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## On a General Reduction Scheme of Fredholm Operators

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### Abstract

In this paper we present a new reduction scheme that we can applied to the study of Fredholm operators. We mainly show in a simple and original manner the stability results for Fredholm operators. We also get a fundamental characterization result of the spectrum of these operators.

**Key words:** Fredholm operators, reduction, stability theorems, spectrum.

**AMS Classification Subject:** 47A53, 47B48

### I. INTRODUCTION

The Fredholm operators have two important properties. First of all the properties of stability under small perturbations, this result has been proved for the first time by F. V. Atkinson in 1951 [1]. It has been generalized successively in 1952 by M. G. Krein and M. A. Krasnoselsky in [4] and by B. Sg. Nagy in [10]. A more general form was introduced thirty five years after by J. P. Labrousse and B. Mercier in [5]. These operators have also another important property of decomposition known as Kato decomposition .

On the other hand, if  $C(H)$  (resp.  $\mathcal{L}(H)$ ) designates the closed linear operators space (resp. bounded) on a dense domain of a Hilbert space  $H$ , we know that  $(A \sim B)$  if and only if  $(A - B)$  is compact on  $H$ , is an equivalence relation in  $\mathcal{L}(H)$ . Many results have been obtained via this relation "  $\sim$  ", we can mentioned in particular the spectral correction by compact perturbation, the Weyl theorem, the Calkin algebra, and the possibility

of extending some results and spectral properties to the class of Fredholm operators of  $C(H)$  (see eg.[5]). We have to note also the use of Fredholm theory in the essential spectrum definition of an operator of  $C(H)$ , a notion which has already been used in different contexts by many authors.

In this paper we present a new and original method adapted for the first time to Fredholm operators of class  $C(H)$ , some stability theorems and spectral results are easily obtained.

Let us briefly describe the essential ideas of our method inspired by the Feshbach method which is a well known technique of reduction, introduced by Feshbach in [2] also used in [6] and [7] to study the spectral properties and resonances of Schrödinger Hamiltonians for polyatomic molecules in the semiclassical limit. The way in which the results of [6] and [7] (see also [9]) have been proved relies on the construction of an operator acting on a grater space by means of eigenfunctions of the considered electronic Hamiltonian.

We present here this method in a slightly different way that can be applied to the class of Fredholm operators and prove our main results, namely the stability theorem and a result of spectral reduction for these operators. Our trick is to reduce

the problem to the inversion of a general matrix operator using no spectral tools but including the classical case of the Feshbach method.

The main goal of the first section is the construction of a matrix operator associated to a continuous family  $(P_z)_{z \in \mathbb{C}}$  of bounded linear operators between two complex Banach spaces  $E$  and  $F$ .

We assume that for a certain  $z_0$ ,  $P_{z_0}$  (noted  $P$ ) is a Fredholm operator with index  $(n_0 - d_0)$ , where  $n_0$  is the nullity and  $d_0$  is the deficiency of  $P$ .

One considers then the "Grushin operator" defined by

$$\mathcal{P}_z = \begin{pmatrix} P_z & R_0^- \\ R_0^+ & 0 \end{pmatrix}$$

and acts on  $E \oplus \mathbb{C}^{d_0}$  to  $F \oplus \mathbb{C}^{n_0}$ .

$R_0^+$  and  $R_0^-$  are bounded linear operators respectively from  $E$  to  $\mathbb{C}^{n_0}$  and from  $\mathbb{C}^{d_0}$  to  $F$ , with maximum rank the ranges of  $P$  and  $R_0^-$  are transverse and  $R_0^+$  is invertible on the kernel of  $P$

Thus,  $\mathcal{P}_z$  becomes invertible for  $z$  sufficiently near to  $z_0$ , its inverse is given by

$$\mathcal{P}_z^{-1} = \begin{pmatrix} A(z) & A^+(z) \\ A^-(z) & A^{-+}(z) \end{pmatrix}$$

where  $A(z)$ ,  $A^\pm(z)$  and  $A^{-+}(z)$  are bounded and depends continuously on  $z$ .

We establish that the equation

$$P_z f = g$$

admits a solution  $f$  in  $E$  if and only if  $A^-(z)g$  belongs to the range of  $A^{-+}(z)$ .

