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SOME PROPERTIES OF FINITE ORDER SOLUTIONS OF A CLASS OF LINEAR DIFFERENTIAL EQUATIONS WITH ENTIRE COEFFICIENTS

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Abstract In this paper, we investigate the growth of solutions of some higher order linear differential equations. We find conditions on the coefficients which will guarantee the existence of an asymptotic value for a transcendental entire solution of finite order and its derivatives.

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1. INTRODUCTION

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory [9], [10], [15]. In addition, we will use the notation $\sigma(f)$ to denote the order of an entire function $f(z)$, $\tau(f)$ to denote the type of $f(z)$ with $0 < \sigma(f) = \sigma < +\infty$, which is defined to be

$$\tau(f) = \limsup_{r\to+\infty} \frac{\log M(r, f)}{r^{\sigma}},$$

where $M(r, f) = \max_{|z|=r} |f(z)|$. The hyper-order $\sigma_2(f)$ of f is defined by ([15])

$$\sigma_2(f) = \limsup_{r\to+\infty} \frac{\log\log T(r, f)}{\log r} = \limsup_{r\to+\infty} \frac{\log\log\log M(r, f)}{\log r},$$

where $T(r, f)$ is the Nevanlinna characteristic function of f.

For $k \geqslant 2$, we consider the linear differential equation

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = 0, \tag{1}$$

where $A_0(z), \cdots, A_{k-1}(z)$ are entire functions with $A_0(z) \not\equiv 0$. It is well known that all solutions of (1) are entire functions. A classical result, due to Wittich [14], tells that all solutions of (1) are of finite order of growth if and only if all coefficients are polynomials. For a complete analysis of possible orders in the polynomial case, see [8]. If some (or all) of the coefficients are transcendental, a natural question is to ask when and how many solutions of finite order may appear. Partial results have been available since a paper of Frei [4]. In its generality, however, the problem remains open.

Throughout this paper, we will use the following notations: Given $\varepsilon > 0$ small enough and $0 \leqslant \theta_1 < \theta_2 < 2\pi$, let $S(\varepsilon)$ denote the sector $S(\varepsilon) = \{z \in \mathbb{C} : \theta_1 + \varepsilon \leqslant \arg z \leqslant \theta_2 - \varepsilon\}$. We also denote $S(0) = \{z \in \mathbb{C} : \theta_1 \leqslant \arg z \leqslant \theta_2\}$.

In [5], Gundersen proved the following results.

Theorem 1.1. *[5] Let $A_0(z) \not\equiv 0$ and $A_1(z)$ be entire functions such that for real constants $\alpha, \beta, \theta_1, \theta_2$ with $\alpha > 0, \beta > 0$ and $\theta_1 < \theta_2$, we have*

$$|A_0(z)| \geqslant \exp\left\{(1 + o(1))\, \alpha\, |z|^{\beta}\right\}$$

and

$$|A_1(z)| \leqslant \exp\left\{o(1)\, |z|^{\beta}\right\}$$

as $z \to \infty$ with $\theta_1 \leqslant \arg z \leqslant \theta_2$. Then every solution $f \not\equiv 0$ of the differential equation

$$f'' + A_1(z) f' + A_0(z) f = 0 \qquad (2)$$

has infinite order.

Theorem 1.2. *[5] Let $A_0(z) \not\equiv 0$ and $A_1(z)$ be entire functions such that for real constants $\alpha, \beta, \theta_1, \theta_2$ with $\alpha > 0, \beta > 0$ and $\theta_1 < \theta_2$, we have*

$$|A_1(z)| \geqslant \exp\left\{(1 + o(1))\, \alpha\, |z|^{\beta}\right\}$$

and

$$|A_0(z)| \leqslant \exp\left\{o(1)\, |z|^{\beta}\right\}$$

as $z \to \infty$ with $\theta_1 \leqslant \arg z \leqslant \theta_2$. If f is a nontrivial solution of (2) of finite order, then the following conclusions hold:
(i) There exists a constant $b \neq 0$ such that $f(z) \to b$ as $z \to \infty$ in $S(\varepsilon)$. Indeed,

$$|f(z) - b| \leqslant \exp\left\{-(1 + o(1))\, \alpha\, |z|^{\beta}\right\}.$$

(ii) For each integer $k \geqslant 1$

$$\left|f^{(k)}(z)\right| \leqslant \exp\left\{-(1 + o(1))\, \alpha\, |z|^{\beta}\right\}$$

as $z \to \infty$ in $S(\varepsilon)$.

