People's Democratic Republic of Algeria
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Abdelhamid ibn Badis University of Mostaganem
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Department of Mathematics and Computer Science

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Riad DIDA

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On the Conjecture of Brück for Solutions of Linear Differential Equations

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Chairperson:	Benharrat BELAÏDI	Professor	UMAB, Mostaganem
Examiner:	Ahmed MEDEGHRI	Professor	NHSM, Algiers
Examiner:	Berrabah BENDOUKHA	Professor	UMAB, Mostaganem
Supervisor:	Abdallah EL FARISSI	Associate Professor	NHSM, Algiers
Co-supervisor:	Zinelaâbidine LATREUCH	Associate Professor	NHSM, Algiers

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THESIS

Abstract

In this thesis, we use Nevanlinna theory to investigate Brück's conjecture for solutions of both homogeneous and non-homogeneous linear differential equations with meromorphic coefficients. We begin by examining Brück's conjecture for solutions of second-order homogeneous linear differential equations. Subsequently, we apply alternative methods to study the conjecture for solutions of higher-order homogeneous linear differential equations. We also consider the non-homogeneous case. Finally, we generalize certain previous results on small function sharing between meromorphic functions and their linear differential polynomials, using these results to confirm Brück's conjecture for entire functions under certain conditions.

2020 Mathematics Subject Classification (MSC2020): 30D35.

Keywords and phrases: Nevanlinna theory, meromorphic functions, conjecture of Brück, linear differential equations, uniqueness theorems, linear differential polynomials, deficiency.

Résumé

Dans cette thèse, nous utilisons la théorie de Nevanlinna pour examiner la conjecture de Brück pour les solutions d'équations différentielles linéaires homogènes et non homogènes avec des coefficients méromorphes. Nous commençons par étudier la conjecture de Brück pour les solutions d'équations différentielles linéaires homogènes d'ordre deux. Par la suite, nous appliquons des méthodes alternatives pour étudier la conjecture pour les solutions d'équations différentielles linéaires homogènes d'ordre supérieur. Nous considérons également le cas non homogène. Enfin, nous généralisons certains résultats antérieurs sur le partage de petites fonctions entre des fonctions méromorphes et leurs polynômes différentiels linéaires, en utilisant ces résultats pour confirmer la conjecture de Brück pour les fonctions entières sous certaines conditions.

Mots-clés : Théorie de Nevanlinna, fonctions méromorphes, conjecture de Brück, équations différentielles linéaires, théorèmes d'unicité, polynômes différentiels linéaires, déficience.

الملخص

في هذه الأطروحة، نستخدم نظرية نيفانلينا لدراسة تخمين بروك فيما يخص حلول كل من المعادلات التفاضلية الخطية المتجانسة وغير المتجانسة ذات المعاملات الميرومورفية. نبدأ بدراسة تخمين بروك فيما يخص حلول المعادلات التفاضلية الخطية المتجانسة من الدرجة الثانية. بعد ذلك، نطبق طرقًا بديلة لدراسة التخمين فيما يخص حلول المعادلات التفاضلية الخطية المتجانسة من الدرجة العليا. كما نأخذ بعين الاعتبار الحالة غير المتجانسة. أخيرًا، نعمل على تعميم بعض النتائج السابقة حول مشاركة الدوال الصغيرة بين الدوال الميرومورفية وكثيرات حدودها التفاضلية الخطية، مستخدمين هذه النتائج لتأكيد تخمين بروك للدوال الشاملة تحت شروط معينة.

الكلمات المفتاحية: نظرية نيفانلينا، الدوال الميرومورفية، تخمين بروك، المعادلات التفاضلية الخطية، نظريات الوحدانية، كثيرات الحدود التفاضلية الخطية، النقص.

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Publications and Communications

Publications and Submitted Manuscripts Included in This Thesis

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2. El Farissi, A. and Dida, R., *Brück's conjecture for solutions of higher-order complex ODE*, Submitted, 2024.

3. Dida, R. and El Farissi, A., *Uniqueness of meromorphic functions sharing one small function with their linear differential polynomials*, Submitted, 2024.

List of Communications

1. Entire solutions of some linear differential equations that share one finite value CM with their first derivative. 8th International Conference on Mathematics, July 11, 2024. Istanbul, Turkey. URL: icomath.com.

Contents

Introduction

Nevanlinna theory, developed by Rolf Nevanlinna [27] in the 1920s, is an important part of complex analysis for studying how meromorphic functions take values in the complex plane. This theory replaces the maximum modulus, unsuitable for meromorphic functions due to their poles, with the characteristic function to describe their growth. It is based on the first and second main theorems which help examine how often meromorphic functions take certain values, and understand their patterns and growth. Nevanlinna theory has many applications and has naturally become the main tool for studying complex differential equations, complex difference equations and in particular the uniqueness theory of meromorphic functions, which investigates conditions under which a meromorphic function is uniquely determined. For instance, Nevanlinna [26] in his five-values theorem (1926) proved that two non-constant meromorphic functions sharing five distinct values IM must be identical, motivating many researchers to minimize shared values under suitable conditions. In 1977, Rubel and Yang [28] proved that an entire function and its derivative sharing two finite values CM must be identical. Brück [2] further extended this field by proposing a conjecture in 1996 on the uniqueness of entire functions sharing a single value with their derivatives. This conjecture has since inspired significant research despite remaining unresolved.

The objective of this thesis is to study Brück's conjecture. Our first aim is to study Brück's conjecture for linear differential equations with meromorphic coefficients. Our second aim is to study the small function sharing problem between meromorphic functions and their linear differential polynomials.

In the first chapter, we present the main results and key notations of Nevanlinna theory such as Poisson-Jensen's formula, Nevanlinna functions and the two main theorems.

In the second chapter, we present some important uniqueness results for combinations of meromorphic functions, which will be required for the proofs of our results. Additionally, we provide an overview of Brück's conjecture and the important results related to it.

In the third chapter, we study Brück's conjecture for entire solutions of linear differential equations. In particular, we provide a rephrasing of the conjecture to accommodate entire solutions of infinite order. In the first section, we confirm our rephrased conjecture for solutions of homogeneous second-order linear differential equations

$$f'' + A(z)f' + B(z)f = 0,$$

where $A(z)$ and $B(z)$ are meromorphic functions of finite order.

In the second section, we use a different method to study polynomial sharing between entire functions and their derivatives for solutions of homogeneous and non-homogeneous higher-order linear differential equations

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0,$$

and

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = F(z),$$

where $k \geq 2$, $A_j(z)$ $(j = 0, 1, \ldots, k-1)$ are meromorphic functions of finite order and $F(z)$ $(\not\equiv 0)$ is a meromorphic function satisfying $\overline{\lambda}(F - \varphi) < \infty$ for some meromorphic function $\varphi(z)$ of finite order. After that, we consider the case where $\overline{\lambda}(F - \varphi)$ is not necessarily finite, and we prove similar results under some conditions on $F(z)$.

In the final chapter, we investigate meromorphic functions that share a small function with their linear differential polynomials. In particular, we generalize the results of Al-Khaladi [1], and we use the obtained results to confirm the rephrased conjecture under certain conditions on the counting function of the distinct zeros of $f - \varphi$ where $\varphi(z)$ is a small function of $f(z)$.

Chapter 1

Nevanlinna Theory

This chapter outlines the fundamental results and standard notations of Nevanlinna theory. For more details, see [15, 16, 19, 31].

Throughout this chapter, we denote the open disc of radius R $(0 < R < \infty)$ centred at the origin as $D(R) = \{z \in \mathbb{C} : |z| < R\}$, and the closed disc of radius R centred at the origin as $\overline{D(R)} = \{z \in \mathbb{C} : |z| \leq R\}$.

1.1 Poisson-Jensen Formula

Theorem 1.1.1 (Poisson formula, [15, 19])**.** *Let u be a harmonic function on the closed disc $\overline{D(R)}$ (i.e., u is harmonic in an open disk $D(R')$ containing $\overline{D(R)}$). Let $z \in D(R)$, then*

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - |z|^2}{|Re^{i\theta} - z|^2} u\left(Re^{i\theta}\right) d\theta. \tag{1.1}$$

Proof. Let $z \in D(R)$, then the Möbius transformation defined by

$$T(w) = \frac{R^2(z - w)}{R^2 - \overline{z}w}$$

is an automorphism of $\overline{D(R)}$ which maps the circle of radius R centred at the origin to itself and 0 to z. Furthermore $T^{-1} = T$.

Given that u is harmonic on $D(R')$, there exists an analytic function f in $D(R')$ such that $u = \operatorname{Re}(f)$. Thus, $f \circ T$ is analytic in $\overline{D(R)}$. Let $\gamma(\theta) = T\left(Re^{i\theta}\right)$, $\theta \in [0, 2\pi]$. Then, by

Cauchy integral formula

$$
\begin{aligned}
f(z)=f\left(T\left(0\right)\right) &= \frac{1}{2\pi i}\int_{\gamma}\frac{f\left(T\left(w\right)\right)}{w}dw \\
&= \frac{1}{2\pi i}\int_{0}^{2\pi}\frac{\gamma'\left(\theta\right)}{\gamma\left(\theta\right)}f\left(T\left(\gamma\left(\theta\right)\right)\right)d\theta \\
&= \frac{1}{2\pi}\int_{0}^{2\pi}\frac{T'\left(Re^{i\theta}\right)}{T\left(Re^{i\theta}\right)}Re^{i\theta}f\left(T\left(T\left(Re^{i\theta}\right)\right)\right)d\theta \\
&= \frac{1}{2\pi}\int_{0}^{2\pi}\frac{T'\left(Re^{i\theta}\right)}{T\left(Re^{i\theta}\right)}Re^{i\theta}f\left(Re^{i\theta}\right)d\theta \\
&= \frac{1}{2\pi}\int_{0}^{2\pi}\frac{R^2-|z|^2}{|Re^{i\theta}-z|^2}f\left(Re^{i\theta}\right)d\theta.
\end{aligned}
$$

Therefore,

$$
u(z)=\frac{1}{2\pi}\int_{0}^{2\pi}\frac{R^2-|z|^2}{|Re^{i\theta}-z|^2}u\left(Re^{i\theta}\right)d\theta. \qquad \square
$$

Theorem 1.1.2 (Poisson-Jensen formula, [15, 19])**.** *Let $f\not\equiv 0$ be a meromorphic function on $\overline{D(R)}$. Let $a_1,a_2,\ldots,a_p$ denote the zeros of f in $D(R)$, each zero repeated according to its multiplicity, and let $b_1,b_2,\ldots,b_q$ denote the poles of f in $D(R)$, each pole repeated according to its multiplicity. Let $z\in D(R)$, if $f(z)\neq 0,\infty$, then*

$$
\log|f(z)|=\frac{1}{2\pi}\int_{0}^{2\pi}\frac{R^2-|z|^2}{|Re^{i\theta}-z|^2}\log\left|f\left(Re^{i\theta}\right)\right|d\theta-\sum_{j=1}^{p}\log\left|\frac{R^2-\overline{a_j}z}{R\left(z-a_j\right)}\right|+\sum_{j=1}^{q}\log\left|\frac{R^2-\overline{b_j}z}{R\left(z-b_j\right)}\right|.
$$

If $z=0$, then we obtain Jensen's formula

$$
\log|f\left(0\right)|=\frac{1}{2\pi}\int_{0}^{2\pi}\log\left|f\left(Re^{i\theta}\right)\right|d\theta-\sum_{j=1}^{p}\log\frac{R}{|a_j|}+\sum_{j=1}^{q}\log\frac{R}{|b_j|}. \tag{1.2}
$$

Before proceeding to the proof, we need the following lemma

Lemma 1.1.1 ([16])**.** *Let $z\in D(R)$ and $a=Re^{i\alpha}$, where $\alpha\in[0,2\pi[$. Then the integral*

$$
\int_{0}^{2\pi}\frac{R^2-|z|^2}{|Re^{i\theta}-z|^2}\log\left|Re^{i\theta}-a\right|d\theta
$$

converges and

$$\log|z-a| = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2-|z|^2}{|Re^{i\theta}-z|^2}\log\left|Re^{i\theta}-a\right|d\theta.$$

Proof. Since the function $\log|z-a|$ is not harmonic on $\overline{D(R)}$, we cannot apply Poisson's formula to it. Let $r\in\,]1,2[$, then the function $\log|z-ra|$ is harmonic on $\overline{D(R)}$. By applying Poisson's formula (1.1), we obtain

$$\log|z-ra| = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2-|z|^2}{|Re^{i\theta}-z|^2}\log\left|Re^{i\theta}-ra\right|d\theta. \tag{1.3}$$

For all $\theta\in[0,2\pi[\,\setminus\{\alpha\}$ we have

$$\lim_{r\to 1}\frac{R^2-|z|^2}{|Re^{i\theta}-z|^2}\log\left|Re^{i\theta}-ra\right| = \frac{R^2-|z|^2}{|Re^{i\theta}-z|^2}\log\left|Re^{i\theta}-a\right|.$$

Noting that for all $\theta\in[0,2\pi[\,\setminus\{\alpha\}$ and $r\in\,]1,2[$, we have

$$\log\left|Re^{i\theta}-ra\right| = \log\left|Re^{i\theta}-rRe^{i\alpha}\right| = \log R+\log\left|e^{i(\theta-\alpha)}-r\right|,$$

and

$$\left|\log\left|e^{i(\theta-\alpha)}-r\right|\right| \le \log 3-\log\left|\sin(\theta-\alpha)\right|,$$

we deduce that

$$\left|\frac{R^2-|z|^2}{|Re^{i\theta}-z|^2}\log\left|Re^{i\theta}-ra\right|\right| \le \frac{R+|z|}{R-|z|}\left(|\log R|+\log 3-\log|\sin(\theta-\alpha)|\right) = g(\theta),$$

since $\lim\limits_{\theta\to\alpha}|\theta-\alpha|^{\frac{1}{2}}g(\theta)=0$, then g is integrable on $[0,2\pi]$. Hence, the assertion follows by letting $r\to 1$ in (1.3) and using Lebesgue's dominated convergence theorem. □

Proof of Theorem 1.1.2. Let $a'_1,a'_2,\ldots,a'_{p'}$ and $b'_1,b'_2,\ldots,b'_{q'}$ respectively denote the zeros and the poles of f in the circle of radius R centred at the origin, each zero and pole repeated according to its multiplicity. Consider the function

$$F(z) = f(z)\frac{\displaystyle\prod_{j=1}^{p}\frac{R^2-\overline{a_j}z}{R(z-a_j)}\prod_{k=1}^{q'}(z-b'_k)}{\displaystyle\prod_{k=1}^{q}\frac{R^2-\overline{b_k}z}{R(z-b_k)}\prod_{j=1}^{p'}(z-a'_j)}. \tag{1.4}$$

Then F is analytic on $\overline{D(R)}$ and does not have any zeros or poles. Therefore, $\log|F(z)|$ is harmonic on $\overline{D(R)}$. Hence, by Poisson's formula (1.1), we have

$$\log|F(z)| = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2-|z|^2}{|Re^{i\theta}-z|^2}\log\left|F\left(Re^{i\theta}\right)\right| d\theta. \tag{1.5}$$

Since for all $w = Re^{i\theta},\ j = 1,2,\dots,p$ and $k = 1,2,\dots,q$

$$\left|\frac{R^2-\overline{a_j}w}{R\left(w-a_j\right)}\right| = \left|\frac{R^2-\overline{b_k}w}{R\left(w-b_k\right)}\right| = 1,$$

then

$$\log\left|F\left(Re^{i\theta}\right)\right| = \log\left|f\left(Re^{i\theta}\right)\right| + \sum_{k=1}^{q'}\log\left|Re^{i\theta}-b'_k\right| - \sum_{j=1}^{p'}\log\left|Re^{i\theta}-a'_j\right|.$$

By this and (1.5), we obtain

$$\begin{aligned}
\log|F(z)| &= \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2-|z|^2}{|Re^{i\theta}-z|^2}\log\left|F\left(Re^{i\theta}\right)\right| d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2-|z|^2}{|Re^{i\theta}-z|^2}\left(\log\left|f\left(Re^{i\theta}\right)\right| + \sum_{k=1}^{q'}\log\left|Re^{i\theta}-b'_k\right| - \sum_{j=1}^{p'}\log\left|Re^{i\theta}-a'_j\right|\right) d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2-|z|^2}{|Re^{i\theta}-z|^2}\log\left|f\left(Re^{i\theta}\right)\right| d\theta + \sum_{k=1}^{q'}\frac{1}{2\pi}\int_0^{2\pi} \frac{R^2-|z|^2}{|Re^{i\theta}-z|^2}\log\left|Re^{i\theta}-b'_k\right| d\theta \\
&\quad - \sum_{j=1}^{p'}\frac{1}{2\pi}\int_0^{2\pi} \frac{R^2-|z|^2}{|Re^{i\theta}-z|^2}\log\left|Re^{i\theta}-a'_j\right| d\theta.
\end{aligned}$$

By this and Lemma 1.1.1, we obtain

$$\log|F(z)| = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2-|z|^2}{|Re^{i\theta}-z|^2}\log\left|f\left(Re^{i\theta}\right)\right| d\theta + \sum_{k=1}^{q'}\log|z-b'_k| - \sum_{j=1}^{p'}\log\left|z-a'_j\right|. \tag{1.6}$$

By (1.4), we obtain

$$\begin{aligned}
\log|F(z)| = \log|f(z)| &+ \sum_{j=1}^{p}\log\left|\frac{R^2-\overline{a_j}z}{R\left(z-a_j\right)}\right| - \sum_{k=1}^{q}\log\left|\frac{R^2-\overline{b_k}z}{R\left(z-b_k\right)}\right| + \sum_{k=1}^{q'}\log|z-b'_k| \\
&- \sum_{j=1}^{p'}\log\left|z-a'_j\right|.
\end{aligned}$$

By this and (1.6), we obtain

$$\log|f(z)| = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2-|z|^2}{|Re^{i\theta}-z|^2}\log\left|f\left(Re^{i\theta}\right)\right| d\theta - \sum_{j=1}^{p}\log\left|\frac{R^2-\overline{a_j}z}{R\left(z-a_j\right)}\right| + \sum_{j=1}^{q}\log\left|\frac{R^2-\overline{b_j}z}{R\left(z-b_j\right)}\right|. \quad \square$$

The next corollary gives an analogue of Jensen's formula when 0 can be a zero or a pole of f. Before that we need to introduce additional notation.

Definition 1.1.1 ([8]). *Let $a \in \mathbb{C}$ and $f \not\equiv 0$ be a meromorphic function in an open neighbourhood of a. Then $f(z)$ has Laurent expansion about a of the form*

$$f(z) = \sum_{n=m}^{\infty} c_n (z-a)^n, \quad m \in \mathbb{Z}, \quad c_m \neq 0.$$

The integer m is called the order of $f(z)$ at a and is denoted by $\mathrm{ord}_a f$, and the non-zero complex number c_m is called the initial Laurent coefficient of $f(z)$ at a and is denoted by $\mathrm{ilc}\,(f,a)$.

Corollary 1.1.1 ([8, 31]). *Let $f \not\equiv 0$ be a meromorphic function on $\overline{D(R)}$. Let $a_1, a_2, \ldots, a_p$ denote the non-zero zeros of $f(z)$ in $D(R)$, each zero repeated according to its multiplicity, and let $b_1, b_2, \ldots, b_q$ denote the non-zero poles of $f(z)$ in $D(R)$, each pole repeated according to its multiplicity. Then*

$$\begin{aligned}
\log|\mathrm{ilc}\,(f,0)| &= \frac{1}{2\pi}\int_0^{2\pi}\log\left|f\left(Re^{i\theta}\right)\right| d\theta - \sum_{j=1}^{p}\log\frac{R}{|a_j|} + \sum_{j=1}^{q}\log\frac{R}{|b_j|} - (\mathrm{ord}_0 f)\log R \\
&= \frac{1}{2\pi}\int_0^{2\pi}\log\left|f\left(Re^{i\theta}\right)\right| d\theta - \sum_{\substack{\zeta\in D(R)\\ \zeta\neq 0}} (\mathrm{ord}_\zeta f)\log\frac{R}{|\zeta|} - (\mathrm{ord}_0 f)\log R. \qquad (1.7)
\end{aligned}$$

Proof. Let

$$g(z) = \frac{f(z)}{z^{\mathrm{ord}_0 f}}.$$

Then $g \not\equiv 0$ is meromorphic on $\overline{D(R)}$ and $g(0) = \mathrm{ilc}\,(f,0) \neq 0, \infty$. Moreover the zeros and poles of $g(z)$ in $\overline{D(R)}$ are exactly the non-zero zeros and poles of $f(z)$ in $\overline{D(R)}$, respectively. Hence, by applying Jensen's formula (1.2) to $g(z)$, we obtain

$$\begin{aligned}
\log|\mathrm{ilc}\,(f,0)| &= \frac{1}{2\pi}\int_0^{2\pi}\log\frac{\left|f\left(Re^{i\theta}\right)\right|}{R^{\mathrm{ord}_0 f}} d\theta - \sum_{j=1}^{p}\log\frac{R}{|a_j|} + \sum_{j=1}^{q}\log\frac{R}{|b_j|} \\
&= \frac{1}{2\pi}\int_0^{2\pi}\log\left|f\left(Re^{i\theta}\right)\right| d\theta - \sum_{j=1}^{p}\log\frac{R}{|a_j|} + \sum_{j=1}^{q}\log\frac{R}{|b_j|} - (\mathrm{ord}_0 f)\log R. \quad \square
\end{aligned}$$

Remark 1.1.1. *Equation (1.7) is also referred to as Jensen's formula.*

1.2 Nevanlinna Functions

Definition 1.2.1 ([15, 19]). *Let $x \geq 0$, we define*

$$\log^+ x = \max\{0, \log x\} = \begin{cases} \log x & \text{if } x > 1 \\ 0 & \text{if } 0 \leq x \leq 1. \end{cases}$$

This function satisfies the following properties for any non-negative real numbers $x, x_1, x_2, \ldots, x_n$

(1) $\log x = \log^+ x - \log^+ \dfrac{1}{x}$.

(2) $|\log x| = \log^+ x + \log^+ \dfrac{1}{x}$.

