



Quantum Birkhoff–Gustavson normal form in some completely resonant cases

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ABSTRACT

The goal of this paper is to present an explicit computation of the Birkhoff–Gustavson normal form in the $1:1$, $1:2$ and $1:3$ resonances, by using the concept of Weyl quantization, especially when it is about the first terms. We use in particular the Weyl symbolic calculus and the abstract theorem of the quantum Birkhoff–Gustavson normal form in infinite dimension.

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1. Introduction

The theory of the normal forms has been introduced by H. Poincaré (1957) in his thesis and since remains a fundamental tool for the treatment of the nonlinear dynamics. G.D. Birkhoff [3] applied for the first time the method of normal forms in the Hamiltonian case where he studies some nonresonant Hamiltonian systems near an equilibrium position. This method has been extended to the resonant case by F.G. Gustavson [11] so that, one should refer to the Birkhoff–Gustavson normal forms (BGNF).

The concept of Birkhoff–Gustavson normal forms led to several applications [2,9,17]. We mention the article [8] as reference to an algorithm of reduction to the Birkhoff normal form. M. Gutzwiller [12] in his book emphasized the importance of the method of normal forms in different semiclassical constructions. For other applications of the Birkhoff normal forms, one can consult the article of S. Zelditch [21].

The normal forms for pseudodifferential operators near a nondegenerate minimum of a symbol, are already used by several authors, let's mention in particular the article [19] of J. Sjöstrand that treats the nonresonant case in a neighborhood of a fixed point of elliptic type and the article of V. Guillemin [10] where he constructs a microlocal Birkhoff normal form in a formal neighborhood of a periodic curve.

A. Iantchenko defines in [13] the Birkhoff normal form for the Fourier integral operators associated to canonical transformations of the real hyperbolic type, microlocally near the fixed point of the transformation.

L. Charles and S. Vu Ngoc treat in [4] the completely resonant case and gives new proofs for many of the results. It is shown that a pseudodifferential operator can be reduced to a Toeplitz operator on a reduced symplectic orbifold. Using this quantum reduction, new spectral asymptotics concerning the fine structure of eigenvalue clusters are obtained.

Concerning quantum and semiclassical mechanics, the Birkhoff normal form can be applied in a very prosperous way. It became an important tool in molecular physics and numerical computations of eigenvalues and resonances (see [20] and

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more recently [15,18]). We suggest the reader to consult [1,5,14,16,21] for further references and connections with other problems.

On the other hand, we are interested in [7] to the calculation of the spectrum in the 1 : 2 resonance relation, it after have calculate the BGNF of the Hamiltonian in this case, while passing by the principle of annihilation and creation operators and Bargmann transform.

The purpose of this paper is to survey in a first step some of the results in this area and to give indications of some of the techniques used to prove them. We present the quantum formal Birkhoff normal form theorem, where the main idea is to find the simplest form of the Schrödinger Hamiltonian after having transformed it in a perturbation of the harmonic oscillator.

In the second step, the Birkhoff–Gustavson normal form is employed to study the resonances of semiclassical Schrödinger Hamiltonians. We explicitly determine the Birkhoff–Gustavson normal form in 1 : 1, 1 : 2 and 1 : 3 resonances. The basic tool used here is the Weyl symbolic calculus and consequently the Weyl quantization.

2. Hypotheses, classical results and quantum formal Birkhoff normal form theorem

Let $T^*\mathbb{R}^n = \mathbb{R}^{2n}$ be the phase space with coordinates $(x, \xi) = (x_1, \dots, x_n, \xi_1, \dots, \xi_n)$, and we consider the Schrödinger operator

$$P(h) = -\frac{h^2}{2}\Delta + V(x) \tag{1}$$

where V is a smooth potential on \mathbb{R}^n . Assume $V \in C^\infty(\mathbb{R}^n, \mathbb{R})$ has a global minimum at a point 0 which we shall call here the origin, and suppose this minimum is nondegenerate. By a linear, unitary change of variables in local coordinates near 0, one can assume that the hessian matrix $V''(0)$ is diagonal, let (v_1^2, \dots, v_n^2) be its eigenvalues, with $v_j > 0$. The rescaling $x_j \rightarrow \sqrt{v_j}x_j$ transforms $P(h)$ into a perturbation of the harmonic oscillator \widehat{H}_2 :

$$P(h) = \widehat{H}_2 + W(x) \tag{2}$$

with

$$\widehat{H}_2 = \sum_{j=1}^n \frac{v_j}{2} \left(-h^2 \frac{\partial^2}{\partial x_j^2} + x_j^2 \right) \tag{3}$$

where $W(x)$ is a smooth potential of order $\mathcal{O}(|x|^3)$ at the origin.