Theorem 1.1 has been generalized to the higher order case by Belaïdi and Hamouda as follows.

Theorem 1.3. *[1] Let* $A_0(z) \not\equiv 0, A_1(z), \cdots, A_{k-1}(z)$ *be entire functions such that for real constants* $\alpha, \beta, \mu, \theta_1, \theta_2$ *with* $0 \leqslant \beta < \alpha$, $\mu > 0$ *and* $\theta_1 < \theta_2$, *we have*

$$|A_0(z)| \geqslant e^{\alpha|z|^{\mu}}$$

and

$$\left|A_j(z)\right| \leqslant e^{\beta|z|^{\mu}} \ (j = 1, \cdots, k-1)$$

as $z \to \infty$ *with* $\theta_1 \leqslant \arg z \leqslant \theta_2$. *Then every solution* $f \not\equiv 0$ *of* (1) *has infinite order.*

Theorem 1.2 has been generalized to the higher order case by Belaïdi and Hamani as follows.

Theorem 1.4. *[2] Let* $A_0(z) \not\equiv 0, A_1(z), \cdots, A_{k-1}(z)$ *be entire functions such that for real constants* $\alpha, \beta, \theta_1, \theta_2$ *with* $\alpha > 0, \beta > 0$ *and* $\theta_1 < \theta_2$, *we have*

$$|A_1(z)| \geqslant \exp\left\{(1 + o(1))\, \alpha\, |z|^{\beta}\right\},$$

$$\left|A_j(z)\right| \leqslant \exp\left\{(o(1))\, |z|^{\beta}\right\}, \quad j = 0, 2, 3, \cdots, k-1$$

as $z \to \infty$ *with* $\theta_1 \leqslant \arg z \leqslant \theta_2$. *If* f *is a nontrivial solution of* (1) *of finite order, then the following conclusions hold:*
(i) There exists a constant $b \neq 0$ *such that* $f(z) \to b$ *as* $z \to \infty$ *in* $S(\varepsilon)$. *Indeed,*

$$|f(z) - b| \leqslant \exp\left\{-(1 + o(1))\, \alpha\, |z|^{\beta}\right\}.$$

(ii) For each integer $m \geqslant 1$

$$\left|f^{(m)}(z)\right| \leqslant \exp\left\{-(1 + o(1))\, \alpha\, |z|^{\beta}\right\}$$

as $z \to \infty$ *in* $S(\varepsilon)$.

This result has been generalized by Laine and Yang by taking $A_s(z)$ instead of $A_1(z)$.

Theorem 1.5. *[11] Let* $\theta_1 < \theta_2$ *be given to fix a sector* $S(0) = \{z \in \mathbb{C} : \theta_1 \leqslant \arg z \leqslant \theta_2\}$, *let* $k \geqslant 2$ *be a natural number, and let* $\delta > 0$ *be any real number such that* $k\delta < 1$. *Suppose that* $A_0(z), A_1(z), \cdots, A_{k-1}(z)$ *with* $A_0(z) \not\equiv 0$ *are entire functions such that for real constants* $\alpha > 0, \beta > 0$, *we have, for some* $s = 1, \cdots, k-1$,

$$|A_s(z)| \geqslant \exp\left\{(1 + \delta)\, \alpha\, |z|^{\beta}\right\}, \tag{3}$$

$$\left|A_j(z)\right| \leqslant \exp\left\{\delta\alpha\, |z|^{\beta}\right\} \tag{4}$$

for all $j = 0, \cdots, s-1, s+1, \cdots, k-1$ *whenever* $|z| = r \geqslant r_\delta$ *in the sector* $S(0)$. *Given* $\varepsilon > 0$ *small enough, if* f *is a transcendental solution of* (1) *of finite order* $\rho < \infty$, *then the following conclusions hold:*
(i) There exists $j \in \{0, \cdots, s-1\}$ *and a complex constant* $b_j \neq 0$ *such that* $f^{(j)}(z) \to b_j$ *as* $z \to \infty$ *in* $S(\varepsilon)$. *More precisely,*

$$\left|f^{(j)}(z) - b_j\right| \leqslant \exp\left\{-(1 - k\delta)\,\alpha\,|z|^\beta\right\}$$

in $S(\varepsilon)$, *provided* $|z|$ *is large enough.*
(ii) For each integer $m \geqslant j + 1$,

$$\left|f^{(m)}(z)\right| \leqslant \exp\left\{-(1 - k\delta)\,\alpha\,|z|^\beta\right\}$$

in $S(3\varepsilon)$ *for all* $|z|$ *large enough.*

Recently in [13], Tu and Yi investigated the case when most coefficients in (1) have the same order with each other and obtained the following result.

