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Some Stronger Topologies for Closed Operators in Hilbert Space

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Abstract

The purpose of this paper is to introduce some new metrics on the set $C(H)$ of closed densely defined operators on a Hilbert space H and we characterize the closure of the subset of bounded elements of $C(H)$ for these metrics. We define the notion of quotient-convergence in $C(H)$ and we show that the toplogy induced by quotient-convergence is strictly stronger than the topology induced from the gap metric.

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1 Introduction

Let H be a separable Hilbert space. Let us denote by $B(H)$ the algebra of bounded linear operators on H and $C(H)$ the set of closed operators of dense domain in H. If $A \in C(H)$, the domain of A is denoted by $D(A)$ and

$G(A) = \{(x, Ax) \; ; \; x \in D(A)\}$ is its graph, in particular $G(A)$ is a closed subspace of $H \oplus H$. The null space and the range of A will be denoted by $N(A)$ and $R(A)$ respectively. $A^*$is the adjoint of A.

$C(H)$ equipped with a distance g called "gap" metric becomes a metric space.

$$A, B \in C(H), \qquad g(A,B) = g(G(A), G(B)) = \left\| P_{G(A)} - P_{G(B)} \right\|_{B(H \oplus H)}$$

where $P_{G(A)}$ and $P_{G(B)}$ denote respectively the orthogonal projection in $H \oplus H$ on the graph $G(A)$ of the operator A and the graph $G(B)$ of the operator B.

Nevertheless, the spectral analysis is incomplete in $C(H)$ with respect to the one in $B(H)$ because of certain insufficiencies of algebraic and topological characters taken in $C(H)$. Precisely, the sum and the product of two closed operators need not be closed on H and $C(H)$ is not complete for the metric g because if we consider for example the sequence of operators $(nI)_{n \in \mathbb{N}^*}$ where I is the identity operator on H, then the nI are bounded operators, therefore closed on H, for all $n \in \mathbb{N}^*$.

$$\begin{cases} G(nI) = \left\{ \left(\frac{1}{n} x, x \right) \; ; \; x \in H \right\}, \text{ for all } n \in \mathbb{N}^* \\ \lim\limits_{n \to +\infty} g(G(nI), \;\; \{0\} \oplus H) = 0 \end{cases}$$

But $\{0\} \oplus H$ cannot be the graph of a linear operator on H, thus $(nI)_{n \in \mathbb{N}^*}$ is a Cauchy sequence without limit in $C(H)$.

We can remedy this defect by considering the completion of $C(H)$ in the set $LR(H)$ of closed linear relations on H , i.e. the set of all closed linear subspaces of $H \oplus H$ of infinite dimension and codimension. Equipped with the metric g, $LR(H)$ is a complete metric space.

Closed linear operators are identified as linear relations via their graphs and hence the inclusions $B(H) \subset C(H) \subset LR(H)$ hold.

Theorem 1 *(see [7]) The closure of $C(H)$ in $LR(H)$ is the complement in $LR(H)$ of semibounded linear relations E such that $ind_2(E) \neq 0$.*

Where $E \in LR(H)$ is semibounded if $E + H_2$ is closed in $H \oplus H$ and $\min(\dim(E \cap H_2), \dim(E^{\perp} \cap H_1)) < +\infty$. $ind_2(E) = \dim(E \cap H_2) - \dim(E^{\perp} \cap H_1)$, $H_1 = H \oplus \{0\}$ and $H_2 = \{0\} \oplus H$.

Other metrics equivalent to g can be defined in term of the operator norm of $R_A = (1 + A^*A)^{-1}$, R_{A^*} and AR_A.

Indeed, if $A \in C(H)$ then $R_A = (1+A^*A)^{-1}$ exists as a bounded self-adjoint operator with domain $D(R_A) = H$, $AR_Ax = R_{A^*}Ax$ for all $x \in D(A)$. It is known (see e.g. [6], [5]) that the function $A \mapsto F(A) = AS_A(I+S_A)^{-1}$ sends elements of $C(H)$ onto the set $C_0(H)$ of contractions T such that $\|T\|_{B(H)} \leq 1$ and $N(I - T^*T) = \{0\}$ where $S_A = \sqrt{R_A}$ is the unique positive definite selfadjoint square root of R_A.

