

Applying the limit as $n \to \infty$, we get $\|\vartheta_{C,n}(t)\| \to 0$ and $\|\vartheta_{N,n}(t)\| \to 0$. This shows the existence of the solutions.

Next, let us assume that there exists another solution to the model (4.9), say $C_1(t)$, $N_1(t)$. Then, it follows from (4.15),

$$\begin{aligned} C(t) - C_1(t) &= \frac{1}{\Gamma(\alpha(t))} \sum_{s=0}^{t-\alpha(t)} (t-\sigma(s))^{(\alpha(t)-1)} (M_1(s, C(s)) - M_1(s, C_1(s))), \\ N(t) - N_1(t) &= \frac{1}{\Gamma(\beta(t))} \sum_{s=0}^{t-\beta(t)} (t-\sigma(s))^{(\beta(t)-1)} (M_2(s, N(s)) - M_2(s, N_1(s))). \end{aligned} \tag{4.22}$$

By taking norm, we obtain

$$\begin{aligned}\|C(t)-C_1(t)\| &= \|\frac{1}{\Gamma(\alpha(t))}\sum_{s=0}^{t-\alpha(t)}(t-\sigma(s))^{(\alpha(t)-1)}(M_1(s,C(s))-M_1(s,C_1(s)))\|,\\ \|N(t)-N_1(t)\| &= \|\frac{1}{\Gamma(\beta(t))}\sum_{s=0}^{t-\beta(t)}(t-\sigma(s))^{(\beta(t)-1)}(M_2(s,N(s))-M_2(s,N_1(s)))\|.\end{aligned} \tag{4.23}$$

This implies

$$\begin{aligned}\|C(t)-C_1(t)\| = &\frac{1}{\Gamma(\alpha(t))}\sum_{s=0}^{t-\alpha(t)}(t-\sigma(s))^{(\alpha(t)-1)}\|(M_1(s,C(s))-M_1(s,C_1(s)))\|,\\ \|N(t)-N_1(t)\| = &\frac{1}{\Gamma(\beta(t))}\sum_{s=0}^{t-\beta(t)}(t-\sigma(s))^{(\beta(t)-1)}\|(M_2(s,N(s))-M_2(s,N_1(s)))\|.\end{aligned} \tag{4.24}$$

Using the Lipschitz condition for the kernels,

$$\|C(t)-C_1(t)\| < \frac{c_1}{\Gamma(\alpha(t))}\sum_{s=0}^{t-\alpha(t)}(t-\sigma(s))^{(\alpha(t)-1)}\|C(t)-C_1(t)\|, \tag{4.25}$$

$$\|N(t)-N_1(t)\| < \frac{c_2}{\Gamma(\beta(t))}\sum_{s=0}^{t-\beta(t)}(t-\sigma(s))^{(\beta(t)-1)}\|N(t)-N_2(t)\|. \tag{4.26}$$

In consequence

$$\|C(t)-C_1(t)\|\left(1-c_1\sum_{s=0}^{t-\alpha(t)}K_{\alpha(t)}(t,s)\right) < 0, \tag{4.27}$$

$$\|N(t)-N_1(t)\|\left(1-c_2\sum_{s=0}^{t-\beta(t)}K_{\beta(t)}(t,s)\right) < 0. \tag{4.28}$$

If $c_1\sum_{s=0}^{t-\alpha(t)}K_{\alpha(t)}(t,s)<1$ and $c_2\sum_{s=0}^{t-\beta(t)}K_{\beta(t)}(t,s)<1$, then it contradicts the property of norm. This proves the uniqueness result. ■

Next, we present the theory of stability for our discrete variable-order mathematical model of diabetes and its complications.

Theorem 4.3.2

The proposed system is locally stable in nature.

