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Research Article

On the Iterated Exponent of Convergence of Solutions of Linear Differential Equations with Entire and Meromorphic Coefficients

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We investigate the zeros of the difference of the derivative of solutions of the higher-order linear differential equations $f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0$ and small functions, where $A_0(z), \ldots, A_{k-1}(z)$ are entire or meromorphic functions of finite iterated p order.

1. Introduction and Main Results

In this paper, a meromorphic function will mean meromorphic in the whole complex plane. We shall use the standard notations in Nevanlinna value distribution of meromorphic functions [1, 2] such as $T(r,f)$, $N(r,f)$, $m(r,f)$. For the definition of the iterated order of a meromorphic function, we use the same definition as in [3], [4, p. 317], [2, p. 129]. For all $r \in \mathbb{R}$, we define $\exp_1 r := e^r$ and $\exp_{p+1} r := \exp(\exp_p r)$, $p \in \mathbb{N}$. We also define for all r sufficiently large $\log_1 r := \log r$ and $\log_{p+1} r := \log(\log_p r)$, $p \in \mathbb{N}$. Moreover, we denote by $\exp_0 r := r$, $\log_0 r := r$, $\log_{-1} r := \exp_1 r$ and $\exp_{-1} r := \log_1 r$.

Definition 1 (see [2, 3]). Let f be a meromorphic function. Then the iterated p order $\rho_p(f)$ of f is defined by

$$\rho_p(f) = \limsup_{r\to+\infty} \frac{\log_p T(r,f)}{\log r}, \quad (p \geqslant 1 \text{ is an integer}). \tag{1}$$

For $p = 1$, this notation is called order and for $p = 2$ hyperorder. If f is an entire function, then the iterated p order $\rho_p(f)$ of f is defined by

$$\rho_p(f) = \limsup_{r\to+\infty} \frac{\log_p T(r,f)}{\log r} = \limsup_{r\to+\infty} \frac{\log_{p+1} M(r,f)}{\log r}, \quad (p \geqslant 1 \text{ is an integer}), \tag{2}$$

where $M(r,f) = \max_{|z|=r} |f(z)|$.

Definition 2 (see [3]). The finiteness degree of the order of a meromorphic function f is defined by

$$i(f) = \begin{cases} 0 & \text{for } f \text{ rational}, \\ \min\{j \in \mathbb{N} : \rho_j(f) < +\infty\}, & \text{for } f \text{ transcendental for which some } j \in \mathbb{N} \text{ with } \rho_j(f) < +\infty \text{ exists}, \\ +\infty, & \text{for } f \text{ with } \rho_j(f) = +\infty\ \forall j \in \mathbb{N}. \end{cases} \tag{3}$$

Definition 3 (see [3]). The iterated convergence exponent of the sequence of zeros of a meromorphic function $f(z)$ is defined by

$$\lambda_p(f) = \limsup_{r\to+\infty} \frac{\log_p N(r,1/f)}{\log r}, \quad (p \geqslant 1 \text{ is an integer}), \tag{4}$$

where $N(r,1/f)$ is the counting function of zeros of $f(z)$ in $\{z : |z| < r\}$. Similarly, the iterated convergence exponent of the sequence of distinct zeros of $f(z)$ is defined by

$$\overline{\lambda}_p(f) = \limsup_{r\to+\infty} \frac{\log_p \overline{N}(r,1/f)}{\log r}, \quad (p \geqslant 1 \text{ is an integer}), \tag{5}$$

where $\overline{N}(r,1/f)$ is the counting function of distinct zeros of $f(z)$ in $\{z : |z| < r\}$.

Definition 4 (see [3]). The finiteness degree of the iterated convergence exponent of the sequence of zeros of a meromorphic function $f(z)$ is defined by

$$i_\lambda(f) = \begin{cases} 0, & \text{if } n\left(r,\dfrac{1}{f}\right) = O(\log r), \\ \min\left\{j\in\mathbb{N} : \lambda_j(f) < \infty\right\}, & \text{if } \lambda_j(f) < \infty \text{ for some } j\in\mathbb{N}, \\ \infty, & \text{if } \lambda_j(f) = \infty\ \forall j\in\mathbb{N}. \end{cases} \tag{6}$$

Remark 5. Similarly, we can define the finiteness degree $i_{\overline{\lambda}}(f)$ of $\overline{\lambda}_p(f)$.

Definition 6 (see [5]). Let f be an entire function. Then the iterated p type of f, with iterated p order $0 < \rho_p(f) < \infty$ is defined by

$$\tau_p(f) = \limsup_{r\to+\infty} \frac{\log_p M(r,f)}{r^{\rho_p(f)}}, \quad (p \geqslant 1 \text{ is an integer}), \tag{7}$$

where $M(r,f) = \max_{|z|=r}|f(z)|$.

We define the linear measure of a set $E \subset [0,+\infty)$ by $m(E) = \int_0^{+\infty} \chi_E(t)dt$ and the logarithmic measure of a set $F \subset (1,+\infty)$ by $lm(F) = \int_1^{+\infty}(\chi_F(t)/t)dt$, where χ_H is the characteristic function of a set H.

Recently, Xu, Tu, and Zheng have investigated the relationship between the small functions and the derivative of solutions of higher-order linear differential equation:

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = 0, \tag{8}$$

where $A_0(z),\ldots,A_{k-1}(z)$ are entire or meromorphic functions and have obtained the following results.

Theorem A (see [6]). *Let $A_j(z)$ $(j = 0,1,\ldots,k-1)$ be entire functions with finite order and satisfy one of the following conditions:*

(i) $\max\{\rho(A_j) : j = 1,2,\ldots,k-1\} < \rho(A_0) < \infty$.

(ii) $0 < \rho(A_{k-1}) = \cdots = \rho(A_1) = \rho(A_0) < \infty$ *and* $\max\{\tau(A_j) : j = 1,2,\ldots,k-1\} = \tau_1 < \tau(A_0) = \tau$.

Then for every solution $f \not\equiv 0$ of (8) *and for any entire function $\varphi(z) \not\equiv 0$ satisfying $\rho_2(\varphi) < \rho(A_0)$, we have*

$$\overline{\lambda}_2\left(f^{(i)} - \varphi\right) = \rho_2(f) = \rho(A_0), \quad (i = 0,1,\ldots). \tag{9}$$

Theorem B (see [6]). *Let $A_j(z)$ $(j = 1,\ldots,k-1)$ be polynomials and $A_0(z)$ be a transcendental entire function. Then for every solution $f \not\equiv 0$ of* (8) *and for any entire function $\varphi(z)$ of finite order, we have*

(i) $\overline{\lambda}(f-\varphi) = \lambda(f-\varphi) = \rho(f) = \infty$,

(ii) $\overline{\lambda}(f^{(i)}-\varphi) = \lambda(f^{(i)}-\varphi) = \rho(f^{(i)}-\varphi) = \infty$, $(i \geqslant 1, i \in \mathbb{N})$.

When the coefficients are meromorphic functions, they have proved the following theorem.