Theorem 1.6. [13] *Let* $A_j(z)$ $(j = 0, \cdots, k-1)$ *be entire functions satisfying* $\sigma(A_0) = \sigma$, $\tau(A_0) = \tau$, $0 < \sigma < \infty$, $0 < \tau < \infty$, *and let* $\sigma\left(A_j\right) \leqslant \sigma$, $\tau\left(A_j\right) < \tau$ *if* $\sigma\left(A_j\right) = \sigma$ $(j = 1, \cdots, k-1)$, *then every solution* $f \not\equiv 0$ *of* (1) *satisfies* $\sigma_2(f) = \sigma(A_0)$.

The remainder of the paper is organized as follows. In Section 2, we shall show our main results which improve and extend many results in the above-mentioned papers. Section 3 is for some lemmas and basic theorems. The other sections are for the proofs of our main results.

2. RESULTS

In this paper, we extend the above results by proving the following two theorems.

Theorem 2.1. *Let* $A_0(z) \not\equiv 0, A_1(z), \cdots, A_{k-1}(z)$ *be entire functions such that for real constants* $\alpha, \beta, \mu, \theta_1, \theta_2$ *with* $0 \leqslant \beta < \alpha$, $\mu > 0$ *and* $(\theta_1, \theta_2) \subset \left[0, \frac{\pi}{2n}\right) \cup \left(\frac{3\pi}{2n}, \frac{2\pi}{n}\right)$, *we have*

$$|A_0(z)| \geqslant \exp\{\alpha\,|z|^\mu\} \tag{5}$$

and

$$\left|A_j(z)\right| \leqslant \exp\{\beta\,|z|^\mu\} \quad (j = 1, \cdots, k-1) \tag{6}$$

for $\arg z \in (\theta_1, \theta_2)$, *with* $|z|$ *large enough. Then every solution* $f \not\equiv 0$ *of the differential equation*

$$f^{(k)} + A_{k-1}(z)\,e^{z^n} f^{(k-1)} + \cdots + A_1(z)\,e^{z^n} f' + A_0(z)\,e^{z^n} f = 0 \tag{7}$$

has infinite order.

Theorem 2.2. *Let* $A_0(z) \not\equiv 0, A_1(z), \cdots, A_{k-1}(z)$ *be entire functions such that for real constants* $\alpha, \beta, \mu, \theta_1, \theta_2$ *with* $0 \leqslant \beta < \alpha$, $\mu > 0$ *and* $(\theta_1, \theta_2) \subset \left[0, \frac{\pi}{2n}\right) \cup \left(\frac{3\pi}{2n}, \frac{2\pi}{n}\right)$, *we have for some* $s = 1, \cdots, k-1$,

$$|A_s(z)| \geqslant \exp\{\alpha |z|^{\mu}\} \tag{8}$$

and

$$\left|A_j(z)\right| \leqslant \exp\{\beta |z|^{\mu}\} \tag{9}$$

for all $j = 0, \cdots, s-1, s+1, \cdots, k-1$ *for* $\arg z \in (\theta_1, \theta_2)$, *with* $|z|$ *large enough. Given* $\varepsilon > 0$ *small enough, if* f *is a transcendental solution of* (7) *of finite order* $\sigma < \infty$, *then the following conclusions hold:*

(i) There exists $j \in \{0, \cdots, s-1\}$ *and a complex constant* $b_j \neq 0$ *such that* $f^{(j)}(z) \to b_j$ *as* $z \to \infty$ *in* $S(\varepsilon)$. *More precisely,*

$$\left|f^{(j)}(z) - b_j\right| \leqslant \exp\{(\beta + \tau - \alpha)|z|^{\mu}\}$$

in $S(\varepsilon)$, *provided* $|z|$ *is large enough, where* $0 < \tau < \frac{\alpha-\beta}{k}$.

(ii) For each integer $m \geqslant j+1$,

$$\left|f^{(m)}(z)\right| \leqslant \exp\{(\beta + \tau - \alpha)|z|^{\mu}\}$$

in $S(3\varepsilon)$, *provided* $|z|$ *is large enough, where* $0 < \tau < \frac{\alpha-\beta}{k}$.