The fundamental properties about R_A and S_A are (see [6]) .

$$R_A, S_A \in B(H),\ \ \|R_A\|_{B(H)} \leq 1 \text{ and } \|S_A\|_{B(H)} \leq 1$$

$$\|AR_A\|_{B(H)} \leq 1 \text{ and } \|AS_A\|_{B(H)} \leq 1$$

$$(AR_A)^* = A^*R_{A^*} \text{ and } (AS_A)^* = A^*S_{A^*} \tag{1}$$

$$A^*S_{A^*}AS_A = I - R_A$$

Furthermore,

$$\begin{array}{l} (F(A))^* = F(A^*) \\ R_{F(A)} = \frac{1}{2}(I+S_A) \\ F(A)R_{F(A)} = \frac{1}{2}AS_A \end{array} \tag{2}$$

The topology induced by g on $C(H)$ has good properties concerning the stability of the index of operators with index. On the other hand the results are much less good as regards the stability of the spectrum of an operator. For application it proves to be for use to have other metrics on $C(H)$, which are more pratical.

The second metric is defined on $C(H)$ by

$$p(A,B) = \sqrt{\|R_A - R_B\|_{B(H)}^2 + \|R_{A^*} - R_{B^*}\|_{B(H)}^2 + 2\,\|AR_A - BR_B\|_{B(H)}^2}$$

Then p is equivalent to g and one has

$$g(A,B) \leq \sqrt{2}p(A,B) \leq 2g(A,B)\ ;\text{ for all } A,B \in C(H) \tag{3}$$

and

$$p(A,B) \leq 4\,\|A - B\|\ \ ;\text{ for all } A,B \in B(H)$$

Moreover we have for all $A, B \in C(H)$ (see [5])

$$g(A,B) \leq 8g(F(A),F(B)) \leq 8\sqrt{2}p(F(A),F(B)) \leq 32\sqrt{2}\,\|F(A) - F(B)\|_{B(H)}$$

Definition 2 *Let $A, B \in C(H)$. We put,*

$$\begin{aligned} d_1(A,B) &= \sqrt{\|R_A - R_B\|^2_{B(H)} + \|AR_A - BR_B\|^2_{B(H)}} \\ d_2(A,B) &= \sqrt{2}\min(d_1(A,B), d_1(A^*,B^*)) \end{aligned}$$

It is evident that d_1 and d_2 is metrics on $C(H)$ and $d_2(A,B) \leq \sqrt{2}d_1(A,B) \leq \sqrt{2}p(A,B) \leq 2g(A,B)$.

Theorem 3 *The topology induced from the metric g on $C(H)$ is strictly stronger than that induced from d_1 which is strictly stronger in his turn than that induced from d_2.*

In order to exclude the possibility that the metrics d_1, d_2 and g generate the same topology, we shall show an exemple of a sequence which converges differently in gap metric g, d_1 and d_2.

Let $H = l^2$ the space of square-summable sequences and put for $n \in \mathbb{N}^*$,

$$A_n(x) = A_n(x_1, x_2, ..., x_k, ...) = ([A_n(x)]_1, [A_n(x)]_2, ..., [A_n(x)]_k, ...)$$

where

$$[A_n(x)]_k = \begin{cases} kx_k & \text{if } \ k < n \\ kx_{k+1} & \text{if } \ k \geq n \end{cases}$$

Then,

$$[A_n^*(x)]_k = \begin{cases} kx_k & \text{if } \ k < n \\ 0 & \text{if } k = n \\ (k-1)x_{k-1} & \text{if } \ k > n \end{cases}$$

and thus,

$$[(I + A_n^* A_n)(x)]_k = \begin{cases} (1+k^2)x_k & \text{if } \ k < n \\ x_n & \text{if } k = n \\ [1+(k-1)^2]x_k & \text{if } \ k > n \end{cases}$$

Consequently,

$$[R_{A_n}(x)]_k = \begin{cases} \frac{1}{(1+k^2)}x_k & \text{if } \ k < n \\ x_n & \text{if } k = n \\ \frac{1}{(1+(k-1)^2)}x_k & \text{if } \ k > n \end{cases}$$

Let us define,

$$[A(x)]_k = kx_k$$

We have $R_A = R_{A^*} = R_{A_n^*}$, $\left\|R_{A^*} - R_{A_n^*}\right\|_{B(H)} = 0$ and

$$[(A^*R_{A^*} - A_n^*R_{A_n^*})(x)]_k = \begin{cases} 0 & \text{if } k < n \\ \frac{n}{(1+n^2)}x_n & \text{if } k = n \\ \frac{k}{(1+k^2)}x_k - \frac{(k-1)}{(1+(k-1)^2)}x_{k-1} & \text{if } k > n \end{cases}$$