Theorem C (see [6]). *Let $A_j(z)$ $(j = 0,1,\ldots,k-1)$ be meromorphic functions satisfying* $\max\{\rho(A_j) : j = 1,2,\ldots,k-1\} < \rho(A_0)$ *and $\delta(\infty, A_0) > 0$. Then for every meromorphic solution $f \not\equiv 0$ of* (8) *and for any meromorphic function $\varphi(z) \not\equiv 0$ satisfying $\rho_2(\varphi) < \rho(A_0)$, we have*

$$\overline{\lambda}_2\left(f^{(i)} - \varphi\right) = \lambda_2\left(f^{(i)} - \varphi\right) \geqslant \rho(A_0), \quad (i = 0,1,\ldots), \text{ where } f^{(0)} = f. \tag{10}$$

In this paper, we improve and extend the above results by considering the iterated order, and we obtain the following theorems. For some closely related to applications of this paper, see the papers of Gupta et al. [7, 8].

Theorem 7. *Let $A_j(z)$ $(j = 0,1,\ldots,k-1)$ be entire functions of finite iterated order with $i(A_0) = p (0 < p < \infty)$ and satisfy one of the following conditions:*

(i) $\max\{\rho_p(A_j) : j = 1,2,\ldots,k-1\} < \rho_p(A_0)$.

(ii) $\max\{\rho_p(A_j) : j = 1,2,\ldots,k-1\} \leqslant \rho_p(A_0) = \rho (0 < \rho < \infty)$ *and* $\max\{\tau_p(A_j) : \rho_p(A_j) = \rho_p(A_0)\} < \tau_p(A_0) = \tau (0 < \tau < \infty)$.

Then for every solution $f \not\equiv 0$ of (8) *and for any entire function $\varphi(z) \not\equiv 0$ satisfying $\rho_{p+1}(\varphi) < \rho_p(A_0)$, we have*

$$\overline{\lambda}_{p+1}\left(f^{(i)} - \varphi\right) = \lambda_{p+1}\left(f^{(i)} - \varphi\right) = \rho_{p+1}(f) = \rho_p(A_0), \quad (i = 0,1,\ldots). \tag{11}$$

Theorem 8. *Let $A_j(z)$ $(j = 1,2,\ldots,k-1)$ be polynomials and $A_0(z)$ be a transcendental entire function with $0 < \rho_p(A_0) < \infty$. Then for every solution $f \not\equiv 0$ of* (8) *and for any entire function $\varphi(z)$ of finite iterated order $\rho_p(\varphi) < \infty$, we have*

(i) $\overline{\lambda}_p(f-\varphi)=\lambda_p(f-\varphi)=\rho_p(f)=\infty$.

(ii) $\overline{\lambda}_p(f^{(i)}-\varphi)=\lambda_p(f^{(i)}-\varphi)=\rho_p(f^{(i)}-\varphi)=\infty, i\geqslant 1, i\in\mathbb{N}$.

Theorem 9. *Let $A_j(z)$ $(j=0,1,\ldots,k-1)$ be meromorphic functions of finite iterated order with $i(A_0)=p(0<p<\infty)$ satisfying $\max\{\rho_p(A_j):j=1,2,\ldots,k-1\}<\rho_p(A_0)$ and $\delta(\infty,A_0)>0$. Then for every meromorphic solution $f\not\equiv 0$ of (8) whose poles are of uniformly bounded multiplicity and for any meromorphic function $\varphi(z)\not\equiv 0$ satisfying $\rho_{p+1}(\varphi)<\rho_p(A_0)$, we have*

$$\overline{\lambda}_{p+1}\left(f^{(i)}-\varphi\right)=\lambda_{p+1}\left(f^{(i)}-\varphi\right)=\rho_p\left(A_0\right),\quad (i=0,1,\ldots). \tag{12}$$

Corollary 10. *Under the assumptions of Theorem 7, if $\varphi(z)=z$, then, for every solution $f\not\equiv 0$ of* (8)*, we have*

$$\overline{\lambda}_{p+1}\left(f^{(i)}-z\right)=\lambda_{p+1}\left(f^{(i)}-z\right)=\rho_{p+1}\left(f\right)=\rho_p\left(A_0\right),\quad (i=0,1,\ldots). \tag{13}$$

Corollary 11. *Under the assumptions of Theorem 8, if $\varphi(z)=z$, then, for every solution $f\not\equiv 0$ of* (8)*, we have*

(i) $\overline{\lambda}_p(f-z)=\lambda_p(f-z)=\rho_p(f)=\infty$.

(ii) $\overline{\lambda}_p(f^{(i)}-z)=\lambda_p(f^{(i)}-z)=\rho_p(f^{(i)}-z)=\infty$, $i\geqslant 1$, $i\in\mathbb{N}$.

Corollary 12. *Under the assumptions of Theorem 9, if $\varphi(z)=z$, then for every meromorphic solution $f\not\equiv 0$ of* (8) *whose poles are of uniformly bounded multiplicity, we have*

$$\overline{\lambda}_{p+1}\left(f^{(i)}-z\right)=\lambda_{p+1}\left(f^{(i)}-z\right)=\rho_p\left(A_0\right),\quad (i=0,1,\ldots). \tag{14}$$

2. Auxiliary Lemmas

In order to prove our theorems, we need the following lemmas.

Lemma 13 (see [6]). *Assume that $f\not\equiv 0$ is a solution of* (8)*, set $g_i=f^{(i)}-\varphi$, and then g_i satisfies the equation*

$$\begin{aligned} g_i^{(k)}+U_{k-1}^i g_i^{(k-1)}+\cdots+U_0^i g_i \\ =-\left[\varphi^{(k)}+U_{k-1}^i\varphi^{(k-1)}+\cdots+U_0^i\varphi\right], \end{aligned} \tag{15}$$

where

$$\begin{gathered} U_j^i=U_{j+1}^{i-1'}+U_j^{i-1}-\frac{U_0^{i-1'}}{U_0^{i-1}}U_{j+1}^{i-1},\qquad U_j^0=A_j,\\ j=0,1,2,\ldots,k-1,\\ U_k^{i-1}\equiv 1,\quad (i\geqslant 1, i\in\mathbb{N}). \end{gathered} \tag{16}$$

Lemma 14 (see [3]). *Let f be a meromorphic function for which $i(f)=p\geqslant 1$ and $\rho_p(f)=\rho$, and let $k\geqslant 1$ be an integer. Then for any $\varepsilon>0$,*

$$m\left(r,\frac{f^{(k)}}{f}\right)=O\left(\exp_{p-2}r^{\rho+\varepsilon}\right) \tag{17}$$

outside of a possible exceptional set E_1 of finite linear measure.

