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ON THE MEROMORPHIC SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS

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Abstract In this paper, we investigate the growth of meromorphic solutions of higher order linear differential equation $f^{(k)}+A_{k-1}(z)e^{P_{k-1}(z)}f^{(k-1)}+\cdots+A_1(z)e^{P_1(z)}f'+A_0(z)e^{P_0(z)}f=0$ $(k\geq 2)$, where $P_j(z)$ $(j=0,1,\cdots,k-1)$ are nonconstant polynomials such that $\deg P_j=n$ $(j=0,1,\cdots,k-1)$ and $A_j(z)(\not\equiv 0)$ $(j=0,1,\cdots,k-1)$ are meromorphic functions with order $\rho(A_j)<n$ $(j=0,1,\cdots,k-1)$.

Key words Linear differential equations, meromorphic solutions, order of growth.

1 Introduction and Statement of Results

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory[1,2]. In this paper a function is said to be meromorphic if it is meromorphic in the whole complex plane. Let f be a meromorphic function, one defines

$$m(r,f)=\frac{1}{2\pi}\int_0^{2\pi}\log^+|f(re^{it})|dt,$$
$$N(r,f)=\int_0^r\frac{(n(t,f)-n(0,f))}{t}dt+n(0,f)\log r,$$

and $T(r,f)=m(r,f)+N(r,f)$ $(r>0)$ is the Nevanlinna characteristic function of f, where $\log^+x=\max(0,\log x)$ for $x\geq 0$ and $n(t,f)$ is the number of the poles of $f(z)$ lying in $|z|\leq t$, counting according to their multiplicity. See [1], [2] for notations and definitions.

We recall the following definition:

Definition 1.1[1,2] Let f be a meromorphic function. Then the order $\rho(f)$ of $f(z)$ is defined by

$$\rho(f)=\overline{\lim_{r\to+\infty}}\frac{\log T(r,\ f)}{\log\ r}. \tag{1}$$

For the second order linear differential equation

$$f''+A_1(z)f'+A_0(z)f=0, \tag{2}$$

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where $A_1(z)$ or $A_0(z)$ is transcendental meromorphic function, it is well-known that if f_1, f_2 are two linearly independent meromorphic solutions of (2), then by [3, Lemma 3], there is at least one of f_1, f_2 of infinite order.

Z. X. Chen[4] and K. H. Kwon[5] have studied the second order linear differential equation

$$f'' + A_1(z)e^{P_1(z)}f' + A_0(z)e^{P_0(z)}f = 0, \tag{3}$$

where $P_1(z), P_0(z)$ are nonconstant polynomials, $A_1(z)$, $A_0(z)$ ($\not\equiv 0$) are entire functions such that $\rho(A_1) < \deg P_1(z)$, $\rho(A_0) < \deg P_0(z)$. Gundersen showed in [5, p.419] that if $\deg P_1(z) \neq \deg P_0(z)$, then every nonconstant solution of (3) is of infinite order. If $\deg P_1(z) = \deg P_0(z)$, then (3) may have nonconstant solutions of finite order. For instance $f(z) = e^z + 1$ satisfies $f'' + e^z f' - e^z f = 0$.

In [7], Z. X. Chen and K. H. Shon have investigated the case when $\deg P_1(z) = \deg P_0(z)$ and have proved the following result:

Theorem A[7] *Let $A_j(z)(\not\equiv 0)$ $(j = 0, 1)$ be meromorphic functions with $\rho(A_j) < 1$ $(j = 0, 1)$, a, b be complex numbers such that $ab \neq 0$ and $\arg a \neq \arg b$ or $a = cb$ $(0 < c < 1)$. Then every meromorphic solution $f(z) \not\equiv 0$ of the equation*

$$f'' + A_1(z)e^{az}f' + A_0(z)e^{bz}f = 0 \tag{4}$$

has infinite order.

In this paper, we shall investigate generalizations of problems of the above type to higher order homogeneous linear differential equations, we obtain the following results which greatly extend the result of Z. X. Chen and K. H. Shon.

