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ESTIMATION OF HYPER-ORDER OF SOLUTIONS TO HIGHER ORDER COMPLEX LINEAR DIFFERENTIAL EQUATIONS WITH ENTIRE COEFFICIENTS OF SLOW GROWTH

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Abstract In this paper, we study the growth of meromorphic solutions of higher order linear differential equations with entire coefficients and we obtain some estimations on the hyper-order and hyper convergence exponent of zeros of these solutions. We extend some results due to C. Y. Zhang, J. Tu [16]; L. Wang, H. Liu [14].

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1. INTRODUCTION

We assume that the reader is familiar with the fundamental results and the standard notations of Nevanlinna's theory (see e.g. [10, 12, 15]).

Definition 1.1. *The order of a meromorphic function f is defined as*

$$\sigma(f) = \limsup_{r\to+\infty} \frac{\log T(r,f)}{\log r},$$

here $T(r,f)$ is the Nevanlinna characteristic function of f which is defined by

$$T(r,f) = N(r,f) + m(r,f), \quad (r>0),$$

where

$$N(r,f) = \int_0^r \frac{[n(t,\infty,f) - n(0,\infty,f)]}{t} dt + n(0,\infty,f)\log r,$$

$$m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left| f\left(re^{i\theta}\right)\right| d\theta$$

and $n(t,\infty,f)$ *denote the number of poles of* f *in the disc* $|z|\leq t$. *If* f *is an entire function, then*

$$\sigma(f)=\limsup_{r\to+\infty}\frac{\log\log M(r,f)}{\log r},$$

where $M(r,f)=\max_{|z|=r}|f(z)|$.

Definition 1.2. *The hyper-order of a meromorphic function* f *is defined as*

$$\sigma_2(f)=\limsup_{r\to+\infty}\frac{\log\log T(r,f)}{\log r}.$$

If $f(z)$ *is an entire function, then*

$$\sigma_2(f)=\limsup_{r\to+\infty}\frac{\log\log\log M(r,f)}{\log r}.$$

Definition 1.3. *The lower order of a meromorphic function* f *is defined as*

$$\mu(f)=\liminf_{r\to+\infty}\frac{\log T(r,f)}{\log r}.$$

If f *is an entire function, then*

$$\mu(f)=\liminf_{r\to+\infty}\frac{\log\log M(r,f)}{\log r}.$$

Definition 1.4. *The convergence exponent of zeros and distinct zeros of a meromorphic function* f *are respectively defined by*

$$\lambda(f)=\limsup_{r\to+\infty}\frac{\log N(r,\frac{1}{f})}{\log r},\qquad \overline{\lambda}(f)=\limsup_{r\to+\infty}\frac{\log \overline{N}(r,\frac{1}{f})}{\log r},$$

where $N\left(r,\frac{1}{f}\right)$ $\left(\overline{N}\left(r,\frac{1}{f}\right)\right)$ *is the integrated counting function of zeros (distinct zeros) of* f *in* $\{z:|z|\leq r\}$.

Definition 1.5. *The hyper convergence exponent of zeros and distinct zeros of a meromorphic function* f *are respectively defined by*

$$\lambda_2(f)=\limsup_{r\to+\infty}\frac{\log\log N(r,\frac{1}{f})}{\log r},\qquad \overline{\lambda}_2(f)=\limsup_{r\to+\infty}\frac{\log\log \overline{N}(r,\frac{1}{f})}{\log r}.$$

Definition 1.6. *Let* $f(z)=\sum\limits_{n=0}^{+\infty}a_nz^n$ *be an entire function. We denote by* $\mu(r)=\max\{|a_n|r^n:\ n=0,1...\}$ *the maximal term of* f. *Then the central* *index of* f *is defined by*

$$\nu_f(r) = \max\{m; \mu(r) = |a_m| r^m\}.$$

In the past years, many authors investigated the growth of solutions of the higher order linear differential equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = F(z), \qquad (1.1)$$

when $A_j(z)$ $(j = 0, 1, \cdots, k-1)$, $F(z)(\not\equiv 0)$ are entire (or meromorphic) functions and obtained some valuable results, (see e.g. [2, 3, 4, 11, 12, 13, 14, 16]). In 2014, Wang and Liu investigated the properties of solutions of equation (1.1) when there exists some coefficient $A_s(z)$ $(0 \le s \le k-1)$ verifying the condition $\mu(A_s) < \frac{1}{2}$ and obtained the following result.