Lemma 15 (see [1]). *Let f be a transcendental meromorphic function and $k\geqslant 1$ be an integer. Then*

$$m\left(r,\frac{f^{(k)}}{f}\right)=O\left(\log\left(rT\left(r,f\right)\right)\right) \tag{18}$$

outside of a possible exceptional set E_2 of finite linear measure, and if f is of finite order of growth, then

$$m\left(r,\frac{f^{(k)}}{f}\right)=O\left(\log r\right). \tag{19}$$

Lemma 16 (see [9]). *Let $\varphi:[0,+\infty)\to\mathbb{R}$ and $\psi:[0,+\infty)\to\mathbb{R}$ be monotone nondecreasing functions such that $\varphi(r)\leqslant\psi(r)$ for all $r\notin E_3\cup[0,1]$, where $E_3\subset(1,+\infty)$ is a set of finite logarithmic measure. Let $\alpha>1$ be a given constant. Then there exists an $r_0=r_0(\alpha)>0$ such that $\varphi(r)\leqslant\psi(\alpha r)$ for all $r>r_0$.*

Lemma 17 (see [10]). *Let f be a meromorphic solution of* (8)*, assuming that not all coefficients A_j are constants. Given a real constant $\gamma>1$, and denoting $T(r)=\sum_{j=0}^{k-1}T(r,A_j)$, we have*

$$\begin{aligned} \log m\left(r,f\right)&<T\left(r\right)\left\{\left(\log r\right)\log T\left(r\right)\right\}^{\gamma}, &&\text{if } p=0,\\ \log m\left(r,f\right)&<r^{2p+\gamma-1}T\left(r\right)\left\{\log T\left(r\right)\right\}^{\gamma}, &&\text{if } p>0 \end{aligned} \tag{20}$$

outside of an exceptional set E_p with $\int_{E_p}t^{p-1}dt<+\infty$.

Remark 18. We note that in the above lemma, $p=1$ corresponds to Euclidean measure and $p=0$ to logarithmic measure.

Lemma 19 (see [11]). *Let $f(z)$ be a meromorphic function of finite iterated order satisfying $i(f)=p$. Then there exists a set $E_4\subset(1,+\infty)$ with infinite logarithmic measure such that for all $r\in E_4$, we have*

$$\lim_{r\to\infty}\frac{\log_p T\left(r,f\right)}{\log r}=\rho. \tag{21}$$

Lemma 20. *Let $A_0(z),A_1(z),\ldots,A_{k-1}(z)$ be entire functions of finite iterated order with $i(A_0)=p(0<p<\infty)$ and satisfy $\max\{\rho_p(A_j):j=1,2,\ldots,k-1\}=\rho_1<\rho_p(A_0)$ and set*

$$\begin{gathered} U_j^1=A_{j+1}'+A_j-\frac{A_0'}{A_0}A_{j+1},\\ U_j^i=U_{j+1}^{i-1'}+U_j^{i-1}-\frac{U_0^{i-1'}}{U_0^{i-1}}U_{j+1}^{i-1}, \end{gathered} \tag{22}$$

where $j = 0, 1, \ldots, k-1$*;* $A_k \equiv 1$*,* $U_k^{i-1} \equiv 1$ *and* $i \geqslant 1$*,* $i \in \mathbb{N}$*. Then there exists a set* E_5 *with infinite logarithmic measure such that for all* $r \in E_5$

$$\lim_{r\to\infty} \frac{\log_p m\left(r, U_0^i\right)}{\log r} = \rho_p\left(A_0\right) > \limsup_{r\to\infty} \frac{\max_{1\leqslant j\leqslant k-1}\left\{\log_p m\left(r, U_j^i\right)\right\}}{\log r} = \rho_1. \tag{23}$$

Proof. We prove this lemma by using mathematical induction. First, when $i = 1$, we have $U_j^1 = A_{j+1}' + A_j - (A_0'/A_0)A_{j+1}$, $j = 0, 1, \ldots, k-1$ and $A_k \equiv 1$.

When $j = 0$, that is, $U_0^1 = A_1' + A_0 - (A_0'/A_0)A_1$. Then we have

$$m\left(r, U_0^1\right) \leqslant m\left(r, A_1\right) + m\left(r, A_0\right) + m\left(r, \frac{A_1'}{A_1}\right) + m\left(r, \frac{A_0'}{A_0}\right) + O(1). \tag{24}$$

From $-A_0 = A_1' - U_0^1 - (A_0'/A_0)A_1$, we have

$$m\left(r, A_0\right) \leqslant m\left(r, A_1\right) + m\left(r, U_0^1\right) + m\left(r, \frac{A_1'}{A_1}\right) + m\left(r, \frac{A_0'}{A_0}\right) + O(1). \tag{25}$$

When $j \neq 0$, from the definitions of U_j^1, we have

$$m\left(r, U_j^1\right) \leqslant m\left(r, A_{j+1}\right) + m\left(r, A_j\right) + m\left(r, \frac{A_{j+1}'}{A_{j+1}}\right) + m\left(r, \frac{A_0'}{A_0}\right) + O(1), \tag{26}$$

$j = 1, 2, \ldots, k-1$. Since $A_j(z)$ are entire functions with $\max\{\rho_p(A_j) : j = 1, 2, \ldots, k-1\} < \rho_p(A_0) < \infty$, then by Lemma 19, there exists a set $E_4 \subset (1, +\infty)$ with infinite logarithmic measure such that for all $r \in E_4$

$$m\left(r, A_{j+1}\right) = o\left(m\left(r, A_0\right)\right), \quad (j = 1, 2, \ldots, k-1). \tag{27}$$

By this, (26), and Lemma 14, we get for all $r \in E_4$

$$\max_{1\leqslant j\leqslant k-1}\left\{m\left(r, U_j^1\right)\right\} \leqslant \max_{1\leqslant j\leqslant k-1}\left\{m\left(r, A_j\right) + o\left(m\left(r, A_0\right)\right) + O\left(\exp_{p-2} r^{\beta}\right) + O(1)\right\} \tag{28}$$

for some constant $\beta < \infty$ outside of a possible exceptional set E_1 of finite linear measure. From (24), (25), and (28) we get for all $r \in E_4 \setminus E_1$

$$\lim_{r\to\infty} \frac{\log_p m\left(r, U_0^1\right)}{\log r} = \rho_p\left(A_0\right) > \rho_1 = \limsup_{r\to\infty} \frac{\max_{1\leqslant j\leqslant k-1}\left\{\log_p m\left(r, A_j\right)\right\}}{\log r} \geqslant \limsup_{r\to\infty} \frac{\max_{1\leqslant j\leqslant k-1}\left\{\log_p m\left(r, U_j^1\right)\right\}}{\log r}. \tag{29}$$

Now, suppose that (23) holds, for $i \leqslant n, n \in \mathbb{N}$; that is, there is a set E_5 with infinite logarithmic measure such that for all $r \in E_5$

$$\limsup_{r\to\infty} \frac{\log_p m\left(r, U_0^n\right)}{\log r} = \rho_p\left(A_0\right) > \limsup_{r\to\infty} \frac{\max_{1\leqslant j\leqslant k-1}\left\{\log_p m\left(r, U_j^n\right)\right\}}{\log r} = \rho_1. \tag{30}$$

Next, we prove that (23) holds for $i = n+1$. Since $i = n+1$, then we have

$$U_j^{n+1} = U_{j+1}^{n'} + U_j^n - \frac{U_0^{n'}}{U_0^n} U_{j+1}^n, \tag{31}$$

where $j = 0, 1, \ldots, k-1$; $U_k^n \equiv 1$. When $j = 0$, we have $U_0^{n+1} = U_1^{n'} + U_0^n - (U_0^{n'}/U_0^n)U_1^n$. Then we obtain that

$$m\left(r, U_0^{n+1}\right) \leqslant m\left(r, U_0^n\right) + m\left(r, U_1^n\right) + m\left(r, \frac{U_0^{n'}}{U_0^n}\right) + m\left(r, \frac{U_1^{n'}}{U_1^n}\right) + O(1). \tag{32}$$