Theorem 1.1 *Let*

$$P_j(z) = \sum_{i=0}^{n} a_{i,j}z^i, \quad j = 0, 1, \cdots, k-1$$

be nonconstant polynomials, where $a_{0,j}, a_{1,j}, \cdots, a_{n,j}$ $(j = 0, 1, \cdots, k-1)$ are complex numbers such that $a_{n,j}a_{n,0} \neq 0$ $(j = 1, 2, \cdots, k-1)$, and $A_j(z)(\not\equiv 0)$ $(j = 0, 1, \cdots, k-1)$ be meromorphic functions. Suppose that $\arg a_{n,j} \neq \arg a_{n,0}$ or $a_{n,j} = ca_{n,0}$ $(0 < c < 1)$ $(j = 1, 2, \cdots, k-1)$, $\rho(A_j) < n$ $(j = 0, 1, \cdots, k-1)$. Then every meromorphic solution $f(z) \not\equiv 0$ of the equation

$$f^{(k)} + A_{k-1}(z)e^{P_{k-1}(z)}f^{(k-1)} + \cdots + A_1(z)e^{P_1(z)}f' + A_0(z)e^{P_0(z)}f = 0, \tag{5}$$

is of infinite order where $k \geq 2$.

Theorem 1.2 *Let*

$$P_j(z) = \sum_{i=0}^{n} a_{i,j}z^i, \quad j = 0, 1, \cdots, k-1$$

be nonconstant polynomials where $a_{0,j}, \cdots, a_{n,j}$ $(j = 0, 1, \cdots, k-1)$ are complex numbers such that $a_{n,j}a_{n,0} \neq 0$ $(j = 1, 2, \cdots, k-1)$, and $A_j(z)(\not\equiv 0)$ $(j = 0, 1, \cdots, k-1)$ be meromorphic functions. Suppose that $a_{n,j} = ca_{n,0}$ $(c > 1)$ and $\deg(P_j - cP_0) = m \geq 1$ $(j = 1, 2, \cdots, k-1)$, $\rho(A_j) < m$ $(j = 0, 1, \cdots, k-1)$. Then every meromorphic solution $f(z) \not\equiv 0$ of the equation (5) *is of infinite order.*

2 Lemmas for the Proofs of Theorems

Our proofs depend mainly upon the following Lemmas.

Lemma 2.1[8] *Let $f(z)$ be a transcendental meromorphic function, and let $\alpha > 1$ and $\varepsilon > 0$ be given constants. Then the following two statements hold:*

(i) *There exist a constant $A>0$ and a set $E_1\subset[0,\infty)$ having finite linear measure such that for all z satisfying $|z|=r\notin E_1$, we have*

$$\left|\frac{f^{(j)}(z)}{f(z)}\right|\le A[T(\alpha r,\,f)r^{\varepsilon}\log T(\alpha r,\,f)]^{j},\quad j\in\boldsymbol{N};\tag{6}$$

(ii) *There exist a constant $B>0$ and a set $E_2\subset[0,2\pi)$ that has linear measure zero, such that if $\psi_0\in[0,2\pi)\backslash E_2$, then there is a constant $R_0=R_0(\psi_0)>1$ such that for all z satisfying $\arg z=\psi_0$ and $|z|=r\ge R_0$, we have*

$$\left|\frac{f^{(j)}(z)}{f(z)}\right|\le B\left[\frac{T(\alpha r,\,f)}{r}(\log^{\alpha} r)\log T(\alpha r,\,f)\right]^{j},\quad j\in\boldsymbol{N}.\tag{7}$$

Lemma 2.2[7] *Let $f(z)$ be a transcendental meromorphic function of order $\rho(f)=\rho<+\infty$. Then for any given $\varepsilon>0$, there exists a set $E_3\subset[0,2\pi)$ that has linear measure zero, such that if $\psi_1\in[0,2\pi)\backslash E_3$, then there is a constant $R_1=R_1(\psi_1)>1$ such that for all z satisfying $\arg z=\psi_1$ and $|z|=r\ge R_1$, we have*

$$\exp\{-r^{\rho+\varepsilon}\}\le|f(z)|\le\exp\{r^{\rho+\varepsilon}\}.\tag{8}$$

Lemma 2.3[9,p.253–255] *Let $P_0(z)=\sum_{i=0}^{n}b_iz^i$ where n is a positive integer, $b_n=\alpha_ne^{i\theta_n}$, $\alpha_n>0$, and $\theta_n\in[0,2\pi)$. For any given ε $(0<\varepsilon<\frac{\pi}{4n})$, we introduce $2n$ closed angles*

$$S_j:-\frac{\theta_n}{n}+(2j-1)\frac{\pi}{2n}+\varepsilon\le\theta\le-\frac{\theta_n}{n}+(2j+1)\frac{\pi}{2n}-\varepsilon,\quad j=0,1,\cdots,2n-1.\tag{9}$$

Then there exists a positive number $R_2=R_2(\varepsilon)$ such that for $|z|=r>R_2$,

$$\mathrm{Re}P_0(z)>\alpha_nr^n(1-\varepsilon)\sin(n\varepsilon),\tag{10}$$

if $z=re^{i\theta}\in S_j$, when j is even; while

$$\mathrm{Re}P_0(z)<-\alpha_nr^n(1-\varepsilon)\sin(n\varepsilon),\tag{11}$$

if $z=re^{i\theta}\in S_j$, when j is odd.