3. PRELIMINARY LEMMAS

Lemma 3.1. [6] *Let* f *be a transcendental entire function of finite order* σ, *let* $\Gamma = \{(k_1, j_1), (k_2, j_2), \cdots, (k_m, j_m)\}$ *denote a finite set of distinct pairs of integers that satisfy* $k_i > j_i \geqslant 0$ $(i = 1, \cdots, m)$, *and let* $\varepsilon > 0$ *be a given constant. Then there exists a set* $E \subset [0, 2\pi)$ *that has linear measure zero, such that if* $\psi_0 \in [0, 2\pi) - E$, *then there is a constant* $R_0 = R_0(\psi_0) > 1$ *with the property that for all* z *satisfying* $\arg z = \psi_0$ *and* $|z| \geqslant R_0$, *and for all* $(k, j) \in \Gamma$, *we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leqslant |z|^{(k-j)(\sigma-1+\varepsilon)}.$$

Remark 3.1. *In this paper, we use Lemma 3.1 several times and the resulting exceptional sets E are not the same each time, although they are denoted by the same letter.*

Lemma 3.2. [11] *Let* $f(z)$ *be an entire function and suppose that* $\left|f^{(k)}(z)\right|$ *is unbounded on some ray* $\arg z = \theta$. *Then there exists an infinite sequence of points* $z_j = r_j e^{i\theta}$ $(j = 1, 2, \cdots)$, *where* $r_j \to +\infty$, *such that* $f^{(k)}\left(z_j\right) \to \infty$ *and*

$$\left|\frac{f^{(q)}\left(z_j\right)}{f^{(k)}\left(z_j\right)}\right| \leqslant \frac{1}{(k-q)!}(1 + o(1))\left|z_j\right|^{k-q} \quad (q = 0, \cdots, k-1).$$

Lemma 3.3. *Given $\mu > 0$ and $h > 0$, the integral $I(r) = \int_r^{+\infty} \exp(-ht^{\mu})\,dt$ converges. More precisely we have*

$$I(r) \leqslant \exp\{(-h+\varepsilon)\, r^{\mu}\}$$

for r large enough and $\varepsilon > 0$.

Proof. It is easy to prove that $\int_r^{+\infty} \exp(-ht^{\mu})\,dt$ converges. For r large enough, we have

$$I(r) = \int_r^{+\infty} \exp(-ht^{\mu})\,dt = \int_r^{+\infty} \frac{t^2}{t^2} \exp(-ht^{\mu})\,dt$$

$$\leqslant r^2 \exp(-hr^{\mu}) \int_r^{+\infty} \frac{1}{t^2} dt = r \exp(-hr^{\mu}) \leqslant \exp\{(-h+\varepsilon)\, r^{\mu}\} \quad (\varepsilon > 0).$$

∎

Lemma 3.4. *(Phragmén-Lindelöf Theorem, see [12], p. 214). Let $f(z)$ be analytic in the sector $D = \{z : \alpha < \arg z < \beta,\ r_0 < |z| < \infty\}$ and continuous on $\overline{D} = D \cup \Gamma$, where Γ is the boundary of D. If for any given small $\varepsilon > 0$, there exists $r_1(\varepsilon) > 0$ such that for $|z| \geqslant r_1(\varepsilon)$, $z \in D$, we have*

$$|f(z)| < \exp\left\{\varepsilon |z|^{\frac{\pi}{\beta-\alpha}}\right\},$$

and for $z \in \Gamma$, we have $|f(z)| \leqslant M$ ($M > 0$ is a constant), then $|f(z)| \leqslant M$ for all $z \in D$. $|f(z)| = M$ if and only if f is a constant.

Remark 3.2. *(see [3]). Now suppose that $g(z)$ is analytic in the sector $D = \{z : \alpha < \arg z < \beta,\ r_0 \leqslant |z| < \infty\}$ and satisfies $|g(z)| \leqslant \exp\{|z|^{\sigma}\}$ for some constant $0 \leqslant \sigma < \infty$. If a subset $E \subset (\alpha,\beta)$ has linear measure zero and for any $\psi_0 \in (\alpha,\beta) - E$, $|g(z)|$ is bounded for all z satisfying $\arg z = \psi_0$ and $|z| \geqslant r_0$, then for any given small $\varepsilon > 0$, there exists $r_1(\varepsilon) > r_0$ such that $|g(z)| \leqslant \exp\left\{\varepsilon |z|^{\sigma+1}\right\}$ for $|z| = r > r_1$. We may choose points $\theta_j \in (\alpha,\beta) - E$ $(j = 1, \cdots, n)$ such that $\theta_1 < \theta_2 < \cdots < \theta_n$ $(\alpha < \theta_1 \leqslant \alpha + \varepsilon,\ \beta - \varepsilon \leqslant \theta_n < \beta)$ and $\max\{\theta_{j+1} - \theta_j : 1 \leqslant j \leqslant n-1\} < \frac{\pi}{\sigma+1}$. Now from Lemma 3.4, $|g(z)| \leqslant M$ holds in the sectors $\{z : \theta_j \leqslant \arg z \leqslant \theta_{j+1}, |z| \geqslant r_0\}$ $(j = 1, \cdots, n-1)$. Hence $|g(z)| \leqslant M$ holds in the sector $\{z : \alpha + \varepsilon \leqslant \arg z \leqslant \beta - \varepsilon, |z| \geqslant r_0\}$.*