First of all, we recall the h -Weyl quantization which is based on a particular grading for formal symbols in x, ξ, h , where the degree in semiclassical parameter counts twice the degree of each other variable x_i or ξ_j .

Let d and m be real numbers, and $S^d(m)$ the set of all families $(a(\cdot, h))_{h \in]0,1]}$ of functions in $\mathcal{C}^\infty(\mathbb{R}^{2n})$ such that for all $\alpha \in \mathbb{N}^{2n}$:

$$|\partial_{(x,\xi)}^\alpha a(x, \xi, h)| \leq C_\alpha h^d (1 + |x|^2 + |\xi|^2)^{m/2}$$

for some constant $C_\alpha > 0$, uniformly in h .

Let $\Psi^d(m, \mathbb{R}^n)$ be the set of all unbounded linear operators A on $L^2(\mathbb{R}^n)$ that are h -Weyl quantizations of symbols $a \in S^d(m)$:

$$(Au)(x) = (Op_h^W(a)u)(x) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^{2n}} e^{\frac{i}{h}(x-y,\xi)} a\left(\frac{x+y}{2}, \xi, h\right) u(y) dy d\xi \tag{4}$$

Now, we introduce the space

$$\mathcal{E} = \mathbb{C}[x, \xi, h] = \mathbb{C}[x_1, \dots, x_n, \xi_1, \dots, \xi_n, h]$$

of formal series of $(2n + 1)$ variables with complex coefficients, where the degree of the monomial $x^\alpha \xi^\beta h^\ell$ is defined by $|\alpha| + |\beta| + 2\ell$, $\alpha, \beta \in \mathbb{N}^n$, $\ell \in \mathbb{N}$.

Let \mathcal{D}_N be the finite dimension vector space spanned by monomials of degree N and \mathcal{O}_N the subspace of \mathcal{E} consisting of formal series whose coefficients of degree $< N$ vanish. Then $(\mathcal{O}_N)_{N \in \mathbb{N}}$ is a filtration:

$$\mathcal{E} = \mathcal{O}_0 \supset \mathcal{O}_1 \supset \dots, \quad \bigcap_N \mathcal{O}_N = \{0\}$$

We define $[\dots]_W$ the Weyl bracket by the Taylor series of the symbol of the operator commutator $[\widehat{f}, \widehat{g}]$. More precisely:

$$[f_T, g_T]_W = \sigma_W[\widehat{f}, \widehat{g}] = \sigma_W(\widehat{f}\widehat{g} - \widehat{g}\widehat{f}) \tag{5}$$

where \hat{f} and \hat{g} are the \hbar -Weyl quantizations of symbols f and g , with formal Taylor series at the origin f_T and g_T in \mathcal{E} . Then, $[\cdot, \cdot]_W$ is bilinear antisymmetrical, and satisfies the Jacobi identity:

$$[[f_T, g_T]_W, v_T]_W + [[v_T, f_T]_W, g_T]_W + [[g_T, v_T]_W, f_T]_W = 0$$

and the Leibniz identity:

$$[f_T, g_T v_T]_W = [f_T, g_T]_W v_T + g_T [f_T, v_T]_W$$

The Weyl bracket defines an abstract Lie algebra structure on \mathcal{E} (called Weyl algebra), and it is the unique bracket for which \hbar is central:

$$[\hbar, x_j]_W = [\hbar, \xi_j]_W = 0 \quad \text{and} \quad [\xi_j, x_j]_W = \frac{\hbar}{i}, \quad \text{for all } j = 1, \dots, n$$

Let $A \in \mathcal{E}$, we define the operator ad_A on \mathcal{E} by:

$$ad_A(P) = [A, P]_W \tag{6}$$

and we shall be particularly interested in the adjoint action of elements of \mathcal{D}_2 , where each P of \mathcal{D}_2 is of the forms:

$$P(\hbar) = \hbar H_0 + H_2$$

where $H_0 \in \mathbb{C}$ and H_2 is a quadratic form in (x, ξ) .