(3) $\log^+ \left(\sum\limits_{j=1}^{n} x_j \right) \leq \sum\limits_{j=1}^{n} \log^+ x_j + \log n$.

(4) $\log^+ \left(\prod\limits_{j=1}^{n} x_j \right) \leq \sum\limits_{j=1}^{n} \log^+ x_j$.

Definition 1.2.2 (Proximity function, [15, 19]). *Let $f(z)$ be a meromorphic function. The proximity function of $f(z)$ is defined by*

$$m(r, \infty, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f\left(re^{i\theta}\right) \right| d\theta,$$

and

$$m(r, a, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \frac{1}{f\left(re^{i\theta}\right) - a} \right| d\theta, \quad a \in \mathbb{C}, \quad f \not\equiv a.$$

$m(r, \infty, f)$ is usually denoted by $m(r, f)$. Hence, $m(r, a, f) = m\left(r, \dfrac{1}{f - a}\right)$.

Example 1.2.1. *Let $f(z) = e^z$. Then, we have*

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| e^{re^{i\theta}} \right| d\theta = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left(e^{r\cos\theta} \right) d\theta = \frac{r}{\pi}.$$

Definition 1.2.3 (Counting functions, [15, 19]). *Let $f(z)$ be a meromorphic function. Let $n(t, \infty, f)$ denote the number of poles of $f(z)$ in the closed disc $\overline{D(t)}$, each pole is counted according to its multiplicity, and let $\overline{n}(t, \infty, f)$ denote the number of poles of $f(z)$ in the closed disc $\overline{D(t)}$, each pole is counted only once. Let $a \in \mathbb{C}$, if $f \not\equiv a$, let $n(t, a, f)$ denote the number of a-points[1] of $f(z)$ in the closed disc $\overline{D(t)}$, each a-point is counted according to its multiplicity,*

[1] An a-point of a function f is a point z where $f(z) = a$.

and let $\overline{n}(t,a,f)$ denote the number of a-points of $f(z)$ in the closed disc $\overline{D(t)}$, each a-point is counted only once. The integrated counting function of $f(z)$ is defined by

$$N(r,f)=\int_0^r \frac{n(t,\infty,f)-n(0,\infty,f)}{t}dt+n(0,\infty,f)\log r,$$

and

$$N(r,a,f)=\int_0^r \frac{n(t,a,f)-n(0,a,f)}{t}dt+n(0,a,f)\log r,\quad a\in\mathbb{C},\quad f\not\equiv a.$$

Similarly, we define

$$\overline{N}(r,f)=\int_0^r \frac{\overline{n}(t,\infty,f)-\overline{n}(0,\infty,f)}{t}dt+\overline{n}(0,\infty,f)\log r,$$

and

$$\overline{N}(r,a,f)=\int_0^r \frac{\overline{n}(t,a,f)-\overline{n}(0,a,f)}{t}dt+\overline{n}(0,a,f)\log r,\quad a\in\mathbb{C},\quad f\not\equiv a.$$

We note that $N(r,a,f)=N\left(r,\dfrac{1}{f-a}\right)$ *and* $\overline{N}(r,a,f)=\overline{N}\left(r,\dfrac{1}{f-a}\right)$.

Definition 1.2.4 ([31])**.** *Let $f(z)$ be a meromorphic function. For $k\in\mathbb{N}$, let $n_{k)}(t,f)$ (resp. $n_{(k}(t,f)$) denote the number of poles of $f(z)$ with multiplicities not greater (resp. not less) than k in the closed disc $\overline{D(t)}$, each pole is counted according to its multiplicity. Similarly, let $\overline{n}_{k)}(t,f)$ (resp. $\overline{n}_{(k}(t,f)$) denote the number of poles of $f(z)$ with multiplicities not greater (resp. not less) than k in the closed disc $\overline{D(t)}$, each pole is counted only once. We define*

$$N_{k)}(r,f)=\int_0^r \frac{n_{k)}(t,f)-n_{k)}(0,f)}{t}dt+n_{k)}(0,f)\log r,$$

and

$$\overline{N}_{k)}(r,f)=\int_0^r \frac{\overline{n}_{k)}(t,f)-\overline{n}_{k)}(0,f)}{t}dt+\overline{n}_{k)}(0,f)\log r.$$

Similarly

$$N_{(k}(r,f)=\int_0^r \frac{n_{(k}(t,f)-n_{(k}(0,f)}{t}dt+n_{(k}(0,f)\log r,$$

and

$$\overline{N}_{(k}(r,f)=\int_0^r \frac{\overline{n}_{(k}(t,f)-\overline{n}_{(k}(0,f)}{t}dt+\overline{n}_{(k}(0,f)\log r.$$

We define

$$N_k(r,f)=N_{k)}(r,f)+k\overline{N}_{(k+1}(r,f).$$

Example 1.2.2. *Let* $f(z)=\dfrac{e^z}{z^4(z+1)^2(z-1)^3}$. *Then, for* $r\geq 1$ *we have*

$$n(r,f)=9 \quad and \quad n(0,f)=4,$$

$$n_{2)}(r,f)=2 \quad and \quad n_{2)}(0,f)=0,$$

$$n_{(3}(r,f)=7 \quad and \quad n_{(3}(0,f)=4.$$

Therefore,

$$N(r,f)=9\log r \quad , \quad N_{2)}(r,f)=2\log r \quad and \quad N_{(3}(r,f)=7\log r.$$

Similarly, we have

$$\overline{n}(r,f)=3 \quad and \quad \overline{n}(0,f)=1,$$

$$\overline{n}_{2)}(r,f)=1 \quad and \quad \overline{n}_{2)}(0,f)=0,$$

$$\overline{n}_{(3}(r,f)=2 \quad and \quad \overline{n}_{(3}(0,f)=1.$$

Hence,

$$\overline{N}(r,f)=3\log r \quad , \quad \overline{N}_{2)}(r,f)=\log r \quad and \quad \overline{N}_{(3}(r,f)=2\log r.$$

We deduce that

$$N_2(r,f)=6\log r.$$

Definition 1.2.5 (Nevanlinna characteristic function, [15, 19]). *Let $f(z)$ be a meromorphic function. The Nevanlinna characteristic function of $f(z)$ is defined by*

$$T\left(r, f\right) = m\left(r, f\right) + N\left(r, f\right),$$

and

$$T\left(r, a, f\right) = m\left(r, a, f\right) + N\left(r, a, f\right), \quad a \in \mathbb{C}, \quad f \not\equiv a.$$

We note that $T\left(r, a, f\right) = T\left(r, \dfrac{1}{f-a}\right)$.

Example 1.2.3. *Let $f(z) = e^z$. We have*

$$m\left(r, f\right) = \frac{r}{\pi} \quad \text{and} \quad N\left(r, f\right) = 0.$$

Hence,

$$T\left(r, f\right) = \frac{r}{\pi}.$$

Proposition 1.2.1 ([15, 19]). *Let $f, f_1, f_2, \ldots, f_p$, be meromorphic functions. Then*

(1) $m\left(r, \sum_{j=1}^{p} f_j\right) \leq \sum_{j=1}^{p} m\left(r, f_j\right) + \log p$.

(2) $m\left(r, \prod_{j=1}^{p} f_j\right) \leq \sum_{j=1}^{p} m\left(r, f_j\right)$.

(3) $N\left(r, \sum_{j=1}^{p} f_j\right) \leq \sum_{j=1}^{p} N\left(r, f_j\right)$.

(4) $N\left(r, \prod_{j=1}^{p} f_j\right) \leq \sum_{j=1}^{p} N\left(r, f_j\right)$.

(5) $T\left(r, \sum_{j=1}^{p} f_j\right) \leq \sum_{j=1}^{p} T\left(r, f_j\right) + \log p$.

(6) $T\left(r, \prod_{j=1}^{p} f_j\right) \leq \sum_{j=1}^{p} T\left(r, f_j\right)$.

(7) $T\left(r, f^n\right) = nT\left(r, f\right), \ \forall n \in \mathbb{N}$.

1.3 The First Main Theorem

Lemma 1.3.1 ([15, 19]). *Let $f \not\equiv 0$ be a meromorphic function. Let $a_1, a_2, \ldots, a_p$ denote the non-zero zeros of $f(z)$ in $D(r)$, each zero repeated according to its multiplicity, and let $b_1, b_2, \ldots, b_q$ denote the non-zero poles of $f(z)$ in $D(r)$, each pole repeated according to its multiplicity. Then*

$$N\left(r, \frac{1}{f}\right) = \sum_{j=1}^{p} \log \frac{r}{|a_j|} + n\left(0, 0, f\right) \log r,$$

and

$$N\left(r, f\right) = \sum_{j=1}^{q} \log \frac{r}{|b_j|} + n\left(0, \infty, f\right) \log r.$$

Proof. Without loss of generality, we suppose that $|a_1| \leq |a_2| \leq \cdots \leq |a_p|$. Denoting $r_j = |a_j|$ for $j = 1, 2, \ldots, p$, we obtain

$$\begin{aligned}
\sum_{j=1}^{p} \log \frac{r}{|a_j|} = \sum_{j=1}^{p} \log \frac{r}{r_j} & \\
&= p \log r - \sum_{j=1}^{p} \log r_j \\
&= \sum_{j=1}^{p-1} j \left(\log r_{j+1} - \log r_j\right) + p \left(\log r - \log r_p\right) \\
&= \sum_{j=1}^{p-1} \int_{r_j}^{r_{j+1}} \frac{j}{t} dt + \int_{r_p}^{r} \frac{p}{t} dt \\
&= \int_{0}^{r} \frac{n\left(t, 0, f\right) - n\left(0, 0, f\right)}{t} dt.
\end{aligned}$$

Similarly,

$$\sum_{j=1}^{q} \log \frac{r}{|b_j|} = \int_{0}^{r} \frac{n\left(t, \infty, f\right) - n\left(0, \infty, f\right)}{t} dt.$$

Hence, we obtain

$$N\left(r, \frac{1}{f}\right) = \sum_{j=1}^{p} \log \frac{r}{|a_j|} + n\left(0, 0, f\right) \log r,$$

and

$$N(r,f)=\sum_{j=1}^{q}\log\frac{r}{|b_j|}+n(0,\infty,f)\log r. \qquad \square$$

Theorem 1.3.1 (First main theorem, [15, 19]). *Let $a\in\mathbb{C}$, and let $f\not\equiv a$ be a meromorphic function. Then*

$$T\left(r,\frac{1}{f-a}\right)=T(r,f)-\log|\mathrm{ilc}(f-a,0)|+\varphi(r,a), \tag{1.8}$$

where

$$|\varphi(r,a)|\le\log^+|a|+\log 2.$$

Proof. First, suppose that $a=0$. Let $a_1,a_2,\dots,a_p$ denote the non-zero zeros of $f(z)$ in $D(r)$, each zero repeated according to its multiplicity, and let $b_1,b_2,\dots,b_q$ denote the non-zero poles of f in $D(r)$, each pole repeated according to its multiplicity. Then, by Jensen's formula (1.7), we have

$$\log|\mathrm{ilc}(f,0)|=\frac{1}{2\pi}\int_0^{2\pi}\log\left|f\left(re^{i\theta}\right)\right|d\theta-\sum_{j=1}^{p}\log\frac{r}{|a_j|}+\sum_{j=1}^{q}\log\frac{r}{|b_j|}-(\mathrm{ord}_0 f)\log r. \tag{1.9}$$

Noting that

$$\log x=\log^+ x-\log^+\frac{1}{x},$$

and that

$$\mathrm{ord}_0 f=n(0,0,f)-n(0,\infty,f),$$

equation (1.9) becomes

$$\begin{aligned}\log|\mathrm{ilc}(f,0)|&=\frac{1}{2\pi}\int_0^{2\pi}\log^+\left|f\left(re^{i\theta}\right)\right|d\theta-\frac{1}{2\pi}\int_0^{2\pi}\log^+\left|\frac{1}{f\left(re^{i\theta}\right)}\right|d\theta-\sum_{j=1}^{p}\log\frac{r}{|a_j|}-n(0,0,f)\log r\\&\quad+\sum_{j=1}^{q}\log\frac{r}{|b_j|}+n(0,\infty,f)\log r.\\&=m(r,f)-m\left(r,\frac{1}{f}\right)-\sum_{j=1}^{p}\log\frac{r}{|a_j|}-n(0,0,f)\log r+\sum_{j=1}^{q}\log\frac{r}{|b_j|}\\&\quad+n(0,\infty,f)\log r.\end{aligned}$$

By this and Lemma 1.3.1, we obtain

$$\log|\mathrm{ilc}(f,0)| = m(r,f) - m\left(r,\frac{1}{f}\right) - N\left(r,\frac{1}{f}\right) + N(r,f)$$
$$= T(r,f) - T\left(r,\frac{1}{f}\right).$$

Hence,

$$T\left(r,\frac{1}{f}\right) = T(r,f) - \log|\mathrm{ilc}(f,0)|,$$

which is equation (1.8) with $\varphi(r,0)\equiv 0$. Suppose now that $a\neq 0$. Then, from the first part of the proof, we have

$$T\left(r,\frac{1}{f-a}\right) = T(r,f-a) - \log|\mathrm{ilc}(f-a,0)|. \tag{1.10}$$

Let

$$\varphi(r,a) = T(r,f-a) - T(r,f).$$

Then, (1.10) becomes

$$T\left(r,\frac{1}{f-a}\right) = T(r,f) - \log|\mathrm{ilc}(f-a,0)| + \varphi(r,a),$$

and by Proposition 1.2.1(5), we see that $\varphi(r,a)$ satisfies

$$|\varphi(r,a)| \leq \log^{+}|a| + \log 2. \qquad \square$$

Remark 1.3.1. *The first main theorem may be expressed simply as*

$$T\left(r,\frac{1}{f-a}\right) = T(r,f) + O(1),\ \ (r\to\infty),$$

where the bounded error term $O(1)$ depends on a.

Proposition 1.3.1 ([15,19]). *Let $f(z)$ be a non-constant meromorphic function, and let $a,b,c,d\in\mathbb{C}$ such that $ad-bc\neq 0$. Define $g(z)$ as*

$$g(z) = \frac{af(z)+b}{cf(z)+d}.$$

Then we have

$$T(r,g) = T(r,f) + O(1).$$

Proof. If $c = 0$, then by Proposition 1.2.1, we obtain

$$T(r, g) = T\left(r, \frac{af + b}{d}\right) \leq T(r, f) + O(1).$$

If $c \neq 0$, we can write $g(z)$ as

$$g(z) = \frac{a}{c} + \frac{bc - ad}{c^2} \frac{1}{f(z) + \frac{d}{c}}.$$

Hence, by Proposition 1.2.1 and the first main theorem, we obtain

$$T(r, g) \leq T(r, f) + O(1).$$

In addition, we have $f(z) = \dfrac{-dg(z) + b}{cg(z) - a}$. By following similar steps as before, we prove that

$$T(r, f) \leq T(r, g) + O(1).$$

Therefore,

$$T(r, g) = T(r, f) + O(1).$$

□

1.4 Growth of Meromorphic Functions

Definition 1.4.1 ([16]). *For all $x \geq 0$, we define $\log_1 x = \log x$ and $\log_{p+1}^+ x = \log^+ \left(\log_p^+ x\right)$, where $p \in \mathbb{N}$.*

Definition 1.4.2 ([16]). *Let $f(z)$ be a meromorphic function and let $p \in \mathbb{N}$. The iterated p-order of $f(z)$ is defined by*

$$\rho_p(f) = \limsup_{r \to \infty} \frac{\log_p^+ T(r, f)}{\log r}.$$

The quantity $\rho_1(f)$ is called the order of $f(z)$, denoted by $\rho(f)$, and $\rho_2(f)$ is called the hyper-order of $f(z)$. The lower-order of $f(z)$ is defined by

$$\mu(f) = \liminf_{r \to \infty} \frac{\log^+ T(r, f)}{\log r}.$$

Example 1.4.1. *Let $f(z) = e^z$. Then*

$$\lim_{r \to \infty} \frac{\log^+ T(r, f)}{\log r} = \lim_{r \to \infty} \frac{\log^+ \frac{r}{\pi}}{\log r} = 1.$$

Therefore,

$$\rho(f) = \mu(f) = 1.$$

Remark 1.4.1. *If $f(z)$ is an entire function, then the p-order and the lower-order of $f(z)$ are also given, respectively, by*

$$\rho_p(f) = \limsup_{r\to\infty} \frac{\log^+_{p+1} M(r,f)}{\log r},$$
$$\mu(f) = \liminf_{r\to\infty} \frac{\log^+ \log^+ M(r,f)}{\log r},$$

where $M(r,f) = \max_{|z|=r} |f(z)|$.

Example 1.4.2. *Let* $f(z) = e^{e^z}$. *We have* $M(r,f) = e^{e^r}$. *Hence,*

$$\rho(f) = \mu(f) = \lim_{r\to\infty} \frac{\log^+ \log^+ e^{e^r}}{\log r} = \infty,$$

and

$$\rho_2(f) = \limsup_{r\to\infty} \frac{\log^+ \log^+ \log^+ e^{e^r}}{\log r} = 1.$$

Definition 1.4.3 ([20])**.** *Let $f(z)$ be an entire function of order $\rho(f)$ such that $0 < \rho(f) < \infty$. Then, the type of $f(z)$ is defined by*

$$\tau(f) = \limsup_{r\to\infty} \frac{\log^+ M(r,f)}{r^{\rho(f)}}.$$

Theorem 1.4.1 ([20])**.** *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *be an entire function. Then,*

$$\rho(f) = \limsup_{n\to\infty} \frac{n \log n}{\log \frac{1}{|a_n|}}.$$

If $0 < \rho(f) < \infty$, *then*

$$\tau(f) = \frac{1}{\rho(f)e} \limsup_{n\to\infty} \left(n \sqrt[n]{|a_n|^{\rho(f)}} \right).$$

Definition 1.4.4 ([17])**.** *Let $f(z)$ be a meromorphic function and let $p \in \mathbb{N}$. The iterated p-exponent of convergence of zeros of $f(z)$ is defined by*

$$\lambda_p(f) = \limsup_{r\to\infty} \frac{\log^+_p N\left(r, \frac{1}{f}\right)}{\log r},$$

and the iterated p-exponent of convergence of distinct zeros of $f(z)$ is defined by

$$\overline{\lambda}_p(f) = \limsup_{r\to\infty} \frac{\log^+_p \overline{N}\left(r, \frac{1}{f}\right)}{\log r}.$$

For $p = 1$, $\lambda_1(f)$ *is called the exponent of convergence of zeros of* f, *denoted by* $\lambda(f)$. *Similarly,* $\overline{\lambda}_1(f)$ *is called the exponent of convergence of distinct zeros of* f, *denoted by* $\overline{\lambda}(f)$.

Example 1.4.3. *Let* $f(z) = e^z$. *Since* e^z *has no zeros, it follows that* $N\left(r, \frac{1}{f}\right) = 0$. *Hence,* $\lambda(f) = 0$. *On the other hand, since* $N\left(r, \frac{1}{f-1}\right) = 2\left\lfloor \dfrac{r}{2\pi} \right\rfloor$, *we have* $\lambda(f-1) = 1$.

1.5 The Second Main Theorem

Definition 1.5.1 ([31]). *Let* $f(z)$ *be a meromorphic function. We denote by* $S(r,f)$ *any quantity satisfying* $S(r,f) = o(T(r,f))$ *as* $r \to \infty$ *outside of a possible exceptional set* $E \subset [0, +\infty[$ *of finite linear measure, i.e.,* $S(r,f) = o(T(r,f))$ *as* $r \to \infty$, $r \notin E$, *where* $\int_E dt < \infty$. *We say that a meromorphic function* $g(z)$ *is a small function of* $f(z)$ *if* $T(r,g) = S(r,f)$.