While using then  $\mathcal{P}_z$  and its inverse, we obtain our first fundamental result that for  $z$  in a small complex neighborhood of  $z_0$ ,  $P_z$  and  $A^{-+}(z)$  are Fredholm operators with the same index equal to the index of  $P$

Consequently, the property of being Fredholm and the index are stable under small perturbations. This theorem has long been known but such a result was proved by using the properties of closed operators with closed range (see eg. [3]). Nevertheless, it appears that the reduction that we have obtained enable us to find directly in a simple and original manner the stability results of Fredholm theory (the Fredholm operator class is open in  $C(H)$ , and the index remains constant on every connexe component of this class). In particular, this stability is new in the form given here.

In the second section we assume that  $E \subset F$ ,  $P_z = P - z$ , where  $z_0 = 0$ , its shown that our

method is also adapted for the spectral analysis of Fredholm operators.

In fact, another interest to consider the matrix  $\mathcal{P}_z$  is that it reduces the spectral study of the Fredholm operator  $P$  to that of a simpler linear operator acting on  $\mathbb{C}^{n_0}$  in the sense

$$z \in Sp(P) \iff z \in Sp(z - A^{-+}(z))$$

where  $Sp$  stands for the spectrum.

This reduction characterize the spectrum of  $P$  through a finite matrix of analytic compact operators with constant index defined on a finite dimensional complex euclidian space.

## II. FREDHOLM THEORY

Let  $E$  and  $F$  be a complex Banach spaces,  $\omega$  an open complex and  $(P_z)_{z \in \omega}$  a continuous family on  $\omega$  of bounded linear operators from  $E$  to  $F$ .

Suppose that for a certain  $z_0$  of  $\omega$ , the corresponding operator  $P_{z_0}$  noted  $P$  is a Fredholm operator. We denote  $n_0 = \dim(\ker P) < +\infty$  and  $d_0 = \text{codim}_F(\text{Im}P) = \dim(F/\text{Im}P) < +\infty$ ,  $(n_0 - d_0)$  is called the index of  $P$ .

By analogy with the Grushin operators introduced into [2], [6], and [7], we consider two linear operators  $R_0^+$  from  $E$  to  $\mathbb{C}^{n_0}$  and  $R_0^-$  from  $\mathbb{C}^{d_0}$  to  $F$ , such that :

(H1)  $R_0^\pm$  are bounded and of maximum rank,

$$\begin{cases} \dim(\text{Im}R_0^-) = d_0 \\ \dim(\text{Im}R_0^+) = n_0 \end{cases}$$

(H2)  $R_0^-(\mathbb{C}^{d_0})$  is transverse with  $P(E)$ ,

$$F = \text{Im}R_0^- \oplus \text{Im}P$$

(H3)  $R_0 = R_0^+_{|\ker P}$  is invertible

### Remarks 1.1:

1) The two nonorthogonal projections  $p_0$  from  $F$  to  $\text{Im}P$  and  $p_0^-$  from  $F$  to  $\text{Im}R_0^-$  are bounded relatively to the decomposition (H2). In particular,  $rg(p_0^-) = \text{corg}(p_0) = d_0$ , (here  $rg$  (resp.  $\text{corg}$ ) stands for the rank (resp. corank or deficiency).

2) If  $\mathcal{E}$  is a complement of  $\ker P$  in  $E$ , then  $P' = P|_{\mathcal{E}}$  is an isomorphism of  $\mathcal{E}$  onto  $\text{Im}P$ .

3)  $R_0^-$  is an isomorphism of  $\mathbb{C}^{d_0}$  onto  $\text{Im}R_0^-$ .

Example of operators  $R_0^\pm$  :

(1) If  $\{e_1, \dots, e_{d_0}\}$  is a fixed basis of the quotient space  $F/ImP$  of  $F$  by  $ImP$ , we put

$$\begin{cases} Pf + R_0^- \alpha = g \\ R_0^+ f = \beta \\ g \in F, \beta \in \mathbf{C}^{n_0} \end{cases}$$

$$R_0^-(\lambda_1, \dots, \lambda_{d_0}) = \sum_{k=1}^{d_0} \lambda_k e_k \in F, \quad (\lambda_1, \dots, \lambda_{d_0}) \in \mathbf{C}^{d_0}$$

Since  $E = \ker P \oplus \mathcal{E}$  we have  $f = f_0 + f_{\mathcal{E}}$  with  $f_0 \in \ker P$  and  $f_{\mathcal{E}} \in \mathcal{E}$ , then

$$\begin{cases} Pf_{\mathcal{E}} + R_0^- \alpha = g \\ R_0^+ f_0 + R_0^+ f_{\mathcal{E}} = \beta \end{cases}$$

We can easily see that  $rg(R_0^-) = d_0$  and that  $ImR_0^-$  is transverse with  $ImP$ .