Proof: We compute the equilibrium point of the proposed model.

$$\Delta_t^{\alpha}(C(t)) = 0, \tag{4.29}$$

$$\Delta_t^{\beta}(N(t)) = 0, \tag{4.30}$$

which implies,

$$-(\delta+\nu)C(t)+\delta N(t) = 0, \tag{4.31}$$

$$I(t)-(\mu+\xi)C(t)-\psi N(t) = 0. \tag{4.32}$$

There exists a unique equilibrium point,

$$C(t) = \frac{I(t)\delta}{(\mu+\xi)\delta + \psi(\delta+\nu)}, \qquad N(t) = \frac{I(t)(\delta+\nu)}{(\mu+\xi)\delta + \psi(\delta+\nu)}. \tag{4.33}$$

Let us define the Jacobian matrix of the system over the unknown variables

$$\Upsilon = \begin{bmatrix} -(\delta+\nu) & \delta \\ -(\mu+\xi) & -\psi \end{bmatrix}.$$

The characteristic polynomial is

$$\Lambda^2 + (\delta+\nu+\psi)\Lambda + (\delta\psi + \nu\psi + \delta\mu + \delta\xi) = 0.$$

Since the described parameters are positive, and from other side the trace of Υ is negative, it yields to the negativity of Υ. This shows the local stability of the discrete variable-order system. ■

4.4 Numerical scheme

In this section, we establish the Adam-type predictor corrector method [?] for the introduced system (4.9) to solve the discrete-time variable-order model.

$$\begin{aligned} \Delta_t^{\alpha(t)} x(t) &= f(t, x(t)), \\ x(0)^k &= x_0^k, 0 < \alpha(t) < 1, 0 < t < \tau, \end{aligned} \tag{4.34}$$

Where $k = 0, 1, \ldots, n-1$. The above variable-order problem is equivalent to Volterra equation

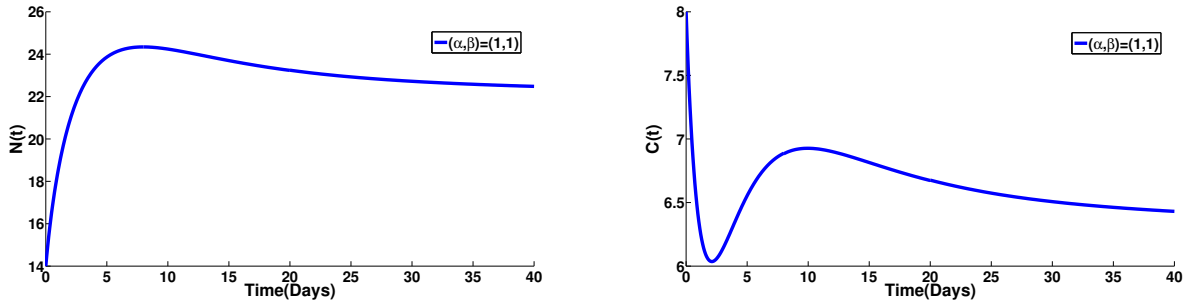
$$x(t) = \sum_{k=0}^{n-1} x_0^k \frac{t^k}{k!} + \frac{1}{\Gamma(\alpha(t))} \sum_{s=0}^{t-\alpha(t)} (t - \sigma(s))^{(\alpha(t)-1)} f(s, x(s)), \tag{4.35}$$

such that $n = [\alpha(t)]$, $\alpha(t) = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ and $\beta(t) = (\beta_1, \beta_2, \ldots, \beta_k)$. By means of letting $x_r \approx x(t_r)$, the discretization leads to