Theorem A [14] *Suppose that* $A_0(z), \cdots, A_{k-1}(z)$, $F(z)$ *are meromorphic functions of finite order. If there exists some* $s \in \{0, 1, \cdots, k-1\}$ *such that*

$$b = \max\left\{\sigma(A_j), (j \ne s), \sigma(F), \lambda\left(\frac{1}{A_s}\right)\right\} < \mu(A_s) < \frac{1}{2},$$

then
(i) *Every transcendental meromorphic solution* f *of* (1.1) *whose poles are of uniformly bounded multiplicities, satisfies* $\mu(A_s) \le \sigma_2(f) \le \sigma(A_s)$. *Furthermore, if* $F \not\equiv 0$, *then we have* $\mu(A_s) \le \bar{\lambda}_2(f) = \lambda_2(f) = \sigma_2(f) \le \sigma(A_s)$.
(ii) *If* $s \ge 2$, *then every non-transcendental meromorphic solution* f *of* (1.1) *is a polynomial with* $\deg f \le s-1$. *If* $s = 0$ *or* 1, *then every nonconstant solution* f *of* (1.1) *is transcendental.*

When $F(z)$ is of infinite order, Wang and Liu considered the linear differential equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = Qe^P, \qquad (1.2)$$

when $A_j(z)$ $(j = 0, 1, \cdots, k-1)$, $Q(z)(\not\equiv 0)$ are meromorphic functions and P is a transcendental entire function and obtained the following result.

Theorem B [14] *Suppose that* $A_0(z), \cdots, A_{k-1}(z)$, $Q(z)(\not\equiv 0)$ *are meromorphic functions of finite order,* P *is a transcendental entire function such that*

$$\max\left\{\sigma(P), \sigma(Q), \sigma(A_j), (1 \le j \le k-1), \lambda\left(\frac{1}{A_0}\right)\right\} < \mu(A_0) < \frac{1}{2}.$$

Then every solution f of (1.2) is transcendental, and every transcendental meromorphic solution f of (1.2) whose poles are of uniformly bounded multiplicities satisfies $\mu(A_0) \leq \bar{\lambda}_2(f) = \lambda_2(f) = \sigma_2(f) \leq \sigma(A_0)$.

For $k \geq 2$, we consider the linear differential equation

$$A_k(z)f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = F(z), \tag{1.3}$$

when $A_j(z)$ $(j = 0, 1, \cdots, k)$, $F(z)$ are entire functions such that $A_0 A_k F \not\equiv 0$. It well-known that if $A_k(z) \equiv 1$, then all solutions of (1.3) are entire functions, but when $A_k(z)$ is a nonconstant entire function, then equation (1.3) can possess meromorphic solutions. For instance the equation

$$zf''' + 4f'' + \left(-1 - \frac{1}{2}z^2 - z\right)e^{-z}f' + \left(\left(1 - \frac{1}{2}z^2 + 2z\right)e^{-2z} + ze^{-3z}\right)f$$

$$= \left(-1 - \frac{1}{2}z^2 - z\right)e^{-z} + \left(z - \frac{1}{2}z^3 + 2z^2\right)e^{-2z} + z^2 e^{-3z}$$

has a meromorphic solution $f(z) = \frac{1}{z^2}e^{e^{-z}} + z$. Thus, there exist two questions. Firstly, can we have the same properties as in Theorem A for the linear differential equation (1.3), when there exists some coefficient $A_s(z)$ $(0 \leq s \leq k)$ verifying the condition $\mu(A_s) < \frac{1}{2}$? Secondly, how about the growth of meromorphic solutions of the linear differential equation

$$A_k(z)f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = Qe^P, \tag{1.4}$$

when $A_j(z)$ $(j = 0, 1, \cdots, k)$, $Q(z) (\not\equiv 0)$ are entire functions and P is a transcendental entire function? In this paper, we proceed this way and we obtain the following results.