Indeed,  $\forall g \in F, \exists (\lambda_1, \dots, \lambda_{d_0}) \in \mathbf{C}^{d_0}$  such that  $g = \sum_{k=1}^{d_0} \lambda_k e_k \in F/ImP$  and  $(g - \sum_{k=1}^{d_0} \lambda_k e_k) \in ImP$ . Then

By the assumption **(H3)**, we can rewrite this system as

$$\begin{cases} Pf_{\mathcal{E}} + R_0^- \alpha = g \\ f_0 = (R_0)^{-1}(\beta - R_0^+ f_{\mathcal{E}}) \end{cases}$$

$$g = \sum_{k=1}^{d_0} \lambda_k e_k + (g - \sum_{k=1}^{d_0} \lambda_k e_k) \in ImR_0^- \oplus ImP$$

or since  $Pf_{\mathcal{E}} = p_0 g$  and  $R_0^- \alpha = p_0^- g$ , we obtain

$$\text{because } ImR_0^- \cap ImP = \{0\}.$$

$$\begin{cases} f_{\mathcal{E}} = (P')^{-1} p_0 g \\ f_0 = (R_0)^{-1}(\beta - R_0^+ (P')^{-1} p_0 g) \\ \alpha = (R_0^-)^{-1} p_0^- g \end{cases}$$

(2) If  $\{v_1, \dots, v_{n_0}\}$  is a basis of  $\ker P$  and  $\mathcal{E}$  is a complement of  $\ker P$  in  $E$ , then every element  $f$  of  $E$  is written in a unique way in the form  $f = \sum_{k=1}^{n_0} \mu_k v_k + f_{\mathcal{E}}$  where  $f_{\mathcal{E}}$  is the projection of  $f$  on  $\mathcal{E}$ . We define  $R_0^+$  from  $E$  to  $\mathbf{C}^{n_0}$  by  $R_0^+ f = (\mu_1, \dots, \mu_{n_0})$ . By construction  $rg(R_0^+) = n_0$ ,  $R_0^+$  is also invertible on  $\ker P$ .

Thus  $\mathcal{P}_0$  is invertible and its inverse is a bounded operator which can be written as the formulae given in the proposition.

To study the properties of the family  $(P_z)_{z \in \omega}$ , in a small enough neighborhood of  $z_0$ , we fixe  $\varepsilon > 0$  such that  $\varepsilon < 1/\|\mathcal{P}_0^{-1}\|_{\mathcal{L}(F \oplus \mathbf{C}^{n_0}, \mathbf{E} \oplus \mathbf{C}^{d_0})}$  and we define the operator  $\mathcal{P}_z$  from  $E \oplus \mathbf{C}^{d_0}$  to  $F \oplus \mathbf{C}^{n_0}$  by:

We consider now the following matrix operator (the so-called Grushin operator) :

$$\mathcal{P}_0 = \begin{pmatrix} P & R_0^- \\ R_0^+ & 0 \end{pmatrix}$$

$$\mathcal{P}_z = \begin{pmatrix} P_z & R_0^- \\ R_0^+ & 0 \end{pmatrix}$$

defined from  $E \oplus \mathbf{C}^{d_0}$  to  $F \oplus \mathbf{C}^{n_0}$ .

By continuity of the family  $(P_z)_{z \in \omega}$ , there exist a complex neighborhood  $\Omega$  of  $z_0$  included in  $\omega$  such that

**Proposition 1.2:**

The operator  $\mathcal{P}_0$  defined above is invertible in  $\mathcal{L}(E \oplus \mathbf{C}^{d_0}, F \oplus \mathbf{C}^{n_0})$  and its inverse is bounded given by

$$\mathcal{P}_0^{-1} = \begin{pmatrix} A_0 & A_0^+ \\ A_0^- & A_0^{-+} \end{pmatrix}$$

$$\|\mathcal{P}_z - P\|_{\mathcal{L}(E, F)} < \varepsilon, \quad \forall z \in \Omega$$