$$\begin{aligned} C_{w+1} &= \sum_{r=0}^{[\alpha(k)]} C_0^{(r)} \frac{t_{w+1}^{(r)}}{r!} + \frac{h^{\alpha(k)}}{\Gamma(\alpha(k)+2)} \sum_{r=0}^{w} (u_{r,w+1})(-(\delta+\nu)C_r + \delta N_r) \\ &\quad + \frac{h^{\alpha(k)}}{\Gamma(\alpha(k)+2)} (u_{w+1,w+1})(-(\delta+\nu)C_{w+1}^{PF} + \delta N_{w+1}^{PF}), \\ N_{w+1} &= \sum_{r=0}^{[\beta(k)]} N_0^{(r)} \frac{t_{w+1}^{(r)}}{r!} + \frac{h^{\beta(k)}}{\Gamma(\beta(k)+2)} \sum_{r=0}^{w} (v_{r,w+1})(I - (\mu+\xi)C_r - \psi N_r) \\ &\quad + \frac{h^{\beta(k)}}{\Gamma(\beta(k)+2)} (v_{w+1,w+1})(I - (\mu+\xi)C_{w+1}^{PF} - \psi N_{w+1}^{PF}), \end{aligned} \tag{4.36}$$

where

$$u_{r,w+1} = \begin{cases} w^{\alpha(k)+1} - (w-\alpha(k))(w+1)^{\alpha(k)}, & \text{if } r = 0 \\ (w-r+2)^{\alpha(k)+1} + (w-r)^{\alpha(k)+1} - 2(w-r+1)^{\alpha(k)+1}, & \text{if } 1 < r < w \\ 1, & \text{if } r = w+1 \ . \end{cases}$$



(a) (b)

Figure 4.2: Plot represents the solution of the system (4.9), for $(\alpha,\beta)=(1,1)$.

and

$$v_{r,w+1} = \begin{cases} w^{\beta(k)+1} - (w-\beta(k))(w+1)^{\beta(k)}, & \text{if } r = 0 \\ (w-r+2)^{\beta(k)+1} + (w-r)^{\beta(k)+1} - 2(w-r+1)^{\beta(k)+1}, & \text{if } 1 < r < w \\ 1, & \text{if } r = w+1 \ . \end{cases}$$

The proposed prediction formula is calculated as follow

$$C_{w+1}^{PF} = \sum_{r=0}^{[\alpha(k)]} C_0^{(r)} \frac{t_{w+1}^{(r)}}{r!} + \frac{h^{\alpha(k)}}{\Gamma(\alpha(k)+1)} \sum_{r=0}^{w} (j_{r,w+1})(-(\delta+\nu)C_r + \delta N_r), \tag{4.37}$$

$$N_{w+1}^{PF} = \sum_{r=0}^{[\beta(k)]} N_0^{(r)} \frac{t_{w+1}^{(r)}}{r!} + \frac{h^{\beta(k)}}{\Gamma(\beta(k)+1)} \sum_{r=0}^{w} (l_{r,w+1})(I - (\mu+\xi)C_r - \psi N_r), \tag{4.38}$$

where

$$j_{r,w+1} = (w+1-r)^{\alpha(k)} - (w-r)^{\alpha(k)} \tag{4.39}$$

and

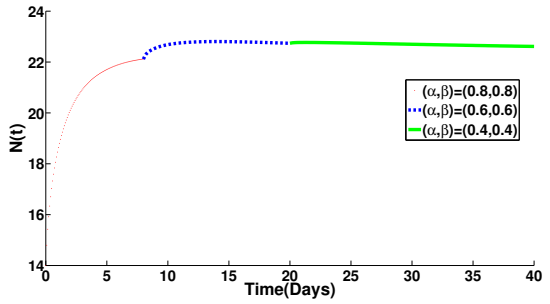
$$l_{r,w+1} = (w+1-r)^{\beta(k)} - (w-r)^{\beta(k)}. \tag{4.40}$$