Theorem 1.1. *Suppose that $A_0(z), \cdots, A_k(z)$, $F(z)$ are entire functions of finite order. If there exists some $s \in \{0, 1, \cdots, k\}$ such that*

$$\alpha = \max\{\sigma(A_j), (j \neq s), \sigma(F)\} < \mu(A_s) < \frac{1}{2}, \tag{1.5}$$

then
(i) Every transcendental meromorphic solution f of (1.3) such that $\lambda\left(\frac{1}{f}\right) < \mu(f)$ satisfies $\mu(A_s) \leq \sigma_2(f) \leq \sigma(A_s)$. Furthermore, if $F \not\equiv 0$, then we have $\mu(A_s) \leq \bar{\lambda}_2(f) = \lambda_2(f) = \sigma_2(f) \leq \sigma(A_s)$.
(ii) If $s \geq 2$, then every rational solution f of (1.3) is a polynomial with $\deg f \leq s - 1$. If $s = 0$ or 1, then every nonconstant solution f of (1.3) is transcendental.

Remark 1.1. Setting $A_k(z) \equiv 1$ in Theorem 1.1 we obtain the result of Zhang and Tu ([16], Theorem 1.8).

Corollary 1.1 *Suppose that* $A_0(z), \cdots, A_k(z)$, $F(z)(\not\equiv 0)$ *are entire functions. If there exists some* $s \in \{0, 1, \cdots, k\}$ *such that*

$$\alpha = \max\{\sigma(A_j), (j \neq s), \sigma(F)\} < \mu(A_s) = \sigma(A_s) < \frac{1}{2},$$

then every transcendental meromorphic solution f *of* (1.3) *such that* $\lambda\left(\frac{1}{f}\right) < \mu(f)$ *satisfies* $\bar{\lambda}_2(f) = \lambda_2(f) = \sigma_2(f) = \sigma(A_s)$, *and every rational solution* f *of* (1.3) *is a polynomial with* $\deg f \leq s - 1$.

Theorem 1.2. *Suppose that* $A_0(z), \cdots, A_k(z)$, $Q(z)(\not\equiv 0)$ *are entire functions of finite order,* P *is a transcendental entire function such that*

$$\max\{\sigma(P), \sigma(Q), \sigma(A_j), (1 \leq j \leq k)\} < \mu(A_0) < \frac{1}{2}. \tag{1.6}$$

Then every solution f *of* (1.4) *is transcendental, and every transcendental meromorphic solution* f *of* (1.4) *such that* $\lambda\left(\frac{1}{f}\right) < \mu(f)$ *satisfies* $\mu(A_0) \leq \bar{\lambda}_2(f) = \lambda_2(f) = \sigma_2(f) \leq \sigma(A_0)$.

Remark 1.2. In Theorems 1.1 and 1.2, we remove the restriction $\lambda\left(\frac{1}{A_s}\right) < \mu(A_s)$.

Corollary 1.2 *Suppose that* $A_0(z), \cdots, A_k(z)$, $Q(z)(\not\equiv 0)$ *are entire functions of finite order,* P *is a transcendental entire function such that*

$$\max\{\sigma(P), \sigma(Q), \sigma(A_j), (1 \leq j \leq k)\} < \mu(A_0) = \sigma(A_0) < \frac{1}{2}.$$

Then every solution f *of* (1.4) *is transcendental, and every transcendental meromorphic solution* f *of* (1.4) *such that* $\lambda\left(\frac{1}{f}\right) < \mu(f)$ *satisfies* $\bar{\lambda}_2(f) = \lambda_2(f) = \sigma_2(f) = \sigma(A_0)$.

Remark 1.3. Obviously, Theorem 1.1 and Theorem 1.2 are generalization of Theorems A, B of Wang and Liu [14] and Theorem 1.8 of Zhang and Tu [16].