Since $-U_0^n = U_1^{n'} - U_0^{n+1} - (U_0^{n'}/U_0^n)U_1^n$, then we have

$$m\left(r, U_0^n\right) \leqslant m\left(r, U_0^{n+1}\right) + m\left(r, U_1^n\right) + m\left(r, \frac{U_0^{n'}}{U_0^n}\right) + m\left(r, \frac{U_1^{n'}}{U_1^n}\right) + O(1). \tag{33}$$

When $j \neq 0$, from the definitions of U_j^{n+1}, $j = 1, 2, \ldots, k-1$ and $U_k^n \equiv 1$, we have

$$m\left(r, U_j^{n+1}\right) \leqslant m\left(r, U_{j+1}^n\right) + m\left(r, U_j^n\right) + m\left(r, \frac{U_{j+1}^{n'}}{U_{j+1}^n}\right) + m\left(r, \frac{U_0^{n'}}{U_0^n}\right) + O(1). \quad (34)$$

From (30)–(34), there exists a set E_5 with infinite logarithmic measure such that for all $r \in E_5$

$$\begin{aligned}\lim_{r\to\infty} \frac{\log_p m\left(r, U_0^{n+1}\right)}{\log r} &= \lim_{r\to\infty} \frac{\log_p m\left(r, U_0^n\right)}{\log r} \\ &= \rho_p\left(A_0\right) > \rho_1 \\ &= \limsup_{r\to\infty} \frac{\max_{1\leqslant j\leqslant k-1}\left\{\log_p m\left(r, U_j^n\right)\right\}}{\log r} \\ &= \limsup_{r\to\infty} \frac{\max_{1\leqslant j\leqslant k-1}\left\{\log_p m\left(r, U_j^{n+1}\right)\right\}}{\log r}.\end{aligned} \quad (35)$$

Thus, the proof of Lemma 20 is completed. □

Lemma 21. *Let $A_0, A_1, \ldots, A_{k-1}$ be meromorphic functions of finite iterated order. If*

$$\limsup_{r\to\infty} \frac{\max_{1\leqslant j\leqslant k-1}\left\{\log_p m\left(r, A_j\right)\right\}}{\log r} = \beta_1 \quad (36)$$

and there exists a set E_6 with infinite logarithmic measure such that

$$\lim_{r\to\infty} \frac{\log_p m\left(r, A_0\right)}{\log r} = \beta_2 > \beta_1 \quad (37)$$

holds for all $r \in E_6$, then every meromorphic solution $f \not\equiv 0$ of (8) *satisfies $\rho_p(f) = \infty$ and $\rho_{p+1}(f) \geqslant \beta_2$.*

Proof. Suppose that $f \not\equiv 0$ is a meromorphic solution of (8). If $\rho_p(f) = \rho < \infty$, then from (8) we have

$$m\left(r, A_0\right) \leqslant \sum_{j=1}^{k} m\left(r, \frac{f^{(j)}}{f}\right) + \sum_{j=1}^{k-1} m\left(r, A_j\right) + O(1). \quad (38)$$

By using Lemma 14, we obtain

$$m\left(r, A_0\right) \leqslant O\left(\exp_{p-2} r^{\rho+\varepsilon}\right) + \sum_{j=1}^{k-1} m\left(r, A_j\right) + O(1) \quad (39)$$

outside of a possible exceptional set E_1 of finite linear measure. By the hypothesis of Lemma 21, there exists a set E_6 having infinite logarithmic measure such that for all $|z| = r \in E_6 \setminus E_1$, we have for any $\varepsilon (0 < 2\varepsilon < \beta_2 - \beta_1)$

$$\exp_{p-1}\left\{r^{\beta_2-\varepsilon}\right\} \leqslant O\left(\exp_{p-2} r^{\rho+\varepsilon}\right) + (k-1)\exp_{p-1}\left\{r^{\beta_1+\varepsilon}\right\} \quad (40)$$

and so by Lemma 16, we obtain

$$\beta_2 \leqslant \beta_1, \quad (41)$$

and from this we obtain a contradiction. Hence $\rho_p(f) = \infty$.

Now, assume that $f \not\equiv 0$ is a meromorphic solution of (8) with $\rho_p(f) = \infty$. By (8) we have

$$m\left(r, A_0\right) \leqslant \sum_{j=1}^{k} m\left(r, \frac{f^{(j)}}{f}\right) + \sum_{j=1}^{k-1} m\left(r, A_j\right) + O(1). \quad (42)$$

By Lemma 15 and (42), we have

$$m\left(r, A_0\right) \leqslant O\left\{\log rT(r, f)\right\} + \sum_{j=1}^{k-1} m\left(r, A_j\right), \quad r \notin E_2, \quad (43)$$

where $E_2 \subset [1, +\infty)$ is a set having finite linear measure. By the hypothesis of Lemma 21, there exists a set E_6 having infinite logarithmic measure such that, for all $|z| = r \in E_6 \setminus E_2$, we have by using (43)

$$\exp_{p-1}\left\{r^{\beta_2-\varepsilon}\right\} \leqslant O\left\{\log rT(r, f)\right\} + (k-1)\exp_{p-1}\left\{r^{\beta_1+\varepsilon}\right\}, \quad (44)$$

where $0 < 2\varepsilon < \beta_2 - \beta_1$. By (44) and Lemma 16, we have

$$\rho_{p+1}(f) \geqslant \beta_2. \quad (45)$$

□

Lemma 22 (see [12, Theorem 3]). *Let $f(z)$ be a transcendental meromorphic function, and let $\alpha > 1$ be a given constant. Then there exists a set $E_7 \subset (1, \infty)$ with finite logarithmic measure and a constant $B > 0$ that depends only on α and $i, j (0 \leqslant i < j \leqslant k)$, such that for all z satisfying $|z| = r \notin [0, 1] \cup E_7$, we have*

$$\left|\frac{f^{(j)}(z)}{f^{(i)}(z)}\right| \leqslant B\left\{\frac{T(\alpha r, f)}{r}\left(\log^{\alpha} r\right)\log T(\alpha r, f)\right\}^{j-i}. \quad (46)$$

From the above lemma, we obtain the following result.