Thus,

$$\left\|A^*R_{A^*} - A_n^*R_{A_n^*}\right\|_{B(H)} \leq \frac{2}{\sqrt{1+n^2}}$$

and $\lim\limits_{n\to+\infty} \|AR_A - A_nR_{A_n}\|_{B(H)} = \lim\limits_{n\to+\infty} \left\|A^*R_{A^*} - A_n^*R_{A_n^*}\right\|_{B(H)} = 0$.
Furthermore,

$$[(R_A - R_{A_n})(x)]_k = \begin{cases} 0 \text{ if } k < n \\ (\frac{1}{(1+n^2)} - 1)x_n & \text{if } k = n \\ \frac{(-2k+1)}{(1+k^2)(1+(k-1)^2)}x_k & \text{if } k > n \end{cases}$$

Then, $\|R_A - R_{A_n}\|_{B(H)} \geq \frac{n^2}{1+n^2}$.

We find finally that, $\lim\limits_{n\to+\infty} d_1(A, A_n) = 1$ and $\lim\limits_{n\to+\infty} d_2(A, A_n) = 0$. If we put $B = A^*$ and $B_n = A_n^*$, we have $\lim\limits_{n\to+\infty} d_1(B, B_n) = 0$ and $\lim\limits_{n\to+\infty} p(B, B_n) = 1$.

Our purpose in this paper is to introduce on $C(H)$ some metrics, in term of the operator norm of R_A, strictly stronger than the gap metric g. We characterize essentially the closure of $B(H)$ in $C(H)$ for these metrics.

Using the quotient representation of bounded operators, we also introduce a metric in the set of almost closed operators on H, the topology induced on $C(H)$ from this metric is strictly stronger than that induced by g.

Finally, we define the notion of quotient-convergence of a sequence in $C(H)$ and we show that the topology induced by quotient- convergence is strictly stronger than the topolgy induced from the gap metric.

2 Strictly stronger metrics than g

To be able to refine the completion of $C(H)$ from g, we construct in this section some metrics strictly stronger than g .

Definition 4 *Let $A, B \in C(H)$. We put,*

$$s(A,B) = \sqrt{\|S_A - S_B\|_{B(H)}^2 + \|AS_A - BS_B\|_{B(H)}^2 + \|S_{A^*} - S_{B^*}\|_{B(H)}^2 + \|A^*S_{A^*} - B^*S_{B^*}\|_{B(H)}^2}$$

$s(A,B)$ is a metric on $C(H)$ and it follows from (1) that $s(A,B) = 2p(F(A), F(B))$.

Let us put $l(A,B) = 2g(F(A), F(B))$. Then it follows from (3) that

$$g(A,B) \leq 4l(A,B) \leq 4\sqrt{2}s(A,B) \leq 8\sqrt{2}l(A,B) \tag{4}$$

The topology induced from the metric s on $C(H)$ is strictly stronger than that induced from the metric g.

Indeed, let us define the operators A_n on l^2 by

$$[A_n(x)]_k = \begin{cases} kx_k & \text{if } \ k < n \\ -kx_k & \text{if } \ k \geq n \end{cases} \tag{5}$$

Then, $A_n^* = A_n$, $[R_{A_n}(x)]_k = [R_{A_n^*}(x)]_k = \frac{x_k}{1+k^2}$ and

$$[A_n R_{A_n}(x)]_k = [A_n^* R_{A_n^*}(x)]_k = \begin{cases} \frac{k}{1+k^2}x_k & \text{if } \ k < n \\ \frac{-k}{1+k^2}x_k & \text{if } \ k \geq n \end{cases}$$

If we define A on l^2 by $[A(x)]_k = kx_k$, we see that $g(A, A_n) = \|S_{A^*}(A - A_n)S_{A_n}\|_{B(H)} \leq \frac{2n}{1+n^2}$. Thus, the sequence $(A_n)_n$ converges to 0 for g but does not converge to the same limit for the metrics s, because $\|AS_A - A_nSA_n\|_{B(H)} \geq \frac{2n}{\sqrt{1+n^2}} \to 2$.

Theorem 5 *$B(H)$ is dense open subset of the metric space $(C(H), s)$.*

Proof. Let $A \in C(H)$ and $A_t = 2tF(A)(1 - t^2F(A^*)F(A))^{-1}$ for $t < 1$.

Then, $A_t \in B(H)$, $F(A_t) = tF(A)$, and

$$\begin{aligned} g(A_t, A) &\leq 8g(F(A_t), F(A)) = 4l(A_t, A) \\ &\leq 32\sqrt{2}\,|1-t|\,\|F(A)\|_{B(H)} \end{aligned} \tag{6}$$

Thus, $\lim\limits_{t\to 1} g(A_t, A) = \lim\limits_{t\to 1} l(A_t, A) = \lim\limits_{t\to 1} s(A_t, A) = 0$.