2. PRELIMINARY LEMMAS

Lemma 2.1 [8] *Let* f *be a transcendental meromorphic function in the plane, and let* $\alpha > 1$ *be a given constant. Then there exist a set* $E_1 \subset (1, +\infty)$ *that* *has a finite logarithmic measure, and a constant $B > 0$ depending only on α and (m, n) $(m, n \in \{0, 1, \cdots, k\})$ $m < n$ such that for all z with $|z| = r \notin [0, 1] \cup E_1$, we have*

$$\left|\frac{f^{(n)}(z)}{f^{(m)}(z)}\right| \leq B \left(\frac{T(\alpha r, f)}{r}(\log^{\alpha} r) \log T(\alpha r, f)\right)^{n-m}.$$

Lemma 2.2 [6] *Let $f(z) = \frac{g(z)}{d(z)}$ be a meromorphic function, where $g(z)$ and $d(z)$ are entire functions satisfying $\mu(g) = \mu(f) = \mu \leq \sigma(g) = \sigma(f) \leq +\infty$ and $\lambda(d) = \sigma(d) = \lambda(\frac{1}{f}) < \mu$. Then there exists a set $E_2 \subset (1, +\infty)$ of finite logarithmic measure, such that for all z satisfying $|z| = r \notin [0, 1] \cup E_2$ and $|g(z)| = M(r, g)$ we have*

$$\left|\frac{f(z)}{f^{(s)}(z)}\right| \leq r^{2s}, \ (s \in \mathbb{N}).$$

Lemma 2.3 [9] *Let $g : [0, +\infty) \to \mathbb{R}$ and $h : [0, +\infty) \to \mathbb{R}$ be monotone nondecreasing functions such that $g(r) \leq h(r)$ for all $r \notin E_3 \cup [0, 1]$, where $E_3 \subset (1, +\infty)$ is a set of finite logarithmic measure. Then for any $\alpha > 1$, there exists an $r_0 = r_0(\alpha) > 0$ such that $g(r) \leq h(\alpha r)$ for all $r > r_0$.*

Lemma 2.4 [6] *Let $f(z) = \frac{g(z)}{d(z)}$ be a meromorphic function, where $g(z)$ and $d(z)$ are entire functions satisfying $\mu(g) = \mu(f) = \mu \leq \sigma(g) = \sigma(f) \leq +\infty$ and $\lambda(d) = \sigma(d) = \lambda(\frac{1}{f}) < \mu$. Then there exists a set $E_4 \subset (1, +\infty)$ of finite logarithmic measure, such that for all z satisfying $|z| = r \notin [0, 1] \cup E_4$ and $|g(z)| = M(r, g)$, we have*

$$\frac{f^{(n)}(z)}{f(z)} = \left(\frac{\nu_g(r)}{z}\right)^n (1 + o(1)), \quad (n \geq 1),$$

where $\nu_g(r)$ denote the central index of $g(z)$.

Lemma 2.5 [5] *Let $g(z)$ be an entire function of order $\sigma(g) = \alpha < \infty$. Then for any $\varepsilon > 0$, there exist a set $E_5 \subset [1, +\infty)$ that has a finite linear measure and finite logarithmic measure, such that for all z satisfying $|z| = r \notin [0, 1] \cup E_5$, we have*

$$\exp\{-r^{\alpha+\varepsilon}\} \leq |g(z)| \leq \exp\{r^{\alpha+\varepsilon}\}.$$

Lemma 2.6 [7] *Let $g(z)$ be an entire function of infinite order, with the hyper-order $\sigma_2(g) = \sigma$, and $\nu_g(r)$ denote the central index of $g(z)$. Then*

$$\limsup_{r\to+\infty} \frac{\log\log \nu_g(r)}{\log r} = \sigma.$$

Lemma 2.7 [1] *Let $g(z)$ be an entire function with $0 \le \mu(g) < 1$. Then for every $\alpha \in (\mu(g), 1)$, there exists a set $E_6 \subset [0,\infty)$ such that*

$$\overline{\log dens} E_6 \ge 1 - \frac{\mu(g)}{\alpha},$$

where $E_6 = \{r \in [0,\infty) : m(r) > M(r)\cos\pi\alpha\}$, $m(r) = \inf_{|z|=r} \log|g(z)|$, $M(r) = \sup_{|z|=r} \log|g(z)|$.