Lemma 23. *Let f be a transcendental meromorphic function with finite iterated order $\rho_p(f) = \rho < +\infty$, $H = \{(k_1, j_1), (k_2, j_2), \ldots, (k_q, j_q)\}$ be a finite set of distinct pairs of integers satisfying $k_i > j_i \geqslant 0$ $(i = 1, \ldots, q)$, and let $\varepsilon > 0$ be a given constant. Then, there exists a set $E_8 \subset (1, +\infty)$ with finite logarithmic measure, such that, for all z satisfying $|z| \notin E_8 \cup [0, 1]$ and for all $(k, j) \in H$, we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leqslant \left\{\exp_{p-1}\left\{r^{\rho-1+\varepsilon}\right\}\right\}^{k-j}. \quad (47)$$

Proof. The definition

$$\rho_p(f) = \limsup_{r\to+\infty} \frac{\log_p T(r,f)}{\log r} \tag{48}$$

implies that for any given $\varepsilon_1 > 0$ there exists $R > 1$ such that for all $r > R$ we have

$$T(r,f) < \exp_{p-1}\{r^{\rho+\varepsilon_1}\}. \tag{49}$$

Combining (49) with Lemma 22, for $\alpha > 0$, there exists a set $E_8 = [1,R] \cup E_7$ that has finite logarithmic measure and a constant $B > 0$, such that if $|z| \notin E_8 \cup [0,1]$, we obtain

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leqslant B\left\{\frac{\exp_{p-1}\{(\alpha r)^{\rho+\varepsilon_1}\}}{r}(\log^\alpha r)\exp_{p-2}\{(\alpha r)^{\rho+\varepsilon_1}\}\right\}^{k-j}. \tag{50}$$

Hence, there exists a constant $\varepsilon > 1 + \varepsilon_1$, such that if $|z| \notin E_8 \cup [0,1]$, we have

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leqslant \left\{\exp_{p-1}\{r^{\rho-1+\varepsilon}\}\right\}^{k-j}. \tag{51}$$

□

Lemma 24 (see [5]). *Let $f(z)$ be an entire function of iterated p order $0 < \rho_p(f) < +\infty$ and iterated p type $0 < \tau_p(f) < \infty$. Then, for every given $\beta < \tau_p(f)$, there exists a set $E_9 \subset [1,+\infty)$ that has infinite logarithmic measure, such that, for all $r \in E_9$, we have*

$$\log_p M(r,f) > \beta r^{\rho_p(f)}. \tag{52}$$

Lemma 25. *Let $A_0(z), A_1(z), \ldots, A_{k-1}(z)$ be entire functions with finite iterated order and satisfy $\max\{\rho_p(A_j) : j = 1,2,\ldots,k-1\} \leqslant \rho_p(A_0) = \rho_2 (0 < \rho_2 < \infty)$, and $\max\{\tau_p(A_j) : \rho_p(A_j) = \rho_p(A_0)\} = \tau_1 < \tau_p(A_0) = \tau (0 < \tau < \infty)$, and let U_j^1, U_j^i be stated as in Lemma 20. Then, for any given $\varepsilon (0 < 2\varepsilon < \tau - \tau_1)$, there exists a set E_{10} with infinite logarithmic measure such that*

$$\left|U_j^i\right| \leqslant \exp_p\{(\tau_1+\varepsilon)r^{\rho_2}\}, \qquad \left|U_0^i\right| \geqslant \exp_p\{(\tau-\varepsilon)r^{\rho_2}\}, \tag{53}$$

where $i \geqslant 1, i \in \mathbb{N}$ and $j = 1,2,\ldots,k-1$.

Proof. We prove this lemma by using mathematical induction.

(i) We first prove that U_j^i $(j = 0,1,\ldots,k-1)$ satisfy (53) when $i = 1$. From the definition of $U_j^1 = A'_{j+1} + A_j - (A'_0/A_0)A_{j+1} (j \neq 0)$ and $U_0^1 = A'_1 + A_0 - (A'_0/A_0)A_1$, we have

$$\begin{aligned}\left|U_0^1\right| &\geqslant |A_0| - |A_1|\left(\left|\frac{A'_1}{A_1}\right| + \left|\frac{A'_0}{A_0}\right|\right),\\ \left|U_j^1\right| &\leqslant \left|A_j\right| + \left|A_{j+1}\right|\left(\left|\frac{A'_{j+1}}{A_{j+1}}\right| + \left|\frac{A'_0}{A_0}\right|\right),\\ &j = 1,2,\ldots,k-1,\ A_k \equiv 1.\end{aligned} \tag{54}$$

From Lemmas 23, 24, and (54), for any $\varepsilon (0 < 4\varepsilon < \tau - \tau_1)$, there exists a set E_{10} with infinite logarithmic measure such that

$$\begin{aligned}\left|U_0^1\right| &\geqslant \exp_p\left\{\left(\tau - \frac{\varepsilon}{4}\right)r^{\rho_2}\right\}\\ &\quad - 2\exp_p\left\{\left(\tau_1 + \frac{\varepsilon}{4}\right)r^{\rho_2}\right\}\exp_{p-1}\{r^M\}\\ &\geqslant \exp_p\left\{\left(\tau - \frac{\varepsilon}{2}\right)r^{\rho_2}\right\},\\ \left|U_j^1\right| &\leqslant \exp_p\left\{\left(\tau_1 + \frac{\varepsilon}{4}\right)r^{\rho_2}\right\}\\ &\quad + 2\exp_p\left\{\left(\tau_1 + \frac{\varepsilon}{4}\right)r^{\rho_2}\right\}\exp_{p-1}\{r^M\}\\ &\leqslant \exp_p\left\{\left(\tau_1 + \frac{\varepsilon}{2}\right)r^{\rho_2}\right\}, \quad j \neq 0,\end{aligned} \tag{55}$$

where $M > 0$ is a constant, not necessarily the same at each occurrence.

(ii) Next, we show that U_j^i $(j = 0,1,\ldots,k-1)$ satisfy (53) when $i = 2$. From

$$\begin{aligned}U_0^2 &= U_1^{1'} + U_0^1 - \frac{U_0^{1'}}{U_0^1}U_1^1,\\ U_j^2 &= U_{j+1}^{1'} + U_j^1 - \frac{U_0^{1'}}{U_0^1}U_{j+1}^1,\\ &j = 1,2,\ldots,k-1,\ U_k^1 \equiv 1,\end{aligned} \tag{56}$$

we have

$$\begin{aligned}\left|U_0^2\right| &\geqslant \left|U_0^1\right| - \left|U_1^1\right|\left(\left|\frac{U_1^{1'}}{U_1^1}\right| + \left|\frac{U_0^{1'}}{U_0^1}\right|\right),\\ \left|U_j^2\right| &\leqslant \left|U_j^1\right| + \left|U_{j+1}^1\right|\left(\left|\frac{U_{j+1}^{1'}}{U_{j+1}^1}\right| + \left|\frac{U_0^{1'}}{U_0^1}\right|\right),\\ &j = 1,2,\ldots,k-1.\end{aligned} \tag{57}$$

By the conclusions of (i) and Lemma 23, (55)–(57), we obtain for all $|z| = r \in E_{10}$

$$\begin{aligned}
\left|U_0^2\right| &\geqslant \exp_p\left\{\left(\tau - \frac{\varepsilon}{2}\right) r^{\rho_2}\right\} \\
&\quad - 2\exp_p\left\{\left(\tau_1 + \frac{\varepsilon}{2}\right) r^{\rho_2}\right\} \exp_{p-1}\left\{r^M\right\} \\
&\geqslant \exp_p\left\{(\tau - \varepsilon) r^{\rho_2}\right\}, \\
\left|U_j^2\right| &\leqslant \exp_p\left\{\left(\tau_1 + \frac{\varepsilon}{2}\right) r^{\rho_2}\right\} \\
&\quad + 2\exp_p\left\{\left(\tau_1 + \frac{\varepsilon}{2}\right) r^{\rho_2}\right\} \exp_{p-1}\left\{r^M\right\} \\
&\leqslant \exp_p\left\{(\tau_1 + \varepsilon) r^{\rho_2}\right\}, \quad j \neq 0.
\end{aligned} \tag{58}$$