Lemma 2.4 *Let $P(z)=a_mz^m+a_{m-1}z^{m-1}+\cdots+a_1z+a_0$ with $a_m\ne0$ be a polynomial. Then for every $\varepsilon>0$, there exists $R_3>0$ such that for all $|z|=r>R_3$ the inequalities*

$$(1-\varepsilon)|a_m|r^m\le|P(z)|\le(1+\varepsilon)|a_m|r^m\tag{12}$$

hold.

Proof Clearly,

$$|P(z)|=|a_m||z|^m\left|1+\frac{a_{m-1}}{a_m}\frac{1}{z}+\cdots+\frac{a_0}{a_m}\frac{1}{z^m}\right|.$$

Denote

$$R_m(z)=\frac{a_{m-1}}{a_m}\frac{1}{z}+\cdots+\frac{a_0}{a_m}\frac{1}{z^m}.$$

Obviously, $|R_m(z)|<\varepsilon$, if $|z|>R_3$ for some $\varepsilon>0$. This means that

$$\begin{aligned}(1-\varepsilon)|a_m|r^m&\le(1-|R_m(z)|)|a_m|r^m\le|1+R_m(z)||a_m|r^m\\&=|P(z)|\le(1+|R_m(z)|)|a_m|r^m\\&\le(1+\varepsilon)|a_m|r^m.\end{aligned}$$

3 Proof of Theorem 1.1

Assume $f(z) \not\equiv 0$ is a meromorphic solution of (5). First of all we prove that every meromorphic solution of (5) is transcendental. If $f(z) \not\equiv 0$ is a rational solution of (5), then by the hypotheses of Theorem 1.1 and

$$f = -\left(\frac{e^{-P_0(z)}}{A_0(z)} f^{(k)} + \frac{A_{k-1}(z)}{A_0(z)} e^{P_{k-1}(z)-P_0(z)} f^{(k-1)} + \cdots + \frac{A_1(z)}{A_0(z)} e^{P_1(z)-P_0(z)} f'\right), \tag{13}$$

we obtain a contradiction since the left side of equation (13) is a rational function but the right side is a transcendental meromorphic function.

Now we prove that equation (5) cannot have nonzero polynomial solution. Suppose first that $\arg a_{n,j} \neq \arg a_{n,0}$ $(j = 1, 2, \cdots, k-1)$. Assume $f(z) \not\equiv 0$ is a polynomial solution of (5). By Lemma 2.3, there exist real numbers $b > 0$, R_2, and $\theta_1 < \theta_2$ such that for all $r \geq R_2$ and $\theta_1 \leq \theta \leq \theta_2$, we have

$$\mathrm{Re} P_j(re^{i\theta}) < 0, \quad j = 1, 2, \cdots, k-1, \quad \mathrm{Re} P_0(re^{i\theta}) > br^n. \tag{14}$$

Let $\max\{\rho(A_j)\ (j = 0, 1, \cdots, k-1)\} = \beta < n$. Then by Lemma 2.2, there exists a set $E_3 \subset [0, 2\pi)$ that has linear measure zero, such that if $\theta \in [0, 2\pi) \backslash E_3$, then there is a constant $R_1 = R_1(\theta) > 1$ such that for all z satisfying $\arg z = \theta$ and $|z| = r \geq R_1$, we have

$$\exp\{-r^{\beta+\varepsilon}\} \leq |A_j(z)| \leq \exp\{r^{\beta+\varepsilon}\}, \quad j = 0, 1, \cdots, k-1. \tag{3.3}$$

By (5) we can write

$$A_0(z)e^{P_0(z)} f = -\left(f^{(k)} + A_{k-1}(z)e^{P_{k-1}(z)} f^{(k-1)} + \cdots + A_1(z)e^{P_1(z)} f'\right). \tag{16}$$