Since \hbar is central, we may restrict here to quadratic forms only.

We have a fundamental property of the Weyl bracket:

Proposition 1.

$$i\hbar^{-1} ad_{H_2}(P) = \{H_2, P\}$$

where the classical Poisson bracket is

$$\{f, g\} = \sum_{j=1}^n \frac{\partial f}{\partial \xi_j} \frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial \xi_j} \quad \text{for all } f, g \in C^\infty(\mathbb{R}^{2n}) \tag{7}$$

Proof. Indeed, first recall the Moyal formula [6] which expresses the relation between the Weyl quantization and Lie bracket. If A and B are two Weyl quantizations of symbols a and b , then we have:

$$[a, b]_W = \sigma_W[A, B] = \frac{2}{i} a \sin\left(\frac{\hbar D}{2}\right) b \tag{8}$$

where

$$D = \sum_{j=1}^n \frac{\overleftarrow{\partial}}{\partial \xi_j} \frac{\overrightarrow{\partial}}{\partial x_j} - \frac{\overleftarrow{\partial}}{\partial x_j} \frac{\overrightarrow{\partial}}{\partial \xi_j}$$

and

$$\sin\left(\frac{\hbar D}{2}\right) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!} \frac{\hbar^{2j+1}}{2^{2j+1}} D^{2j+1}$$

hence,

$$\begin{aligned} [a, b]_W &= \sigma_W[A, B] = \frac{2}{i} a \left(\frac{\hbar D}{2}\right) b + \frac{2}{i} a \left(\frac{-1}{3!} \left(\frac{\hbar}{2}\right)^3 D^3\right) b + \dots \\ &= \frac{2}{i} \frac{\hbar}{2} \left(\sum_{j=1}^n \frac{\overleftarrow{\partial} a}{\partial \xi_j} \frac{\overrightarrow{\partial} b}{\partial x_j} - \frac{\overleftarrow{\partial} a}{\partial x_j} \frac{\overrightarrow{\partial} b}{\partial \xi_j}\right) + \frac{2}{i} a \left(\frac{-1}{3!} \left(\frac{\hbar}{2}\right)^3 D^3\right) b + \dots \\ &= \frac{\hbar}{i} \{a, b\} + \mathcal{O}(\hbar^3) \end{aligned}$$

If $a = H_2 = \sum_{j=1}^n \frac{v_j}{2} (x_j^2 + \xi_j^2)$, then

$$H_2 D^k P = 0, \quad \text{for all } k \geq 3$$

hence the result. \square

Another very important result that is going to serve us thereafter:

Proposition 2. ad_{H_2} is diagonal on the $\mathbb{C}[z, \bar{z}, h]$ -basis $\{z^\beta \bar{z}^\gamma\}$, where β and γ are multi-indices in \mathbb{N}^n , $z = (z_1, \dots, z_n)$, $z_j = x_j + i\xi_j$ and we have:

$$ad_{H_2}(z^\beta \bar{z}^\gamma) = h(\gamma - \beta, \nu)z^\beta \bar{z}^\gamma \tag{9}$$