Theorem 1.5.1 ([19]). *Suppose that* $f(z)$ *is a non-constant meromorphic function and let* $R(z,f) = \dfrac{P(z,f)}{Q(z,f)}$, *where*

$$P(z,f) = \sum_{j=0}^{p} a_j(z) f^j \quad \text{and} \quad Q(z,f) = \sum_{j=0}^{q} b_j(z) f^j$$

are two relatively prime polynomials in $f(z)$ *with small meromorphic coefficients of* $f(z)$ *and* $a_p \not\equiv 0$, $b_q \not\equiv 0$. *Then*

$$T(r, R(z,f)) = \max\{p,q\} T(r,f) + S(r,f).$$

Theorem 1.5.2 (Lemma on the logarithmic derivative, [15, 31]). *Let* $f(z)$ *be a non-constant meromorphic function. If* $f(z)$ *is of finite order, then*

$$m\left(r, \frac{f'}{f}\right) = O(\log r), \ (r \to \infty).$$

If $f(z)$ *is of infinite order, then*

$$m\left(r, \frac{f'}{f}\right) = O(\log(rT(r,f))),$$

as $r \to \infty$ *outside of a possible exceptional set of finite linear measure.*

Remark 1.5.1. *The lemma on the logarithmic derivative can be stated as*

$$m\left(r, \frac{f'}{f}\right) = S(r,f).$$

Theorem 1.5.3 (Second main theorem, [19]). *Let $f(z)$ be a non-constant meromorphic function. Let $a_1, a_2, \ldots, a_q \in \mathbb{C}$, where $q \geq 2$, be distinct points. Then*

$$m(r,f) + \sum_{j=1}^{q} m\left(r, \frac{1}{f-a_j}\right) \leq 2T(r,f) - N'(r,f) + S(r,f), \tag{1.11}$$

where

$$N'(r,f) = 2N(r,f) - N(r,f') + N\left(r, \frac{1}{f'}\right). \tag{1.12}$$

Proof. Let

$$P(z,f) = \prod_{j=1}^{q} (f-a_j).$$

Then there exist constants $A_1, A_2, \ldots, A_q \in \mathbb{C}$ such that

$$\frac{1}{P(z,f)} = \sum_{j=1}^{q} \frac{A_j}{f-a_j}.$$

Therefore,

$$\begin{aligned} T\left(r, \frac{1}{P(z,f)}\right) &= m\left(r, \frac{1}{P(z,f)}\right) + N\left(r, \frac{1}{P(z,f)}\right) \\ &= m\left(r, \sum_{j=1}^{q} \frac{A_j}{f-a_j}\right) + N\left(r, \prod_{j=1}^{q} \frac{1}{f-a_j}\right). \end{aligned} \tag{1.13}$$

We have

$$\begin{aligned} m\left(r, \sum_{j=1}^{q} \frac{A_j}{f-a_j}\right) &= m\left(r, \frac{1}{f'} \sum_{j=1}^{q} A_j \frac{f'}{f-a_j}\right) \\ &\leq m\left(r, \frac{1}{f'}\right) + \sum_{j=1}^{q} m\left(r, \frac{f'}{f-a_j}\right) + O(1) \\ &= m\left(r, \frac{1}{f'}\right) + S(r,f), \end{aligned} \tag{1.14}$$

and by the first main theorem

$$\begin{aligned} m\left(r, \frac{1}{f'}\right) &= T\left(r, \frac{1}{f'}\right) - N\left(r, \frac{1}{f'}\right) \\ &= T(r,f') - N\left(r, \frac{1}{f'}\right) + O(1) \\ &= m(r,f') + N(r,f') - N\left(r, \frac{1}{f'}\right) + O(1), \end{aligned}$$

it follows that

$$\begin{aligned}
m\left(r,\frac{1}{f'}\right) &= m\left(r,f\frac{f'}{f}\right)+N\left(r,f'\right)-N\left(r,\frac{1}{f'}\right)+O\left(1\right)\\
&\leq m\left(r,f\right)+m\left(r,\frac{f'}{f}\right)+N\left(r,f'\right)-N\left(r,\frac{1}{f'}\right)+O\left(1\right)\\
&= m\left(r,f\right)+N\left(r,f'\right)-N\left(r,\frac{1}{f'}\right)+S\left(r,f\right)\\
&= T\left(r,f\right)-N\left(r,f\right)+N\left(r,f'\right)-N\left(r,\frac{1}{f'}\right)+S\left(r,f\right). \qquad (1.15)
\end{aligned}$$

On the other hand, by Proposition 1.2.1(4), we have

$$N\left(r,\prod_{j=1}^{q}\frac{1}{f-a_j}\right)\leq\sum_{j=1}^{q}N\left(r,\frac{1}{f-a_j}\right). \qquad (1.16)$$

By (1.13), (1.14), (1.15) and (1.16), it follows that

$$\begin{aligned}
T\left(r,\frac{1}{P\left(z,f\right)}\right) &\leq T\left(r,f\right)-N\left(r,f\right)+N\left(r,f'\right)-N\left(r,\frac{1}{f'}\right)+\sum_{j=1}^{q}N\left(r,\frac{1}{f-a_j}\right)\\
&\quad +S\left(r,f\right). \qquad (1.17)
\end{aligned}$$

By the first main theorem and Theorem 1.5.1, we have

$$\begin{aligned}
T\left(r,\frac{1}{P\left(z,f\right)}\right) &= T\left(r,P\left(z,f\right)\right)+O\left(1\right)\\
&= qT\left(r,f\right)+S\left(r,f\right)\\
&= \sum_{j=1}^{q}T\left(r,\frac{1}{f-a_j}\right)+S\left(r,f\right)\\
&= \sum_{j=1}^{q}m\left(r,\frac{1}{f-a_j}\right)+\sum_{j=1}^{q}N\left(r,\frac{1}{f-a_j}\right)+S\left(r,f\right).
\end{aligned}$$

Substituting this into (1.17), we obtain

$$\sum_{j=1}^{q}m\left(r,\frac{1}{f-a_j}\right)\leq T\left(r,f\right)-N\left(r,f\right)+N\left(r,f'\right)-N\left(r,\frac{1}{f'}\right)+S\left(r,f\right),$$

and by adding $m\left(r,f\right)$ to both sides of the last inequality, we obtain

$$\begin{aligned}
m\left(r,f\right)+\sum_{j=1}^{q}m\left(r,\frac{1}{f-a_j}\right) &\leq m\left(r,f\right)+T\left(r,f\right)-N\left(r,f\right)+N\left(r,f'\right)-N\left(r,\frac{1}{f'}\right)+S\left(r,f\right)\\
&= 2T\left(r,f\right)-2N\left(r,f\right)+N\left(r,f'\right)-N\left(r,\frac{1}{f'}\right)+S\left(r,f\right).
\end{aligned}$$

Hence,

$$m(r,f)+\sum_{j=1}^{q} m\left(r,\frac{1}{f-a_j}\right)\le 2T(r,f)-N'(r,f)+S(r,f). \qquad \square$$

Corollary 1.5.1 ([19])**.** *Let $f(z)$ be a non-constant meromorphic function. Let $a_1,a_2,\dots,a_q\in\mathbb{C}$, where $q\ge 2$, be distinct points. Then*

$$(q-1)T(r,f)\le N(r,f)+\sum_{j=1}^{q} N\left(r,\frac{1}{f-a_j}\right)-N'(r,f)+S(r,f),$$

where $N'(r,f)$ is defined by (1.12).

Proof. By the first main theorem, we have for $j=1,2,\dots,q$

$$m\left(r,\frac{1}{f-a_j}\right)=T(r,f)-N\left(r,\frac{1}{f-a_j}\right)+O(1).$$

In addition, we have

$$m(r,f)=T(r,f)-N(r,f).$$

Substituting both equations into (1.11) yields the result. $\square$

Corollary 1.5.2 ([19])**.** *Let $f(z)$ be a non-constant meromorphic function. Let $a_1,a_2,\dots,a_q\in\mathbb{C}$, where $q\ge 2$, be distinct points. Then*

$$(q-1)T(r,f)\le \overline{N}(r,f)+\sum_{j=1}^{q}\overline{N}\left(r,\frac{1}{f-a_j}\right)+S(r,f). \tag{1.18}$$

Proof. By Corollary 1.5.1, we have

$$(q-1)T(r,f)\le N(r,f)+\sum_{j=1}^{q} N\left(r,\frac{1}{f-a_j}\right)-N'(r,f)+S(r,f), \tag{1.19}$$

where $N'(r,f)$ is defined by (1.12). In addition,

$$N'(r,f)=\int_0^r \frac{n'(t,f)-n'(0,f)}{t}dt+n'(0,f)\log r,$$

where

$$n'(t,f)=2n(t,f)-n(t,f')+n\left(t,\frac{1}{f'}\right).$$

If f has a pole of order $p\ge 1$ at z_0 in $\overline{D(t)}$, then z_0 is counted $p-1$ times by $n'(t,f)$. Similarly, if $f-a_j$ has a zero of order $k\ge 1$ at z_0 in $\overline{D(t)}$ for some $j=1,2,\dots,q$, then z_0 is counted

$k-1$ times by $n'(t,f)$. Hence,

$$N(r,f)+\sum_{j=1}^{q} N\left(r,\frac{1}{f-a_j}\right)-N'(r,f)\leq \overline{N}(r,f)+\sum_{j=1}^{q}\overline{N}\left(r,\frac{1}{f-a_j}\right),$$

by this inequality and (1.19), we obtain (1.18). □

Definition 1.5.2 ([31]). *Let $f(z)$ and $g(z)$ be meromorphic functions, and let $a\in\mathbb{C}$. If $f(z)-a$ and $g(z)-a$ have the same zeros, we say that $f(z)$ and $g(z)$ share the value a IM (ignoring multiplicities). If $f(z)-a$ and $g(z)-a$ have the same zeros with the same multiplicities, we say that $f(z)$ and $g(z)$ share the value a CM (counting multiplicities).*
Similarly, we say that $f(z)$ and $g(z)$ share the value ∞ IM (ignoring multiplicities) if $f(z)$ and $g(z)$ have the same poles, and we say that $f(z)$ and $g(z)$ share the value ∞ CM (counting multiplicities) if $f(z)$ and $g(z)$ have the same poles with the same multiplicities.

Example 1.5.1.

(1) *Let*

$$f(z)=z^2(z+1)(z-1)^3e^z,\quad g(z)=z^2(z+1)(z-1)^3e^{e^z},\quad h(z)=z(z+1)(z-1)e^{z^2}.$$

The functions $f(z)$ and $g(z)$ share the value 0 CM. On the other hand, the functions $f(z)$ (or $g(z)$) and $h(z)$ share the value 0 IM, but not CM.

(2) *The functions e^z and e^{-z} share the values $0,-1,1,\infty$ CM.*

Definition 1.5.3. *Let $f(z)$, $g(z)$ and $a(z)$ be meromorphic functions. We say that $f(z)$ and $g(z)$ share $a(z)$ CM (resp. IM) if $f(z)-a(z)$ and $g(z)-a(z)$ share the value 0 CM (resp. IM). If the zeros of $f(z)-a(z)$ are also zeros of $g(z)-a(z)$, we write $f=a\Rightarrow g=a$.*

Theorem 1.5.4 (Nevanlinna's five-value theorem, [26]). *If two non-constant meromorphic functions $f(z)$ and $g(z)$ share five distinct values IM, then $f\equiv g$.*

Proof. Suppose that $f(z)$ and $g(z)$ share five distinct values a_1,a_2,a_3,a_4,a_5 IM. First, assume that all these values are finite, then by Corollary 1.5.2, we have

$$3T(r,f)\leq\sum_{j=1}^{5}\overline{N}\left(r,\frac{1}{f-a_j}\right)+S(r,f),\tag{1.20}$$

and

$$3T(r,g)\leq\sum_{j=1}^{5}\overline{N}\left(r,\frac{1}{g-a_j}\right)+S(r,g).\tag{1.21}$$

Suppose that $f \not\equiv g$. Since f and g share the values a_1, a_2, a_3, a_4, a_5 IM, we have

$$\begin{aligned}\sum_{j=1}^{5} \overline{N}\left(r, \frac{1}{f-a_j}\right) = \sum_{j=1}^{5} \overline{N}\left(r, \frac{1}{g-a_j}\right) &\le N\left(r, \frac{1}{f-g}\right)\\ &\le T\left(r, f-g\right) + O\left(1\right)\\ &\le T\left(r, f\right) + T\left(r, g\right) + O\left(1\right).\end{aligned}$$

From this, along with (1.20) and (1.21), we obtain

$$3\left(T\left(r, f\right) + T\left(r, g\right)\right) \le 2\left(T\left(r, f\right) + T\left(r, g\right)\right) + S\left(r, f\right) + S\left(r, g\right).$$

Hence, it follows that

$$T\left(r, f\right) + T\left(r, g\right) \le S\left(r, f\right) + S\left(r, g\right),$$

which is a contradiction. Therefore, $f \equiv g$.
Assume that one of the values a_1, a_2, a_3, a_4, a_5 is infinite. Without loss of generality, assume that $a_5 = \infty$. Let a be any finite value different from a_1, a_2, a_3, a_4. Define

$$F(z) = \frac{1}{f(z) - a} \quad \text{and} \quad G(z) = \frac{1}{g(z) - a},$$

and

$$b_j = \frac{1}{a_j - a} \text{ for } j = 1, 2, 3, 4, \quad \text{and} \quad b_5 = 0.$$

Then $F(z)$ and $G(z)$ share the values b_1, b_2, b_3, b_4, b_5 IM. From the first part of the proof, we deduce that $F \equiv G$. Therefore, $f \equiv g$. □

Definition 1.5.4. *Let $f(z)$ be a non-constant meromorphic function, and let $\varphi(z)$ be a meromorphic function such that $T\left(r, \varphi\right) = o\left(T\left(r, f\right)\right)$ as $r \to \infty$. The deficiency of $\varphi(z)$ with respect to $f(z)$ is defined by*

$$\delta\left(\varphi, f\right) = \liminf_{r\to\infty} \frac{m\left(r, \frac{1}{f-\varphi}\right)}{T\left(r, f\right)} = 1 - \limsup_{r\to\infty} \frac{N\left(r, \frac{1}{f-\varphi}\right)}{T\left(r, f\right)}.$$

We also define

$$\delta_{m)}\left(\varphi, f\right) = 1 - \limsup_{r\to\infty} \frac{N_{m)}\left(r, \frac{1}{f-\varphi}\right)}{T\left(r, f\right)}, \quad \delta_{m}\left(\varphi, f\right) = 1 - \limsup_{r\to\infty} \frac{N_{m}\left(r, \frac{1}{f-\varphi}\right)}{T\left(r, f\right)},$$

and

$$\Theta_{m)}\left(\varphi,f\right)=1-\limsup_{r\to\infty}\frac{\overline{N}_{m)}\left(r,\frac{1}{f-\varphi}\right)}{T\left(r,f\right)},\quad \Theta\left(\infty,f\right)=1-\limsup_{r\to\infty}\frac{\overline{N}\left(r,f\right)}{T\left(r,f\right)}.$$

Obviously, we have

$$0\leq\delta\left(\varphi,f\right)\leq\delta_{m}\left(\varphi,f\right)\leq\delta_{m)}\left(\varphi,f\right)\leq\Theta_{m)}\left(\varphi,f\right)\leq 1.$$

Chapter 2

Uniqueness Theorems of Meromorphic Functions and Brück's Conjecture

This chapter focuses on some important uniqueness theorems of meromorphic functions, with particular attention to Brück's conjecture. In the first section, we review several key theorems related to combinations of meromorphic functions, along with their detailed proofs, which are needed later and are taken from [27,31]. In the second section, we focus on Brück's conjecture, providing an overview of the conjecture itself and summarizing significant results related to it.

2.1 Combinations of Meromorphic Functions

Definition 2.1.1 ([31]). *Let $f_1, f_2, \ldots, f_n$ be meromorphic functions. The determinant*

$$W(f_1, f_2, \ldots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

is called the Wronskian of $f_1, f_2, \ldots, f_n$.

Theorem 2.1.1 ([31]). *Let $f_1, f_2, \ldots, f_n$, where $n \geq 2$, be linearly independent meromorphic functions such that*

$$\sum_{j=1}^{n} f_j \equiv 1. \tag{2.1}$$

Then for all $j = 1, 2, \ldots, n$, we have

$$T(r, f_j) \le \sum_{k=1}^{n} N\left(r, \frac{1}{f_k}\right) + N(r, f_j) + N(r, W)$$
$$- \sum_{k=1}^{n} N(r, f_k) - N\left(r, \frac{1}{W}\right) + S(r),$$

where W is the Wronskian $W(f_1, f_2, \ldots, f_n)$, and

$$S(r) = o(T(r)), \ (r \to \infty, \ r \notin E),$$

here,

$$T(r) = \max_{1 \le k \le n} \{T(r, f_k)\},$$

and E is a set of finite linear measure.

Proof. By taking derivatives on both sides of (2.1), we obtain for $k = 1, 2, \ldots, n-1$

$$\sum_{j=1}^{n} f_j^{(k)} \equiv 0. \tag{2.2}$$

Let $j = 1, 2, \ldots, n$. By adding to the jth column the other columns and using (2.1) and (2.2), we obtain

$$W = W_j, \tag{2.3}$$

where W_j is the algebraic cofactor of f_j in W. Let

$$\Delta = \frac{W}{f_1 f_2 \cdots f_n} = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \frac{f_1'}{f_1} & \frac{f_2'}{f_2} & \cdots & \frac{f_n'}{f_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{f_1^{(n-1)}}{f_1} & \frac{f_2^{(n-1)}}{f_2} & \cdots & \frac{f_n^{(n-1)}}{f_n} \end{vmatrix}.$$

From this, we see that

$$\Delta_j = \frac{W_j}{f_1 f_2 \cdots f_{j-1} f_{j+1} \cdots f_n},$$

where Δ_j is the algebraic cofactor of the element at the jth column and the first row in Δ. From this and (2.3), we have

$$f_j = \frac{\Delta_j}{\Delta},$$

and hence we obtain

$$\begin{aligned} m(r, f_j) &\le m(r, \Delta_j) + m\left(r, \frac{1}{\Delta}\right) \\ &= m(r, \Delta_j) + T\left(r, \frac{1}{\Delta}\right) - N\left(r, \frac{1}{\Delta}\right) \\ &= m(r, \Delta_j) + m(r, \Delta) + N(r, \Delta) - N\left(r, \frac{1}{\Delta}\right) + O(1). \end{aligned} \tag{2.4}$$

From

$$\Delta = \frac{W}{f_1 f_2 \cdots f_n},$$

we have

$$N(r, \Delta) - N\left(r, \frac{1}{\Delta}\right) = \sum_{k=1}^{n} N\left(r, \frac{1}{f_k}\right) - \sum_{k=1}^{n} N(r, f_k) + N(r, W) - N\left(r, \frac{1}{W}\right). \tag{2.5}$$

We note that for all $j = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, n-1$

$$m\left(r, \frac{f_j^{(k)}}{f_j}\right) = S(r, f_j) = S(r).$$

Therefore,

$$m(r, \Delta) = S(r) \quad \text{and} \quad m(r, \Delta_j) = S(r)$$

for all $j = 1, 2, \ldots, n$. From this, (2.4) and (2.5), we obtain

$$\begin{aligned} T(r, f_j) \le &\sum_{k=1}^{n} N\left(r, \frac{1}{f_k}\right) + N(r, f_j) + N(r, W) \\ &- \sum_{k=1}^{n} N(r, f_k) - N\left(r, \frac{1}{W}\right) + S(r). \end{aligned}$$ □

Theorem 2.1.2 ([31])**.** *Let $f_1, f_2, \ldots, f_n$, where $n \ge 2$, be meromorphic functions satisfying:*

(1) $\sum_{j=1}^{n} C_j f_j \equiv 0$, *where $C_1, C_2, \ldots, C_n \in \mathbb{C}$.*

(2) $f_j \not\equiv 0$ *for $j = 1, 2, \ldots, n$, and $\frac{f_j}{f_k}$ is not constant for $1 \le j < k \le n$.*

(3) $\sum_{j=1}^{n} \left(N(r, f_j) + N\left(r, \frac{1}{f_j}\right)\right) = o(\omega(r)),\ (r \to \infty,\ r \notin E)$,
here, $\omega(r) = \min_{1 \le j < k \le n} \left\{ T\left(r, \frac{f_j}{f_k}\right)\right\}$ and E is a set of finite linear measure.

Then $C_j = 0$ for all $j = 1, 2, \ldots, n$.

Proof. We prove the theorem by induction on n. First, suppose that $n = 2$. Then

$$C_1 f_1 + C_2 f_2 \equiv 0.$$

Assume that $C_1 \neq 0$. Then, from the last equation, we obtain

$$\frac{f_1}{f_2} \equiv -\frac{C_2}{C_1},$$

which contradicts condition (2). Hence, $C_1 = 0$, and consequently, $C_2 = 0$. Thus, Theorem 2.1.2 holds for $n = 2$.

Suppose that Theorem 2.1.2 is true for $n = m \geq 2$ and we will now prove it for $n = m+1$. Suppose that $f_1, f_2, \ldots, f_{m+1}$ are meromorphic functions satisfying the conditions of Theorem 2.1.2, then we have

$$\sum_{j=1}^{m+1} C_j f_j \equiv 0. \tag{2.6}$$

Suppose that $C_j \neq 0$ for all $j = 1, 2, \ldots, m+1$. For $j = 1, 2, \ldots, m$, let

$$g_j = -\frac{C_j f_j}{C_{m+1} f_{m+1}}. \tag{2.7}$$

From (2.6), we have

$$\sum_{j=1}^{m} g_j \equiv 1.$$

Suppose there exist scalars $a_1, a_2, \ldots, a_m$, such that

$$\sum_{j=1}^{m} a_j g_j \equiv 0.$$

From this and (2.7), we see that

$$\sum_{j=1}^{m} a_j C_j f_j \equiv 0.$$

Since Theorem 2.1.2 is supposed true for $n = m$, then we deduce that $a_j C_j = 0$ for all $j = 1, 2, \ldots, m$. Since $C_1, C_2, \ldots, C_m$ are all non-zero, we deduce that $a_1 = a_2 = \cdots = a_m = 0$. Hence, $g_1, g_2, \ldots, g_m$ are linearly independent.

Let $T(r) = \max_{1 \leq k \leq m} \{T(r, g_k)\}$. From (2.7), we have

$$N\left(r, \frac{1}{g_j}\right) + N(r, g_j) \leq N\left(r, \frac{1}{f_j}\right) + N(r, f_j) + N\left(r, \frac{1}{f_{m+1}}\right) + N(r, f_{m+1})$$

for all $j = 1, 2, \ldots, m$. From this and condition (3), we obtain

$$N\left(r, \frac{1}{g_j}\right) + N(r, g_j) = o\left(T\left(r, \frac{f_j}{f_{m+1}}\right)\right) = o\left(T(r, g_j)\right), \ (r \to \infty, \ r \notin E)$$

for all $j = 1, 2, \ldots, m$. From this, we deduce that

$$N\left(r, \frac{1}{g_j}\right) + N(r, g_j) = S(r) \tag{2.8}$$

for all $j = 1, 2, \ldots, m$, where $S(r) = o(T(r)), \ (r \to \infty, \ r \notin E)$. By applying Theorem 2.1.1 to $g_1, g_2, \ldots, g_m$ and using (2.8), we obtain

$$T(r, g_k) = S(r)$$

for all $k = 1, 2, \ldots, m$. Hence, $T(r) = S(r)$, which is a contradiction. Therefore, at least one of $C_1, C_2, \ldots, C_{m+1}$ is zero. Without loss of generality, suppose that $C_{m+1} = 0$. From (2.6), we have

$$\sum_{j=1}^{m} C_j f_j \equiv 0.$$

Hence, $f_1, f_2, \ldots, f_m$ satisfy the conditions of Theorem 2.1.2 for $n = m$ which is true by the induction hypothesis. Therefore, $C_j = 0$ for all $j = 1, 2, \ldots, m$. □

Theorem 2.1.3 ([31]). *Let $f_1, f_2, \ldots, f_n$ be meromorphic functions and $g_1, g_2, \ldots, g_n$ be entire functions, where $n \geq 2$, satisfying:*

(1) $\sum_{j=1}^{n} f_j e^{g_j} \equiv 0.$

(2) *$g_j - g_k$ is not constant for $1 \leq j < k \leq n$.*

(3) $T(r, f_j) = o\left(T\left(r, e^{g_h - g_k}\right)\right), \ (r \to \infty, \ r \notin E)$
for all $j = 1, 2, \ldots, n$ and $1 \leq h < k \leq n$, where E is a set of finite linear measure.