4.5 Numerical simulations and results

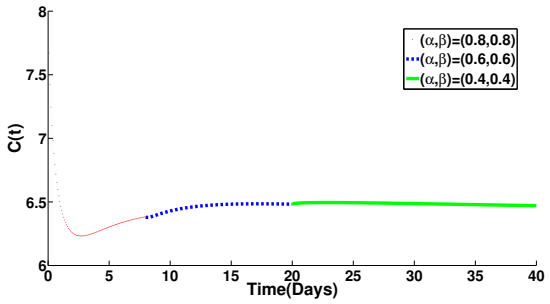
In this section, we shown the effects of variable fractional-order derivative for the proposed model (4.9), by using the numerical scheme described in the above section 4.4. The model parameters are assumed as $\mu = 0.0207$; $\xi = 0.693$; $\psi = 0.2$; $\delta = 0.2$; $\nu = 0.5$; $I(t) = 5(1+1/(t+1))$. Additionally, we establish the initial value conditions for the system of equation (4.9) as follows:

$$C_0 = 8, \ N_0 = 14.$$

Here, we solve the proposed model (4.9) of fractional variable order in discrete time using the method outlined in the previous section. Figures (4.2)-(4.5) illustrates the importance of the variable order of α and β over the unknown variables $C(t)$ and $N(t)$.
Figure (4.2) shows the solution plot for a fixed α and β value for 1. Here the system, take more time to stabilize and to attain the equilibrium point. Figure (4.3) shows the solution plot for a α and β value varies as (0.8,0.6,0.4) in a every interval time. Here the solution attains

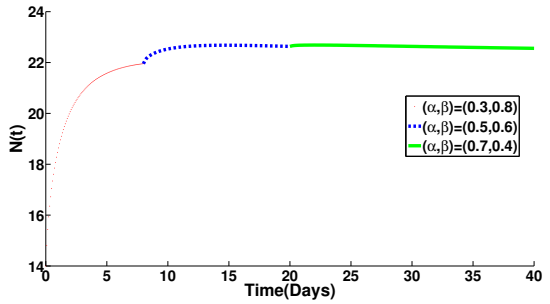


(a)

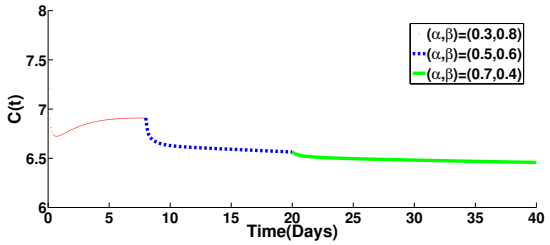


(b)

Figure 4.3: Plot represents the solution of the system (4.9), for (α,β)=((0.8,0.6,0.4),(0.8,0.6,0.4)).

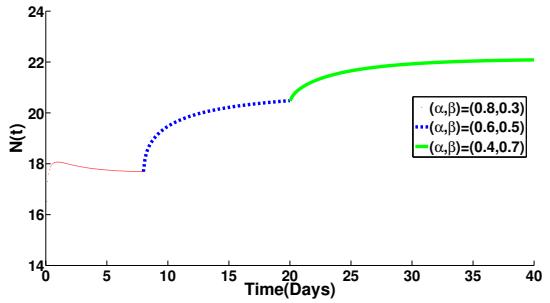


(a)

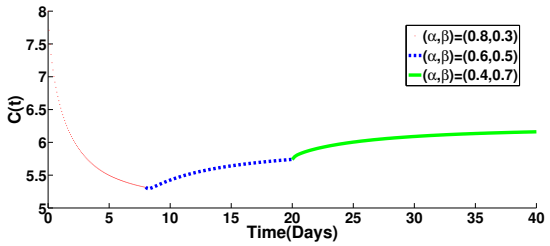


(b)

Figure 4.4: Plot represents the solution of the system (4.9), for (α,β)=((0.3,0.5,0.7),(0.8,0.6,0.4)).