Proof. We have:

$$H_2 = \sum_{j=1}^n \frac{\nu_j}{2}(x_j^2 + y_j^2), \quad \frac{\partial H_2}{\partial x_j} = \nu_j x_j, \quad \frac{\partial H_2}{\partial \xi_j} = \nu_j \xi_j$$

$$\frac{\partial z^\beta \bar{z}^\gamma}{\partial x_j} = \beta_j z^{\beta - e_j} \bar{z}^\gamma + \gamma_j \bar{z}^{\gamma - e_j} z^\beta$$

$$\frac{\partial z^\beta \bar{z}^\gamma}{\partial \xi_j} = i\beta_j z^{\beta - e_j} \bar{z}^\gamma - i\gamma_j \bar{z}^{\gamma - e_j} z^\beta$$

where $e_j = (0, \dots, 0, 1, 0, \dots, 0)$, $j = 1, \dots, n$. Thus,

$$\begin{aligned} \{H_2, z^\beta \bar{z}^\gamma\} &= \sum_{j=1}^n \nu_j \xi_j (\beta_j z^{\beta - e_j} \bar{z}^\gamma + \gamma_j \bar{z}^{\gamma - e_j} z^\beta) - \nu_j x_j (i\beta_j z^{\beta - e_j} \bar{z}^\gamma - i\gamma_j \bar{z}^{\gamma - e_j} z^\beta) \\ &= \sum_{j=1}^n -i\nu_j \beta_j z^{\beta - e_j} \bar{z}^\gamma (x_j + i\xi_j) + i\nu_j \gamma_j \bar{z}^{\gamma - e_j} z^\beta (x_j - i\xi_j) \\ &= \sum_{j=1}^n i\nu_j (\gamma_j - \beta_j) z^\beta \bar{z}^\gamma \end{aligned}$$

and therefore

$$ad_{H_2}(z^\beta \bar{z}^\gamma) = \frac{h}{i} \{H_2, z^\beta \bar{z}^\gamma\} = h(\gamma - \beta, \nu)z^\beta \bar{z}^\gamma \quad \square$$

All necessary ingredients are there, and then the formal quantum Birkhoff–Gustavson normal form can be expressed as follows:

Theorem 3. If $L \in \mathcal{O}_3$, then there exists $A \in \mathcal{O}_3$ and $K \in \mathcal{O}_3$ such that

$$\begin{aligned} e^{ih^{-1}ad_A}(H_2 + L) &= H_2 + K \\ [H_2, K]_W &= 0 \end{aligned} \tag{BGNF}$$

Moreover if H_2 and L have real coefficients then A and K can be chosen to have real coefficients as well.

Notice that the sum:

$$e^{ih^{-1}ad_A}(H_2 + L) = \sum_{l=0}^{\infty} \frac{1}{l!} \left(\frac{i}{h}ad_A\right)^l (H_2 + L) \tag{10}$$

is convergent in \mathcal{E} because $\frac{i}{h}ad_A$ sends \mathcal{O}_N into \mathcal{O}_{N+1} .

(BGNF) is the Birkhoff–Gustavson normal form of the operator $P(h)$ near the origin.

Since our goal is to compute the BGNF taking counts the resonances, so let's recall the definitions of the different relations of resonances:

Definition 4. Let's consider the harmonic oscillator $H_2 = \sum_{j=1}^n \frac{\nu_j}{2}(x_j^2 + \xi_j^2)$ where the frequencies $\nu_j > 0$.

1) One says that a resonance relation exists if there exist integer numbers k_j not all vanishing such that:

$$k_1 \nu_1 + \dots + k_n \nu_n = 0$$

i.e. the $(\nu_j)_{1 \leq j \leq n}$ are dependent over \mathbb{Z} . $k = \sum_{j=1}^n |k_j|$ is called the order of the resonance.

2) One says that there is no resonance relation if the frequencies $(\nu_j)_j$ are linearly independent over \mathbb{Z} (or \mathbb{Q}):

$$\sum_{j=1}^n k_j \nu_j = 0 \Rightarrow k_j = 0, \quad j = 1, \dots, n$$

3) and we say that the frequencies $(\nu_j)_j$ are completely resonant, if there exist a real number $\nu_c > 0$ and coprime positive integers p_1, \dots, p_n such that $\nu_j = \nu_c p_j$, for all $j = 1, \dots, n$.

Remark 5. It's clear that if the coefficients $(\nu_j)_{1 \leq j \leq n}$ are completely resonant then there are resonant.