Lemma 2.8 *Let $f(z)$ be an entire function such that $\mu(f) < \frac{1}{2}$. Then for any given $\varepsilon > 0$, there exists a set $E_7 \subset (1, +\infty)$ with $\overline{\log dens} E_7 > 0$, such that for all z satisfying $|z| = r \in E_7$, we have*

$$|f(z)| \ge \exp\{r^{\mu(f)-\varepsilon}\}.$$

Proof. Set $\alpha_0 = \frac{\frac{1}{2}+\mu(f)}{2}$. Then, by Lemma 2.7, there exists a set H with $\overline{\log dens} H \ge 1 - \frac{\mu(f)}{\alpha_0}$, such that for all z satisfying $|z| = r \in H$, we have

$$\log|f(z)| \ge \cos(\pi\alpha_0)\log M(r,f). \tag{2.1}$$

By the definition of the lower order, for any given $\varepsilon > 0$, there exists $r_1 > 0$ such that

$$\log M(r,f) \ge r^{\mu(f)-\frac{\varepsilon}{2}}, \tag{2.2}$$

holds for $r > r_1$. Since

$$\frac{\cos(\pi\alpha_0) r^{\mu(f)-\frac{\varepsilon}{2}}}{r^{\mu(f)-\varepsilon}} \to +\infty, \ (r \to +\infty), \tag{2.3}$$

then, by (2.1) − (2.3), there exists $r_2 (\ge r_1)$, such that for all z satisfying $|z| = r \in H \setminus [0, r_2]$, we have

$$|f(z)| \ge \exp\left\{\cos(\pi\alpha_0) r^{\mu(f)-\frac{\varepsilon}{2}}\right\} \ge \exp\{r^{\mu(f)-\varepsilon}\}.$$

Setting $E_7 = H \cap [r_2, +\infty]$, then $\overline{\log dens} E_7 > 0$. ∎

Lemma 2.9 [15] *Let f, g be nonconstant meromorphic functions with $\sigma(f)$ as order and $\mu(g)$ as lower order. Then we have*

$$\mu(f+g) \leq \max\{\sigma(f), \mu(g)\}$$

and

$$\mu(fg) \leq \max\{\sigma(f), \mu(g)\}.$$

Furthermore, if $\mu(g) > \sigma(f)$, then we obtain

$$\mu(f+g) = \mu(fg) = \mu(g).$$

3. PROOF OF THEOREM 1.1

(i) Assume that f is a transcendental meromorphic solution of (1.3) such that $\lambda\left(\frac{1}{f}\right) < \mu(f)$. From (1.3), we obtain

$$|A_s(z)| \leq \left|\frac{f}{f^{(s)}}\right| \left[|A_k(z)| \left|\frac{f^{(k)}}{f}\right| + |A_{k-1}(z)| \left|\frac{f^{(k-1)}}{f}\right| + \cdots + |A_{s+1}(z)| \left|\frac{f^{(s+1)}}{f}\right| \right.$$

$$\left. + |A_{s-1}(z)| \left|\frac{f^{(s-1)}}{f}\right| + \cdots + |A_1(z)| \left|\frac{f'}{f}\right| + |A_0(z)| + \left|\frac{F}{f}\right|\right]. \tag{3.1}$$

By Lemma 2.1, there exists a constant $B > 0$ and a set $E_1 \subset (1, +\infty)$ of finite logarithmic measure such that for all z satisfying $|z| = r \notin [0,1] \cup E_1$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq B\left(T(2r, f)\right)^{k+1}, \quad 1 \leq j \leq k. \tag{3.2}$$

Since $\lambda\left(\frac{1}{f}\right) < \mu(f)$, then by Hadamard's factorization theorem, we can write f as $f(z) = \frac{g(z)}{d(z)}$, where $g(z)$ and $d(z)$ are entire functions satisfying

$$\mu(g) = \mu(f) = \mu \leq \sigma(g) = \sigma(f),\ \sigma(d) = \lambda\left(\frac{1}{f}\right) < \mu.$$

Then by Lemma 2.2, there exists a set E_2 of finite logarithmic measure such that for all $|z| = r \notin [0,1] \cup E_2$ and $|g(z)| = M(r, g)$ and for r sufficiently large, we have

$$\left|\frac{f(z)}{f^{(s)}(z)}\right| \leq r^{2s}. \tag{3.3}$$

By (1.5), for any given ε with $0 < 2\varepsilon < \mu(A_s) - \alpha$, we have for sufficiently large r

$$|A_j(z)| \le \exp\{r^{\alpha+\varepsilon}\},\ (j \ne s),\ \ |F(z)| \le \exp\{r^{\alpha+\varepsilon}\}. \tag{3.4}$$