Then $f_j \equiv 0$ for all $j = 1, 2, \ldots, n$.

Proof. We prove the theorem by induction on n. Suppose that $n = 2$. Then

$$f_1 e^{g_1} + f_2 e^{g_2} \equiv 0.$$

Assume that $f_1 \not\equiv 0$. Then, from the last equation, we obtain

$$e^{g_1 - g_2} = -\frac{f_2}{f_1}.$$

By this and condition (3), we obtain

$$\begin{aligned} T\left(r, e^{g_1-g_2}\right) &= T\left(r, -\frac{f_2}{f_1}\right) \\ &\leq T\left(r, f_2\right) + T\left(r, f_1\right) + O\left(1\right) \\ &= o\left(T\left(r, e^{g_1-g_2}\right)\right), \ \left(r \to \infty, \ r \notin E\right), \end{aligned}$$

which is a contradiction. Hence, $f_1 \equiv 0$, and then $f_2 \equiv 0$. Therefore, Theorem 2.1.3 is true for $n = 2$.

Assume that Theorem 2.1.3 is true for $n = m \geq 2$ and we will now prove it for $n = m + 1$. Suppose that $f_1, f_2, \ldots, f_{m+1}$ and $g_1, g_2, \ldots, g_{m+1}$ satisfy the conditions of Theorem 2.1.3. Then we have

$$\sum_{j=1}^{m+1} f_j e^{g_j} \equiv 0. \tag{2.9}$$

Suppose that $f_j \not\equiv 0$ for all $j = 1, 2, \ldots, m + 1$, and let

$$F_j = f_j e^{g_j}, \quad C_j = 1. \tag{2.10}$$

From (2.9), we obtain

$$\sum_{j=1}^{m+1} C_j F_j \equiv 0.$$

Obviously, $F_j \not\equiv 0$ for all $j = 1, 2, \ldots, m + 1$, and we can easily prove that $\dfrac{F_j}{F_k}$ is not constant for $1 \leq j < k \leq m + 1$. From condition (3), we have

$$\begin{aligned} N\left(r, F_j\right) + N\left(r, \frac{1}{F_j}\right) &= N\left(r, f_j\right) + N\left(r, \frac{1}{f_j}\right) \\ &\leq 2T\left(r, f_j\right) + O\left(1\right) \\ &= o\left(T\left(r, e^{g_h-g_k}\right)\right), \ \left(r \to \infty, \ r \notin E\right) \end{aligned} \tag{2.11}$$

for all $j = 1, 2, \ldots, m + 1$ and $1 \leq h < k \leq m + 1$. For $1 \leq h < k \leq m + 1$, we have

$$\frac{F_h}{F_k} = \frac{f_h}{f_k} e^{g_h - g_k}.$$

Thus,

$$\begin{aligned}
T\left(r, e^{g_h - g_k}\right) &= T\left(r, \frac{f_k}{f_h}\frac{F_h}{F_k}\right) \\
&\leq T\left(r, f_k\right) + T\left(r, f_h\right) + T\left(r, \frac{F_h}{F_k}\right) + O\left(1\right) \\
&= T\left(r, \frac{F_h}{F_k}\right) + o\left(T\left(r, e^{g_h - g_k}\right)\right), \ \left(r \to \infty, \ r \notin E\right).
\end{aligned}$$

Hence,

$$T\left(r, e^{g_h - g_k}\right) = O\left(T\left(r, \frac{F_h}{F_k}\right)\right), \ \left(r \to \infty, \ r \notin E\right) \tag{2.12}$$

for all $1 \leq h < k \leq m+1$. From (2.11) and (2.12), we obtain for $j = 1, 2, \ldots, m+1$

$$N\left(r, F_j\right) + N\left(r, \frac{1}{F_j}\right) = o\left(\omega\left(r\right)\right), \ \left(r \to \infty, \ r \notin E\right),$$

here, $\omega\left(r\right) = \min\limits_{1 \leq h < k \leq m+1}\left\{T\left(r, \frac{F_h}{F_k}\right)\right\}$. We see that $F_1, F_2, \ldots, F_{m+1}$ satisfy the conditions of Theorem 2.1.2. Hence, $C_j = 0$ for all $j = 1, 2, \ldots, m+1$ which contradicts (2.10). Therefore, at least one of $f_1, f_2, \ldots, f_{m+1}$ is identically zero. Without loss of generality, suppose that $f_{m+1} \equiv 0$. From (2.9), we have

$$\sum_{j=1}^{m} f_j e^{g_j} \equiv 0.$$

Hence, $f_1, f_2, \ldots, f_m$ and $g_1, g_2, \ldots, g_m$ satisfy the conditions of Theorem 2.1.3 for $n = m$. Therefore, $f_j \equiv 0$ for all $j = 1, 2, \ldots, m$. □

Corollary 2.1.1 ([31]). *Let $f_1, f_2, \ldots, f_{n+1}$ be meromorphic functions and $g_1, g_2, \ldots, g_n$ be entire functions, where $n \geq 1$, satisfying:*

(1) $\sum\limits_{j=1}^{n} f_j e^{g_j} \equiv f_{n+1}$.

(2) *$g_j - g_k$ and g_h are not constant for $1 \leq j < k \leq n$ and $h = 1, 2, \ldots, n$.*

(3) *$T\left(r, f_j\right) = o\left(T\left(r, e^{g_h - g_k}\right)\right), \ \left(r \to \infty, \ r \notin E\right)$ for all $j = 1, 2, \ldots, n+1$ and $1 \leq h < k \leq n$, $T\left(r, f_j\right) = o\left(T\left(r, e^{g_k}\right)\right), \ \left(r \to \infty, \ r \notin E\right)$ for all $j = 1, 2, \ldots, n+1$ and $k = 1, 2, \ldots, n$, where E is a set of finite linear measure.*

Then $f_j \equiv 0$ for all $j = 1, 2, \ldots, n+1$.

Proof. We can express condition (1) as

$$\sum_{j=1}^{n} f_j e^{g_j} - f_{n+1} e^{g_{n+1}} \equiv 0,$$

where $g_{n+1} \equiv 0$. We see that $f_1, f_2, \ldots, f_{n+1}$ and $g_1, g_2, \ldots, g_{n+1}$ satisfy the conditions of Theorem 2.1.3. Hence, we deduce that $f_j \equiv 0$ for all $j = 1, 2, \ldots, n+1$. □

Corollary 2.1.2 ([31]). *Let $f_1, f_2, \ldots, f_{n+1}$ be meromorphic functions and $g_1, g_2, \ldots, g_n$ be entire functions, where $n \geq 1$, satisfying:*

(1) $\sum_{j=1}^{n} f_j e^{g_j} \equiv f_{n+1}$.

(2) *$g_j - g_k$ and g_h are not constant for $1 \leq j < k \leq n$ and $h = 1, 2, \ldots, n$.*

(3) *$\rho(f_j) < \rho(e^{g_h - g_k})$ for all $j = 1, 2, \ldots, n+1$ and $1 \leq h < k \leq n$.*
$\rho(f_j) < \rho(e^{g_k})$ for all $j = 1, 2, \ldots, n+1$ and $k = 1, 2, \ldots, n$.

Then $f_j \equiv 0$ for all $j = 1, 2, \ldots, n+1$.

To prove Corollary 2.1.2 we need the following theorems.

Theorem 2.1.4 ([31]). *Let $g(z)$ be a non-constant entire function. Then $\mu(e^g) = \rho(e^g) \in \mathbb{N} \cup \{\infty\}$, and we have:*

(1) *If $\rho(e^g) < \infty$, then $\rho(e^g) \in \mathbb{N}$, and $g(z)$ is a polynomial of degree $\rho(e^g)$.*

(2) *If $\rho(e^g) = \infty$, then $g(z)$ is transcendental.*

Theorem 2.1.5 ([31]). *Let $f(z)$ and $g(z)$ be meromorphic functions. If $\rho(f) < \mu(g)$, then*

$$T(r, f) = o(T(r, g)), \ (r \to \infty).$$

Proof of Corollary 2.1.2. Condition (1) can be expressed as

$$\sum_{j=1}^{n} f_j e^{g_j} - f_{n+1} e^{g_{n+1}} \equiv 0,$$

where $g_{n+1} \equiv 0$. By Theorems 2.1.4 and 2.1.5 we see that $f_1, f_2, \ldots, f_{n+1}$ and $g_1, g_2, \ldots, g_{n+1}$ satisfy the conditions of Corollary 2.1.1. Thus, it follows that $f_j \equiv 0$ for $j = 1, 2, \ldots, n+1$. □

2.2 Brück's Conjecture

The classical five-value theorem of Nevanlinna, as stated in Theorem 1.5.4, claims that if two non-constant meromorphic functions $f(z)$ and $g(z)$ share five distinct values IM, then $f \equiv g$. If $f(z)$ and $g(z)$ share four distinct values CM, then by Nevanlinna's four value theorem [27], $f(z)$ is a Möbius transformation of $g(z)$. These results have been generalized and refined by numerous authors in various directions [31]. Specifically, the number of shared values can be reduced if $f(z)$ and $g(z)$ are related. Rubel and Yang [28] were the first to investigate the value-sharing between an entire function and its derivative. They proved the following result.

Theorem 2.2.1 ([28]). *Let $f(z)$ be a non-constant entire function. If $f(z)$ and $f'(z)$ share two finite values CM, then $f \equiv f'$.*

Mues and Steinmetz [23] extended the previous result to the case of IM sharing values and obtained the following result.

Theorem 2.2.2 ([23]). *Let $f(z)$ be a non-constant entire function. If $f(z)$ and $f'(z)$ share two finite values IM, then $f \equiv f'$.*

Furthermore, Mues and Steinmetz [24] and independently Gundersen [13] extended Theorem 2.2.1 to meromorphic functions. There are many generalizing results related to Theorems 2.2.1 and 2.2.2, mainly in two directions. One is extending the shared values to non-constant functions, such as polynomials or small entire functions with respect to $f(z)$ (see, e.g., [36]), and the other direction is replacing the first derivative of $f(z)$ to a higher-order derivative or a linear differential polynomial (see, e.g., [12]). For example, Yang [32] proved the following result.

Theorem 2.2.3 ([32]). *Let $f(z)$ be a non-constant entire function. If $f(z)$ and $f^{(k)}(z)$ $(k \geq 1)$ share two finite values CM, then $f \equiv f^{(k)}$.*

It is natural to consider whether Theorems 2.2.1 and 2.2.2 can be extended to the case of single-value sharing. However, by considering the function $f(z) = e^{e^z} \int_0^z e^{-e^t} \left(1 - e^t\right) dt$ we can see that $f'(z) - 1 = e^z (f(z) - 1)$. Hence, $f(z)$ and $f'(z)$ share the value 1 CM but $f' \not\equiv f$. Additional counterexamples are provided by the solutions of the following differential equations

$$f'(z) - 1 = e^{z^n} (f(z) - 1), \quad n \in \mathbb{N}, \tag{2.13}$$

and

$$f'(z) - 1 = e^{e^z} (f(z) - 1). \tag{2.14}$$

From these equations, we observe that $\rho_2(f) = n$ and $\rho_2(f) = \infty$, respectively. Based on these remarks and the previous results, Brück [2] raised the following question:
What conclusion can be reached if $f(z)$ and $f'(z)$ share a single value, and if an appropriate growth restriction on $f(z)$ is assumed? In light of this, he [2] proposed the following conjecture.

Conjecture 1 (Brück's conjecture, [2]). *Let $f(z)$ be a non-constant entire function such that $\rho_2(f) \notin \mathbb{N} \cup \{\infty\}$. If $f(z)$ and $f'(z)$ share one finite value a CM, then*

$$f' - a = c(f - a), \tag{2.15}$$

for some non-zero constant c.

For the case when $a = 0$, Brück [2] proved the following result.

Theorem 2.2.4 ([2]). *Let $f(z)$ be a non-constant entire function such that $\rho_2(f) \notin \mathbb{N} \cup \{\infty\}$. If $f(z)$ and $f'(z)$ share the value 0 CM, then*

$$f' = cf,$$

for some non-zero constant c.

Proof. Since $f(z)$ and $f'(z)$ share 0 CM, then $f'(z) = e^{h(z)}f(z)$, where $h(z)$ is an entire function. Consequently, both $f(z)$ and $f'(z)$ has no zeros. Thus, $f(z)$ can be expressed as $f(z) = e^{g(z)}$, where $g(z)$ is an entire function. Substituting this into $f'(z)$, we obtain $f'(z) = g'(z)e^{g(z)} = e^{h(z)}f(z)$. From this, it follows that $g'(z) = e^{h(z)}$. Suppose that $h(z)$ is not constant. By Theorem 2.1.4, we have $\rho(g') = \rho(e^h) \in \mathbb{N} \cup \{\infty\}$. On the other hand, $\rho(g') = \rho(g) = \rho_2(e^g) = \rho_2(f) \notin \mathbb{N} \cup \{\infty\}$, which is a contradiction. Therefore, $h(z)$ must be constant, and we conclude that $f' = cf$ for some $c \in \mathbb{C} \setminus \{0\}$. $\square$

The most challenging case is when $a \neq 0$. To this day, Conjecture 1 remains unresolved in its full generality. The first attempt to resolve Conjecture 1 for the case $a \neq 0$ was made by Brück under certain hypotheses and without any growth restrictions on $f(z)$. He [2] established the following result.

Theorem 2.2.5 ([2]). *Let $f(z)$ be a non-constant entire function such that $N\left(r, \frac{1}{f'}\right) = S(r, f)$. If $f(z)$ and $f'(z)$ share the value 1 CM, then*

$$f' - 1 = c(f - 1),$$

for some non-zero constant c.

The following two examples given by Brück [2] show that Conjecture 1 does not hold if $f(z)$ and $f'(z)$ share one finite value IM.

Example 2.2.1.

(1) *The function* $f(z) = z^2$ *and its derivative* $f'(z) = 2z$ *share the value* 0 *IM and the conclusion of Conjecture 1 does not hold.*

(2) *Let* $f(z) = e^{\frac{z^2}{2}}\left(\int_0^z e^{-\frac{t^2}{2}}(1-t)\,dt + 1\right)$. *Then, we have* $f'(z) - 1 = z\,(f(z) - 1)$ *and* $f'(0) = f(0) = 1$. *Here,* 0 *is a simple* 1*-point of* $f(z)$ *and a double* 1*-point of* $f'(z)$. *Hence,* $f(z)$ *and* $f'(z)$ *share the value* 1 *IM and the conclusion of Conjecture 1 does not hold.*

Gundersen and Yang [14] made the first attempt to resolve Brück's conjecture for non-zero value sharing without any assumptions about the zeros of $f(z)$ or $f'(z)$. They proved the conjecture for entire functions of finite order. Indeed, they obtained the following result.

Theorem 2.2.6 ([14])**.** *Let* $f(z)$ *be a non-constant entire function of finite order. If* $f(z)$ *and* $f'(z)$ *share a non-zero finite value* a *CM, then*

$$f' - a = c\,(f - a)\,,$$

for some non-zero constant c.

Yang [33] generalized the previous theorem to entire functions that share a non-zero value with their higher order derivatives and proved the following result.

Theorem 2.2.7 ([33])**.** *Let* $f(z)$ *be a non-constant entire function of finite order. If* $f(z)$ *and* $f^{(k)}(z)$ $(k \geq 1)$ *share a non-zero finite value* a *CM, then*

$$f^{(k)} - a = c\,(f - a)\,,$$

for some non-zero constant c.

The previous theorem has been extended to polynomial sharing instead of value sharing. Indeed, Wang [30] proved the following result.

Theorem 2.2.8 ([30])**.** *Let* $f(z)$ *be a non-constant entire function of finite order and* $Q(z)$ *be a polynomial of degree* $q \geq 1$. *If* $f(z)$ *and* $f^{(k)}(z)$ $(k \geq 1)$ *share* $Q(z)$ *CM, then*

$$f^{(k)} - Q = c\,(f - Q)\,,$$

for some non-zero constant c.

Furthermore, Li and Gao [21] proved the following result.

Theorem 2.2.9 ([21]). *Suppose that $Q_1(z)$ and $Q_2(z)$ are two non-zero polynomials and $P(z)$ is a polynomial. If $f(z)$ is a non-constant entire solution of*

$$f^{(k)} - Q_1 = (f - Q_2)\, e^P,$$

then $\rho_2(f) = \deg(P)$.

By considering the solutions of (2.14), we can see that the condition on the order of $f(z)$ cannot be omitted from Theorem 2.2.6 without any additional assumptions. In [5], Chen and Shon improved Theorem 2.2.6 and established the following result.

Theorem 2.2.10 ([5]). *Let $f(z)$ be a non-constant entire function such that $\rho_2(f) < \frac{1}{2}$. If $f(z)$ and $f'(z)$ share a non-zero finite value a CM, then*

$$f' - a = c\,(f - a)\,,$$

for some non-zero constant c.

Cao [3] extended the previous result to the case where $\rho_2(f) = \frac{1}{2}$. Later, Chen et al. [4] improved the previous theorem and proved the following result.

Theorem 2.2.11 ([4]). *Let $f(z)$ be a non-constant entire function such that $\rho(f) \neq 1$ and $\rho_2(f) < 1$, and let $L(f) = a_k f^{(k)} + \cdots + a_1 f' + a_0 f$, where $k \geq 1$ and $a_0, a_1, \ldots, a_k\ (\neq 0)$ are constants. If $f(z)$ and $L(f)$ share a finite value a CM, then*

$$L(f) - a = c\,(f - a)\,,$$

for some non-zero constant c.

The previous theorem shows that Brück's conjecture holds for entire functions with hyper-order less than 1.

Chapter 3

Brück's Conjecture for Solutions of Linear Differential Equations

In this chapter, we study Brück's conjecture for the solutions of linear differential equations. We begin by providing a rephrasing of Brück's conjecture for entire functions of infinite order. After that we consider entire solutions of homogeneous second-order linear differential equations. Finally, we generalize the results obtained in the first section to polynomial sharing between entire solutions of higher-order linear differential equations and their derivatives.

3.1 Motivation

Generally, the proof strategy of the results related to Brück's conjecture is to assume that

$$\frac{f'-a}{f-a} = e^{g(z)},$$

where $g(z)$ is a non-constant entire function in order to derive a contradiction, which means that $g(z)$ must be constant, and the claim of Conjecture 1 follows. Since Brück's conjecture holds for entire functions of finite order, as indicated by Theorem 2.2.6, we next suppose that $f(z)$ is of infinite order. However, the solutions of (2.15) take the form

$$f(z) = \alpha e^{cz} + a - \frac{a}{c},$$

where $\alpha, c \in \mathbb{C} \setminus \{0\}$. Thus, $f(z)$ is of finite order $\rho(f) = 1$. Based on this remark, we suggest to rephrase Conjecture 1 as follows.

Conjecture 2 ([9]). *Let $f(z)$ be an entire function of infinite order such that $\rho_2(f) \notin \mathbb{N} \cup \{\infty\}$. Then $f(z)$ and $f'(z)$ cannot share any finite value CM.*

Similar to Conjecture 1, Conjecture 2 does not hold when $\rho_2(f) \in \mathbb{N} \cup \{\infty\}$. In this case, we only have to consider the solutions of (2.13) and (2.14).

We note that any entire function is a solution of a second-order linear differential equation with entire coefficients as shown by the following result.

Theorem 3.1.1 ([19, Theorem 4.4]). *Let $f(z)$ be an entire function. Then there exist entire functions $a_0(z)$, $a_1(z)$ and $a_2(z)$ such that*

$$a_2(z)f'' + a_1(z)f' + a_0(z)f = 0.$$

If all zeros of $f(z)$ are simple, then $a_2 \equiv 1$.

3.2 Brück's Conjecture for Solutions of Second-Order Linear Differential Equations

Inspired by Theorem 3.1.1 and to provide a partial answer to Conjecture 2, in this section we investigate the class of non-trivial entire solutions of

$$f'' + A(z)f' + B(z)f = 0, \tag{3.1}$$

where $A(z)$ and $B(z)$ are meromorphic functions.

Main Results

We proceed to prove that Conjecture 2 holds for all solutions of (3.1) with finite order meromorphic coefficients. Additionally, we present several consequences of our main result.

Theorem 3.2.1 ([9]). *Let $f(z)$ be an entire function of infinite order such that $\rho_2(f) \notin \mathbb{N} \cup \{\infty\}$. If $f(z)$ is a solution of (3.1), where $A(z)$ and $B(z)$ are meromorphic functions of finite order, then $f(z)$ cannot share any finite value CM with its derivative.*

Proof. Suppose, contrary to the claim, that $f(z)$ and $f'(z)$ share a finite value a CM. We may assume $a \neq 0$ because the case $a = 0$ has been established in Theorem 2.2.4. Therefore,

$$f' = e^{g(z)}(f - a) + a, \tag{3.2}$$

where $g(z)$ is an entire function. First, we assume that $g(z)$ is transcendental. From (3.2), we observe that $\rho(g) \leq \rho_2(f)$. We can write (3.2) as

$$F' - e^{g(z)}F = 1, \tag{3.3}$$

where $F(z) = \dfrac{f(z)}{a} - 1$. Differentiating (3.2) yields

$$f'' = (f - a)\, e^{2g(z)} + (g'(z)f - ag'(z) + a)\, e^{g(z)}. \tag{3.4}$$

Substituting (3.2) and (3.4) into (3.1) results in

$$\mathcal{A}(z)F = e^{g(z)} + A(z) + B(z),$$

where

$$\mathcal{A}(z) = -\left(e^{2g(z)} + (A(z) + g'(z))e^{g(z)} + B(z)\right).$$

It is evident that $\mathcal{A} \not\equiv 0$, otherwise, $\rho\,(A + B) = \rho\,(e^g) = \infty$, which leads to a contradiction. Therefore,

$$F = \frac{e^{g(z)} + A(z) + B(z)}{\mathcal{A}(z)}.$$

By substituting this into (3.3), we obtain

$$B(z)e^{3g(z)} + H_2(z)e^{2g(z)} + H_1(z)e^{g(z)} = H_0(z),$$

where

$$H_0(z) = B(z)A'(z) - B'(z)A(z) + B^2(z),$$

$$\begin{aligned} H_1(z) \;&=\; (A(z) + B(z))\, g''(z) - (A'(z) + B'(z) + 3B(z) - (A(z) + B(z))\, A(z))\, g'(z) \\ &+\; (A(z) + B(z))\, (g'(z))^2 - A(z)B'(z) + A'(z)B(z) + B'(z) - A(z)B(z) + B^2(z), \end{aligned}$$

and

$$H_2(z) = g''(z) + (A(z) + 3B(z))\, g'(z) - (g'(z))^2 + A(z)B(z) - B(z) - B'(z).$$

Clearly $H_0(z)$,$H_1(z)$ and $H_2(z)$ are meromorphic functions of finite order. By applying Corollary 2.1.2, we find that $B \equiv 0$, $H_2 \equiv 0$, $H_1 \equiv 0$ and $H_0 \equiv 0$. Hence,

$$g''(z) + A(z)g'(z) - (g'(z))^2 = 0, \tag{3.5}$$

and

$$A(z)g''(z) - \left(A'(z) - A^2(z)\right) g'(z) + A(z)\, (g'(z))^2 = 0. \tag{3.6}$$

From (3.5), we obtain

$$g''(z) = (g'(z))^2 - A(z)g'(z),$$

and substituting this into (3.6) yields

$$\frac{A'(z)}{A(z)} = 2g'(z).$$

Therefore, $A(z) = ce^{2g(z)}$, where $c \in \mathbb{C} \setminus \{0\}$. This is a contradiction, as $A(z)$ is of finite order. Consequently, $g(z)$ cannot be a transcendental entire function.