3. Application to 1 : 1, 1 : 2 and 1 : 3 resonances

This third and last section is the goal of this paper, so we intend, to apply the quantum BGNF theorem in three completely resonant cases 1 : 1, 1 : 2 and 1 : 3 resonances, where K is computed in the $\mathbb{C}[z, \bar{z}, \hbar]$ -basis, then we compute its symmetric Weyl quantization, and finally we deduct the one of the Hamiltonian $P(\hbar)$.

Let's notice that the general idea is the same in the three cases but the results are different according to the different relations of resonances.

3.1. The 1 : 1 resonance (completely resonant case)

Consider the following harmonic oscillator

$$\widehat{H}_2 = \frac{1}{2} \left(-\hbar^2 \frac{\partial^2}{\partial x_1^2} + x_1^2 \right) + \frac{1}{2} \left(-\hbar^2 \frac{\partial^2}{\partial x_2^2} + x_2^2 \right) \tag{11}$$

with symbol

$$H_2 = \frac{1}{2} (|z_1|^2 + |z_2|^2) \tag{12}$$

where $z_j = x_j + i\xi_j$, $j = 1, 2$.

We propose to look for $K_3 \in \mathcal{D}_3$ such as

$$[H_2, K_3]_W = 0$$

Since $[H_2, K_3]_W = 0$ is equivalent to $\{H_2, K_3\} = 0$, it is sufficient to calculate the Weyl symbol

$$K_3 = \sum_{2\ell + |\beta| + |\gamma| = 3} h^\ell z^\alpha \bar{z}^\beta \quad \text{such that} \quad \langle \nu, \beta - \alpha \rangle = 0$$

then,

$$\langle \nu, \beta - \alpha \rangle = 0 \iff \alpha_1 + \alpha_2 = \beta_1 + \beta_2 \tag{res 1 : 1}$$

where $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$.

To calculate K_3 , it is necessary to look for all monomials of order 3 of type

$$K_3 = z_1^{\alpha_1} z_2^{\alpha_2} \bar{z}_1^{\beta_1} \bar{z}_2^{\beta_2}$$

that satisfy the resonance relation (res 1 : 1).

We notice that no monomial exists in \mathcal{D}_3 verifying $|\alpha| + |\beta| = 3$ and the resonance relation (res 1 : 1), so $K_3 = 0$, but one can calculate $K_4 \in \mathcal{D}_4$.

Indeed, the couples $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$ that verify $|\alpha| + |\beta| = 4$ and the resonance relation (res 1 : 1) are:

- $\alpha = \beta = (1, 1)$
- $\alpha = \beta = (2, 0)$
- $\alpha = \beta = (0, 2)$
- $\alpha = (2, 0) \quad \text{and} \quad \beta = (0, 2)$
- $\alpha = (0, 2) \quad \text{and} \quad \beta = (2, 0)$

Thus, K_4 is generated by the monomials:

$$|z_1|^2 |z_2|^2, |z_1|^4, |z_2|^4, z_1^2 \bar{z}_2^2, \bar{z}_1^2 z_2^2$$

Finally, we obtain the symmetric Weyl quantization \widehat{K}_4 of K_4 :

$$\widehat{K}_4 \in \text{Op}_\hbar^W [\text{vect}(|z_1|^2 |z_2|^2, |z_1|^4, |z_2|^4, \text{Re}(z_1^2 \bar{z}_2^2))]$$

where $\text{vect}(|z_1|^2 |z_2|^2, |z_1|^4, |z_2|^4, \text{Re}(z_1^2 \bar{z}_2^2))$ is the subspace spanned by $|z_1|^2 |z_2|^2, |z_1|^4, |z_2|^4$ and $\text{Re}(z_1^2 \bar{z}_2^2)$.