Hence

$$\|\mathcal{P}_0^{-1}\|_{\mathcal{L}(F \oplus \mathbf{C}^{n_0}, \mathbf{E} \oplus \mathbf{C}^{d_0})} \|\mathcal{P}_z - \mathcal{P}_0\|_{\mathcal{L}(E \oplus \mathbf{C}^{d_0}, F \oplus \mathbf{C}^{n_0})} < 1$$

Consequently, for all  $z$  in  $\Omega$

$$\mathcal{P}_z = \mathcal{P}_0 [1 - \mathcal{P}_0^{-1}(\mathcal{P}_0 - \mathcal{P}_z)]$$

is invertible in  $\mathcal{L}(E \oplus \mathbf{C}^{d_0}, F \oplus \mathbf{C}^{n_0})$  and its inverse is a bounded operator given by the Neumann series:

Where

$$\begin{cases} A_0 = (1 - (R_0)^{-1} R_0^+) (P')^{-1} p_0 \in \mathcal{L}(F, E) \\ A_0^+ = (R_0)^{-1} \in \mathcal{L}(\mathbf{C}^{n_0}, \mathbf{E}) \\ A_0^- = (R_0^-)^{-1} p_0^- \in \mathcal{L}(F, \mathbf{C}^{d_0}) \\ A_0^{-+} = 0 \in \mathcal{L}(\mathbf{C}^{n_0}, \mathbf{C}^{d_0}) \end{cases}$$

$$\mathcal{P}_z^{-1} = [1 - \mathcal{P}_0^{-1}(\mathcal{P}_0 - \mathcal{P}_z)]^{-1} \mathcal{P}_0^{-1}$$

Here we have denoted by  $(P')^{-1}$  the inverse of  $P|_{\mathcal{E}}$ .

$$= \sum_{n=0}^{\infty} [1 - \mathcal{P}_0^{-1} \mathcal{P}_z]^n \mathcal{P}_0^{-1}$$

**proof 1 :** Showing that  $\mathcal{P}_0$  is invertible is equivalent to solving the following system where the unknown  $f \oplus \alpha$  is in  $E \oplus \mathbf{C}^{d_0}$  :

We denote for  $z \in \Omega$

$$\mathcal{E}(z) = \mathcal{P}_z^{-1} = \begin{pmatrix} A(z) & A^+(z) \\ A^-(z) & A^{-+}(z) \end{pmatrix}$$

**Remark 1.3:**

$A(z) \in \mathcal{L}(F, E)$ ,  $A^+(z) \in \mathcal{L}(\mathbf{C}^{n_0}, \mathbf{E})$ ,  $A^-(z) \in \mathcal{L}(F, \mathbf{C}^{d_0})$  and  $A^{-+}(z) \in \mathcal{L}(\mathbf{C}^{n_0}, \mathbf{C}^{d_0})$  depends continuously on  $z$  in  $\Omega$  and  $A^{-+}(z_0) = A_0^{-+} = 0$ , in addition the operators  $A^\pm(z)$  remains with maximal rank in  $\Omega$ ,

$$\begin{cases} \text{rg}(A^+(z)) = n_0 \\ \text{rg}(A^-(z)) = d_0 \end{cases}$$

For  $z \in \Omega$ , let us now study the equation  $P_z f = g$  and observe the interest of the operator  $\mathcal{P}_z$  in the solutions of this equation through the following result:

**Proposition 1.4:**

For a given  $g \in F$ , the operational equation  $P_z f = g$ , admits a solution  $f$  in  $E$  if and only if  $A^-(z)g \in \text{Im}A^{-+}(z)$ .

**proof 2 :**

By introducing the auxiliary variable  $R_0^+ f \in \mathbf{C}^{n_0}$ , we have

$$\begin{aligned} P_z f &= g \\ \Leftrightarrow \mathcal{P}_z(f \oplus \theta) &= (g \oplus R_0^+ f) \\ \Leftrightarrow \mathcal{E}(z)(g \oplus R_0^+ f) &= (f \oplus \theta) \end{aligned}$$

Then

$$\begin{cases} A(z)g + A^+(z)R_0^+ f = f & (1) \\ A^-(z)g + A^{-+}(z)R_0^+ f = 0 & (2) \end{cases}$$