By Lemma 2.8, for any given $\varepsilon > 0$, there exists a set $E_7 \subset (1, +\infty)$ with $\overline{\log dens} E_7 > 0$, such that for all z satisfying $|z| = r \in E_7$, we have

$$|A_s(z)| \ge \exp\{r^{\mu(A_s)-\varepsilon}\}. \tag{3.5}$$

Since $\sigma(d) = \lambda\left(\frac{1}{f}\right) < \mu(f) = \mu(g)$, then for any ε with $0 < 2\varepsilon < \mu(f) - \lambda\left(\frac{1}{f}\right)$ and for sufficiently large r we have

$$\left|\frac{F(z)}{f(z)}\right| = \left|\frac{d(z)}{g(z)}\right| |F(z)| = \left|\frac{d(z)}{M(r,g)}\right| |F(z)|$$

$$\le \frac{\exp\{r^{\lambda\left(\frac{1}{f}\right)+\varepsilon}\}}{\exp\{r^{\mu(f)-\varepsilon}\}} \exp\{r^{\alpha+\varepsilon}\} \le \exp\{r^{\alpha+\varepsilon}\}. \tag{3.6}$$

Let $E_8 = E_7 \setminus ([0,1] \cup E_1 \cup E_2)$, then we have $\overline{\log dens} E_8 > 0$. Then, by substituting (3.2) – (3.6) into (3.1), for all z satisfying $|z| = r \in E_8$ and $|g(z)| = M(r,g)$, we obtain

$$\exp\{r^{\mu(A_s)-\varepsilon}\} \le B(k+1) r^{2s} (T(2r,f))^{k+1} \exp\{r^{\alpha+\varepsilon}\}. \tag{3.7}$$

By (3.7) and Lemma 2.3, we get $\mu(A_s) - \varepsilon \le \sigma_2(f)$. Since $\varepsilon > 0$ is arbitrary, we have $\mu(A_s) \le \sigma_2(f)$. Now, we prove that $\sigma_2(f) \le \sigma(A_s)$. We can write (1.3) as

$$-A_k(z)\frac{f^{(k)}}{f} = A_{k-1}(z)\frac{f^{(k-1)}}{f} + \cdots + A_{s+1}(z)\frac{f^{(s+1)}}{f}$$

$$+A_s(z)\frac{f^{(s)}}{f} + A_{s-1}(z)\frac{f^{(s-1)}}{f} + \cdots + A_1(z)\frac{f'}{f} + A_0(z) - \frac{F(z)}{f(z)}. \tag{3.8}$$

By Lemma 2.4, there exists a set $E_4 \subset (1, +\infty)$ of finite logarithmic measure such that for all $|z| = r \notin [0,1] \cup E_4$ and $|g(z)| = M(r,g)$, we have

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{\nu_g(r)}{z}\right)^j (1 + o(1)), \quad (j = 1, \cdots, k). \tag{3.9}$$

For any given $\varepsilon > 0$, for sufficiently large r we have

$$|A_j(z)| \le \exp\{r^{\sigma(A_s)+\varepsilon}\},\ j = 0, \cdots, k-1. \tag{3.10}$$

By Lemma 2.5, for any given $\varepsilon > 0$, there exists a set $E_5 \subset (1, +\infty)$ of finite logarithmic measure such that for all z satisfying $|z| = r \notin [0, 1] \cup E_5$, we have

$$|A_k(z)| \geq \exp\{-r^{\sigma(A_k)+\varepsilon}\} \geq \exp\{-r^{\sigma(A_s)+\varepsilon}\}. \tag{3.11}$$