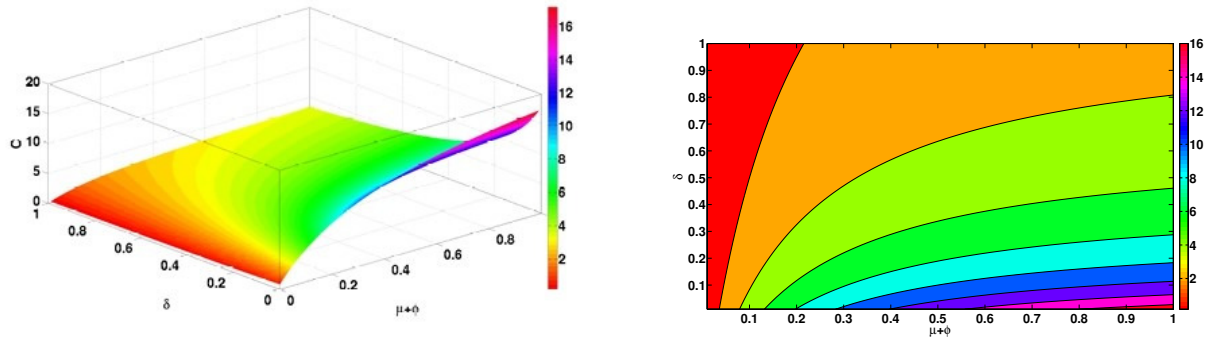


(a)



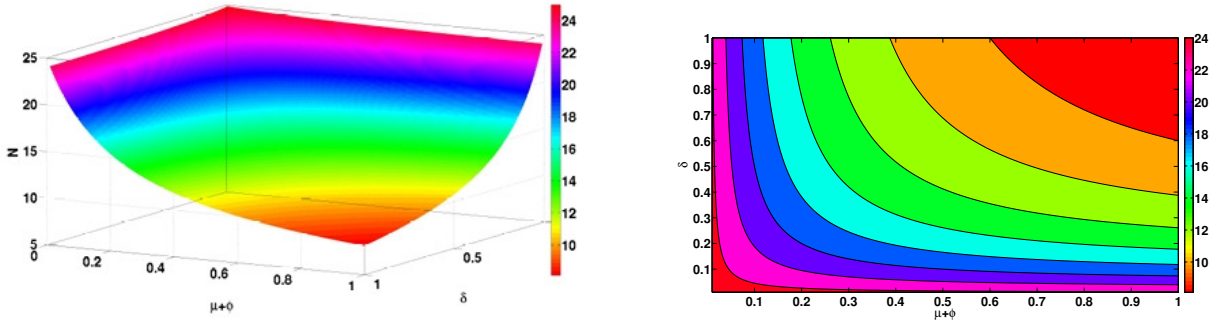
(b)

Figure 4.5: Plot represents the solution of the system (4.9), for (α,β)=((0.8,0.6,0.4),(0.3,0.5,0.7)).



(a) (b)

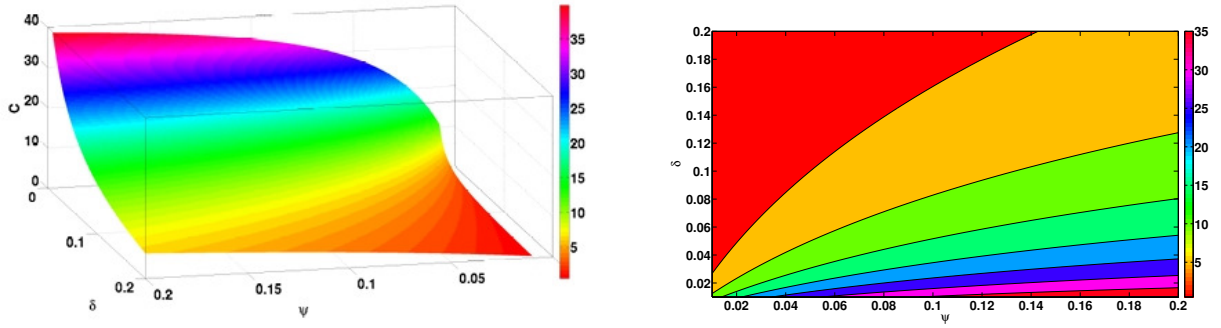
Figure 4.6: Plot represents the sensitivity of the C(Population with complications) of the model (4.9) with respect to the model parameters δ and $\mu + \xi$