By using (14)–(16) and Lemma 2.4 we obtain for any given ε $(0 < \varepsilon < \min\{1, n-\beta\})$ and for $z = re^{i\theta}$ with $|z| = r \geq \max\{R_1, R_2, R_3\}$, $\theta \in (\theta_1, \theta_2) \backslash E_3$,

$$\begin{aligned} &\exp\{-r^{\beta+\varepsilon}\} \exp\{br^n\}(1-\varepsilon)|a_m| r^m \\ \leq &\left|A_0(z)e^{P_0(z)} f\right| \\ = &\left|f^{(k)} + A_{k-1}(z)e^{P_{k-1}(z)} f^{(k-1)} + \cdots + A_1(z)e^{P_1(z)} f'\right| \\ \leq &(1 + (k-1)\exp\{r^{\beta+\varepsilon}\})(1+\varepsilon)|a_m| m r^{m-1} \\ \leq &M \exp\{r^{\beta+\varepsilon}\}(1+\varepsilon)|a_m| m r^{m-1}, \end{aligned} \tag{17}$$

where $M > 0$ is some constant and $f(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0$ with $a_m \neq 0$. From (17) we get for $z = re^{i\theta}$ with $|z| = r \geq \max\{R_1, R_2, R_3\}$ and $\theta \in (\theta_1, \theta_2) \backslash E_3$

$$\exp\{br^n - 2r^{\beta+\varepsilon}\} \leq M \frac{1+\varepsilon}{1-\varepsilon} m \frac{1}{r} \tag{18}$$

and this is a contradiction since $\exp\{br^n - 2r^{\beta+\varepsilon}\} \to +\infty$ $(\beta + \varepsilon < n)$ and $M \frac{1+\varepsilon}{1-\varepsilon} m \frac{1}{r} \to 0$ as $r \to +\infty$. By using similar reasoning as above we can prove that if $a_{n,j} = c a_{n,0}$ $(0 < c < 1)$, then equation (5) cannot have nonzero polynomial solution.

Now we prove that every solution of (5) is of infinite order. Suppose first that $\arg a_{n,j} \neq \arg a_{n,0}$ $(j = 1, 2, \cdots, k-1)$. By Lemma 2.1 (ii), there exist real numbers $\theta_0 \in (\theta_1, \theta_2)$, $R_0 > 1$ such that for all z satisfying $z = re^{i\theta_0}$ and $r \geq R_0$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq B[T(2r, f)]^{k+1}, \quad j = 1, 2, \cdots, k, \quad B > 0. \tag{19}$$

It follows from (5) that

$$\begin{aligned}\left|A_0(z)e^{P_0(z)}\right| \le \left|\frac{f^{(k)}(z)}{f(z)}\right| + \left|A_{k-1}(z)e^{P_{k-1}(z)}\right|\left|\frac{f^{(k-1)}(z)}{f(z)}\right| \\ + \cdots + \left|A_1(z)e^{P_1(z)}\right|\left|\frac{f'(z)}{f(z)}\right|. \qquad (20)\end{aligned}$$

Hence by (14), (15), (19), and (20), we get for $z = re^{i\theta_0}$ with $|z| = r \ge \max\{R_0, R_1, R_2\}$ and $\theta_0 \in (\theta_1, \theta_2)\backslash E_3$

$$\exp\{-r^{\beta+\varepsilon}\}\exp\{br^n\} \le (1 + (k-1)\exp\{r^{\beta+\varepsilon}\})B[T(2r, f)]^{k+1}. \qquad (21)$$

Thus $\beta+\varepsilon < n$ implies $\rho(f) = +\infty$. Suppose now $a_{n,j} = ca_{n,0}$ $(0 < c < 1)$ $(j = 1, 2, \cdots, k-1)$. Since $\deg P_0 > \deg(P_j - cP_0)$ $(j = 1, 2, \cdots, k-1)$, by Lemma 2.3, there exist real numbers $b > 0$, λ, R_4, and $\theta_3 < \theta_4$ such that for all $r \ge R_4$ and $\theta_3 \le \theta \le \theta_4$, we have

$$\mathrm{Re}P_0(re^{i\theta}) > br^n, \quad \mathrm{Re}(P_j(re^{i\theta}) - cP_0(re^{i\theta})) < \lambda, \quad j = 1, 2, \cdots, k-1. \qquad (22)$$