Consequently,

$$\widehat{K}_4 = \lambda_1 Op_h^W(|z_1|^4) + \lambda_2 Op_h^W(|z_2|^4) + \lambda_3 Op_h^W(|z_1|^2|z_2|^2) + \lambda_4 Op_h^W(\operatorname{Re}(z_1^2 \bar{z}_2^2)) \tag{13}$$

where,

$$\begin{aligned} |z_1|^4 &= x_1^4 + \xi_1^4 + 2x_1^2 \xi_1^2 \\ |z_2|^4 &= x_2^4 + \xi_2^4 + 2x_2^2 \xi_2^2 \\ |z_1|^2|z_2|^2 &= x_1^2 x_2^2 + x_1^2 \xi_2^2 + x_2^2 \xi_1^2 + \xi_1^2 \xi_2^2 \\ \operatorname{Re}(z_1^2 \bar{z}_2^2) &= x_1^2 x_2^2 - x_1^2 \xi_2^2 - x_2^2 \xi_1^2 + \xi_1^2 \xi_2^2 + 4x_1 x_2 \xi_1 \xi_2 \end{aligned}$$

and $(\lambda_j)_{1 \leq j \leq 4}$ are the coefficients of \widehat{K}_4 .

The Weyl quantization of every monomial gives us,

$$\begin{aligned} Op_h^W(|z_1|^4) &= x_1^4 + h^4 \frac{\partial^4}{\partial x_1^4} - h^2 \left[x_1^2 \frac{\partial^2}{\partial x_1^2} + 2 \right] \\ Op_h^W(|z_2|^4) &= x_2^4 + h^4 \frac{\partial^4}{\partial x_2^4} - h^2 \left[x_2^2 \frac{\partial^2}{\partial x_2^2} + 2 \right] \\ Op_h^W(|z_1|^2|z_2|^2) &= x_1^2 x_2^2 - h^2 \left[x_1^2 \frac{\partial^2}{\partial x_2^2} + x_2^2 \frac{\partial^2}{\partial x_1^2} - h^2 \frac{\partial^4}{\partial x_1^2 \partial x_2^2} \right] \\ Op_h^W(\operatorname{Re}(z_1^2 \bar{z}_2^2)) &= x_1^2 x_2^2 - h^2 \left[-x_1^2 \frac{\partial^2}{\partial x_2^2} - x_2^2 \frac{\partial^2}{\partial x_1^2} - h^2 \frac{\partial^4}{\partial x_1^2 \partial x_2^2} + 2x_1 x_2 \frac{\partial^2}{\partial x_1 \partial x_2} + 2 \right] \end{aligned}$$

and therefore, we get the following BGNF of the Hamiltonian $P(h)$ in the 1 : 1 resonance,

$$\begin{aligned} \widehat{H}_2 + \widehat{K}_4 &= \sum_{j=1}^2 \frac{1}{2} \left(-h^2 \frac{\partial^2}{\partial x_j^2} + x_j^2 \right) + \lambda_1 \left[x_1^4 + h^4 \frac{\partial^4}{\partial x_1^4} - h^2 \left(x_1^2 \frac{\partial^2}{\partial x_1^2} + 2 \right) \right] \\ &\quad + \lambda_2 \left[x_2^4 + h^4 \frac{\partial^4}{\partial x_2^4} - h^2 \left(x_2^2 \frac{\partial^2}{\partial x_2^2} + 2 \right) \right] \\ &\quad + \lambda_3 \left[x_1^2 x_2^2 - h^2 \left(x_1^2 \frac{\partial^2}{\partial x_2^2} + x_2^2 \frac{\partial^2}{\partial x_1^2} - h^2 \frac{\partial^4}{\partial x_1^2 \partial x_2^2} \right) \right] \\ &\quad + \lambda_4 \left[x_1^2 x_2^2 - h^2 \left(-x_1^2 \frac{\partial^2}{\partial x_2^2} - x_2^2 \frac{\partial^2}{\partial x_1^2} - h^2 \frac{\partial^4}{\partial x_1^2 \partial x_2^2} + 2x_1 x_2 \frac{\partial^2}{\partial x_1 \partial x_2} + 2 \right) \right] \end{aligned} \tag{14}$$