Now, assume that $g(z)$ is a non-constant polynomial of degree n. According to Theorem 2.2.9, this implies that $\rho_2(f) = n$, which contradicts the fact that $\rho_2(f) \notin \mathbb{N}$. Consequently, $g(z)$ must be a constant, and thus $f(z)$ satisfies (2.15), which again contradicts the fact that $f(z)$ is of infinite order, as all solutions of (2.15) are finite order equal to 1. Therefore, $f(z)$ cannot share any finite value a CM with $f'(z)$. □

A simple application of Theorem 3.2.1 is given in the following example.

Example 3.2.1. *Consider the function* $f(z) = e^{\cosh(z\sqrt{z})} + e^{e^z}$. *We have* $\rho_2(f) = \frac{3}{2}$, *and* $f(z)$ *solves the differential equation* (3.1) *with*

$$A(z) = -\frac{\frac{9}{2}z\sinh^2\left(z\sqrt{z}\right) + \frac{9}{2}z\cosh\left(z\sqrt{z}\right) + \frac{3}{2}\frac{\sinh\left(z\sqrt{z}\right)}{\sqrt{z}} - 2e^{2z} - 2e^z}{3\sqrt{z}\sinh\left(z\sqrt{z}\right) - 2e^z},$$

and

$$B(z) = \frac{\left(\frac{9}{2}z\sinh^2\left(z\sqrt{z}\right) + \frac{9}{2}z\cosh\left(z\sqrt{z}\right) + \frac{3}{2}\frac{\sinh\left(z\sqrt{z}\right)}{\sqrt{z}}\right)e^z - 3\sqrt{z}\sinh\left(z\sqrt{z}\right)\left(e^{2z} + e^z\right)}{3\sqrt{z}\sinh\left(z\sqrt{z}\right) - 2e^z}.$$

Therefore, $f(z)$ *does not share any finite value CM with its derivative.*

The following result can be easily deduced from the proof of Theorem 3.2.1.

Corollary 3.2.1 ([9]). *Suppose that* $f(z)$ *is a non-trivial entire solution of* (3.1) *such that* $\rho_2(f) \notin \mathbb{N} \cup \{\infty\}$, *where* $A(z)$ *and* $B(z)$ *are meromorphic functions of finite order. If* $f(z)$ *and its derivative share one finite value* a *CM, then* (2.15) *holds.*

Next, we provide sufficient conditions to guarantee that every non-trivial solution of (3.1) with entire coefficients has a non-integer hyper-order. This requires the following lemma.

Lemma 3.2.1 ([29, Theorem 1]). *Suppose that* $A_0, A_1, \ldots, A_{k-1}$, *where* $k \geq 2$, *are entire functions such that* $0 < \rho(A_0) < \infty$, $0 < \tau(A_0) < \infty$ *and for* $j = 1, 2, \ldots, k-1$ *either* $\rho(A_j) < \rho(A_0)$ *or* $\rho(A_j) = \rho(A_0)$ *and* $\tau(A_j) < \tau(A_0)$. *Then, any non-trivial solution* $f(z)$ *of*

$$f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_0 f = 0$$

satisfies $\rho_2(f) = \rho(A_0)$.

By combining Theorem 3.2.1 and Lemma 3.2.1, we obtain the following result.

Corollary 3.2.2 ([9]). *Suppose that $A(z)$ and $B(z)$ are entire functions such that $\rho(B) \notin \mathbb{N} \cup \{\infty\}$ and $0 < \tau(B) < \infty$. If either $\rho(A) < \rho(B)$ or $\rho(A) = \rho(B)$ and $\tau(A) < \tau(B)$, then every non-trivial solution $f(z)$ of (3.1) has a hyper-order $\rho_2(f) \notin \mathbb{N} \cup \{\infty\}$ and cannot share any finite value CM with $f'(z)$.*

Example 3.2.2. *Let $B(z) = \sum_{n=1}^{\infty} \left(\frac{e\pi}{n}\right)^{\frac{n}{\pi}} z^n$ and $f(z)$ be a non-trivial solution of the differential equation*

$$f'' + e^{z^2} f' + B(z) f = 0.$$

Here, $\rho(e^{z^2}) = 2 < \rho(B) = \pi \notin \mathbb{N} \cup \{\infty\}$. Therefore, by Corollary 3.2.2 $f(z)$ and $f'(z)$ cannot share any finite value CM.

The next result shows that Conjecture 2 holds for entire functions with few distinct zeros in the sense $\overline{\lambda}(f) < \infty$.

Corollary 3.2.3 ([9]). *Let $f(z)$ be an entire function of infinite order and hyper-order $\rho_2(f) \notin \mathbb{N} \cup \{\infty\}$. If $\overline{\lambda}(f) < \infty$, then $f(z)$ cannot share any finite value CM with its derivative.*

For the proof we use the following lemma.

Lemma 3.2.2 ([19, Lemma 1.1.1]). *Suppose that $T_1(r)$ and $T_2(r)$ are two non-decreasing real functions defined in $r \geq 0$. If $T_1(r) \leq T_2(r)$ outside of an exceptional set E of finite linear measure, then for any $\alpha > 1$ we have $T_1(r) \leq T_2(\alpha r)$ for sufficiently large r.*

Proof of Corollary 3.2.3. Let $A(z) = \frac{f'(z)}{f(z)}$, then

$$\begin{aligned} T(r, A) &= N\left(r, \frac{f'}{f}\right) + m\left(r, \frac{f'}{f}\right) \\ &= \overline{N}\left(r, \frac{1}{f}\right) + O\left(\log\left(rT(r,f)\right)\right), \ (r \to \infty, \ r \notin E), \end{aligned}$$

where E is a set of finite linear measure. By using Lemma 3.2.2 we may remove the exceptional set to obtain

$$\rho(A) \leq \max\left\{\rho_2(f), \overline{\lambda}(f)\right\} < \infty.$$

On the other hand, it is evident that $f(z)$ satisfies the following differential equation

$$f'' - A(z) f' - A'(z) f = 0.$$

Now, the assertion of Corollary 3.2.3 follows by applying Theorem 3.2.1. $\square$

Clearly, Conjecture 2 holds for entire functions with finite exponent of convergence.

Example 3.2.3. *Consider the function* $f(z) = \left(e^{z^n} - 1\right) e^{\cos\left(z\sqrt{z}\right)}$, $(n \in \mathbb{N})$. *We have*

$$\rho_2(f) = \frac{3}{2} \quad \textit{and} \quad \lambda\left(f\right) = \lambda\left(e^{z^n} - 1\right) = n < \infty.$$

Therefore, $f(z)$ does not share any finite value CM with its derivative.

Corollary 3.2.4 ([9])**.** *Let $f(z)$ be an entire function of infinite order and hyper-order $\rho_2\left(f\right) \notin \mathbb{N} \cup \{\infty\}$. If $\overline{\lambda}\left(f'\right) < \infty$, then $f(z)$ cannot share any finite value CM with its derivative.*

Proof. Let $A(z) = \dfrac{f''(z)}{f'(z)}$. As in the proof of Corollary 3.2.3, we prove that $A(z)$ is of finite order and that $f(z)$ satisfies the following differential equation

$$f'' - A(z)f' = 0.$$

The assertion of Corollary 3.2.4 follows by applying Theorem 3.2.1. □

Now, we consider the case when $\overline{\lambda}\left(f\right)$ is not necessarily finite. We begin by assuming that $f(z) = \pi(z)e^{h(z)}$, where $\pi(z)$ and $h(z)$ are entire functions. Indeed, the next result shows that Conjecture 2 holds when $\pi(z)$ is a solution of (3.1).

Corollary 3.2.5 ([9])**.** *Let $f(z) = \pi(z)e^{h(z)}$, where $h(z)$ is an entire function of finite order and $\pi(z)$ is a non-trivial entire solution of (3.1), where $A(z)$ and $B(z)$ are meromorphic functions of finite order. If $f(z)$ is of infinite order and hyper-order $\rho_2\left(f\right) \notin \mathbb{N} \cup \{\infty\}$, then $f(z)$ cannot share any finite value CM with its derivative.*

Proof. By substituting $\pi(z) = f(z)e^{-h(z)}$ into the differential equation

$$\pi''(z) + A(z)\pi'(z) + B(z)\pi(z) = 0,$$

we obtain

$$f'' + \left(A(z) - 2h'(z)\right) f' + \left(\left(h'(z)\right)^2 - h''(z) - A(z)h'(z) + B(z)\right) f = 0.$$

According to the assumptions, the coefficients of the above equation are meromorphic functions of finite order, and since $f(z)$ is of infinite order and $\rho_2\left(f\right) \notin \mathbb{N} \cup \{\infty\}$, we can apply Theorem 3.2.1 to reach the conclusion of Corollary 3.2.5. □

Our final result is an immediate consequence of Corollary 3.2.5. In fact, we provide some sufficient conditions on $A(z)$ and $B(z)$ to guarantee that $f(z)$ cannot share any finite value CM with its derivative.

Corollary 3.2.6 ([9]). *Let $f(z) = \pi(z)e^{h(z)}$, where $h(z)$ is an entire function and $\pi(z)$ is a non-trivial solution of (3.1), where $A(z)$ and $B(z)$ are entire functions satisfying one of the following conditions:*

(i) $\max\{\rho(h), \rho(A)\} < \rho(B) \notin \mathbb{N} \cup \{\infty\}$.

(ii) $\max\{\rho(A), \rho(B)\} < \rho(h) \notin \mathbb{N} \cup \{\infty\}$.

Then $f(z)$ cannot share any finite value CM with its derivative.

Before proceeding to the proof, we recall the following lemmas.

Lemma 3.2.3 ([7, Theorem 4]). *Let $A_0, A_1, \ldots, A_{k-1}$, where $k \geq 2$, be entire functions of finite order. If $\rho(A_j) < \rho(A_0) < \infty$ for $j = 1, 2, \ldots, k-1$, then every non-trivial solution $f(z)$ of*

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_0(z)f = 0$$

satisfies $\rho_2(f) = \rho(A_0)$.

Lemma 3.2.4 ([6, Lemma 6]). *Suppose that $A_0, A_1, \ldots, A_{k-1}$, where $k \geq 2$, are entire functions of finite order. If $f(z)$ is a solution of*

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_0(z)f = 0,$$

then $\rho_2(f) \leq \max\{\rho(A_j) : j = 0, \ldots, k-1\}$.

Proof of Corollary 3.2.6. First, suppose that $\max\{\rho(h), \rho(A)\} < \rho(B)$, it follows from Lemma 3.2.3 that every non-trivial solution of (3.1) is an entire function of infinite order and $\rho_2(\pi) = \rho(B) \notin \mathbb{N} \cup \{\infty\}$. On the other hand, since

$$\rho_2\left(e^h\right) = \rho(h) < \rho(B) = \rho_2(\pi),$$

we deduce that $\rho_2(f) = \rho_2(\pi) \notin \mathbb{N} \cup \{\infty\}$. This, combined with Corollary 3.2.5, implies that $f(z)$ cannot share any finite value CM with its derivative.

Now, Assume that $\max\{\rho(A), \rho(B)\} < \rho(h)$ and that $h(z)$ is of finite non-integer order. This, together with Lemma 3.2.4, implies that

$$\rho_2(\pi) \leq \max\{\rho(A), \rho(B)\} < \rho(h) = \rho_2\left(e^h\right).$$

Thus, $\rho_2(f) = \rho(h) \notin \mathbb{N} \cup \{\infty\}$, and $f(z)$ is of infinite order. The assertion then follows from Corollary 3.2.5. $\square$

The following examples illustrate Corollary 3.2.6.

Example 3.2.4. *Let $\pi(z)$ be a non-trivial solution of*

$$\pi'' + \cosh(z)\pi' + \sqrt{z}\sin\left(z^2\sqrt{z}\right)\pi = 0,$$

and $f(z) = \pi(z)e^{\frac{\sin(\sqrt{z})}{\sqrt{z}}}$. Here, $A(z) = \cosh(z)$, $B(z) = \sqrt{z}\sin(z^2\sqrt{z})$ and $h(z) = \dfrac{\sin(\sqrt{z})}{\sqrt{z}}$ are entire functions. Clearly

$$\rho(A) = 1 < \rho(B) = \frac{5}{2} \notin \mathbb{N} \cup \{\infty\} \quad \text{and} \quad \rho(h) = \frac{1}{2} < \rho(B).$$

From Corollary 3.2.6(i), it follows that $f(z)$ cannot share any finite value CM with its derivative.

Example 3.2.5. *Let $\pi(z)$ be a non-trivial solution of (3.1), where*

$$A(z) = \sum_{n=1}^{\infty}\left(\frac{e\log n}{2n}\right)^{2n} z^n \quad \text{and} \quad B(z) = \sum_{n=1}^{\infty}\frac{n^2}{n!}z^n.$$

Then, the function $f(z) = \pi(z)e^{\cos\left(z\sqrt{z}\right)}$ cannot share any finite value CM with its derivative. Noting that $\rho(A) = \frac{1}{2}$, $\rho(B) = 1$ and $\rho(\cos(z\sqrt{z})) = \frac{3}{2} \notin \mathbb{N} \cup \{\infty\}$, which satisfies the assumption of Corollary 3.2.6(ii).

3.3 Brück's Conjecture for Solutions of Higher-Order Linear Differential Equations

In this section, we investigate polynomial sharing between entire functions and their derivatives for solutions of homogeneous and non-homogeneous higher order linear differential equations with meromorphic coefficients under certain conditions. As a consequence of this, we confirm Conjecture 2 for these solutions.

Main Results

First, we investigate the non-trivial entire solutions of

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0, \tag{3.7}$$

where $k \geq 2$, and $A_j(z)$ $(j = 0, 1, \ldots, k-1)$ are meromorphic functions. Indeed, we obtain the following result.

Theorem 3.3.1. *Suppose that $f(z)$ is an entire function of infinite order such that $\rho_2(f) \notin \mathbb{N} \cup \{\infty\}$. If $f(z)$ is a solution of (3.7), where $A_j(z)$ $(j = 0, 1, \ldots, k-1)$ are meromorphic functions of finite order, then $f(z)$ cannot share any polynomial $a(z)$ CM with its derivative.*

The following lemma is needed for the proof of Theorem 3.3.1.

Lemma 3.3.1. *Let $f(z)$ be a non-constant entire function of finite hyper-order $\rho_2(f) < \infty$, $g(z)$ be an entire function and $a(z)$ be a polynomial. If*

$$f' - a = e^{g(z)}\left(f - a\right), \tag{3.8}$$

then for all $l \geq 2$ we have

$$f^{(l)} = \left(\sum_{j=1}^{l} H_{j,l} e^{jg}\right)(f-a) + \sum_{j=0}^{l-1} K_{j,l} e^{jg}, \tag{3.9}$$

where $H_{j,l}$ $(j = 1, 2, \ldots, l)$ and $K_{j,l}$ $(j = 0, 1, 2, \ldots, l-1)$ are entire functions of finite order and $H_{l,l} \equiv 1, K_{l-1,l} \equiv a - a'$.

Proof. We will prove the lemma by induction on l. From (3.8), we observe that $\rho(g) \leq \rho_2(f) < \infty$. Hence, $g(z)$ is of finite order. By differentiating (3.8), we obtain

$$\begin{aligned} f'' &= g' e^g (f-a) + e^g \left(f' - a'\right) + a' \\ &= g' e^g (f-a) + e^g \left[e^g (f-a) + a - a'\right] + a' \\ &= \left(e^{2g} + g' e^g\right)(f-a) + \left(a - a'\right) e^g + a'. \end{aligned}$$

Therefore, the result is true for $l = 2$. Now, assuming that Lemma 3.3.1 is true for $l = n \geq 2$, we have

$$f^{(n)} = \left(\sum_{j=1}^{n} H_{j,n} e^{jg}\right)(f-a) + \sum_{j=0}^{n-1} K_{j,n} e^{jg}, \tag{3.10}$$

where $H_{n,n} \equiv 1$ and $K_{n-1,n} \equiv a - a'$. By differentiating (3.10), we obtain

$$f^{(n+1)} = \left(\sum_{j=1}^{n} H_{j,n} e^{jg}\right)(f' - a') + \left(\sum_{j=1}^{n} (jg' H_{j,n} + H'_{j,n}) e^{jg}\right)(f-a) + \sum_{j=0}^{n-1} (jg' K_{j,n} + K'_{j,n}) e^{jg}. \tag{3.11}$$

From (3.8), we have

$$f' = e^g (f-a) + a,$$

by substituting this into (3.11), we obtain

$$\begin{aligned} f^{(n+1)} = &\left(\sum_{j=1}^{n} H_{j,n} e^{jg}\right)\left(e^g (f-a) + a - a'\right) + \left(\sum_{j=1}^{n} (jg' H_{j,n} + H'_{j,n}) e^{jg}\right)(f-a) \\ &+ \sum_{j=0}^{n-1} (jg' K_{j,n} + K'_{j,n}) e^{jg}. \end{aligned}$$

Hence,

$$f^{(n+1)} = \left((g'H_{1,n} + H'_{1,n})e^{g} + \sum_{j=2}^{n} (H_{j-1,n} + jg'H_{j,n} + H'_{j,n})e^{jg} + e^{(n+1)g} \right)(f-a)$$
$$+ K'_{0,n} + \sum_{j=1}^{n-1} \left((a-a')\, H_{j,n} + jg'K_{j,n} + K'_{j,n} \right) e^{jg} + (a-a')\, e^{ng}.$$

Thus, the result is true for $l = n+1$. □

Proof of Theorem 3.3.1. Assume, contrary to the assertion, that $f(z)$ and $f'(z)$ share a polynomial $a(z)$ CM. We assume that $a \not\equiv 0$ since the case $a \equiv 0$ is already covered by Theorem 2.2.4. Then, we have

$$f' = e^{g}(f-a) + a, \tag{3.12}$$

where $g(z)$ is an entire function. First, suppose that $g(z)$ is a transcendental entire function. By Lemma 3.3.1, for all $l \geq 2$, we have

$$f^{(l)} = \left(\sum_{j=1}^{l} H_{j,l} e^{jg} \right)(f-a) + \sum_{j=0}^{l-1} K_{j,l} e^{jg}, \tag{3.13}$$

where $H_{j,l}$ $(j = 1, 2, \ldots, l)$, $K_{j,l}$ $(j = 0, 1, 2, \ldots, l-1)$ are entire functions of finite order, $H_{l,l} \equiv 1$ and $K_{l-1,l} \equiv a - a'$. Substituting (3.12) and (3.13) into (3.7) gives

$$\begin{aligned}
&\left(\sum_{j=1}^{k} H_{j,k} e^{jg} \right)(f-a) + \sum_{j=0}^{k-1} K_{j,k} e^{jg} \\
&+ A_{k-1} \left[\left(\sum_{j=1}^{k-1} H_{j,k-1} e^{jg} \right)(f-a) + \sum_{j=0}^{k-2} K_{j,k-1} e^{jg} \right] \\
&+ A_{k-2} \left[\left(\sum_{j=1}^{k-2} H_{j,k-2} e^{jg} \right)(f-a) + \sum_{j=0}^{k-3} K_{j,k-2} e^{jg} \right] \\
&\quad \vdots \\
&+ A_{2} \left[\left(\sum_{j=1}^{2} H_{j,2} e^{jg} \right)(f-a) + \sum_{j=0}^{1} K_{j,2} e^{jg} \right] \\
&+ A_1 \left[e^{g}(f-a) + a \right] + A_0\, (f-a) = -aA_0.
\end{aligned}$$

This equation takes the following form

$$\left(\sum_{j=0}^{k} M_j e^{jg} \right)\left(\frac{f}{a} - 1 \right) = \left(\sum_{j=0}^{k-1} N_j e^{jg} \right), \tag{3.14}$$

where M_j $(j = 0, 1, \ldots, k)$ and N_j $(j = 0, 1, \ldots, k-1)$ are meromorphic functions of finite order with $M_k \equiv 1$ and $N_{k-1} \equiv \dfrac{a'}{a} - 1$.