If  $A^-(z)g = \beta \in \text{Im}A^{-+}(z) \subset \mathbf{C}^{d_0}$ , then there exist  $\alpha \in \mathbf{C}^{n_0}$  such that  $\beta = A^{-+}(z)\alpha$ . Replacing this value of  $\beta$  in (2), we find

$$A^{-+}(z)[\alpha + R_0^+ f] = 0$$

We can choose  $f$  such that

$$R_0^+ f = -\alpha$$

While substituting this expression in (1), one has the solution

$$f = A(z)g - A^+(z)\alpha$$

Conversely, if  $f \in E$  is a solution of the equation  $P_z f = g$  then (2) permits to write

$$A^-(z)g = -A^{-+}(z)R_0^+ f \in \text{Im}A^{-+}(z)$$

On one hand, since  $\text{rg}(A^-(z)) = d_0$  if we assume that for  $g \in F$ , there exist  $\varepsilon > 0$  such that  $B_F(g, \varepsilon) \subset F \setminus \text{Im}P_z$  (where  $B_F(g, \varepsilon)$  indicates the open ball of  $F$  centered in  $g$  and of radius  $\varepsilon$ ), then  $\forall h \in B_F(g, \varepsilon)$  the equation  $P_z f = h$  does not have any solution in  $E$ , consequently  $A^-(z)h \notin \text{Im}A^{-+}(z)$ , and  $\text{Im}A^{-+}(z) \subsetneq \mathbf{C}^{d_0}$ , which is absurd. We deduce that  $\text{Im}P_z$  is closed in  $F$ .

On the other hand, let us consider the following commutative diagram :

$$\begin{array}{ccc} F & \xrightarrow{A^-(z)} & \mathbf{C}^{d_0} \\ \downarrow s & t \searrow & \downarrow s' \\ F/\text{Im}P_z & \xrightarrow{\rho} & \mathbf{C}^{d_0}/\text{Im}A^{-+}(z) \end{array}$$

Where  $s(g) = \dot{g}$ ,  $g \in F$ ;  $s'(\alpha) = \hat{\alpha}$ ,  $\alpha \in \mathbf{C}^{d_0}$ ;  $\rho(\dot{g}) = \overline{A^-(z)g}$ ,  $t(g) = \overline{A^-(z)g}$ ;  $\cdot$  is the equivalence class in  $F/\text{Im}P_z$  and  $\hat{\cdot}$  is the class in  $\mathbf{C}^{d_0}/\text{Im}A^{-+}(z)$ .

$$\begin{aligned} \rho(\dot{g}) &= \rho(\dot{h}) \iff A^-(z)(g - h) \in \text{Im}A^{-+}(z) \\ \iff P_z f &= (g - h) \text{ admits a solution } f \text{ in } E \\ \iff (g - h) &\in \text{Im}P_z \iff \dot{g} = \dot{h} \end{aligned}$$

$\rho$  maps  $F/\text{Im}P_z$  onto  $\mathbf{C}^{d_0}/\text{Im}A^{-+}(z)$  because  $\rho \circ s = t$  and  $\text{Im}t = \mathbf{C}^{d_0}/\text{Im}A^{-+}(z)$ .  $\rho$  is then a linear isomorphism.

Consequently,

$$\dim(F/\text{Im}P_z) = \dim(\mathbf{C}^{d_0}/\text{Im}A^{-+}(z)) < +\infty \tag{3}$$

Similarly, if  $\alpha \in \mathbf{C}^{n_0}$ , then for all  $z$  in  $\Omega$

$$\begin{aligned} \beta &= A^{-+}(z)\alpha \iff \mathcal{E}(z)(\theta \oplus \alpha) = (A^+(z)\alpha \oplus \beta) \\ \iff \mathcal{P}_z(A^+(z)\alpha \oplus \beta) &= (\theta \oplus \alpha) \end{aligned}$$

and

$$\begin{cases} P_z A^+(z)\alpha + R_0^- \beta = 0 & (4) \\ R_0^+ A^+(z)\alpha = \alpha & (5) \end{cases}$$

From (5), we deduce that if  $A^+(z)\alpha = 0$  then  $\alpha = 0$ , thus  $\ker A^+(z) = \{0\}$  and  $\dim(\text{Im}A^+(z)) = n_0$  (remark 1.3).

Let  $l$  be the application defined by  $l(\alpha) = A^+(z)\alpha$  from  $\ker A^{-+}(z)$  to  $\ker P_z$ , then

- $l$  is well defined because if  $A^{-+}(z)\alpha = 0$ , we have from (4),  $P_z A^+(z)\alpha = P_z(l(\alpha)) = 0$  and thus  $l(\alpha) \in \ker P_z$ .
- $\ker l = \{0\}$ .  $l$  maps  $\ker A^{-+}(z)$  onto  $\ker P_z$  or  $\text{Im}l = \ker P_z$ , because the equation  $A^+(z)\alpha = f$  has to be solved in  $\ker A^{-+}(z)$  for a given  $f$  in  $\ker P_z$ .