3.2. The 1 : 2 resonance (Fermi resonance)

In this case we consider the following harmonic oscillator:

$$\widehat{H}_2 = \frac{1}{2} \left(-h^2 \frac{\partial^2}{\partial x_1^2} + x_1^2 \right) + \left(-h^2 \frac{\partial^2}{\partial x_2^2} + x_2^2 \right) \tag{15}$$

with symbol

$$H_2 = \frac{1}{2}|z_1|^2 + |z_2|^2 \tag{16}$$

by the same way, we look for $K_3 \in \mathcal{D}_3$ such that $[H_2, K_3]_W = 0$ and thus $\{H_2, K_3\} = 0$, then it is sufficient to calculate the Weyl symbol

$$K_3 = \sum_{2\ell + |\beta| + |\gamma| = 3} h^\ell z^\alpha \bar{z}^\beta$$

such that

$$\langle \nu, \beta - \alpha \rangle = 0 \Leftrightarrow \alpha_1 + 2\alpha_2 = \beta_1 + 2\beta_2 \tag{res 1 : 2}$$

The couples $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$ that verify $|\alpha| + |\beta| = 3$ and the resonance relation (res 1 : 2) are

$$\alpha = (0, 1) \quad \text{and} \quad \beta = (2, 0)$$

$$\alpha = (2, 0) \quad \text{and} \quad \beta = (0, 1)$$

Thus,

$$K_3 = \mu \operatorname{Re}(z_1^2 \bar{z}_2) = \frac{\mu}{2} (z_2 \bar{z}_1^2 + z_1^2 \bar{z}_2), \quad \mu \in \mathbb{R} \quad (17)$$

Now, we calculate \widehat{K}_3 the symmetric Weyl quantization of K_3 ,

$$\begin{aligned} \widehat{K}_3 &= \mu \operatorname{Op}_h^W(\operatorname{Re}(z_1^2 \bar{z}_2)) \\ &= \mu \operatorname{Op}_h^W(x_1^2 x_2 + 2x_1 \xi_1 \xi_2 - \xi_1^2 x_2) \\ &= \mu \operatorname{Op}_h^W(x_1^2 x_2 + 2x_1 \xi_1 \xi_2 - \xi_1^2 x_2) \\ &= \mu \left[x_1^2 x_2 - h^2 \left(2x_1 \frac{\partial^2}{\partial x_1 \partial x_2} - x_2 \frac{\partial^2}{\partial x_1^2} + \frac{\partial}{\partial x_2} \right) \right] \end{aligned}$$

and finally, we get the BGNF of $P(h)$ in the 1 : 2 resonance,

$$\begin{aligned} \widehat{H}_2 + \widehat{K}_3 &= \frac{1}{2} \left(-h^2 \frac{\partial^2}{\partial x_1^2} + x_1^2 \right) + \left(-h^2 \frac{\partial^2}{\partial x_2^2} + x_2^2 \right) \\ &\quad + \mu \left[x_1^2 x_2 - h^2 \left(2x_1 \frac{\partial^2}{\partial x_1 \partial x_2} - x_2 \frac{\partial^2}{\partial x_1^2} + \frac{\partial}{\partial x_2} \right) \right] \end{aligned} \quad (18)$$

Contrary to the last article [7], and in the goal to facilitate us the calculation of the spectrum in this case, we calculated the BGNF of the Hamiltonian, while passing by the annihilation and creation operators and Bargmann transform and not by the principle of the Weyl quantization as here.

3.3. The 1 : 3 resonance

Now, let us take

$$\widehat{H}_2 = \frac{1}{2} \left(-h^2 \frac{\partial^2}{\partial x_1^2} + x_1^2 \right) + \frac{3}{2} \left(-h^2 \frac{\partial^2}{\partial x_2^2} + x_2^2 \right) \quad (19)$$

with symbol

$$H_2 = \frac{1}{2} |z_1|^2 + \frac{3}{2} |z_2|^2 \quad (20)$$

To calculate K_3 , it is necessary to look for all monomials of order 3 that verify the resonance relation:

$$\langle v, \beta - \alpha \rangle = 0 \quad \Leftrightarrow \quad \alpha_1 + 3\alpha_2 = \beta_1 + 3\beta_2 \quad (\text{res } 1 : 3)$$

As the first case of the 1 : 1 resonance, we see that no monomial exists in \mathcal{D}_3 verifying $|\alpha| + |\beta| = 3$ and the resonance relation (res 1 : 3).

On the other hand, one can calculate $K_4 \in \mathcal{D}_4$, that verifies the previous resonance relation.