We claim that $\sum\limits_{j=0}^{k} M_j e^{jg} \not\equiv 0$. Assume that

$$\sum_{j=0}^{k} M_j e^{jg} \equiv 0.$$

Since $g(z)$ is a transcendental entire function and M_j $(j = 0, 1, \ldots, k)$ are of finite order, then by applying Corollary 2.1.2, we obtain $M_k \equiv 0$, which is a contradiction. Hence, (3.14) can be rewritten as

$$F = \frac{\sum\limits_{j=0}^{k-1} N_j e^{jg}}{\sum\limits_{j=0}^{k} M_j e^{jg}} = \frac{P(e^g)}{Q(e^g)}, \tag{3.15}$$

where $F = \dfrac{f}{a} - 1$ and P and Q are relatively prime polynomials in e^g over the field of finite order meromorphic functions. Furthermore

$$P(e^g) = \sum_{j=0}^{d-1} S_j e^{jg} \quad \text{and} \quad Q(e^g) = \sum_{j=0}^{d} L_j e^{jg},$$

where $d = \deg Q \geq 1$, S_j $(j = 0, 1, \ldots, d-1)$ and L_j $(j = 0, 1, \ldots, d)$ are finite order meromorphic functions with $L_d \equiv 1$ and $S_{d-1} \equiv \dfrac{a'}{a} - 1$. From (3.12), we have

$$F' + \left(\frac{a'}{a} - e^g\right) F = 1 - \frac{a'}{a}. \tag{3.16}$$

By substituting (3.15) into (3.16), we obtain

$$P(e^g)\left(Q(e^g)\right)' = \left[\left(P(e^g)\right)' + \left(\frac{a'}{a} - e^g\right) P(e^g) + \left(\frac{a'}{a} - 1\right) Q(e^g)\right] Q(e^g). \tag{3.17}$$

We have

$$\left(Q(e^g)\right)' = \left(\sum_{j=0}^{d} L_j e^{jg}\right)' = \sum_{j=0}^{d} \left(jg'L_j + L_j'\right) e^{jg},$$

and if we define the polynomial of indeterminate Z

$$\widetilde{Q}(Z) = \sum_{j=0}^{d} \left(jg'L_j + L_j'\right) Z^j,$$

then, it is clear that $\deg \widetilde{Q} = \deg Q$ and

$$(Q(e^g))' = \widetilde{Q}(e^g).$$

From this and (3.17), we deduce that Q is a divisor of $P\widetilde{Q}$, and since P and Q are relatively prime, it follows that Q divides $\widetilde{Q}$. Therefore,

$$\widetilde{Q}(e^g) = (Q(e^g))' = hQ(e^g), \tag{3.18}$$

where h is a meromorphic function of finite order. From this, we obtain

$$\sum_{j=0}^{d} \left(jg'L_j + L_j' - hL_j\right) e^{jg} \equiv 0.$$

Since g is a transcendental entire function of finite order and $jg'L_j + L_j' - hL_j$ $(j = 0, 1, \ldots, d)$ are of finite order, then by applying Corollary 2.1.2, we obtain $jg'L_j + L_j' - hL_j \equiv 0$ $(j = 0, 1, \ldots, d)$. In particular, for $j = d$, we obtain

$$h \equiv dg'.$$

From this and (3.18), we deduce that

$$Q(e^g) = ce^{dg},$$

where $c \in \mathbb{C}^*$. Since $Q(e^g) = \displaystyle\sum_{j=0}^{d} L_j e^{jg}$, where $L_d \equiv 1$ we deduce that $c = 1$, and consequently, $Q(e^g) = e^{dg}$. Substituting this into (3.17), it follows that

$$(P(e^g))' - \left(e^g + dg' - \frac{a'}{a}\right) P(e^g) \equiv \left(1 - \frac{a'}{a}\right) e^{dg}.$$

From this, we obtain

$$S_0' - \left(dg' - \frac{a'}{a}\right) S_0 + \sum_{j=1}^{d-1} \left(\left((j-d)g' + \frac{a'}{a}\right) S_j + S_j' - S_{j-1}\right) e^{jg} \equiv 0.$$

Since g is a transcendental entire function of finite order and $S_0' - \left(dg' - \frac{a'}{a}\right) S_0$ and $\left((j-d)g' + \frac{a'}{a}\right) S_j + S_j' - S_{j-1}$ $(j = 1, \ldots, d-1)$ are of finite order, then by applying Corollary 2.1.2, we obtain

$$\begin{cases} S_0' - \left(dg' - \dfrac{a'}{a}\right) S_0 \equiv 0 \\ \left((j-d)g' + \dfrac{a'}{a}\right) S_j + S_j' - S_{j-1} \equiv 0, \ (j = 1, \ldots, d-1). \end{cases}$$

Let $m = \min\{0 \le j \le d-1 \mid S_j \not\equiv 0\}$. If $m = 0$, then $S_0 \not\equiv 0$, and from $S_0' - \left(dg' - \frac{a'}{a}\right) S_0 \equiv 0$, we deduce that $S_0 = ce^{dg}/a$, where $c \in \mathbb{C}^*$. This is a contradiction since S_0 is of finite order. If $1 \le m \le d-1$, then $S_m \not\equiv 0$ and $S_{m-1} \equiv 0$. Therefore, $\left((m-d)g' + \frac{a'}{a}\right) S_m + S_m' \equiv 0$. From this, we deduce that $S_m = ce^{(d-m)g}/a$, where $c \in \mathbb{C}^*$. This is again a contradiction since S_m is of finite order. Hence, $g(z)$ cannot be a transcendental entire function.
Assume now that $g(z)$ is a non-constant polynomial of degree n. From this and Theorem 2.2.9, we conclude that $\rho_2(f) = n \in \mathbb{N}$, which is a contradiction. This leaves us with the possibility that $g(z)$ is a constant c, which implies that $f(z)$ satisfies $f' - a = c(f-a)$. Hence, $f(z) = R(z) + \alpha e^{cz}$, where $\alpha \in \mathbb{C}$ and $R(z)$ is a polynomial, which again contradicts the fact that $f(z)$ is of infinite order. Therefore, $f(z)$ and $f'(z)$ cannot share any polynomial $a(z)$ CM. □

In the following example, we apply Theorem 3.3.1 and show that Theorem 3.2.1 cannot be applied to the function presented.

Example 3.3.1. *Let*

$$g_1(z) = \frac{\sinh\left(z^2\sqrt{z}\right)}{\sqrt{z}}, \quad g_2(z) = \frac{1}{2}\left(\frac{\sinh\left(z^2\sqrt{z}\right)}{\sqrt{z}} + e^z\right), \quad g_3(z) = e^z,$$

and define

$$f(z) = e^{g_1(z)} + e^{g_2(z)} + e^{g_3(z)}.$$

For $r > 0$, we have

$$M(r, f) = f(r) \sim e^{g_1(r)} = M\left(r, e^{g_1}\right), \quad (r \to \infty).$$

Hence,

$$\rho(f) = \rho\left(e^{g_1}\right) = \infty \quad \text{and} \quad \rho_2(f) = \rho_2\left(e^{g_1}\right) = \frac{5}{2}.$$

Suppose that $f(z)$ is a solution of (3.1), where $A(z)$ and $B(z)$ are meromorphic functions of finite order. Then

$$\sum_{j=1}^{3}\left[\left(g_j'\right)^2 + g_j'' + Ag_j' + B\right] e^{g_j} = 0.$$

By applying Corollary 2.1.2, we obtain

$$\begin{cases} \left(g_1'\right)^2 + g_1'' + Ag_1' + B = 0 \\ \left(g_2'\right)^2 + g_2'' + Ag_2' + B = 0 \\ \left(g_3'\right)^2 + g_3'' + Ag_3' + B = 0. \end{cases}$$

From this, we obtain

$$A = \frac{(g_1')^2 - (g_2')^2 + g_1'' - g_2''}{g_2' - g_1'} = \frac{(g_2')^2 - (g_3')^2 + g_2'' - g_3''}{g_3' - g_2'}.$$

Noting that $g_2 = \dfrac{g_1 + g_3}{2}$, *we deduce from the previous equation that* $g_1 - g_3$ *must be constant, which is a contradiction. Therefore,* $f(z)$ *cannot be a solution of* (3.1) *with meromorphic coefficients of finite order, and Theorem 3.2.1 cannot be applied to* $f(z)$.
Now, we search for meromorphic coefficients $A(z)$, $B(z)$ *and* $C(z)$ *such that* $f(z)$ *is a solution of*

$$f''' + A(z)f'' + B(z)f' + C(z)f = 0. \tag{3.19}$$

In fact, $f(z)$ *is a solution of* (3.19) *if and only if*

$$\sum_{j=1}^{3} \left[(g_j')^3 + g_j''' + 3g_j'g_j'' + A\left((g_j')^2 + g_j'' \right) + Bg_j' + C \right] e^{g_j} = 0.$$

By applying Corollary 2.1.2, we find that the previous equation is equivalent to the following system of equations

$$\begin{cases} \left((g_1')^2 + g_1'' \right) A + g_1'B + C = -\left((g_1')^3 + g_1''' + 3g_1'g_1'' \right) \\ \left((g_2')^2 + g_2'' \right) A + g_2'B + C = -\left((g_2')^3 + g_2''' + 3g_2'g_2'' \right) \\ \left((g_3')^2 + g_3'' \right) A + g_3'B + C = -\left((g_3')^3 + g_3''' + 3g_3'g_3'' \right). \end{cases} \tag{3.20}$$

By Theorem 2.1.3, we conclude that e^{g_1}, e^{g_2} *and* e^{g_3} *are linearly independent. Therefore,*

$$\begin{aligned} \begin{vmatrix} (g_1')^2 + g_1'' & g_1' & 1 \\ (g_2')^2 + g_2'' & g_2' & 1 \\ (g_3')^2 + g_3'' & g_3' & 1 \end{vmatrix} &= e^{-(g_1+g_2+g_3)} \begin{vmatrix} (e^{g_1})'' & (e^{g_1})' & e^{g_1} \\ (e^{g_2})'' & (e^{g_2})' & e^{g_2} \\ (e^{g_3})'' & (e^{g_3})' & e^{g_3} \end{vmatrix} \\ &= -e^{-(g_1+g_2+g_3)} W\left(e^{g_1}, e^{g_2}, e^{g_3}\right) \not\equiv 0. \end{aligned}$$

Hence, the system of equations (3.20) *has a unique solution for* A, B *and* C. *By solving this system, we deduce that these functions are meromorphic functions of finite order. Therefore,* $f(z)$ *is a solution of* (3.19) *with* A, B *and* C *uniquely determined by* (3.20). *Furthermore, since* $f(z)$ *is of infinite order and* $\rho_2(f) = \frac{5}{2} \notin \mathbb{N} \cup \{\infty\}$, *Theorem 3.3.1 implies that* $f(z)$ *and* $f'(z)$ *cannot share any polynomial* $a(z)$ *CM.*

The proof of Theorem 3.3.1 allows us to easily derive the following result.

Corollary 3.3.1. *Let* $f(z)$ *be an entire function with hyper-order* $\rho_2(f) \notin \mathbb{N} \cup \{\infty\}$. *Suppose that* $f(z)$ *is a non-trivial solution of* (3.7), *where* $A_j(z)$ $(j = 0, 1, \ldots, k-1)$ *are meromorphic* *functions of finite order. If $f(z)$ and its derivative share one polynomial $a(z)$ CM, then*

$$f' - a = c\,(f - a)\,,$$

for some non-zero constant c.

We now present sufficient conditions under which every non-trivial solution of (3.7) with entire coefficients has a non-integer hyper-order. Using Theorem 3.3.1 together with Lemma 3.2.1, we derive the following result.

Corollary 3.3.2. *Let $A_j(z)$ $(j = 0, 1, \dots, k-1)$ be entire functions satisfying $\rho(A_0) \notin \mathbb{N} \cup \{\infty\}$ and $0 < \tau(A_0) < \infty$. If $\rho(A_j) < \rho(A_0)$ or $\rho(A_j) = \rho(A_0)$ and $\tau(A_j) < \tau(A_0)$ $(j = 1, \dots, k-1)$, then every non-trivial solution $f(z)$ of (3.7) is of hyper-order $\rho_2(f) \notin \mathbb{N} \cup \{\infty\}$ and cannot share any polynomial CM with its derivative.*

Next, we investigate the entire solutions of

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = F(z), \tag{3.21}$$

where $k \geq 2$, and $A_j(z)$ $(j = 0, 1, \dots, k-1)$ are meromorphic functions of finite order and $F(z)$ is a meromorphic function.

Corollary 3.3.3. *Suppose that $f(z)$ is an entire function of infinite order and hyper-order $\rho_2(f) \notin \mathbb{N} \cup \{\infty\}$. If $f(z)$ is a solution of (3.21), where $A_j(z)$ $(j = 0, 1, \dots, k-1)$ are meromorphic functions of finite order and $F(z) \, (\not\equiv 0)$ is a meromorphic function satisfying $\overline{\lambda}(F - \varphi) < \infty$ for some meromorphic function $\varphi(z)$ of finite order, then $f(z)$ cannot share any polynomial CM with its derivative.*

Proof. First we prove the corollary when $\overline{\lambda}(F) < \infty$.
Take $A(z) = \dfrac{F'(z)}{F(z)}$, then by elementary Nevanlinna theory

$$\begin{aligned} T\,(r, A) &= m\left(r, \frac{F'}{F}\right) + N\left(r, \frac{F'}{F}\right) \\ &= \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\,(r, F) + O\left(\log\left(rT\,(r, F)\right)\right) \end{aligned} \tag{3.22}$$

possibly outside an exceptional set of finite linear measure. From (3.21), we see that the poles of $F(z)$ are among the poles of $A_j(z)$ for $j = 0, 1, \dots, k-1$. Therefore, we have

$$\begin{aligned} \overline{N}\,(r, F) &\leq \sum_{j=0}^{k-1} \overline{N}\,(r, A_j) \\ &\leq \sum_{j=0}^{k-1} T\,(r, A_j)\,. \end{aligned}$$

Using this, along with (3.22), we deduce that

$$T(r, A) \leq \overline{N}\left(r, \frac{1}{F}\right) + \sum_{j=0}^{k-1} T(r, A_j) + O(\log(rT(r, F))), \tag{3.23}$$

possibly outside an exceptional set of finite linear measure. From (3.21), we observe that $\rho_2(F) \leq \rho_2(f) < \infty$. By applying Lemma 3.2.2 to (3.23), we can remove the exceptional set to obtain

$$\rho(A) \leq \max\left\{\overline{\lambda}(F), \rho_2(F), \rho(A_j) \ \ (j = 0, \ldots, k-1)\right\} < \infty.$$

On the other hand, from $F' = A(z)F$, it is easy to see that $f(z)$ satisfies the following differential equation

$$\begin{aligned} f^{(k+1)} + (A_{k-1} - A) f^{(k)} + \left(A'_{k-1} + A_{k-2} - AA_{k-1}\right) f^{(k-1)} \\ + \cdots + (A'_1 + A_0 - AA_1) f' + (A'_0 - AA_0) f = 0. \end{aligned}$$

By applying Theorem 3.3.1, we conclude that the assertion of Corollary 3.3.3 holds when $\overline{\lambda}(F) < \infty$.

Now, suppose that $\overline{\lambda}(F - \varphi) < \infty$ for some meromorphic function $\varphi \not\equiv 0$ of finite order. Take $A(z) = \dfrac{(F/\varphi)'(z)}{(F/\varphi)(z) - 1}$, by the same reasoning as in the case $\overline{\lambda}(F) < \infty$ we can prove that

$$\rho(A) \leq \max\left\{\overline{\lambda}(F - \varphi), \rho_2(F), \rho(\varphi), \rho(A_j) \ \ (j = 0, \ldots, k-1)\right\} < \infty,$$

and that $f(z)$ satisfies the following differential equation

$$\begin{aligned} f^{(k+1)} + \left(A_{k-1} - A - \frac{\varphi'}{\varphi}\right) f^{(k)} + \left(A'_{k-1} + A_{k-2} - \left(A + \frac{\varphi'}{\varphi}\right) A_{k-1}\right) f^{(k-1)} \\ + \cdots + \left(A'_1 + A_0 - \left(A + \frac{\varphi'}{\varphi}\right) A_1\right) f' + \left(A'_0 - \left(A + \frac{\varphi'}{\varphi}\right) A_0\right) f = -\varphi A. \end{aligned}$$

If $A \equiv 0$, we apply Theorem 3.3.1. If $A \not\equiv 0$, the assertion follows from the result proved in the first part of the proof. In both cases the assertion of Corollary 3.3.3 holds. □

Example 3.3.2. *The function $f(z) = e^{\sqrt{z}\sinh\left(z^2\sqrt{z}\right)} + e^{e^z} + e^z$ does not share any polynomial CM with its derivative. Indeed, $f(z)$ is of hyper-order $\frac{5}{2}$ and solves the differential equation (3.21) for $k = 2$ with*

$$A_1(z) = -\frac{5}{2}z^2\cosh\left(z^2\sqrt{z}\right) - \frac{1}{2\sqrt{z}}\sinh\left(z^2\sqrt{z}\right),$$

$$A_0(z) = -\frac{25}{4}z\cosh\left(z^2\sqrt{z}\right) - \frac{25z^5 - 1}{4z\sqrt{z}}\sinh\left(z^2\sqrt{z}\right),$$

and

$$F(z) = \left(-\frac{5}{4}z\left(2ze^z+5\right)\cosh\left(z^2\sqrt{z}\right) - \frac{25z^5+2ze^z-1}{4z\sqrt{z}}\sinh\left(z^2\sqrt{z}\right) + e^{2z} + e^z\right)e^{e^z} + \varphi(z),$$

where

$$\varphi(z) = \left(-\frac{5}{4}z\left(2z+5\right)\cosh\left(z^2\sqrt{z}\right) - \frac{25z^5+2z-1}{4z\sqrt{z}}\sinh\left(z^2\sqrt{z}\right) + 1\right)e^z.$$

Clearly, $A_1(z)$, $A_0(z)$ and $\varphi(z)$ are of finite order, and $\lambda\left(F-\varphi\right) < \infty$.

Obviously, Corollary 3.3.3 holds if $F(z)$ is of finite order.

Corollary 3.3.4. *Suppose that $f(z)$ is an entire function of infinite order and hyper-order $\rho_2(f) \notin \mathbb{N}\cup\{\infty\}$. If $\overline{\lambda}(f^{(k)}-\varphi) < \infty$ for some $k \geq 0$, where φ is a meromorphic function of finite order, then $f(z)$ cannot share any polynomial CM with its derivative.*

Proof. For $k=0$ or $k=1$, we use the same proof as in Corollary 3.3.3. If $k \geq 2$, by applying Corollary 3.3.3 with $A_j \equiv 0$ $(j=0,\dots,k-1)$ and $F(z) = f^{(k)}(z)$, the result follows. $\square$

Example 3.3.3. *The function $f(z) = e^{g(z)} + \sin(z)$, where $g(z) = \sum_{n=1}^{\infty}\frac{1}{n^{2n}}z^n$ does not share any polynomial CM with its derivative. Indeed $\varphi(z) = \sin(z)$ is of finite order and*

$$\rho_2(f) = \frac{1}{2} \notin \mathbb{N}\cup\{\infty\} \quad \text{and} \quad \lambda(f-\varphi) = \lambda(e^g) = 0 < \infty.$$

Clearly, Corollary 3.3.4 is true for entire functions with a finite Borel or Picard exceptional value.

Next, we consider the case where $\overline{\lambda}\left(F-\varphi\right)$ is not necessarily finite for any meromorphic function $\varphi(z)$ of finite order. To begin, assume that $f(z)$ is a solution of (3.21) such that $F(z) = \pi(z)e^{h(z)} + \varphi(z)$, where $h(z)$ is an entire function, $\varphi(z)$ is a meromorphic function and $\pi(z)$ is a meromorphic solution of

$$\pi^{(n)} + B_{n-1}(z)\pi^{(n-1)} + \cdots + B_1(z)\pi' + B_0(z)\pi = G(z), \tag{3.24}$$

where $n \geq 2$, and $B_j(z)$ $(j=0,1,\dots,n-1)$, $G(z)$ are meromorphic functions.

Corollary 3.3.5. *Suppose that $f(z)$ is an entire function, $h(z)$ is an entire function of finite order and $\pi(z)$ is a meromorphic solution of (3.24), where $B_j(z)$ $(j=0,1,\dots,n-1)$ are meromorphic functions of finite order and $G(z)$ is a meromorphic function such that $G \equiv 0$ or $G \not\equiv 0$ and $\overline{\lambda}(G) < \infty$. If $f(z)$ is a solution of (3.21) of infinite order and hyper-order $\rho_2(f) \notin \mathbb{N}\cup\{\infty\}$, where $A_j(z)$ $(j=0,1,\dots,k-1)$, $\varphi(z)$ are meromorphic functions of finite order and $F(z) = \pi(z)e^{h(z)} + \varphi(z)\,(\not\equiv 0)$, then $f(z)$ cannot share any polynomial CM with its derivative.*

Proof. From (3.21) and $F(z) = \pi(z)e^{h(z)} + \varphi(z)$, we obtain

$$\pi(z) = \sum_{i=0}^{k} A_i(z)e^{-h(z)} f^{(i)} - \varphi(z)e^{-h(z)}, \tag{3.25}$$

where $A_k \equiv 1$. By induction, we can easily prove that for all $0 \leq i \leq k$ and $j \geq 0$

$$\begin{aligned} \left(A_i(z)e^{-h(z)}\right)^{(j)} &= \widetilde{A}_{i,j}(z)e^{-h(z)}, \\ \left(\varphi(z)e^{-h(z)}\right)^{(j)} &= \widetilde{\varphi}_j(z)e^{-h(z)}, \end{aligned}$$

where $\widetilde{A}_{i,j}(z)$ and $\widetilde{\varphi}_j(z)$ are meromorphic functions of finite order. In particular $\widetilde{A}_{k,0} \equiv 1$. Using this, (3.25) and Leibniz formula, we obtain the following for all $j \geq 0$

$$\begin{aligned} \pi^{(j)}(z) &= \sum_{i=0}^{k} \sum_{l=0}^{j} \binom{j}{l} \left(A_i(z)e^{-h(z)}\right)^{(j-l)} f^{(i+l)} - \widetilde{\varphi}_j(z)e^{-h(z)} \\ &= \sum_{i=0}^{k} \sum_{l=0}^{j} \binom{j}{l} \widetilde{A}_{i,j-l}(z)e^{-h(z)} f^{(i+l)} - \widetilde{\varphi}_j(z)e^{-h(z)}. \end{aligned}$$

Substituting this into (3.24) gives

$$\sum_{j=0}^{n} \sum_{i=0}^{k} \sum_{l=0}^{j} \binom{j}{l} B_j(z)\widetilde{A}_{i,j-l}(z)e^{-h(z)} f^{(i+l)} - \sum_{j=0}^{n} B_j(z)\widetilde{\varphi}_j(z)e^{-h(z)} = G(z),$$

where $B_n \equiv 1$. From this, we deduce that $f(z)$ satisfies the following differential equation

$$f^{(k+n)} + \sum_{\substack{0\leqslant i\leqslant k \\ 0\leqslant l\leqslant j\leqslant n \\ (i,l)\neq(k,n)}} \binom{j}{l} B_j(z)\widetilde{A}_{i,j-l}(z) f^{(i+l)} = G(z)e^{h(z)} + \psi(z),$$

where $\psi(z) = \sum_{j=0}^{n} B_j(z)\widetilde{\varphi}_j(z)$. Obviously for $0 \leq i \leq k$, $0 \leq l \leq j \leq n$ and $(i,l) \neq (k,n)$ $\binom{j}{l} B_j(z)\widetilde{A}_{i,j-l}(z)$ and $\psi(z)$ are meromorphic functions of finite order. If $Ge^h + \psi \equiv 0$ we apply Theorem 3.3.1, if not, we apply Corollary 3.3.3. In both cases, the assertion follows. □

Based on Corollary 3.3.5, the next result provides conditions on $B_j(z)$ $(j = 0, 1, \ldots, n-1)$ to ensure that $f(z)$ and its derivative cannot share any polynomial CM.