From (3.8) and (3.9), we have

$$-\left(\frac{\nu_g(r)}{z}\right)^k (1 + o(1)) = \frac{1}{A_k(z)} \left[\sum_{j=1}^{k-1} A_j(z) \left(\frac{\nu_g(r)}{z}\right)^j (1 + o(1)) + A_0(z) - \frac{F(z)}{f(z)}\right],$$

it follows

$$\left|\left(\frac{\nu_g(r)}{z}\right)^k\right| |1 + o(1)| \leq \frac{1}{|A_k(z)|} \left[\sum_{j=1}^{k-1} |A_j(z)| \left|\left(\frac{\nu_g(r)}{z}\right)^j\right| |1 + o(1)| + |A_0(z)| + \left|\frac{F(z)}{f(z)}\right|\right]. \tag{3.12}$$

By (3.6) and (3.10) – (3.12) for all z satisfying $|z| = r \notin [0, 1] \cup E_4 \cup E_5$ and $|g(z)| = M(r, g)$, we have

$$\left(\frac{\nu_g(r)}{r}\right) |1 + o(1)| \leq (k + 1) |1 + o(1)| \exp\{r^{\sigma(A_s)+\varepsilon}\},$$

so,

$$\limsup_{r\to+\infty} \frac{\log\log \nu_g(r)}{\log r} \leq \sigma(A_s) + \varepsilon. \tag{3.13}$$

Since $\varepsilon > 0$ is arbitrary, then by (3.13), Lemma 2.3 and Lemma 2.6, we have $\sigma_2(g) \leq \sigma(A_s)$, that is $\sigma_2(f) \leq \sigma(A_s)$. Therefore, we get

$$\mu(A_s) \leq \sigma_2(f) \leq \sigma(A_s).$$

Let $F \not\equiv 0$. Now, we prove $\bar{\lambda}_2(f) = \lambda_2(f) = \sigma_2(f)$. By (1.3), we have

$$\frac{1}{f} = \frac{1}{F}\left(A_k \frac{f^{(k)}}{f} + A_{k-1} \frac{f^{(k-1)}}{f} + \cdots + A_1 \frac{f'}{f} + A_0\right). \tag{3.14}$$

If f has a zero at z_0 of order $\gamma > k$, then F has a zero at z_0 of order $\gamma - k$. Hence we have

$$n(r, \frac{1}{f}) \leq k\overline{n}(r, \frac{1}{f}) + n(r, \frac{1}{F}),$$

$$N(r, \frac{1}{f}) \leq k\overline{N}(r, \frac{1}{f}) + N(r, \frac{1}{F}). \tag{3.15}$$

By (3.14), we have by the lemma of logarithmic derivative [10]

$$m(r, \frac{1}{f}) \leq m(r, \frac{1}{F}) + \sum_{j=0}^{k} m(r, A_j) + O(\log rT(r, f)),\ (r \notin E), \tag{3.16}$$

where E is a set of a finite linear measure. By (3.15) and (3.16), we get

$$T(r, f) \leq k\overline{N}(r, \frac{1}{f}) + T(r, F) + \sum_{j=0}^{k} T(r, A_j) + O(\log rT(r, f)),\ (r \notin E). \tag{3.17}$$

For sufficiently large r and any given $\varepsilon > 0$, we have

$$O(\log rT(r, f)) = o\left(T\left(r, f\right)\right), \tag{3.18}$$

$$T(r, F) + \sum_{j=0}^{k} T(r, A_j) \leq (k+2)\, r^{\sigma(A_s)+\varepsilon}. \tag{3.19}$$

Hence, from (3.17), (3.18) and (3.19) , for sufficiently large $r \notin E$, we get that

$$(1 - o(1))\, T(r, f) \leq k\overline{N}(r, \frac{1}{f}) + (k+2)\, r^{\sigma(A_s)+\varepsilon},$$

so $\sigma_2(f) \leq \bar{\lambda}_2(f)$. Since $\bar{\lambda}_2(f) \leq \sigma_2(f)$, we get

$$\mu(A_s) \leq \bar{\lambda}_2(f) = \lambda_2(f) = \sigma_2(f) \leq \sigma(A_s).$$