(iii) Now, suppose that (53) holds for $i \leqslant n$, $n \in \mathbb{N}$; that is, for any given $\varepsilon (0 < 4\varepsilon < \tau - \tau_1)$, there exists a set E_{10} with infinite logarithmic measure such that

$$\begin{gathered}
\left|U_j^i\right| \leqslant \exp_p\left\{(\tau_1 + \varepsilon) r^{\rho_2}\right\}, \qquad \left|U_0^i\right| \geqslant \exp_p\left\{(\tau - \varepsilon) r^{\rho_2}\right\}, \\
i \leqslant n, \ j = 1, 2, \ldots, k-1.
\end{gathered} \tag{59}$$

From $U_0^{n+1} = U_1^{n'} + U_0^n - (U_0^{n'}/U_0^n) U_1^n$ and $U_j^{n+1} = U_{j+1}^{n'} + U_j^n - (U_0^{n'}/U_0^n) U_{j+1}^n$ $(j = 1, \ldots, k-1)$, $U_k^n \equiv 1$, we have

$$\begin{aligned}
\left|U_0^{n+1}\right| &\geqslant \left|U_0^n\right| - \left|U_1^n\right| \left(\left|\frac{U_1^{n'}}{U_1^n}\right| + \left|\frac{U_0^{n'}}{U_0^n}\right|\right), \\
\left|U_j^{n+1}\right| &\leqslant \left|U_j^n\right| + \left|U_{j+1}^n\right| \left(\left|\frac{U_{j+1}^{n'}}{U_{j+1}^n}\right| + \left|\frac{U_0^{n'}}{U_0^n}\right|\right), \\
& \qquad j = 1, 2, \ldots, k-1.
\end{aligned} \tag{60}$$

Then, from Lemma 23 and (59)-(60), for all $|z| = r \in E_{10}$,

$$\begin{aligned}
\left|U_j^{n+1}\right| &\leqslant \exp_p\left\{(\tau_1 + \varepsilon) r^{\rho_2}\right\} \\
&\quad + 2\exp_p\left\{(\tau_1 + \varepsilon) r^{\rho_2}\right\} \exp_{p-1}\left\{r^M\right\} \\
&\leqslant \exp_p\left\{(\tau_1 + 2\varepsilon) r^{\rho_2}\right\}
\end{aligned} \tag{61}$$

for $j \neq 0$, and

$$\begin{aligned}
\left|U_0^{n+1}\right| &\geqslant \exp_p\left\{(\tau - \varepsilon) r^{\rho_2}\right\} \\
&\quad - 2\exp_p\left\{(\tau_1 + \varepsilon) r^{\rho_2}\right\} \exp_{p-1}\left\{r^M\right\} \\
&\geqslant \exp_p\left\{(\tau - 2\varepsilon) r^{\rho_2}\right\}.
\end{aligned} \tag{62}$$

Thus, the proof of Lemma 25 is complete. □

Lemma 26. *Let $A_0, A_1, \ldots, A_{k-1}$ be meromorphic functions of finite iterated order with $i(A_0) = p (0 < p < \infty)$ such that $\max\{\rho_p(A_j) : j = 1, 2, \ldots, k-1\} = \rho_4 < \rho_p(A_0) = \rho_3$ and $\delta = \delta(\infty, A_0) = \liminf_{r \to \infty} (m(r, A_0))/(T(r, f)) > 0$. Then every meromorphic solution $f \not\equiv 0$ of (8) satisfies $\rho_{p+1}(f) \geqslant \rho_p(A_0) = \rho_3$.*

Proof. Assume that $f(z) \not\equiv 0$ is a meromorphic solution of (8). By (8) we have

$$m(r, A_0) \leqslant \sum_{j=1}^{k} m\left(r, \frac{f^{(j)}}{f}\right) + \sum_{j=1}^{k-1} m\left(r, A_j\right) + O(1). \tag{63}$$

By Lemma 15, we obtain

$$m(r, A_0) \leqslant O\{\log rT(r, f)\} + \sum_{j=1}^{k-1} m\left(r, A_j\right), \quad r \notin E_2, \tag{64}$$

where $E_2 \subset [1, +\infty)$ is a set having finite linear measure. By Lemma 19, there exists a set E_4 having infinite logarithmic measure such that, for all $|z| = r \in E_4$, we have

$$\lim_{r \to \infty} \frac{\log_p T(r, A_0)}{\log r} = \rho_3. \tag{65}$$

Since $\delta(\infty, A_0) > 0$, then for any given $\varepsilon (0 < 2\varepsilon < \min\{\delta, \rho_3 - \rho_4\})$ and for all $r \in E_4$, by (65), we have

$$m(r, A_0) \geqslant (\delta - \varepsilon) T(r, A_0) \geqslant (\delta - \varepsilon) \exp_{p-1}\left\{r^{\rho_3 - \varepsilon}\right\}. \tag{66}$$

From (64) and (66), we have for any given $\varepsilon (0 < 2\varepsilon < \min\{\delta, \rho_3 - \rho_4\})$ and for all $|z| = r \in E_4 \setminus E_2$

$$\begin{aligned}
&(\delta - \varepsilon) \exp_{p-1}\left\{r^{\rho_3 - \varepsilon}\right\} \\
&\quad \leqslant O\{\log rT(r, f)\} + (k-1) \exp_{p-1}\left\{r^{\rho_4 + \varepsilon}\right\}.
\end{aligned} \tag{67}$$

By (67) and Lemma 16, we obtain

$$\rho_{p+1}(f) \geqslant \rho_3. \tag{68}$$

□

Lemma 27. *Let $A_0, A_1, \ldots, A_{k-1}$ be meromorphic functions of finite iterated order. If there exist positive constants $\rho_5, \alpha, \beta (0 < \alpha < \beta)$ and a set E_{11} with infinite logarithmic measure such that*

$$\begin{gathered}
\left|A_0(z)\right| \geqslant \exp_p\left\{\beta r^{\rho_5}\right\}, \\
\max\left\{\left|A_j(z)\right| : j = 1, 2, \ldots, k-1\right\} \leqslant \exp_p\left\{\alpha r^{\rho_5}\right\}
\end{gathered} \tag{69}$$

hold for all $|z| = r \in E_{11}$, then every meromorphic solution $f \not\equiv 0$ of (8) satisfies $\rho_{p+1}(f) \geqslant \rho_5$.