Here, we give a special case of the result due to G. G. Gundersen and E. Steinbart in [7]:

Lemma 3.5. *Let $k \geqslant 2$ be an integer. Suppose that $w(z)$ is an entire function, where $\sigma(w) < +\infty$. Let $\lambda, \mu, \eta, \theta_1$ and θ_2 be real constants satisfying $\lambda > 0, \mu > 0, 0 < \eta < \frac{\lambda}{k}$*

and $\theta_1 < \theta_2$. *Suppose that there exists a set* $E \subset \mathbb{R}$ *that has linear measure zero such that for any* $\theta \in (\theta_1, \theta_2) - E$, *we have*

$$|w(z)| \leqslant \exp\{(-\lambda + \eta)|z|^{\mu}\} \tag{10}$$

as $z \longrightarrow \infty$ *along* $\arg z = \theta$. *Then for any* $\theta \in (\theta_1, \theta_2)$, (10) *holds as* $z \longrightarrow \infty$ *along* $\arg z = \theta$.

4. PROOF OF THEOREM 2.1

Suppose that $f \not\equiv 0$ is a solution of (7) with $\sigma(f) = \sigma < \infty$. From Lemma 3.1, there exists a set $E \subset [0, 2\pi)$ that has linear measure zero, such that if $\psi_0 \in (\theta_1, \theta_2) - E$, then

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leqslant |z|^{k\sigma}, \quad j = 1, \cdots, k \tag{11}$$

as $z \to \infty$ with $\arg z = \psi_0 \in (\theta_1, \theta_2) \subset \left[0, \frac{\pi}{2n}\right) \cup \left(\frac{3\pi}{2n}, \frac{2\pi}{n}\right)$. From (7), we obtain

$$1 \leqslant \left|\frac{1}{A_0(z)e^{z^n}}\right|\left|\frac{f^{(k)}}{f}\right| + \sum_{j=1}^{k-1}\frac{|A_j(z)|}{|A_0(z)|}\left|\frac{f^{(j)}}{f}\right|. \tag{12}$$

By using (5), (6) and (11), we get

$$\left|\frac{A_j(z)}{A_0(z)}\right|\left|\frac{f^{(j)}(z)}{f(z)}\right| \leqslant \frac{1}{e^{(\alpha-\beta)|z|^{\mu}}}|z|^{k\sigma} \;\; (j = 1, \cdots, k-1).$$

So

$$\lim_{z\to\infty}\left|\frac{A_j(z)}{A_0(z)}\right|\left|\frac{f^{(j)}(z)}{f(z)}\right| = 0 \;\; (j = 1, \cdots, k-1). \tag{13}$$

Also from (5) and (11), we have

$$\left|\frac{1}{A_0(z)e^{z^n}}\right|\left|\frac{f^{(k)}(z)}{f(z)}\right| \leqslant \frac{1}{e^{\alpha|z|^{\mu}+|z|^n\cos(n\arg z)}}|z|^{k\sigma}$$

and by taking $z \to \infty$ with $\arg z \in (\theta_1, \theta_2) \subset \left[0, \frac{\pi}{2n}\right) \cup \left(\frac{3\pi}{2n}, \frac{2\pi}{n}\right)$, we obtain

$$\lim_{z\to\infty}\left|\frac{1}{A_0(z)e^{z^n}}\right|\left|\frac{f^{(k)}(z)}{f(z)}\right| = 0. \tag{14}$$

Using (13) and (14) in (12) we get a contradiction. So, every solution $f \not\equiv 0$ of (7) has infinite order.