Corollary 3.3.6. *Suppose that $f^{(k)}(z) = \pi(z)e^{h(z)} + \varphi(z)$, where $k \geq 0$, $h(z)$ and $\varphi(z)$ are entire functions of finite order and $\pi(z)$ is a non-trivial solution of (3.24), where $G \equiv 0$ and $B_j(z)$ $(j = 0, 1, \ldots, n-1)$ are entire functions satisfying one of the following conditions:*

(i) $\max\{\rho(h), \rho(B_j)\ (j = 1, \ldots, n-1)\} < \rho(B_0) \notin \mathbb{N} \cup \{\infty\}$.

(ii) $\max\{\rho(B_j)\ (j = 0, \ldots, n-1)\} < \rho(h) \notin \mathbb{N} \cup \{\infty\}$.

Then $f(z)$ and its derivative cannot share any polynomial CM.

Proof. First, assume that $\max\{\rho(h), \rho(B_j)\ (j = 1, \ldots, n-1)\} < \rho(B_0) \notin \mathbb{N} \cup \{\infty\}$. It follows from Lemma 3.2.3 that every non-trivial solution of (3.24) with $G \equiv 0$ is of infinite order, and $\rho_2(\pi) = \rho(B_0) \notin \mathbb{N} \cup \{\infty\}$. On the other hand, since

$$\rho_2\left(e^h\right) = \rho\left(h\right) < \rho(B_0) = \rho_2(\pi),$$

we deduce that $\rho_2(f) = \rho_2(\pi) \notin \mathbb{N} \cup \{\infty\}$. From this and Corollary 3.3.5, we deduce that $f(z)$ and its derivative cannot share any polynomial CM.

Now, assume that $\max\{\rho(B_j)\ (j = 0, \ldots, n-1)\} < \rho\left(h\right) \notin \mathbb{N} \cup \{\infty\}$, it follows from Lemma 3.2.4 that

$$\rho_2(\pi) \leq \max\{\rho(B_j)\ (j = 0, \ldots, n-1)\} < \rho(h) = \rho_2(e^h).$$

Therefore, $\rho_2(f) = \rho\left(h\right) \notin \mathbb{N} \cup \{\infty\}$ and $f(z)$ is of infinite order. The assertion follows directly from Corollary 3.3.5. $\square$

Chapter 4

Uniqueness of Meromorphic Functions Sharing a Small Function with Their Linear Differential Polynomials

In this chapter, we investigate the problem of small function sharing between meromorphic functions and their linear differential polynomials. By generalizing earlier results given by Al-Khaladi [1], we prove a new result related to Brück's conjecture.

4.1 Preliminary Results

Inspired by the first attempts to solve Brück's conjecture for non-zero value sharing, as represented by Theorems 2.2.5 and 2.2.6, Yu [34] asked whether there exist any similar results for entire or meromorphic functions $f(z)$ if we keep the condition $N\left(r, \frac{1}{f'}\right) = S\left(r, f\right)$ from Theorem 2.2.5, or replace it by $N\left(r, \frac{1}{f}\right) = S\left(r, f\right)$, or with some conditions on $\delta\left(0, f\right)$. In fact, he [34] proved the following result.

Theorem 4.1.1 ([34]). *Suppose that $f(z)$ is a non-constant, non-entire meromorphic function, and $a(z)\,(\not\equiv 0)$ is a small function of $f(z)$ that has no poles in common with $f(z)$. If $f(z)$ share $a(z)$ CM with $f^{(k)}(z)$, where $k \geq 1$, and $4\delta\left(0, f\right) + 2\left(k+8\right)\Theta\left(\infty, f\right) > 2k + 19$, then $f^{(k)} \equiv f$.*

Liu and Gu [22] improved the previous result by establishing the following theorem.

Theorem 4.1.2 ([22]). *Suppose that $f(z)$ is a non-constant meromorphic function, and $a(z)\,(\not\equiv 0)$ is a small function of $f(z)$ that has no poles in common with $f^{(k)}(z)$, where $k \geq 1$, with the same multiplicity. If $f(z)$ share $a(z)$ CM with $f^{(k)}(z)$, and $2\delta\left(0, f\right) + 4\Theta\left(\infty, f\right) > 5$, then $f^{(k)} \equiv f$.*

Zhang and Yang [35] generalized and improved the previous results by replacing $f^{(k)}$ with a linear differential polynomial

$$Q(f) = f^{(k)} + a_{k-1}(z)f^{(k-1)} + \cdots + a_0(z)f, \tag{4.1}$$

where $k \geq 1$ and $a_j(z)$ $(j = 0, 1, \ldots, k-1)$ are polynomials, and obtained the following results.

Theorem 4.1.3 ([35]). *Suppose that $f(z)$ is a non-constant meromorphic function and $Q(f)$ is defined by (4.1). If $f(z)$ share a small function $a(z)\,(\not\equiv 0)$ IM with $Q(f)$, and $5\delta(0,f) + (2k+6)\,\Theta(\infty,f) > 2k+10$, then $Q(f) \equiv f$.*

Theorem 4.1.4 ([35]). *Suppose that $f(z)$ is a non-constant meromorphic function and $Q(f)$ is defined by (4.1). If $f(z)$ share a small function $a(z)\,(\not\equiv 0)$ CM with $Q(f)$, and $2\delta(0,f) + 3\Theta(\infty,f) > 4$, then $Q(f) \equiv f$.*

In [1], Al-Khaladi generalized and improved Theorems 4.1.3 and 4.1.4 by replacing $Q(f)$, defined by (4.1), with

$$L(f) = a_k(z)f^{(k)} + a_{k-1}(z)f^{(k-1)} + \cdots + a_0(z)f, \tag{4.2}$$

where $k \geq 1$, $a_k(z)\,(\not\equiv 0)$ and $a_j(z)$ $(j = 0, 1, \ldots, k)$ are small functions of $f(z)$ and obtained the following results.

Theorem 4.1.5 ([1]). *Suppose that $f(z)$ is a non-constant meromorphic function and $L(f)$ is defined by (4.2). If $f(z)$ share a small function $a(z)\,(\not\equiv 0)$ CM with $L(f)$, and $\delta(0,f) + \Theta(\infty,f) > 1$, then $L(f) \equiv f$.*

Theorem 4.1.6 ([1]). *Suppose that $f(z)$ is a non-constant meromorphic function and $L(f)$ is defined by (4.2). If $f = a \Rightarrow L(f) = a$, where $a(z)\,(\not\equiv 0)$ is a small function of $f(z)$, and $2\delta(0,f) + (k+1)\,\Theta(\infty,f) > k+2$, then $L(f) \equiv f$.*

Theorem 4.1.7 ([1]). *Suppose that $f(z)$ is a non-constant meromorphic function and $L(f)$ is defined by (4.2). If $L(f) = a \Rightarrow f = a$, where $a(z)\,(\not\equiv 0)$ is a small function of $f(z)$, and $2\delta(0,f) + (k+1)\,\Theta(\infty,f) > k+2$, then $L(f) \equiv f$.*

Given Theorems 4.1.5, 4.1.6 and 4.1.7, the following question naturally arises:
Can the conditions $\delta(0,f) + \Theta(\infty,f) > 1$ and $2\delta(0,f) + (k+1)\,\Theta(\infty,f) > k+2$ in Theorem 4.1.5, and Theorems 4.1.6 and 4.1.7, be replaced by $\delta(\varphi,f) + \Theta(\infty,f) > 1$ and $2\delta(\varphi,f) + (k+1)\,\Theta(\infty,f) > k+2$, respectively, where $\varphi(z)$ is a meromorphic function such that $T(r,\varphi) = o(T(r,f))$ as $r \to \infty$?

4.2 Main Results

In this section, we prove theorems that answer our question and give a result related to Brück's conjecture.

In the following, $L(f)$ refers to the linear differential polynomial defined by (4.2). First, we consider the case when $f(z)$ and $L(f)$ share a small function $a(z)$ of $f(z)$ CM.

Theorem 4.2.1. *Let $f(z)$ be a non-constant meromorphic function, and $a(z)$ be a small function of $f(z)$. If $f(z)$ and $L(f)$ share $a(z)$ CM, then for any small function $\varphi(z)$ of $f(z)$ such that $\varphi \not\equiv a$, $L(\varphi) \not\equiv a$ and $\frac{L(f)-a}{f-a} \not\equiv \frac{L(\varphi)-a}{\varphi-a}$, we have*

$$T(r,f) \leq N_{k+1}\left(r, \frac{1}{f-\varphi}\right) + \overline{N}(r,f) + S(r,f). \tag{4.3}$$

Proof. Set

$$\phi = \frac{(\varphi-a)(L(\varphi)-a)}{f-\varphi}\left[\frac{(L(f)-a)'}{L(f)-a} - \frac{(f-a)'}{f-a}\right] - \frac{(\varphi-a)(L(\varphi)-a)' - (\varphi-a)'(L(\varphi)-a)}{f-\varphi} \tag{4.4}$$

$$= (\varphi-a)\frac{L(f)-L(\varphi)}{f-\varphi}\left[\frac{(L(f)-L(\varphi))'}{L(f)-L(\varphi)} - \frac{(L(f)-a)'}{L(f)-a}\right] - (L(\varphi)-a)\left[\frac{(f-\varphi)'}{f-\varphi} - \frac{(f-a)'}{f-a}\right]. \tag{4.5}$$

If $\phi \equiv 0$, then (4.4) implies that

$$\frac{(L(f)-a)'}{L(f)-a} - \frac{(f-a)'}{f-a} = \frac{(L(\varphi)-a)'}{L(\varphi)-a} - \frac{(\varphi-a)'}{\varphi-a}.$$

By integrating this equation, we obtain $L(f) - a = \psi(f-a)$, where $\psi = c\frac{L(\varphi)-a}{\varphi-a}$ and c is a non-zero constant. It follows that $\overline{N}(r,f) = S(r,f)$. Since $\frac{L(f)-a}{f-a} \not\equiv \frac{L(\varphi)-a}{\varphi-a}$, we deduce that $c \neq 1$, and thus we obtain

$$\frac{1}{f-\varphi} = \frac{1}{(c-1)(L(\varphi)-a)}\left[\frac{L(f-\varphi)}{f-\varphi} - \psi\right]. \tag{4.6}$$

Let $z_0 \in \mathbb{C}$ satisfying $a_j(z_0) \neq 0, \infty$ for all $j = 0, 1, \ldots, k$ with $a_j \not\equiv 0$, and $\varphi(z_0) \neq \infty$. If z_0 is a zero of $f - \varphi$ of multiplicity $n \geq k+1$, then z_0 is a pole of multiplicity k of $\frac{L(f-\varphi)}{f-\varphi}$. If z_0 is a zero of $f - \varphi$ of multiplicity $n \leq k$, then z_0 is a pole of multiplicity at most n of $\frac{L(f-\varphi)}{f-\varphi}$. If z_0 is a pole of multiplicity n of f, then z_0 is a pole of multiplicity k of $\frac{L(f-\varphi)}{f-\varphi}$. Hence, from this, the lemma on the logarithmic derivative and (4.6), we obtain

$$\begin{aligned}T(r,f)=T\left(r,\frac{1}{f-\varphi}\right)+S(r,f)\leq T\left(r,\frac{L(f-\varphi)}{f-\varphi}\right)&+S(r,f)\\
&=N\left(r,\frac{L(f-\varphi)}{f-\varphi}\right)+m\left(r,\frac{L(f-\varphi)}{f-\varphi}\right)+S(r,f)\\
&\leq N_{k)}\left(r,\frac{1}{f-\varphi}\right)+k\overline{N}_{(k+1}\left(r,\frac{1}{f-\varphi}\right)+k\overline{N}(r,f)\\
&\quad+S(r,f)\\
&=N_{k}\left(r,\frac{1}{f-\varphi}\right)+S(r,f)\\
&\leq N_{k+1}\left(r,\frac{1}{f-\varphi}\right)+\overline{N}(r,f)+S(r,f).\end{aligned}$$

Then, (4.3) holds.

If $\phi\not\equiv 0$, then by (4.5) and the lemma on the logarithmic derivative, we obtain

$$\begin{aligned}m(r,\phi)\leq&m(r,\varphi-a)+m\left(r,\frac{L(f)-L(\varphi)}{f-\varphi}\right)+m\left(r,\frac{(L(f)-L(\varphi))'}{L(f)-L(\varphi)}\right)\\
&+m\left(r,\frac{(L(f)-a)'}{L(f)-a}\right)+m(r,L(\varphi)-a)+m\left(r,\frac{(f-\varphi)'}{f-\varphi}\right)\\
&+m\left(r,\frac{(f-a)'}{f-a}\right)+O(1)\\
=&S(r,f)+S(r,L(f))\\
=&S(r,f).\end{aligned}$$

Therefore,

$$m(r,\phi)=S(r,f). \tag{4.7}$$

Similarly, from (4.4) and the lemma on the logarithmic derivative, we obtain

$$m(r,(f-\varphi)\phi)=S(r,f).$$

Consequently,

$$\begin{aligned}m(r,f)&\leq m(r,f-\varphi)+S(r,f)\\
&=m\left(r,\frac{(f-\varphi)\phi}{\phi}\right)+S(r,f)\\
&\leq m\left(r,\frac{1}{\phi}\right)+S(r,f).\end{aligned}\tag{4.8}$$

If z_∞ is a pole of f of multiplicity $p\geq 1$ satisfying $a(z_\infty)\neq\infty$, $\varphi(z_\infty)\neq\infty$ and $a_j(z_\infty)\neq 0,\infty$

for all $j = 0, 1, \ldots, k$ such that $a_j \not\equiv 0$, then from (4.4) we deduce that z_∞ is a zero of multiplicity at least $p-1$ of ϕ. Hence, we have

$$N(r, f) - \overline{N}(r, f) \leq N\left(r, \frac{1}{\phi}\right) + S(r, f). \tag{4.9}$$

Let $z_0 \in \mathbb{C}$ satisfying $(\varphi - a)(z_0) \neq 0, \infty$, $(L(\varphi) - a)(z_0) \neq 0, \infty$ and $a_j(z_0) \neq 0, \infty$ for all $j = 0, 1, \ldots, k$ with $a_j \not\equiv 0$.
We note that $f - a$ (or $L(f) - a$) cannot have z_0 as a common zero with $f - \varphi$ or with $L(f) - L(\varphi)$ since $(\varphi - a)(z_0) \neq 0$ and $(L(\varphi) - a)(z_0) \neq 0$.
If z_0 is a zero of $f - a$, then, since $f - a$ and $L(f) - a$ share 0 CM, it follows from (4.4) that ϕ is holomorphic at z_0.
If z_0 is a zero of $f - \varphi$ of multiplicity $n \geq k+1$, then, from (4.5), we deduce that z_0 is a pole of multiplicity $k+1$ of ϕ.
If z_0 is a zero of $f - \varphi$ of multiplicity $n \leq k$, then, from (4.4), we deduce that z_0 is a pole of multiplicity at most n of ϕ. Hence, we obtain

$$\begin{aligned} N(r, \phi) &\leq (k+1)\overline{N}_{(k+1}\left(r, \frac{1}{f-\varphi}\right) + N_{k)}\left(r, \frac{1}{f-\varphi}\right) + S(r, f) \\ &= N_{k+1}\left(r, \frac{1}{f-\varphi}\right) + S(r, f). \end{aligned} \tag{4.10}$$

By summing (4.8) and (4.9), we obtain

$$\begin{aligned} T(r, f) - \overline{N}(r, f) &\leq T\left(r, \frac{1}{\phi}\right) + S(r, f) \\ &= T(r, \phi) + S(r, f) \\ &= N(r, \phi) + m(r, \phi) + S(r, f). \end{aligned}$$

From this, (4.7) and (4.10), we obtain

$$T(r, f) \leq N_{k+1}\left(r, \frac{1}{f-\varphi}\right) + \overline{N}(r, f) + S(r, f).$$

This completes the proof. □

The following results are immediate consequences of Theorem 4.2.1.

Corollary 4.2.1. *Let $f(z)$ be a non-constant meromorphic function, and $a(z)$ and $\varphi(z)$ be two small functions of $f(z)$ such that $a \not\equiv \varphi$ and $a \not\equiv L(\varphi)$. If $f(z)$ and $L(f)$ share $a(z)$ CM, and*

$$N_{k+1}\left(r, \frac{1}{f-\varphi}\right) + \overline{N}(r, f) \leq (\lambda + o(1))\, T(r, f),$$

as $r \to \infty$ and $r \in I$, where $\lambda < 1$ and $I \subset [0, \infty)$ is a set of infinite linear measure, then

$$\frac{L(f) - a}{f - a} \equiv \frac{L(\varphi) - a}{\varphi - a}.$$

Corollary 4.2.2. *Let $f(z)$ be a non-constant meromorphic function, $a(z)$ be a small function of $f(z)$, and $\varphi(z)$ be a meromorphic function such that $T(r, \varphi) = o(T(r, f))$ as $r \to \infty$, with $\varphi \not\equiv a$ and $L(\varphi) \not\equiv a$. If $f(z)$ and $L(f)$ share $a(z)$ CM, and $\delta_{k+1}(\varphi, f) + \Theta(\infty, f) > 1$, then*

$$\frac{L(f) - a}{f - a} \equiv \frac{L(\varphi) - a}{\varphi - a}.$$

Corollary 4.2.3. *Let $f(z)$ be a non-constant meromorphic function, $a(z)$ be a small function of $f(z)$, and $\varphi(z)$ be a meromorphic function such that $T(r, \varphi) = o(T(r, f))$ as $r \to \infty$, with $\varphi \not\equiv a$ and $L(\varphi) \not\equiv a$. If $f(z)$ and $L(f)$ share $a(z)$ CM, and $\delta(\varphi, f) + \Theta(\infty, f) > 1$, then*

$$\frac{L(f) - a}{f - a} \equiv \frac{L(\varphi) - a}{\varphi - a}.$$

Remark 4.2.1. *If we set $\varphi \equiv 0$ in Theorem 4.2.1, Corollaries 4.2.1 and 4.2.3, we obtain [1, Theorem 1.7 ; Corollary 1.8] and Theorem 4.1.5, respectively.*

The following example shows the sharpness of the inequalities $\delta_{k+1}(\varphi, f) + \Theta(\infty, f) > 1$ and $\delta(\varphi, f) + \Theta(\infty, f) > 1$ in Corollaries 4.2.2 and 4.2.3, respectively.

Example 4.2.1. *Let*

$$f(z) = \frac{z\left(e^{2z} - 2z + 2\right)}{2\left(e^{z} - 1\right)^{2}}, \quad L(f) = f' + \left(2 - \frac{1}{z}\right) f, \quad a(z) = z \quad \text{and} \quad \varphi(z) = z - z^{2}.$$

Here, $f(z)$ and $L(f)$ share $a(z)$ CM, $\delta_2(\varphi, f) = \delta(\varphi, f)$, and $\delta(\varphi, f) + \Theta(\infty, f) = \frac{1}{2} + \frac{1}{2} = 1$. However,

$$\frac{L(f) - a}{f - a} = \frac{-2}{e^{z} - 1} \not\equiv 2 = \frac{L(\varphi) - a}{\varphi - a}.$$

Our next result deals with the case when $f = a \Rightarrow L(f) = a$, where $a(z)$ is a small function of $f(z)$.

Theorem 4.2.2. *Let $f(z)$ be a non-constant meromorphic function, and $a(z)$ be a small function of $f(z)$. If $f = a \Rightarrow L(f) = a$, then for any small function $\varphi(z)$ of $f(z)$ such that $\varphi \not\equiv a$, $L(\varphi) \not\equiv a$ and $\frac{L(f)-a}{f-a} \not\equiv \frac{L(\varphi)-a}{\varphi-a}$, we have*

$$T(r, f) \leq N_{k+1}\left(r, \frac{1}{f - \varphi}\right) + \overline{N}_{k)}\left(r, \frac{1}{f - \varphi}\right) + (k+1)\,\overline{N}(r, f) + S(r, f). \tag{4.11}$$

Proof. We set

$$\alpha = \frac{\left[\frac{(f-\varphi)'}{f-\varphi} - \frac{(\varphi-a)'}{\varphi-a}\right]\left[L(f-\varphi) - \frac{L(\varphi)-a}{\varphi-a}(f-\varphi)\right]}{f-a} \tag{4.12}$$

$$= \frac{\left[\frac{(f-a)'}{f-a} - \frac{(\varphi-a)'}{\varphi-a}\right]\left[L(f) - a - \frac{L(\varphi)-a}{\varphi-a}(f-a)\right]}{f-\varphi} \tag{4.13}$$

$$= \left[\frac{(f-a)'}{f-a} - \frac{(\varphi-a)'}{\varphi-a}\right]\left[\frac{L(f-\varphi)}{f-\varphi} - \frac{L(\varphi)-a}{\varphi-a}\right]. \tag{4.14}$$

From (4.14) and the lemma on the logarithmic derivative, it follows that

$$m(r,\alpha) = S(r,f). \tag{4.15}$$

If $\alpha \equiv 0$, then since $L(f) - a - \frac{L(\varphi)-a}{\varphi-a}(f-a) \not\equiv 0$, it follows from (4.13) that $\frac{(f-a)'}{f-a} \equiv \frac{(\varphi-a)'}{\varphi-a}$. Hence, $f - a = c(\varphi - a)$, where c is a non-zero constant. This leads to the contradiction $T(r,f) = S(r,f)$. Therefore, $\alpha \not\equiv 0$.