We make according to equation (4),  $\alpha = R_0^+ f$ . By using (2) we find  $A^{-+}(z)\alpha = -A^-(z)P_z f = 0$  and then  $l(\alpha) = A^+(z)\alpha = f$ , by virtue of (1).

Consequently,

$$\dim(\ker P_z) = \dim(\ker A^{-+}(z)) < +\infty \quad , \quad \forall z \in \Omega \quad (6)$$

We then have established the following first main result relatively to the stability on  $\Omega$  of the Fred-

holm character and the corresponding index.

**Theorem 1.5:**

For any  $z$  in  $\Omega$ ,  $P_z$  and  $A^{-+}(z)$  are Fredholm operators of the same index independent of  $z$ .

$$index(P_z) = index(A^{-+}(z)) = n_0 - d_0 \quad , \quad \forall z \in \Omega$$

**III. SPECTRAL REDUCTION OF FREDHOLM OPERATORS**

Now let us suppose that  $\Omega$  is an open connexe neighborhood of 0 in  $\mathbf{C}$  verifying the conditions of the previous section.

$E \subset F$  with a dense inclusion, and suppose  $P_z = P - z$ , where  $P$  is the Fredholm operator such that  $n_0 = \dim(\ker P) < +\infty$  and  $d_0 = \dim(F/ImP) < +\infty$ . We suppose also that  $P_z$  is invertible for at least a point of  $\Omega$  different from 0. Then according to (3) and (6),  $A^{-+}(z)$  is also invertible at the same point and thus  $n_0 = d_0$ .

A principal interest to reconsider the operator  $P_z$  is that it reduces in fact the spectral study of the Fredholm operator  $P$  to that of a simpler linear operator acting on  $\mathbf{C}^{n_0}$ . Our second main result is

**Theorem 2.1:**

For any  $z$  in  $\Omega$ , we have the following equivalence

$$z \in Sp(P) \iff 0 \in Sp(A^{-+}(z)) \iff \det(A^{-+}(z)) = 0$$

where  $\det$  stands for the determinant.

**proof 3 :** If  $0 \notin Sp(A^{-+}(z))$ , then the equations (1) and (2) gives the following equivalence

$$\iff \begin{cases} P_z f = g \\ R_0^+ f = -(A^{-+}(z))^{-1} A^-(z)g \\ f = A(z)g - A^+(z)(A^{-+}(z))^{-1} A^-(z)g \end{cases}$$

Thus

$$P_z^{-1} = A(z) - A^+(z)(A^{-+}(z))^{-1} A^-(z) \quad (7)$$

which proves that  $z \notin Sp(P)$ .

Conversely, if  $z \notin Sp(P)$ , then equation (4) and (5) permits to write :

$$A^{-+}(z)\alpha = \beta \iff \alpha = -R_0^+ P_z^{-1} R_0^- \beta$$

Then,  $0 \notin Sp(A^{-+}(z))$  and

$$(A^{-+}(z))^{-1} = -R_0^+ P_z^{-1} R_0^- \quad (8)$$

**Remarks 2.2:**

1) The formulae (7) and (8) shows owing to the fact that by construction  $A(z)$ ,  $A^\pm(z)$  and  $A^{-+}(z)$  depend analytically on  $z$ , that a singularity of  $P_z^{-1}$  is necessarily a singularity of  $(A^{-+}(z))^{-1}$  and conversely.

2) Since  $A^{-+}(0) = A_0^{-+} = 0$ , then  $\det(A_0^{-+}) = 0$  and  $0 \in Sp(P)$ .

Since the zeros of  $\det(A^{-+}(z))$  form a discrete set in  $\Omega$ , if we reduce  $\Omega$  we can assume that  $P_z$  is invertible (and thus  $A^{-+}(z)$ ) with a bounded inverse for all  $z$  in  $\Omega \setminus \{0\}$ . 0 is then a pole of multiplicity  $N_0$  of the matrix  $(A^{-+}(z))^{-1}$ .

Let  $\gamma$  be a simple complex closed contour around the singularity 0 in  $\Omega$ . Note that :

$$\pi \stackrel{\text{def}}{=} \frac{-1}{2\pi i} \oint_{\gamma} P_z^{-1} dz \quad (9)$$

The operator  $\pi$  gives some supplementary informations on  $P$  and permits to specify the spectral result of the Theorem 2.1.