We show now that $B(H)$ is open in $(C(H), s)$. More precisely, if $A \in C(H)$ and B is a bounded operator on H such that $s(A,B) < \frac{1}{\sqrt{2(1+\|B\|_{B(H)}^2)}}$, then A is bounded on H.

Let $x \in D(A)$, then for all $y \in H$,

$$\begin{aligned} \langle Ax, y\rangle - \langle x, B^*y\rangle &= \langle (x, Ax), (-B^*y, y)\rangle_{H\oplus H} \\ &= \left\langle P_{G(A)}(x, Ax), (I - P_{G(B)})(-B^*y, y)\right\rangle_{H\oplus H} \end{aligned}$$

and hence by using Schwarz inequality,

$$|\langle (A-B)x, y\rangle| \leq g(A,B)\,\|(x,Ax)\|_{H\oplus H}\,\|(-B^*y,y)\|_{H\oplus H} \tag{7}$$

Setting $y=(A-B)x$ in (7), it follows from (4)

$$\|(A-B)x\| \leq 4\sqrt{2}s(A,B)\sqrt{\|x\|^2+\|Ax\|^2}\sqrt{1+\|B\|^2_{B(H)}}$$

Let us put $\mu = 4\sqrt{2}s(A,B)\sqrt{1+\|B\|^2_{B(H)}} = 1-\varepsilon$, $\varepsilon > 0$. Thus,

$$\|Ax\| \leq \|Bx\| + (1-\varepsilon)[\|x\|+\|Ax\|]$$

or finally

$$\|Ax\| \leq \frac{1}{\varepsilon}[1+\|B\|_{B(H)}]\,\|x\|$$

what shows that A is bounded. ■

Remark 6 *$B(H)$ is open and dense respectively in the metric spaces $(C(H),g)$, $(C(H),p)$ and $(C(H),l)$.*

3 Topologies induced on $C(H)$ from almost closed operators

In this section we recall the definition of almost closed operators and their fundamental properties obtained in [4] and we introduce a metric in the set $AC(H)$ of all almost closed operators on H. The topology induced on $C(H)$ from this metric is strictly stronger than that induced from g. We also define a new topology on $C(H)$ strictly stronger than that induced from g by using the quotient representation of almost cloed operators.

Let A be an operator with domain $D(A)$ in H. A is called almost closed on H if there exists an inner product $[.,.]_A$ on $D(A)$ such that $H_A=(D(A),[.,.]_A)$ is complete, H_A is continuously embedded in H and A is continuous from H_A onto H (ie. $A \in \mathcal{B}(H_A,H)$).

The class $AC(H)$ contain $C(H)$ and is invariant under addition, composition and limits. Almost closed operators verify also the usual properties of the adjoint of linear operators.

In particular, if A is almost closed then $D(A)$ is an operator range, namely there is a unique positive bounded operator A_+ on H such that $D(A) = A_+H$ and $\|x\|_A = \sqrt{[x,x]_A} = \|x\|_{A_+}$ for $x \in D(A)$ where $\|x\|_{A_+} = \|Pt\|$ if $x = At$ and P is the orthogonal projection onto the orthogonal complement $N(A_+)^\perp$ of $N(A_+)$ [1].

Then, A is almost closed if and only if A is represented by a quotient $\frac{B}{A_+}$ of bounded operators, namely there exists a bounded operator $B \in B(H)$ such that $N(A_+) \subseteq N(B)$, $D(A) = A_+H$, $R(A) = BH$ and $AA_+x = Bx$, for all $x \in H$ [3]. For $A = \frac{B}{A_+} \in AC(H)$, we have $\|A\|_{B(H_A,H)} = \|B\|_{B(H)}$.

Consider α be the correpondence between almost closed operators A and the associated positive bounded operators A_+. Such operator $A \in AC(H)$ is uniquely represented up to α by a quotient $\frac{B}{A_+}$, so that we denote $A \overset{\alpha}{=} \frac{B}{A_+}$.