Proof. Assume that $f \not\equiv 0$ is a meromorphic solution of (8). By (8), we obtain

$$\left|A_0\right| \leqslant \left|\frac{f^{(k)}}{f}\right| + \sum_{j=1}^{k-1} \left|A_j\right| \left|\frac{f^{(j)}}{f}\right|. \tag{70}$$

By Lemma 22, there exists a set E_7 having a finite logarithmic measure such that, for all $|z| = r \notin E_7$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leqslant B[T(2r, f)]^{k+1}, \quad j = 1, 2, \ldots, k, \tag{71}$$

where $B > 0$ is a constant. Substituting (69) and (71) into (70), we obtain for all $|z| = r \in E_{11} - E_7$

$$\exp_p\{\beta r^{\rho_5}\} \leqslant Bk[T(2r, f)]^{k+1} \exp_p\{\alpha r^{\rho_5}\}. \tag{72}$$

Since $0 < \alpha < \beta$, then from (72), Definitions 1, 2, and Lemma 16, we can deduce that $i(f) \geqslant p + 1$ and

$$\rho_{p+1}(f) \geqslant \rho_5. \tag{73}$$

□

Lemma 28 (see [13]). *Let $p \geqslant 1$ be an integer, and let $f(z)$ be a meromorphic solution of the differential equation*

$$f^{(k)} + A_{k-1} f^{(k-1)} + \cdots + A_0 f = F, \tag{74}$$

where $A_0, A_1, \ldots, A_{k-1}$ and $F \not\equiv 0$ are meromorphic functions:

(i) *if* $\max\{\rho_p(F), \rho_p(A_j)\,(j = 0, 1, 2, \ldots, k-1)\} < \rho_p(f) = \rho \leqslant \infty$, *then* $\overline{\lambda}_p(f) = \lambda_p(f) = \rho_p(f)$,

(ii) *if* $\max\{\rho_{p+1}(F), \rho_{p+1}(A_j)\,(j = 0, 1, 2, \ldots, k-1)\} < \rho_{p+1}(f) = \rho < \infty$, *then* $\overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \rho_{p+1}(f)$.

Lemma 29 (see [3]). *Let $A_0(z), \ldots, A_{k-1}(z)$ be entire functions such that $i(A_0) = p(0 < p < \infty)$. If either $\max\{i(A_j)\colon j = 1, 2, \ldots, k-1\} < p$ or $\max\{\rho_p(A_j) : j = 1, 2, \ldots, k-1\} < \rho_p(A_0)$, then every solution $f \not\equiv 0$ of (8) satisfies $i(f) = p + 1$ and $\rho_{p+1}(f) = \rho_p(A_0)$.*

Lemma 30 (see [5]). *Let $A_0, A_1, \ldots, A_{k-1}$ be entire functions of finite iterated order, and let $i(A_0) = p(0 < p < \infty)$. Assume that $\max\{\rho_p(A_j) : j = 1, 2, \ldots, k-1\} \leqslant \rho_p(A_0) = \rho(0 < \rho < \infty)$ and $\max\{\tau_p(A_j) : \rho_p(A_j) = \rho_p(A_0)\} < \tau_p(A_0) = \tau(0 < \tau < \infty)$. Then every solution $f \not\equiv 0$ of (8) satisfies $i(f) = p + 1$ and $\rho_{p+1}(f) = \rho_p(A_0) = \rho$.*

3. Proof of the Theorems

Proof of Theorem 7. We consider two cases.

Case 1. Suppose that $\max\{\rho_p(A_j) : j = 1, 2, \ldots, k-1\} < \rho_p(A_0) < \infty$.

(i) First, we prove that $\overline{\lambda}_{p+1}(f - \varphi) = \lambda_{p+1}(f - \varphi) = \rho_{p+1}(f) = \rho_p(A_0)$. Assume that $f \not\equiv 0$ is a solution of (8). From Lemma 29, we have $\rho_{p+1}(f) = \rho_p(A_0)$. Set $g = f - \varphi$. Since $\rho_{p+1}(\varphi) < \rho_p(A_0)$, then $\rho_{p+1}(g) = \rho_{p+1}(f) = \rho_p(A_0)$ and $\overline{\lambda}_{p+1}(g) = \overline{\lambda}_{p+1}(f - \varphi), \lambda_{p+1}(g) = \lambda_{p+1}(f - \varphi)$. By substituting $f = g + \varphi$ into (8), we get that g satisfies the following equation:

$$\begin{aligned} &g^{(k)} + A_{k-1} g^{(k-1)} + \cdots + A_0 g \\ &\quad = -\left[\varphi^{(k)} + A_{k-1}\varphi^{(k-1)} + \cdots + A_0\varphi\right]. \end{aligned} \tag{75}$$

Set $F = \varphi^{(k)} + A_{k-1}\varphi^{(k-1)} + \cdots + A_0\varphi$. If $F \equiv 0$, then, from Lemma 29, we have $\rho_{p+1}(\varphi) = \rho_p(A_0)$, which is a contradiction. Hence $F \not\equiv 0$. From the assumptions of Theorem 7, we get

$$\begin{aligned} &\max\left\{\rho_{p+1}(F), \rho_{p+1}(A_j)\,(j = 0, 1, 2, \ldots, k-1)\right\} \\ &\quad < \rho_{p+1}(g) = \rho_p(A_0). \end{aligned} \tag{76}$$

From Lemma 28 (ii), we have $\overline{\lambda}_{p+1}(f - \varphi) = \lambda_{p+1}(f - \varphi) = \rho_{p+1}(f) = \rho_p(A_0)$.

(ii) Second, we prove that $\overline{\lambda}_{p+1}(f' - \varphi) = \lambda_{p+1}(f' - \varphi) = \rho_{p+1}(f) = \rho_p(A_0)$. Set $g_1 = f' - \varphi$, then $\rho_{p+1}(g_1) = \rho_{p+1}(f) = \rho_p(A_0)$. By Lemma 13, we get that g_1 satisfies the (15). Set $F_1 = \varphi^{(k)} + U_{k-1}^1\varphi^{(k-1)} + \cdots + U_0^1\varphi$, where U_j^1 $(j = 0, 1, \ldots, k-1)$ are stated as in Lemma 13. If $F_1 \equiv 0$, then, from Lemmas 20 and 21, we have $\rho_{p+1}(\varphi) \geqslant \rho_p(A_0)$, a contradiction with $\rho_{p+1}(\varphi) < \rho_p(A_0)$. Hence $F_1 \not\equiv 0$. By Lemma 28 (ii), we have

$$\overline{\lambda}_{p+1}\left(f' - \varphi\right) = \lambda_{p+1}\left(f' - \varphi\right) = \rho_{p+1}(f) = \rho_p(A_0). \tag{77}$$

(iii) Now, we prove that $\overline{\lambda}_{p+1}(f'' - \varphi) = \lambda_{p+1}(f'' - \varphi) = \rho_{p+1}(f) = \rho_p(A_0)$. Set $g_2 = f'' - \varphi$, then $\rho_{p+1}(g_2) = \rho_{p+1}(f) = \rho_p(A_0)$. By Lemma 13, we get that g_2 satisfies (15). Set $F_2 = \varphi^{(k)} + U_{k-1}^2\varphi^{(k-1)} + \cdots + U_0^2\varphi$, where U_j^2 $(j = 0, 1, \ldots, k-1)$ are stated as in Lemma 13. If $F_2 \equiv 0$, then, from Lemmas 20 and 21, we have $\rho_{p+1}(\varphi) \geqslant \rho_p(A_0)$, a contradiction with $\rho_{p+1}(\varphi) < \rho_p(A_0)$. Hence $F_2 \not\equiv 0$. By Lemma 28 (ii), we have

$$\overline{\lambda}_{p+1}\left(f'' - \varphi\right) = \lambda_{p+1}\left(f'' - \varphi\right) = \rho_{p+1}(f) = \rho_p(A_0). \tag{78}$$

(iv) Set $g_3 = f''' - \varphi$, and then $\rho_{p+1}(g_3) = \rho_{p+1}(f) = \rho_p(A_0)$. From Lemmas 13, 20, 21, and 28 (ii), using the same argument as in Case 1 (iii), we can get $\overline{\lambda}_{p+1}(f''' - \varphi) = \lambda_{p+1}(f''' - \varphi) = \rho_{p+1}(f) = \rho_p(A_0)$.