5. PROOF OF THEOREM 2.2

First we prove that $f^{(s)}(z)$ is bounded in $S(\varepsilon)$. From Lemma 3.1, there exists a set $E \subset [0, 2\pi)$ that has linear measure zero, such that for all $j = s+1, \cdots, k$

$$\left|\frac{f^{(j)}(z)}{f^{(s)}(z)}\right| \leqslant |z|^{(j-s)(\sigma-1+\varepsilon)} \leqslant |z|^{k\sigma} \tag{15}$$

along any ray $\arg z = \varphi \in (\theta_1, \theta_2) - E$ with $|z| \geqslant r_1$ large enough, provided $0 < \varepsilon < 1$. If we suppose that $f^{(s)}(z)$ is unbounded on some ray $\arg z = \phi \in S(0) \setminus E$, then by Lemma 3.2 there exists an infinite sequence of points $z_q = r_q e^{i\phi}$ $(q = 1, 2, \cdots)$, where $r_q \to +\infty$, such that $f^{(s)}\left(z_q\right) \to \infty$ and

$$\left|\frac{f^{(j)}\left(z_q\right)}{f^{(s)}\left(z_q\right)}\right| \leqslant \frac{1}{(s-j)!}(1+o(1))\left|z_q\right|^{s-j} \leqslant 2\left|z_q\right|^k \tag{16}$$

for all $j = 0, \cdots, s-1$ when $\left|z_q\right| \geqslant r_2$ large enough. From (7), we can write

$$1 \leqslant \frac{1}{|A_s(z)|}\left|\frac{1}{e^{z^n}}\right|\left|\frac{f^{(k)}(z)}{f^{(s)}(z)}\right| + \frac{|A_{k-1}(z)|}{|A_s(z)|}\left|\frac{f^{(k-1)}(z)}{f^{(s)}(z)}\right|$$
$$+ \cdots + \frac{|A_{s+1}(z)|}{|A_s(z)|}\left|\frac{f^{(s+1)}(z)}{f^{(s)}(z)}\right| + \frac{|A_{s-1}(z)|}{|A_s(z)|}\left|\frac{f^{(s-1)}(z)}{f^{(s)}(z)}\right|$$
$$+ \cdots + \frac{|A_0(z)|}{|A_s(z)|}\left|\frac{f(z)}{f^{(s)}(z)}\right|. \tag{17}$$

Combining (8), (9), (15) and (16) with (17) we obtain a contradiction as $r_q \to +\infty$. Therefore, $f^{(s)}(z)$ remains bounded on all rays $\arg z = \phi \in S(0) \setminus E$. By Remark 3.2, we conclude that $f^{(s)}(z)$ is bounded, say $\left|f^{(s)}(z)\right| \leqslant M$, in the whole sector $S(\varepsilon)$. By s-fold iterated integration along the line segment $[0, z]$, we have

$$f(z) = f(0) + f'(0)z + \cdots + \frac{1}{(s-1)!}f^{(s-1)}(0)z^{s-1}$$
$$+ \int_0^z \cdots \int_0^\zeta \int_0^\xi f^{(s)}(t)\,dt d\xi \cdots du.$$

So, we get

$$|f(z)| \leqslant M'|z|^s. \tag{18}$$

Now, from (7), we can write

$$\left|f^{(s)}(z)\right| \leqslant \frac{|f(z)|}{|A_s(z)|}\left(\left|\frac{1}{e^{z^n}}\right|\left|\frac{f^{(k)}(z)}{f(z)}\right| + |A_{k-1}(z)|\left|\frac{f^{(k-1)}(z)}{f(z)}\right|\right.$$

$$+\cdots+|A_1(z)|\left|\frac{f'(z)}{f(z)}\right|+|A_0(z)|\Bigg) \tag{19}$$

and recalling Lemma 3.1, (18) and the assumptions (8) and (9), we conclude

$$\left|f^{(s)}(z)\right| \leqslant \frac{C|z|^{s+k\sigma}\exp(\beta|z|^{\mu})}{\exp(\alpha|z|^{\mu})} = C|z|^{s+k\sigma}\exp\{(\beta-\alpha)|z|^{\mu}\},$$

where $C > 0$ is a constant, so

$$\left|f^{(s)}(z)\right| \leqslant \exp\{(\beta+\tau-\alpha)|z|^{\mu}\}, \tag{20}$$