**Theorem 2.3:**

- (1)  $\pi$  is a projector of a finite rank
- (2)  $Im\pi = \pi(E) = \pi(F)$ .  $\ker \pi$  and  $Im\pi$  are invariant by  $P$
- (3)  $Sp(P) \cap \overset{\circ}{\gamma} = Sp(P|_{Im\pi}) \cap \overset{\circ}{\gamma}$ ,  $\overset{\circ}{\gamma}$  is the interior of the contour  $\gamma$   
 $Sp(P|_{\ker \pi}) \cap \overset{\circ}{\gamma} = \emptyset$  and  $\#(Sp(P) \cap \overset{\circ}{\gamma}) = rg(\pi)$

**proof 4 :**

(1) Let  $\gamma'$  be an other simple closed complex contour around 0 included in  $\gamma$ . By analyticity of the resolvent  $P_z^{-1}$  in  $\Omega \setminus \{0\}$ , we have by the Cauchy theorem:

$$\begin{aligned} \pi^2 &= -\frac{1}{4\pi^2} \oint_{\gamma} \oint_{\gamma'} P_z^{-1} P_t^{-1} dz dt \\ &= -\frac{1}{4\pi^2} \oint_{\gamma} \oint_{\gamma'} \frac{1}{(t-z)} [P_t^{-1} - P_z^{-1}] dz dt \\ &= \frac{1}{4\pi^2} \oint_{\gamma} \left[ \oint_{\gamma'} \frac{dt}{(t-z)} \right] P_z^{-1} dz \\ &\quad - \frac{1}{4\pi^2} \oint_{\gamma'} \left[ \oint_{\gamma} \frac{dz}{(t-z)} \right] P_t^{-1} dt \\ &= \frac{-1}{2\pi i} \oint_{\gamma'} P_t^{-1} dt = \pi \end{aligned}$$

From (7) we get

$$\pi = \frac{1}{2\pi i} \oint_{\gamma} A^+(z) (A^{-+}(z))^{-1} A^-(z) dz \tag{10}$$

In particular,  $(A^{-+}(z))^{-1}$  is a meromorphic function and possesses a Laurent series in  $\Omega$  expressed as

$$(A^{-+}(z))^{-1} = \frac{R_{-N_0}}{z^{N_0}} + \dots + \frac{R_{-1}}{z} + \tilde{R}(z) \tag{11}$$

where  $\tilde{R}(z)$  is analytical function in  $z$ .

$A^{\pm}(z)$  admits in  $\Omega$  a Taylor series given by:

$$A^{\pm}(z) = \sum_{j=0}^{\infty} \frac{d^j A^{\pm}}{dz^j} (0) z^j \tag{12}$$

Moreover, by using the residue formula, we also have

$$\pi = - \sum_{\substack{j+l-k=-1 \\ 1 \leq k \leq N_0}} \frac{d^j A^+}{dz^j} (0) R_{-k} \frac{d^l A^-}{dz^l} (0) \tag{13}$$

Since the operators  $\frac{d^j A^{\pm}}{dz^j}$  are of finite rank ( $\leq n_0$ ), then the expression (13) shows that  $\pi$  is of finite rank ( $\leq n_0 N_0$ ).

2) We also see in (9) that  $Im\pi \subset E$ . With  $E$  being dense in  $F$ ,  $\pi$  is of finite rank and  $\pi(E) \subset \pi(F)$ , we have  $\pi(E) = \pi(F) = Im\pi$ .

$\forall f \in E, \forall z \in \gamma, PP_z^{-1}f = P_z^{-1}Pf = f + zP_z^{-1}f$ , integrating on  $\gamma$ , we obtain

$$\pi Pf = P\pi f, \forall f \in E \tag{14}$$

In particular, we deduce that  $Im\pi$  and  $\ker \pi$  are invariant by  $P$ .

3) Since  $\pi$  is a projector then  $F = \ker \pi \oplus Im\pi$ .