(v) We prove that $\overline{\lambda}_{p+1}(f^{(i)} - \varphi) = \lambda_{p+1}(f^{(i)} - \varphi) = \rho_{p+1}(f) = \rho_p(A_0)$, $i > 3$, $i \in \mathbb{N}$. Set $g_i = f^{(i)} - \varphi$, and then $\rho_{p+1}(g_i) = \rho_{p+1}(f) = \rho_p(A_0)$. By Lemma 13, we get that g_i satisfies (15). Set $F_i = \varphi^{(k)} + U_{k-1}^i\varphi^{(k-1)} + \cdots + U_0^i\varphi$, where U_j^i $(j = 0, 1, \ldots, k-1, i > 3, i \in \mathbb{N})$ are stated as in Lemma 13. If $F_i \equiv 0$, then from Lemmas 20 and 21, we have $\rho_{p+1}(\varphi) \geqslant \rho_p(A_0)$, a contradiction with $\rho_{p+1}(\varphi) < \rho_p(A_0)$. Hence $F_i \not\equiv 0$. By Lemma 28 (ii), we have $\overline{\lambda}_{p+1}(f^{(i)} - \varphi) = \lambda_{p+1}(f^{(i)} - \varphi) = \rho_{p+1}(f) = \rho_p(A_0)$ $(i > 3, i \in \mathbb{N})$.

Case 2. Suppose that $\max\{\rho_p(A_j) : j = 1, 2, \ldots, k-1\} \leqslant \rho_p(A_0) = \rho(0 < \rho < \infty)$ and $\max\{\tau_p(A_j) : \rho_p(A_j) = \rho_p(A_0)\} < \tau_p(A_0) = \tau(0 < \tau < \infty)$.

(i) First, we prove that $\overline{\lambda}_{p+1}(f-\varphi)=\lambda_{p+1}(f-\varphi)=\rho_{p+1}(f)=\rho_p(A_0)$. Assume that $f\not\equiv 0$ is a solution of (8). From Lemma 30, we have $\rho_{p+1}(f)=\rho_p(A_0)$. Set $g=f-\varphi$. Since $\varphi\not\equiv 0$ is an entire function satisfying $\rho_{p+1}(\varphi)<\rho_p(A_0)$, then $\rho_{p+1}(g)=\rho_{p+1}(f-\varphi)=\rho_p(A_0)$ and $\overline{\lambda}_{p+1}(g)=\overline{\lambda}_{p+1}(f-\varphi)$, $\lambda_{p+1}(g)=\lambda_{p+1}(f-\varphi)$. By substituting $f=g+\varphi$ into (8), we get that g satisfies (75). Set $F=\varphi^{(k)}+A_{k-1}\varphi^{(k-1)}+\cdots+A_0\varphi$. If $F\equiv 0$, then, from Lemma 30, we have $\rho_{p+1}(\varphi)=\rho_p(A_0)$, which is a contradiction. Hence $F\not\equiv 0$. From the assumptions of Theorem 7, we get

$$\max\left\{\rho_{p+1}(F),\rho_{p+1}\left(A_j\right)(j=0,1,2,\ldots,k-1)\right\}<\rho_{p+1}(g)=\rho_p(A_0). \tag{79}$$

From Lemma 28 (ii), we have $\overline{\lambda}_{p+1}(f-\varphi)=\lambda_{p+1}(f-\varphi)=\rho_{p+1}(f)=\rho_p(A_0)$.

(ii) Now, we prove that $\overline{\lambda}_{p+1}(f'-\varphi)=\lambda_{p+1}(f'-\varphi)=\rho_{p+1}(f)=\rho_p(A_0)$. Set $g_1=f'-\varphi$. Since $\varphi\not\equiv 0$ is an entire function satisfying $\rho_{p+1}(\varphi)<\rho_p(A_0)$, then $\rho_{p+1}(g_1)=\rho_{p+1}(f)=\rho_p(A_0)$. By Lemma 13, we get that g_1 satisfies the (15). If $F_1\equiv 0$, then, from Lemmas 25 and 27, we have $\rho_{p+1}(\varphi)\geqslant\rho_p(A_0)$, a contradiction with $\rho_{p+1}(\varphi)<\rho_p(A_0)$. Hence $F_1\not\equiv 0$. By Lemma 28 (ii), we have $\overline{\lambda}_{p+1}(f'-\varphi)=\lambda_{p+1}(f'-\varphi)=\rho_{p+1}(f)=\rho_p(A_0)$. Similar to the arguments as in Case 1 (iii)–(v) and by using Lemmas 13, 25, and 27, we can get $\overline{\lambda}_{p+1}(f^{(i)}-\varphi)=\lambda_{p+1}(f^{(i)}-\varphi)=\rho_{p+1}(f)=\rho_p(A_0)$, $i>1$, $i\in\mathbb{N}$. Thus, the proof of Theorem 7 is completed. □

Proof of Theorem 8. Since $A_j(z)$ $(j=1,2,\ldots,k-1)$ are polynomials and A_0 is a transcendental entire function with $0<\rho_p(A_0)<\infty$; that is, $A_j(z)$ $(j=0,1,2,\ldots,k-1)$ satisfy the conditions of Theorem 7 (i). By using the same argument as in Theorem 7 and Lemma 28 (i), we can get the conclusions of Theorem 8 easily. □

Proof of Theorem 9. Assume that $f\not\equiv 0$ is a meromorphic solution of (8). By (8) we get that the poles of $f(z)$ can only occur at the poles of $A_0,A_1,\ldots,A_{k-1}$. Note that the multiplicities of poles of f are uniformly bounded, and thus we have

$$N(r,f)\leqslant M_1\overline{N}(r,f)\leqslant M_1\sum_{j=0}^{k-1}\overline{N}\left(r,A_j\right)\leqslant M\max\left\{N\left(r,A_j\right):j=0,1,\ldots,k-1\right\}, \tag{80}$$

where M_1 and M are some suitable positive constants. This gives

$$T(r,f)=m(r,f)+O\left(\max\left\{N\left(r,A_j\right):j=0,1,\ldots,k-1\right\}\right). \tag{81}$$

Applying now (81) with Lemma 17, we obtain

$$\rho_{p+1}(f)\leqslant\rho_p(A_0). \tag{82}$$

According to the conditions of Theorem 9, we can easily get the conclusions of Theorem 9 by using the similar argument as in Theorem 7, the estimation (82), and Lemmas 26 and 28 (ii). □

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