where $0 < \tau < \dfrac{\alpha-\beta}{k}$, along any ray $\arg z = \phi \in (\theta_1+\varepsilon, \theta_2-\varepsilon)\setminus E$, provided $|z| \geqslant r_3$ large enough. By combining $\rho\left(f^{(s)}\right) < +\infty$ and Lemma 3.5, we obtain that (20) remains valid in the sector $S(2\varepsilon)$. For $m > s$, we take $z = re^{i\theta} \in S(3\varepsilon)$ such that the disk $\Gamma(z)$ centered at z and of radius at most $\rho = ((m-s)!)^{1/(m-s)}$ is contained in $S(2\varepsilon)$, i.e., we must take $r \geqslant \rho/\sin\varepsilon$. By the Cauchy formula

$$f^{(m)}(z) = \frac{(m-s)!}{2\pi i}\int\limits_{\Gamma(z)} \frac{f^{(s)}(w)}{(w-z)^{m-s+1}}dw$$

and using (20), we get

$$\left|f^{(m)}(z)\right| \leqslant \exp\{(\beta+\tau-\alpha)|z|^{\mu}\} \tag{21}$$

as $|z|$ is large enough along any ray $\arg z = \phi \in S(2\varepsilon)$. Until now, we have proved the second assertion for $m \geqslant s$. We start to prove the first assertion for $j = s-1$. Set

$$a_{s-1} = \int\limits_0^{+\infty} f^{(s)}\left(te^{i\theta}\right)e^{i\theta}dt.$$

By (20), it is easy to see that $\int\limits_0^{+\infty} f^{(s)}\left(te^{i\theta}\right)e^{i\theta}dt$ converges for $\theta \in S(2\varepsilon)$. Moreover, a_{s-1} is independent of θ, because by using (20), the integral of $f^{(s)}(z)$ over the arc $Re^{i\theta}, \theta \in (\phi, \varphi)$ in $S(2\varepsilon)$ tends to zero as $R \to +\infty$. Define now $b_{s-1} = f^{(s-1)}(0)+a_{s-1}$, and suppose that $b_{s-1} \neq 0$. Let $z = re^{i\theta}$ (r large enough) be an arbitrary point in $S(2\varepsilon)$. Then by applying (20), and Lemma 3.3, we get

$$\left|f^{(s-1)}(z) - b_{s-1}\right| = \left|f^{(s-1)}(z) - f^{(s-1)}(0) - a_{s-1}\right|$$

$$= \left|\int\limits_0^{z} f^{(s)}(u)\,du - \int\limits_0^{+\infty} f^{(s)}\left(te^{i\theta}\right)e^{i\theta}dt\right|$$

$$= \left| \int_0^{|z|} f^{(s)}\left(te^{i\theta}\right) e^{i\theta} dt - \left(\int_0^{|z|} f^{(s)}\left(te^{i\theta}\right) e^{i\theta} dt + \int_{|z|}^{+\infty} f^{(s)}\left(te^{i\theta}\right) e^{i\theta} dt \right) \right|$$

$$= \left| - \int_{|z|}^{+\infty} f^{(s)}\left(te^{i\theta}\right) e^{i\theta} dt \right| \leqslant \int_{|z|}^{+\infty} \left| f^{(s)}\left(te^{i\theta}\right) \right| dt \leqslant \exp\left\{ (\beta + 2\tau - \alpha) |z|^{\mu} \right\} \tag{22}$$

as $|z|$ large enough along any ray $\arg z = \theta \in S(2\varepsilon)$. Thus, we have completed the proof in the case $b_{s-1} \neq 0$. Now if $b_{s-1} = 0$, we define $a_{s-2} = \int_0^{+\infty} f^{(s-1)}\left(te^{i\theta}\right) e^{i\theta} dt$ and $b_{s-2} = f^{(s-2)}(0) + a_{s-2}$. To estimate $f^{(s-2)}(z) - b_{s-2}$, we apply Lemma 3.3 and

$$\left| f^{(s-1)}(z) \right| \leqslant \exp\left\{ (\beta + 2\tau - \alpha) |z|^{\mu} \right\}$$

in place of (20) exactly as in (22) to obtain

$$\left| f^{(s-2)}(z) - b_{s-2} \right| \leqslant \exp\left\{ (\beta + 3\tau - \alpha) |z|^{\mu} \right\}$$

as $|z|$ is large enough in $S(2\varepsilon)$. By the same method, if $b_{s-1} = b_{s-2} = \cdots = b_{j+1} = 0$ and $b_j \neq 0$ $(j \in \{0, \cdots, s-1\})$, then

$$\left| f^{(j)}(z) - b_j \right| \leqslant \exp\left\{ (\beta + (s - j + 1)\tau - \alpha) |z|^{\mu} \right\}. \tag{23}$$