(ii) Assume that f is a rational solution of (1.3). If either f is a rational function, which has a pole at z_0 of degree $m \geq 1$, or f is a polynomial with $\deg f \geq s$, then $f^{(s)}(z) \not\equiv 0$. From (1.3), we obtain

$$|A_s(z)| \leq \left[|A_k(z)| \left| \frac{f^{(k)}}{f^{(s)}} \right| + |A_{k-1}(z)| \left| \frac{f^{(k-1)}}{f^{(s)}} \right| + \cdots + |A_{s+1}(z)| \left| \frac{f^{(s+1)}}{f^{(s)}} \right| \right.$$

$$\left. + |A_{s-1}(z)| \left| \frac{f^{(s-1)}}{f^{(s)}} \right| + \cdots + |A_1(z)| \left| \frac{f'}{f^{(s)}} \right| + |A_0(z)| + \left| \frac{1}{f^{(s)}} \right| |F| \right]. \tag{3.20}$$

Then, by substituting (3.4) and (3.5) into (3.20), we obtain

$$\exp\{r^{\mu(A_s)-\varepsilon}\} \leq (k+1)r^M \exp\{r^{\alpha+\varepsilon}\},$$

where M is a constant. This is a contradiction. Therefore, f must be a polynomial with $\deg f \leq s-1$.

If $s=0$ or 1 and f is a polynomial solution of (1.3), then we get that $\deg f \leq s-1$. Thus, f must be a constant. Therefore, every nonconstant solution $f(z)$ of (1.3) is transcendental.

4. PROOF OF THEOREM 1.2

By hypothesis, it is known that every meromorphic solution of (1.4) is of infinite order. Then, every meromorphic solution of (1.4) is transcendental. Assume that f is a transcendental meromorphic solution, such that $\lambda\left(\frac{1}{f}\right) < \mu(f)$. Set $f = ge^P$. Then, we get that

$$\bar{\lambda}_2(g) = \bar{\lambda}_2(f), \quad \lambda_2(g) = \lambda_2(f). \tag{4.1}$$

By substituting $f = ge^P$ into (1.4), we have

$$g^{(k)} + B_{k-1}(z)g^{(k-1)} + \cdots + B_1(z)g' + B_0(z)g = \frac{Q}{A_k(z)}, \tag{4.2}$$

where

$$B_{k-1} = \frac{A_{k-1}}{A_k} + kP', \tag{4.3}$$

$$B_{k-j} = \frac{A_{k-j}}{A_k} + (k-j+1)\frac{A_{k-j+1}}{A_k}P'$$
$$+\sum_{m=2}^{j} \frac{A_{k-j+m}}{A_k}\left[\binom{k-j+m}{m}(P')^m + D_{m-1}(P')\right], \ j=2,...,k \tag{4.4}$$

and $D_{m-1}(P')$ is a differential polynomial in P' of degree $m-1$, its coefficients are constants. By (4.4) , we get

$$B_0 = \frac{A_0}{A_k} + \frac{A_1}{A_k}P' + \sum_{m=2}^{k} \frac{A_m}{A_k}\left[(P')^m + D_{m-1}(P')\right]$$
$$= \frac{1}{A_k}\left[A_0 + A_1P' + \sum_{m=2}^{k} A_m\left[(P')^m + D_{m-1}(P')\right]\right]. \tag{4.5}$$

Using (1.6) , (4.3) , (4.4) , (4.5) and Lemma 2.9, we obtain

$$\mu(B_0) = \max\{\mu(A_0), \sigma(A_j)\ \ (1 \le j \le k)\} = \mu(A_0) \tag{4.6}$$

and

$$\sigma\left(\frac{Q}{A_k}\right) \le \max\{\sigma(A_k), \sigma(Q)\} < \mu(A_0),\ \sigma(B_j) < \mu(A_0),\ j = 1, \cdots, k-1. \tag{4.7}$$

By (4.2) , (4.6) , (4.7) and applying Theorem 1.1 for $A_k(z) \equiv 1$ and $s = 0$, we get

$$\mu(A_0) \le \bar{\lambda}_2(g) = \lambda_2(g) = \sigma_2(g) \le \sigma(A_0). \tag{4.8}$$

Since $\sigma_2(e^P) = \sigma(P) < \mu(A_0) \le \sigma_2(g)$, then we obtain $\sigma_2(f) = \sigma_2(g)$. Hence, by (4.1) and (4.8) , we have

$$\mu(A_0) \le \bar{\lambda}_2(f) = \lambda_2(f) = \sigma_2(f) \le \sigma(A_0).$$

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