Hirasawa [2] defined an $\delta-$neighborhood for $\delta > 0$ of an almost closed operator $A \overset{\alpha}{=} \frac{B}{A_+}$ on H by

$$\begin{aligned} \mathcal{V}(A,\alpha,\delta) &= \{T \in AC(H) \; ; \; T \overset{\alpha}{=} \frac{C}{A_+} \; , \; \|B-C\|_{B(H)} < \delta\} \\ &= \{T \in AC(H) \; ; \; D(A) = D(T) \; , \; \|A-T\|_{B(H_A,H)} < \delta\} \end{aligned} \tag{8}$$

and considered the topology τ induced from the neighborhood system as above. τ is a locally convex Hausdorf topology in the set $AC(H)$ and is independent from the correspondence α. In fact, $AC(H)$ becomes metrizable by means of the metrics

$$\rho(A,T) = \begin{cases} 1 \;\; \text{if} \;\; D(A) \neq D(T) \\ \frac{\|A-T\|_{B(H_A,H)}}{1+\|A-T\|_{B(H_A,H)}} \;\; \text{if} \;\; D(A) = D(T) \end{cases} \tag{9}$$

Theorem 7 *([2]) In $C(H)$, the topology induced from the metric ρ is strictly stronger than that induced from the gap metric g.*

$B(H)$ is a connected component of $AC(H)$ and $C(H)$ is open in $AC(H)$.

In particular, it is also shown that the addition and the scalar multiplication in the set $AC(H)$ are continuous, and that the multiplication from the left side is continuous.

Let $A \in C(H)$, then we can write $A = \Gamma(B) = \frac{B}{(I-B^*B)^{1/2}}$ with a unique positive contraction $B \in C_0(H)$, where Γ is a reversible function from $C_0(H)$ onto $C(H)$ with inverse function defined by $\Gamma^{-1}(A) = A(I + A^*A)^{-1/2} = AS_A$ (see [3]). The related convergence in the space $C(H)$, called quotient-convergence, is defined as follows : $A_n = \frac{B_n}{(I-B_n^*B_n)^{1/2}}$ converges to $A = \frac{B}{(I-B^*B)^{1/2}}$ if B_n converges to B in $B(H)$ where $B_n, B \in C_0(H)$.

The orthogonal projection $P_{G(A)} : H \oplus H \to H \oplus H$ on the graph $G(A)$ of the operator $A = \frac{B}{(I-B^*B)^{1/2}}$ can be described through the following matrix (see Lemma 3.9, [2]) :

$$P_{G(A)} = \begin{pmatrix} (I - B^*B) & (I - B^*B)^{1/2}B^* \\ B(I - BB^*)^{1/2} & BB^* \end{pmatrix} \tag{10}$$

Consequently, if B_n converges to B in $B(H)$, then we have $(I-B_n^*B_n)^{1/2}B^* \longrightarrow (B(I-BB^*)^{1/2})^*$, and this assures the convergence $P_{G(A_n)} \longrightarrow P_{G(A)}$ as $n \longrightarrow \infty$, or $\lim\limits_{n\to+\infty} g(A, A_n) = 0$.

Reconsider now on l^2 the sequence of operators A_n given in (5) and the linear operator $[A(x)]_k = kx_k$. Then, $A_n = \Gamma(B_n)$ and $A = \Gamma(B)$, where $B_n = \Gamma^{-1}(A_n) = A_nS_{A_n}$ and $B = \Gamma^{-1}(A) = AS_A$ are contractions corresponding to A_n and A respectively. Let us notice that by virtue of (1), $\|B_n\|_{B(H)} \leq 1$, $\|B\|_{B(H)} \leq 1$, $N(I - B_n^*B_n) = N(I - A_n^*S_{A_n^*}A_nS_{A_n}) = N(R_{A_n}) = \{0\}$ and $N(I - B^*B) = N(I - A^*S_{A^*}AS_A) = N(R_A) = \{0\}$.Thus, $B, B_n \in C_0(l^2)$, for all $n \in \mathbb{N}$.

We have already shown that the sequence $(A_n)_n$ converges to A for g.

$$[B_n(x)]_k = [B_n^*(x)]_k = \begin{cases} \frac{k}{\sqrt{1+k^2}}x_k & \text{if } k < n \\ -\frac{k}{\sqrt{1+k^2}}x_k & \text{if } k \geq n \end{cases} \quad \text{and} \quad [B(x)]_k = \frac{k}{\sqrt{1+k^2}}x_k$$

As, $\|B_n - B\|_{B(H)} = \|A_nS_{A_n} - AS_A\|_{B(H)} \geq \frac{2n}{\sqrt{1+n^2}} \longrightarrow 2$, then $(A_n)_n$ does not converge to the same limit for quotient-convergence.

We have then shown the following fundamental result :

Theorem 8 *The topology induced on $C(H)$ by quotient-convergence is strictly stronger than the topolgy induced from the gap metric.*

Remark 9 *By virtue of theorem 1, it becomes interesting to compare the completion of $C(H)$ for the metrics g, s and ρ with $AC(H\ )$. In other words, Is it possible to determine a metrics d on $C(H)$ such as the completion of $C(H)$ for d coincide with $AC(H)$?.*

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