(a) (b)

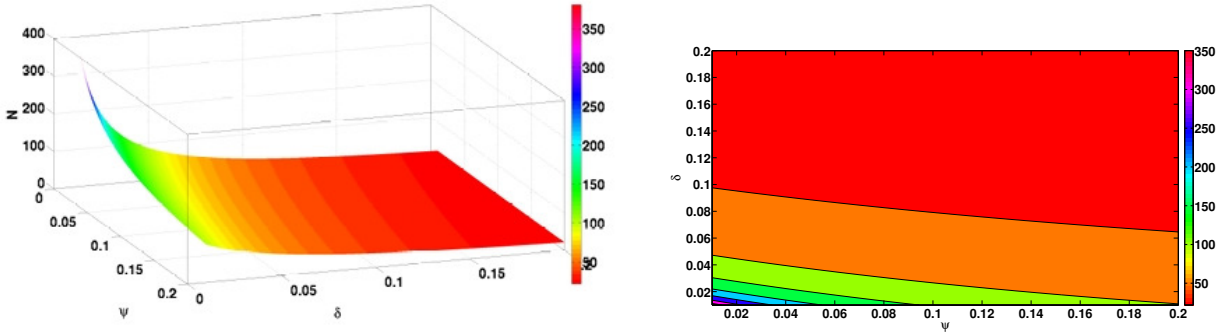
Figure 4.7: Plot represents the sensitivity of the N(total population) of the model (4.9) with respect to the model parameters δ and $\mu + \xi$

in stability in a shorter time. Similarly figure (4.4) and (4.5) shows the solution of the system (4.9), by increasing thing one fractional derivative and decreasing other one. here the solution are forming in a step wise manner and take larger time frame to reach equilibrium point.
Next, we shown the sensitivity of the steady state solutions of the proposed model (4.9), with respect to the model parameters in the Figure (4.6)-(4.9). Moreover we shown the counter plot of the same to understand more detailed of the sensitivity of the model parameters over the unknown variables. For this study we considered, δ $\mu + \xi$ and ψ values varying from 0 to 1, and I is considered as 5. Figure (4.6) shows the sensitivity of the C with respect to δ and $\mu + \xi$. From the figure, we able to notice that, by increasing the rate of $\mu + \psi$ make the system to attain a high risky state. Figure (4.7) shows the sensitivity of the N with respect to δ and $\mu + \xi$. From the figure, we able to notice that, by increasing the rate of δ the total population of the system will be reduced.
Figure (4.8) shows the sensitivity of the C with respect to δ and ψ. From the figure, we able to notice that, by increasing the rate of δ the total population of the system will reduce. Figure (4.9) shows the sensitivity of the N with respect to δ and ψ. From the figure, we able to notice that, by increasing the rate of ψ the total population of the system will be reduced. Sensitivity study towards the system (4.9), shows the importance of δ and ψ.



(a) (b)

Figure 4.8: Plot represents the sensitivity of C(Population with complications) of the model (4.9) with respect to the model parameters δ and ψ



(a) (b)

Figure 4.9: Plot represents the sensitivity of N(total population) of the model (4.9) with respect to the model parameters δ and ψ

Conclusion

This Chapter (4) we proposed a discrete variable-order diabetes model. We explored the potential application of discrete variable-order calculus in our analysis. We proved the existence and uniqueness of the solution for the introduced model. Moreover, we calculated the equilibrium point and studied the stability nature of the obtained point. Numerical scheme of Adam's predictor-corrector method for the proposed model is studied. Finally, the numerical values are taken from the existing literature, we solved the discrete variable-order model using the MATLAB and the results were discussed.

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