Let  $P_1 = P|_{\ker \pi}$ ,  $P_2 = P|_{Im\pi}$  and  $R(z) = \frac{1}{2\pi i} \oint_{\gamma} (t-z)^{-1} P_t^{-1} dt$ ,  $z \in \overset{\circ}{\gamma}$ . We obtain

$$\begin{aligned} \pi R(z) &= \frac{1}{4\pi^2} \oint_{\gamma} \oint_{\gamma'} (t-z)^{-1} P_{\lambda}^{-1} P_t^{-1} dt d\lambda \\ &= \frac{1}{4\pi^2} \oint_{\gamma'} \left[ \oint_{\gamma} \frac{dt}{(t-z)(\lambda-t)} \right] P_{\lambda}^{-1} d\lambda \\ &\quad - \frac{1}{4\pi^2} \oint_{\gamma} \left[ \oint_{\gamma'} \frac{d\lambda}{(\lambda-t)} \right] \frac{P_t^{-1}}{(t-z)} dt \\ &= \frac{1}{4\pi^2} \oint_{\gamma'} [2\pi i (\frac{1}{(\lambda-z)} + \frac{1}{(z-\lambda)})] P_{\lambda}^{-1} d\lambda = 0 \end{aligned}$$

Thus,  $ImR(z) \subset \ker \pi$ .

Consequently,

$$\begin{aligned} P_z R(z) &= \frac{1}{2\pi i} \oint_{\gamma} (t-z)^{-1} P_z P_t^{-1} dt \\ &= \frac{1}{2\pi i} \oint_{\gamma} \frac{dt}{(t-z)} + \frac{1}{2\pi i} \oint_{\gamma} P_t^{-1} dt = (1 - \pi) \end{aligned}$$

Hence

$$(P_1 - z)R(z) |_{\ker \pi = 1} |_{\ker \pi \cap F} \tag{15}$$

Similarly

$$R(z)(P_1 - z) |_{\ker \pi = 1} |_{\ker \pi \cap E} \tag{16}$$

Then, if  $z \in \overset{\circ}{\gamma}$ ,  $(P_1 - z)$  is invertible from  $\ker \pi \cap E$  into  $\ker \pi \cap F$  with bounded inverse equal to  $R(z) |_{\ker \pi}$  and thus  $z \notin Sp(P_1)$ , or else  $Sp(P) \cap \overset{\circ}{\gamma} = \emptyset$ .

On the other hand,  $\forall z \in \Omega \setminus \{0\}$ ,

$$P_z^{-1} = (P_1 - z)^{-1} \oplus (P_2 - z)^{-1} \tag{17}$$

Thus

$$Sp(P) \cap \overset{\circ}{\gamma} = Sp(P_2) \cap \overset{\circ}{\gamma} \tag{18}$$

However,  $P_2 = \pi P|_{Im\pi}$  is defined on a finite dimension space, it possesses then a finite discrete spectrum of a cardinal equal to rank of  $\pi$ .

**Remarks 2.4:**

(1) If  $z_1 \neq 0$  is a second point where  $\det(A^{-+}(z_1)) = 0$ ,  $P_{z_1}$  is not invertible and if  $\pi$  and  $\pi_1$  are the corresponding projectors given by (9), then  $\pi\pi_1 = \pi_1\pi = 0$ .

(2) The use of equation (2) with  $\alpha = R_0^+ f$  and  $f = A^+(z)\alpha$ , we have  $A^-(z)P_z A^+(z) + A^{-+}(z) = 0$  on  $\mathbf{C}^{n_0}$ . Then

$$\frac{dA^{-+}(z)}{dz} = A^-(z)A^+(z)$$

Consequently, if  $\det(A^-(z)A^+(z)) \neq 0$ , the equation  $A^{-+}(z) = 0$  admits the unique solution  $z = 0$  in  $\Omega$ . In this case, 0 is then a simple pole for  $P_z^{-1}$ .

(3) According to equations (4) and (5),  $A_0^- A_0^+$  is invertible if and only if  $\ker P \cap \text{Im} P = \{0\}$ , which excludes the existence of generalized eigenvectors for  $P$ .

(4) If  $P$  is a pseudodifferential operator on  $\mathbf{R}^n$  with an elliptical symbol  $\sigma(x, \xi)$  for which there exist a constant  $\varepsilon > 0$ , such that  $\Omega_\varepsilon = \{(x, \xi) \in \mathbf{R}^{2n}; |\sigma(x, \xi)| < \varepsilon\}$  is bounded in  $\mathbf{R}^{2n}$ ,  $P$  is a Fredholm operator on  $L^2(\mathbf{R}^n)$  (cf. [11], [8]), then it can be spectrally reduced by the previous methods.

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