Indeed, the couples $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$ that satisfy the relation (res 1 : 3) are:

$$\alpha = \beta = (1, 1)$$

$$\alpha = \beta = (2, 0)$$

$$\alpha = \beta = (0, 2)$$

$$\alpha = (3, 0) \quad \text{and} \quad \beta = (0, 1)$$

$$\alpha = (0, 1) \quad \text{and} \quad \beta = (3, 0)$$

Thus, K_4 is generated by the monomials:

$$|z_1|^4, |z_2|^4, |z_1|^2 |z_2|^2, \bar{z}_1^3 z_2, z_1^3 \bar{z}_2$$

The symmetric Weyl quantization

$$\widehat{K}_4 \in \operatorname{Op}_h^W[\operatorname{vect}(|z_1|^4, |z_2|^4, |z_1|^2 |z_2|^2, \operatorname{Re}(z_1^3 \bar{z}_2))]$$

Then,

$$\widehat{K}_4 = \gamma_1 Op_h^W(|z_1|^4) + \gamma_2 Op_h^W(|z_2|^4) + \gamma_3 Op_h^W(|z_1|^2|z_2|^2) + \gamma_4 Op_h^W(\operatorname{Re}(z_1^3 \bar{z}_2))$$

where

$$\operatorname{Re}(z_1^3 \bar{z}_2) = x_1^3 x_2 + 3x_1^2 \xi_1 \xi_2 - 3x_1 x_2 \xi_1^2 - \xi_1^3 \xi_2$$

and $(\gamma_j)_{1 \leq j \leq 4}$ are the coefficients of \widehat{K}_4 .

$$Op_h^W(\operatorname{Re}(z_1^3 \bar{z}_2)) = x_1^3 x_2 - h^2 \left[\frac{3}{2} x_1^2 \frac{\partial^2}{\partial x_1 \partial x_2} + 3 \frac{\partial}{\partial x_2} - \frac{3}{2} x_1 x_2 \frac{\partial^2}{\partial x_1^2} - h^2 \frac{\partial^4}{\partial x_1^3 \partial x_2} \right]$$

and the Weyl quantizations of the other terms have already been calculated in the case of the 1 : 1 resonance.

Therefore, we get the BGNF of $P(h)$ in the 1 : 3 resonance,

$$\begin{aligned} \widehat{H}_2 + \widehat{K}_4 &= \frac{1}{2} \left(-h^2 \frac{\partial^2}{\partial x_1^2} + x_1^2 \right) + \frac{3}{2} \left(-h^2 \frac{\partial^2}{\partial x_2^2} + x_2^2 \right) \\ &+ \gamma_1 \left[x_1^4 + h^4 \frac{\partial^4}{\partial x_1^4} - h^2 \left(x_1^2 \frac{\partial^2}{\partial x_1^2} + 2 \right) \right] \\ &+ \gamma_2 \left[x_2^4 + h^4 \frac{\partial^4}{\partial x_2^4} - h^2 \left(x_2^2 \frac{\partial^2}{\partial x_2^2} + 2 \right) \right] \\ &+ \gamma_3 \left[x_1^2 x_2^2 - h^2 \left(x_1^2 \frac{\partial^2}{\partial x_2^2} + x_2^2 \frac{\partial^2}{\partial x_1^2} - h^2 \frac{\partial^4}{\partial x_1^2 \partial x_2^2} \right) \right] \\ &+ \gamma_4 x_1^3 x_2 - h^2 \left[\frac{3}{2} x_1^2 \frac{\partial^2}{\partial x_1 \partial x_2} + 3 \frac{\partial}{\partial x_2} - \frac{3}{2} x_1 x_2 \frac{\partial^2}{\partial x_1^2} - h^2 \frac{\partial^4}{\partial x_1^3 \partial x_2} \right] \end{aligned} \tag{21}$$

Remark 6. The coefficients $(\lambda_j)_{1 \leq j \leq 4}$, μ and $(\gamma_j)_{1 \leq j \leq 4}$ can be easily found there and the uniqueness of the BGNF, in the three previous resonance relations cases, comes because these coefficients only depend on the initial potential V . Their explicit computations will appear in a paper in preparation.

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