It follows from (5) that

$$\begin{aligned}\left|A_0(z)e^{(1-c)P_0(z)}\right| \le \left|e^{-cP_0(z)}\right|\left|\frac{f^{(k)}(z)}{f(z)}\right| + \left|A_{k-1}(z)e^{P_{k-1}(z)-cP_0(z)}\right|\left|\frac{f^{(k-1)}(z)}{f(z)}\right| \\ + \cdots + \left|A_1(z)e^{P_1(z)-cP_0(z)}\right|\left|\frac{f'(z)}{f(z)}\right|. \qquad (23)\end{aligned}$$

Hence by (15), (19), (22), and (23), we get for $z = re^{i\theta_0}$ with $|z| = r \ge \max(R_0, R_1, R_4)$ and $\theta_0 \in (\theta_3, \theta_4)\backslash E_3$

$$\begin{aligned}&\exp\{-r^{\beta+\varepsilon}\}\exp\{(1-c)br^n\} \\ \le &[\exp\{-cbr^n\} + (k-1)\exp\{r^{\beta+\varepsilon}\}\exp\{\lambda\}]B[T(2r, f)]^{k+1}. \qquad (24)\end{aligned}$$

Thus, $\beta+\varepsilon < n$ and $0 < c < 1$ implies $\rho(f) = +\infty$.

4 Proof of Theorem 1.2

Assume $f(z) \not\equiv 0$ is a meromorphic solution of (5). By using similar reasoning as in the proof of Theorem 1.1, it follows that $f(z)$ must be a transcendental meromorphic solution. From (5), we have

$$\begin{aligned}\left|A_0(z)e^{(1-c)P_0(z)}\right| \le \left|e^{-cP_0(z)}\right|\left|\frac{f^{(k)}(z)}{f(z)}\right| + \left|A_{k-1}(z)e^{P_{k-1}(z)-cP_0(z)}\right|\left|\frac{f^{(k-1)}(z)}{f(z)}\right| \\ + \cdots + \left|A_1(z)e^{P_1(z)-cP_0(z)}\right|\left|\frac{f'(z)}{f(z)}\right|. \qquad (25)\end{aligned}$$

By Lemma 2.1 (i), there exist a constant $A > 0$ and a set $E_1 \subset [0, \infty)$ having finite linear measure such that for all z satisfying $|z| = r \notin E_1$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \le Ar[T(2r, f)]^{k+1}, \quad j = 1, 2, \cdots, k. \qquad (26)$$

By (25) and (26), we have for all z satisfying $|z| = r \notin E_1$

$$\begin{aligned}\left|A_0(z)e^{(1-c)P_0(z)}\right| \le \Big[\left|e^{-cP_0(z)}\right| + \left|A_{k-1}(z)e^{P_{k-1}(z)-cP_0(z)}\right| + \cdots \\ + \left|A_1(z)e^{P_1(z)-cP_0(z)}\right|\Big] Ar[T(2r,\, f)]^{k+1}. \qquad (27)\end{aligned}$$

Since $\deg(P_j - cP_0) = m < \deg P_0 = n$ $(j = 1, 2, \cdots, k-1)$, by Lemma 2.3 (see also [5, p.385]) there exist a positive real number b and a curve Γ tending to infinity such that for all $z \in \Gamma$ with $|z| = r$, we have

$$\mathrm{Re}P_0(z) = 0, \quad \mathrm{Re}(P_j(z) - cP_0(z)) \le -br^m, \quad j = 1, 2, \cdots, k-1. \qquad (28)$$

Let $\max\{\rho(A_j)\ (j = 0, 1, \cdots, k-1)\} = \beta < m$. Then by Lemma 2.2, there exists a set $E_3 \subset [0, 2\pi)$ that has linear measure zero, such that if $\theta \in [0, 2\pi)\backslash E_3$, there is a constant $R_1 = R_1(\theta) > 1$ such that for all z satisfying $\arg z = \theta$ and $|z| = r \ge R_1$, we have

$$\exp\{-r^{\beta+\varepsilon}\} \le |A_j(z)| \le \exp\{r^{\beta+\varepsilon}\}\ (j = 0, 1, \cdots, k-1). \qquad (29)$$

Hence by (27)–(29), we get for all $z \in \Gamma$ with $|z| \notin E_1,\ |z| = r \ge R_1$, and $\theta \in [0, 2\pi)\backslash E_3$

$$\exp\{-r^{\beta+\varepsilon}\} \le (1 + (k-1)\exp\{r^{\beta+\varepsilon}\}\exp\{-br^m\})Ar[T(2r,\, f)]^{k+1}. \qquad (30)$$

Thus $\beta + \varepsilon < m$ implies $\rho(f) = +\infty$.

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