Now it remains to show that the case $b_{s-1} = b_{s-2} = \cdots = b_0 = 0$ is not possible. In this case we have

$$\left| f^{(m)}(z) \right| \leqslant \exp\left\{ (\beta + (s - m + 1)\tau - \alpha) |z|^{\mu} \right\} \quad (\text{ for all } 0 \leqslant m \leqslant s) \tag{24}$$

and for $m \geqslant s$ we have formula (21). From (7), we can write

$$\left| \frac{f^{(s)}(z)}{f(z)} \right| \leqslant \frac{1}{|A_s(z)|} \left| \frac{1}{e^{z^n}} \right| \frac{\left| f^{(k)}(z) \right|}{|f(z)|} + \frac{|A_{k-1}(z)|}{|A_s(z)|} \frac{\left| f^{(k-1)}(z) \right|}{|f(z)|}$$

$$+ \cdots + \frac{|A_1(z)|}{|A_s(z)|} \frac{|f'(z)|}{|f(z)|} + \frac{|A_0(z)|}{|A_s(z)|}. \tag{25}$$

By using (8), (9) and Lemma 3.1 in (25), we obtain

$$\left| \frac{f^{(s)}(z)}{f(z)} \right| \leqslant \exp\left\{ (\beta + \tau - \alpha) |z|^{\mu} \right\}. \tag{26}$$

Also by using (24) for $m = 0$ in (26), we get

$$\left| f^{(s)}(z) \right| \leqslant \exp\left\{ \left[2(\beta - \alpha) + (s + 2)\tau \right] |z|^{\mu} \right\} \tag{27}$$

in $S(2\varepsilon)\setminus E$, hence in $S\left(2\varepsilon+\frac{\varepsilon}{2}\right)$ by Lemma 3.5. If we repeat the steps (22) and (23) until $j=0$ by using (27), we will get

$$|f(z)| \leqslant \exp\{[2(\beta-\alpha)+(2s+2)\tau]|z|^{\mu}\}. \tag{28}$$

Then (28) with (26) give

$$\left|f^{(s)}(z)\right| \leqslant \exp\{[3(\beta-\alpha)+(2s+3)\tau]|z|^{\mu}\}$$

in $S\left(2\varepsilon+\frac{\varepsilon}{2}\right)\setminus E$, hence in $S\left(2\varepsilon+\frac{\varepsilon}{2}+\frac{\varepsilon}{2^2}\right)$ by Lemma 3.5. Now inductively, suppose that we have

$$\left|f^{(s)}(z)\right| \leqslant \exp\{[T(\beta-\alpha)+((T-1)s+T)\tau]|z|^{\mu}\} \tag{29}$$

holds in the sector $S\left(2\varepsilon+\sum_{j=1}^{T-1}\frac{\varepsilon}{2^j}\right)$. Repeating the steps (22), (23) until $j=0$ with (26), we get

$$\left|f^{(s)}(z)\right| \leqslant \exp\{[(T+1)(\beta-\alpha)+(Ts+T+1)\tau]|z|^{\mu}\}$$

in $S\left(2\varepsilon+\sum_{j=1}^{T-1}\frac{\varepsilon}{2^j}\right)\setminus E$ provided $|z|$ large enough. By Lemma 3.5, this inequality remains valid in the whole sector $S\left(2\varepsilon+\sum_{j=1}^{T}\frac{\varepsilon}{2^j}\right)$. Thus we have proved in this special case of $b_{s-1}=b_{s-2}=\cdots=b_0=0$, that (29) is valid in $S\left(2\varepsilon+\sum_{j=1}^{+\infty}\frac{\varepsilon}{2^j}\right)=S(3\varepsilon)$ for all $T\in\mathbb{N}^*$, provided $|z|$ large enough. Fix now a finite line segment in $S(3\varepsilon)$ with $|z|$ large enough. Since $(s+1)\tau\leqslant k\tau<\alpha-\beta$, it follows that

$$T(\beta-\alpha)+((T-1)s+T)\tau=-\left[(\alpha-\beta)-(s+1)\tau\right]T-s\tau\to-\infty$$

as $T\to+\infty$. By (29), we will obtain that $f^{(s)}(z)$ vanishes identically on such line segment. Therefore, by the standard uniqueness theorem of entire functions, f has to be a polynomial, a contradiction. So, the case $b_{s-1}=b_{s-2}=\cdots=b_0=0$ is not possible, which completes the proof.

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