From (4.14), we obtain

$$\frac{1}{f-\varphi} = \frac{1}{\alpha}\left[\frac{1}{\varphi-a}\frac{(f-\varphi)'}{f-\varphi} - \frac{1}{\varphi-a}\frac{(f-a)'}{f-a}\right]\left[\frac{L(f-\varphi)}{f-\varphi} - \frac{L(\varphi)-a}{\varphi-a}\right].$$

From this and the lemma on the logarithmic derivative, we obtain

$$m\left(r, \frac{1}{f-\varphi}\right) \le m\left(r, \frac{1}{\alpha}\right) + S(r,f). \tag{4.16}$$

If z_∞ is a pole of f of multiplicity $p \ge 1$ satisfying $(\varphi - a)(z_\infty) \ne 0, \infty$, $(L(\varphi) - a)(z_\infty) \ne 0, \infty$ and $a_j(z_\infty) \ne 0, \infty$ for all $j = 0, 1, \ldots, k$ such that $a_j \not\equiv 0$, then from (4.14) we deduce that z_∞ is a pole of multiplicity $k+1$ of α.

Let $z_0 \in \mathbb{C}$ satisfying $(\varphi - a)(z_0) \ne 0, \infty$, $(L(\varphi) - a)(z_0) \ne 0, \infty$ and $a_j(z_0) \ne 0, \infty$ for all $j = 0, 1, \ldots, k$ with $a_j \not\equiv 0$.

We note that $f - a$ and $f - \varphi$ cannot have z_0 as a common zero since $(\varphi - a)(z_0) \ne 0$.

If z_0 is a zero of $f - a$, then, since $f = a \Rightarrow L(f) = a$, it follows from (4.13) that α is holomorphic at z_0.

If z_0 is a zero of $f - \varphi$ of multiplicity $n \ge k+1$, then from (4.12) we see that z_0 is a zero of multiplicity $n - k - 1$ of α.

If z_0 is a zero of $f - \varphi$ of multiplicity $n \le k$, then from (4.12) we deduce that z_0 is a pole of multiplicity at most 1 of α.

Hence, we obtain

$$N(r,\alpha) \le \overline{N}_{k)}\left(r, \frac{1}{f-\varphi}\right) + (k+1)\overline{N}(r,f) + S(r,f), \tag{4.17}$$

and

$$N_{(k+1}\left(r,\frac{1}{f-\varphi}\right)-(k+1)\,\overline{N}_{(k+1}\left(r,\frac{1}{f-\varphi}\right)\leq N\left(r,\frac{1}{\alpha}\right)+S\left(r,f\right). \tag{4.18}$$

By summing (4.16) and (4.18) and using (4.15), we obtain

$$\begin{aligned} m\left(r,\frac{1}{f-\varphi}\right)+N_{(k+1}\left(r,\frac{1}{f-\varphi}\right)-(k+1)\,\overline{N}_{(k+1}\left(r,\frac{1}{f-\varphi}\right) &\leq T\left(r,\alpha\right)+S\left(r,f\right) \\ &= N\left(r,\alpha\right)+S\left(r,f\right). \end{aligned} \tag{4.19}$$

Since

$$N\left(r,\frac{1}{f-\varphi}\right)=N_{k)}\left(r,\frac{1}{f-\varphi}\right)+N_{(k+1}\left(r,\frac{1}{f-\varphi}\right),$$

and

$$(k+1)\,\overline{N}_{(k+1}\left(r,\frac{1}{f-\varphi}\right)+N_{k)}\left(r,\frac{1}{f-\varphi}\right)=N_{k+1}\left(r,\frac{1}{f-\varphi}\right),$$

then, from (4.17) and (4.19), we obtain

$$T\left(r,f\right)\leq N_{k+1}\left(r,\frac{1}{f-\varphi}\right)+\overline{N}_{k)}\left(r,\frac{1}{f-\varphi}\right)+(k+1)\,\overline{N}\left(r,f\right)+S\left(r,f\right).$$

Thus, the proof is complete. □

The following results are a direct consequence of Theorem 4.2.2.

Corollary 4.2.4. *Let $f(z)$ be a non-constant meromorphic function, $a(z)$ be a small function of $f(z)$, and $\varphi(z)$ be a meromorphic function such that $T\left(r,\varphi\right)=o\left(T\left(r,f\right)\right)$ as $r\to\infty$, with $\varphi\not\equiv a$ and $L\left(\varphi\right)\not\equiv a$. If $f=a\Rightarrow L\left(f\right)=a$, and $\delta_{k+1}\left(\varphi,f\right)+\Theta_{k)}\left(\varphi,f\right)+(k+1)\,\Theta\left(\infty,f\right)>k+2$, then*

$$\frac{L\left(f\right)-a}{f-a}\equiv\frac{L\left(\varphi\right)-a}{\varphi-a}.$$

Corollary 4.2.5. *Let $f(z)$ be a non-constant meromorphic function, $a(z)$ be a small function of $f(z)$, and $\varphi(z)$ be a meromorphic function such that $T\left(r,\varphi\right)=o\left(T\left(r,f\right)\right)$ as $r\to\infty$, with $\varphi\not\equiv a$ and $L\left(\varphi\right)\not\equiv a$. If $f=a\Rightarrow L\left(f\right)=a$, and $2\delta\left(\varphi,f\right)+(k+1)\,\Theta\left(\infty,f\right)>k+2$, then*

$$\frac{L\left(f\right)-a}{f-a}\equiv\frac{L\left(\varphi\right)-a}{\varphi-a}.$$

Remark 4.2.2. *If we set $\varphi\equiv0$ in Theorem 4.2.2 and Corollary 4.2.5, we obtain [1, Theorem 1.10] and Theorem 4.1.6, respectively.*

Similar to Theorem 4.2.2, the next result deals with the case when $L(f) = a \Rightarrow f = a$, where $a(z)$ is a small function of $f(z)$.

Theorem 4.2.3. *Let $f(z)$ be a non-constant meromorphic function, and $a(z)$ be a small function of $f(z)$. If $L(f) = a \Rightarrow f = a$, then for any small function $\varphi(z)$ of $f(z)$ such that $\varphi \not\equiv a$, $L(\varphi) \not\equiv a$ and $\frac{L(f)-a}{f-a} \not\equiv \frac{L(\varphi)-a}{\varphi-a}$, we have*

$$T(r,f) \le N_{2k+1}\left(r, \frac{1}{f-\varphi}\right) + N_{2k)}\left(r, \frac{1}{f-\varphi}\right) + (k+1)\overline{N}(r,f) + S(r,f). \tag{4.20}$$

Proof. We set

$$\beta = \frac{\left[\frac{(L(f)-L(\varphi))'}{f-\varphi} - \frac{(L(\varphi)-a)'}{L(\varphi)-a}\frac{L(f)-L(\varphi)}{f-\varphi}\right]\left[L(f-\varphi) - \frac{L(\varphi)-a}{\varphi-a}(f-\varphi)\right]}{L(f)-a} \tag{4.21}$$

$$= \frac{\left[\frac{(L(f)-a)'}{L(f)-a} - \frac{(L(\varphi)-a)'}{L(\varphi)-a}\right]\left[L(f) - a - \frac{L(\varphi)-a}{\varphi-a}(f-a)\right]}{f-\varphi} \tag{4.22}$$

$$= \left[\frac{(L(f)-a)'}{L(f)-a} - \frac{(L(\varphi)-a)'}{L(\varphi)-a}\right]\left[\frac{L(f-\varphi)}{f-\varphi} - \frac{L(\varphi)-a}{\varphi-a}\right]. \tag{4.23}$$

From (4.23) and the lemma on the logarithmic derivative, it follows that

$$m(r,\beta) = S(r,f). \tag{4.24}$$

If $\beta \equiv 0$, then since $L(f) - a - \frac{L(\varphi)-a}{\varphi-a}(f-a) \not\equiv 0$, it follows from (4.22) that $\frac{(L(f)-a)'}{L(f)-a} \equiv \frac{(L(\varphi)-a)'}{L(\varphi)-a}$. Hence, $L(f) - a = c(L(\varphi) - a)$, where c is a non-zero constant. Therefore, $T(r, L(f)) = S(r,f)$. From this, we obtain

$$\overline{N}(r,f) = S(r,f), \tag{4.25}$$

and

$$\begin{aligned} T(r,f) &= T\left(r, \frac{1}{f-\varphi}\right) + S(r,f) = N\left(r, \frac{1}{f-\varphi}\right) + m\left(r, \frac{1}{f-\varphi}\right) + S(r,f) \\ &\le N_{k)}\left(r, \frac{1}{f-\varphi}\right) + N_{(k+1}\left(r, \frac{1}{f-\varphi}\right) + m\left(r, \frac{L(f-\varphi)}{f-\varphi}\right) + m\left(r, \frac{1}{L(f-\varphi)}\right) \\ &\quad + S(r,f) \\ &\le N_{k)}\left(r, \frac{1}{f-\varphi}\right) + N_{(k+1}\left(r, \frac{1}{f-\varphi}\right) + m\left(r, \frac{L(f-\varphi)}{f-\varphi}\right) + T(r, L(f)) \\ &\quad + S(r,f) \\ &= N_{k)}\left(r, \frac{1}{f-\varphi}\right) + N_{(k+1}\left(r, \frac{1}{f-\varphi}\right) + S(r,f). \end{aligned} \tag{4.26}$$

Let z_0 be a zero of multiplicity $n \ge k+1$ of $f - \varphi$ satisfying $a_j(z_0) \ne 0, \infty$ for all $j = 0, 1, \ldots, k$

such that $a_j \not\equiv 0$, then z_0 is counted n times by $N_{(k+1}\left(r, \frac{1}{f-\varphi}\right)$ and $n-k$ times by $N\left(r, \frac{1}{L(f-\varphi)}\right)$. Since $n \leq (k+1)(n-k)$, we deduce that

$$\begin{aligned} N_{(k+1}\left(r, \frac{1}{f-\varphi}\right) &\leq (k+1)\, N\left(r, \frac{1}{L(f-\varphi)}\right) + S(r,f) \\ &\leq (k+1)\, T(r, L(f)) + S(r,f) \\ &= S(r,f). \end{aligned} \tag{4.27}$$

In addition, we have

$$N_{k)}\left(r, \frac{1}{f-\varphi}\right) \leq N_{2k+1}\left(r, \frac{1}{f-\varphi}\right) + N_{2k)}\left(r, \frac{1}{f-\varphi}\right). \tag{4.28}$$

By (4.25), (4.26), (4.27) and (4.28), it follows that

$$T(r,f) \leq N_{2k+1}\left(r, \frac{1}{f-\varphi}\right) + N_{2k)}\left(r, \frac{1}{f-\varphi}\right) + (k+1)\,\overline{N}(r,f) + S(r,f).$$

Therefore, (4.20) holds.

If $\beta \not\equiv 0$, then, by simple calculation, we obtain the following from (4.23)

$$\frac{1}{f-\varphi} = \frac{1}{\beta\,(L(\varphi)-a)} \frac{L(f-\varphi)}{f-\varphi}\left[\frac{(L(f)-L(\varphi))'}{L(f)-L(\varphi)} - \frac{(L(f)-a)'}{L(f)-a}\right]\left[\frac{L(f-\varphi)}{f-\varphi} - \frac{L(\varphi)-a}{\varphi-a}\right].$$

From this and the lemma on the logarithmic derivative, we obtain

$$m\left(r, \frac{1}{f-\varphi}\right) \leq m\left(r, \frac{1}{\beta}\right) + S(r,f). \tag{4.29}$$

If z_∞ is a pole of f of multiplicity $p \geq 1$ satisfying $(\varphi - a)(z_\infty) \neq 0, \infty$, $(L(\varphi)-a)(z_\infty) \neq 0, \infty$ and $a_j(z_\infty) \neq 0, \infty$ for all $j = 0, 1, \ldots, k$ such that $a_j \not\equiv 0$, then from (4.23) we deduce that z_∞ is a pole of multiplicity $k+1$ of β.

Let $z_0 \in \mathbb{C}$ satisfying $(\varphi - a)(z_0) \neq 0, \infty$, $(L(\varphi)-a)(z_0) \neq 0, \infty$ and $a_j(z_0) \neq 0, \infty$ for all $j = 0, 1, \ldots, k$ with $a_j \not\equiv 0$.

We note that $L(f) - a$ and $f - \varphi$ cannot have z_0 as a common zero since $L(f) = a \Rightarrow f = a$ and $(\varphi - a)(z_0) \neq 0$.

If z_0 is a zero of $L(f) - a$, then since $L(f) = a \Rightarrow f = a$, it follows from (4.22) that β is holomorphic at z_0.

If z_0 is a zero of $f - \varphi$ of multiplicity $n \geq 2k+1$, then from (4.21) we see that z_0 is a zero of multiplicity $n - 2k - 1$ of β.

If z_0 is a zero of $f - \varphi$ of multiplicity $n \leq 2k$, then from (4.21) we deduce that z_0 is a pole of multiplicity at most n of β.

Hence, we obtain

$$N(r,\beta) \le N_{2k)}\left(r,\frac{1}{f-\varphi}\right) + (k+1)\,\overline{N}(r,f) + S(r,f), \tag{4.30}$$

and

$$N_{(2k+1}\left(r,\frac{1}{f-\varphi}\right) - (2k+1)\,\overline{N}_{(2k+1}\left(r,\frac{1}{f-\varphi}\right) \le N\left(r,\frac{1}{\beta}\right) + S(r,f). \tag{4.31}$$

By summing (4.29) and (4.31) and using (4.24), we obtain

$$\begin{aligned} m\left(r,\frac{1}{f-\varphi}\right) + N_{(2k+1}\left(r,\frac{1}{f-\varphi}\right) - (2k+1)\,\overline{N}_{(2k+1}\left(r,\frac{1}{f-\varphi}\right) &\le T(r,\beta) + S(r,f) \\ &= N(r,\beta) + S(r,f). \end{aligned} \tag{4.32}$$

Since

$$N\left(r,\frac{1}{f-\varphi}\right) = N_{2k)}\left(r,\frac{1}{f-\varphi}\right) + N_{(2k+1}\left(r,\frac{1}{f-\varphi}\right),$$

and

$$(2k+1)\,\overline{N}_{(2k+1}\left(r,\frac{1}{f-\varphi}\right) + N_{2k)}\left(r,\frac{1}{f-\varphi}\right) = N_{2k+1}\left(r,\frac{1}{f-\varphi}\right),$$

then, from (4.30) and (4.32), we obtain

$$T(r,f) \le N_{2k+1}\left(r,\frac{1}{f-\varphi}\right) + N_{2k)}\left(r,\frac{1}{f-\varphi}\right) + (k+1)\,\overline{N}(r,f) + S(r,f).$$

Thus, Theorem 4.2.3 is proved. □

Theorem 4.2.3 leads to the following consequences.

Corollary 4.2.6. *Let $f(z)$ be a non-constant meromorphic function, $a(z)$ be a small function of $f(z)$, and $\varphi(z)$ be a meromorphic function such that $T(r,\varphi) = o(T(r,f))$ as $r \to \infty$, with $\varphi \not\equiv a$ and $L(\varphi) \not\equiv a$. If $L(f) = a \Rightarrow f = a$, and $\delta_{2k+1}(\varphi,f) + \delta_{2k)}(\varphi,f) + (k+1)\,\Theta(\infty,f) > k+2$, then*

$$\frac{L(f)-a}{f-a} \equiv \frac{L(\varphi)-a}{\varphi-a}.$$

Corollary 4.2.7. *Let $f(z)$ be a non-constant meromorphic function, $a(z)$ be a small function of $f(z)$, and $\varphi(z)$ be a meromorphic function such that $T(r,\varphi) = o(T(r,f))$ as $r \to \infty$, with $\varphi \not\equiv a$ and $L(\varphi) \not\equiv a$. If $L(f) = a \Rightarrow f = a$, and $2\delta(\varphi,f) + (k+1)\,\Theta(\infty,f) > k+2$, then*

$$\frac{L(f)-a}{f-a} \equiv \frac{L(\varphi)-a}{\varphi-a}.$$

Remark 4.2.3. *If we set $\varphi \equiv 0$ in Theorem 4.2.3 and Corollary 4.2.7, we obtain [1, Theorem 1.12] and Theorem 4.1.7, respectively.*

Next, we state consequences of Theorems 4.2.2 and 4.2.3 in the case where $f(z)$ is an entire function.

Corollary 4.2.8. *Let $f(z)$ be a non-constant entire function, $a(z)$ and $\varphi(z)$ be two small functions of $f(z)$ such that $\varphi \not\equiv a$ and $L(\varphi) \not\equiv a$. If either $f = a \Rightarrow L(f) = a$ or $L(f) = a \Rightarrow f = a$, and if $\overline{N}\left(r, \frac{1}{f-\varphi}\right) = S(r, f)$, then*

$$\frac{L(f) - a}{f - a} \equiv \frac{L(\varphi) - a}{\varphi - a}.$$

Corollary 4.2.9. *Let $f(z)$ be a non-constant entire function, $a(z)$ and $\varphi(z)$ be two small functions of $f(z)$ such that $\varphi \not\equiv a$ and $L(\varphi) \not\equiv a$. If either $f = a \Rightarrow L(f) = a$ or $L(f) = a \Rightarrow f = a$, and if $\overline{\lambda}_p(f - \varphi) < \rho_p(f)$ for some $p \in \mathbb{N}$, then*

$$\frac{L(f) - a}{f - a} \equiv \frac{L(\varphi) - a}{\varphi - a}.$$

Clearly, the conclusion of Corollaries 4.2.8 and 4.2.9 remains valid if the condition "$f = a \Rightarrow L(f) = a$ or $L(f) = a \Rightarrow f = a$" is replaced by "$L(f)$ and $f(z)$ share $a(z)$ IM (or CM)".

Since Brück's conjecture has been proved for entire functions of finite order, next we suppose that $f(z)$ is of infinite order. From Corollaries 4.2.8 and 4.2.9, we obtain the following result related to Brück's conjecture.

Corollary 4.2.10. *Let $f(z)$ be an entire function of infinite order with hyper-order $\rho_2(f) \notin \mathbb{N} \cup \{\infty\}$. If there exists a small function $\varphi(z)$ of $f(z)$, not of the form $az + b$, where $a, b \in \mathbb{C}$, that satisfies one of the following conditions:*

(i) $\overline{N}\left(r, \frac{1}{f-\varphi}\right) = S(r, f)$.

(ii) $\overline{\lambda}(f - \varphi) < \infty$.

(iii) $\overline{\lambda}_2(f - \varphi) < \rho_2(f)$.

Then $f(z)$ and $f'(z)$ cannot share any finite value CM.

Proof. Assume that $f(z)$ and $f'(z)$ share a finite value a CM. Then

$$\frac{f' - a}{f - a} = e^g, \tag{4.33}$$

where $g(z)$ is an entire function. Since $\varphi \not\equiv a$ and $\varphi' \not\equiv a$, it follows from the conditions (i), (ii) and (iii), along with Corollaries 4.2.8 and 4.2.9, that

$$\frac{f'-a}{f-a} = \frac{\varphi'-a}{\varphi-a}.$$

From this and (4.33), we obtain

$$\frac{(f-\varphi)'}{f-\varphi} = e^{g}. \tag{4.34}$$

By integrating this equation, we find that

$$f = be^{G} + \varphi, \tag{4.35}$$

where $b \in \mathbb{C} \setminus \{0\}$ and G is a primitive of e^g. Since $\varphi(z)$ is a small function of $f(z)$, we deduce that $\rho\left(e^g\right) = \rho_2\left(e^G\right) = \rho_2\left(f\right) \notin \mathbb{N} \cup \{\infty\}$. Therefore, $g(z)$ is constant. Hence, $e^g \equiv c \in \mathbb{C} \setminus \{0\}$. From this and (4.35), it follows that $f = \alpha e^{cz} + \varphi$, where $\alpha \in \mathbb{C} \setminus \{0\}$. From this, we see that $\rho\left(f\right) = 1 < \infty$, which is a contradiction. Therefore, $f(z)$ and its derivative cannot share any finite value CM. $\square$

Conclusion

In this thesis, we considered Brück's conjecture for entire solutions of linear differential equations. To establish our results, we employed tools from Nevanlinna theory and uniqueness theory of meromorphic functions. As the conjecture was proved for entire functions of finite order, we restricted our study to the case of infinite order. Accordingly, we rephrased the conjecture as follows: *If an entire function $f(z)$ satisfies*

$$\rho(f) = \infty \quad \text{and} \quad \rho_2(f) \notin \mathbb{N} \cup \{\infty\}, \tag{$\star$}$$

then $f(z)$ and $f'(z)$ cannot share any finite value CM.

In general, this conjecture remains open. In our study, we gave a positive answer to this conjecture for a "large" class of entire functions satisfying ($\star$) and being solutions of homogeneous linear differential equations with meromorphic coefficients of finite order.

This work is done in three steps: First, we address entire solutions of homogeneous second-order linear differential equations. Then, we extended the second-order case to cover entire solutions of both homogeneous and non-homogeneous higher-order linear differential equations in the context of polynomial sharing. We demonstrated that while certain functions do not satisfy the conditions of the second-order case, they satisfy the condition of the higher-order case, which shows that the higher-order case is indeed a generalization of the second-order case. The proofs provided for the second-order are simpler than the higher-order case.

Furthermore, we generalized earlier results on meromorphic functions sharing a small function with one of their linear differential polynomials, and used these results to confirm the above conjecture for a certain class of entire functions.

Knowing that any entire function is a solution of a linear differential equation, it is natural to ask whether it is always possible (or not) that $f(z)$ is a solution of a homogeneous linear differential equation with meromorphic coefficients of finite order and satisfies ($\star$).

It is also natural to ask whether we can prove the above conjecture for entire functions $f(z)$ satisfying ($\star$) and being solutions of homogeneous linear differential equations whose coefficients are small meromorphic functions of $f(z)$.

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