International Journal of Modern Mathematical Sciences, 2013, 7(2): 195-211

Modern Scientific Press

International Journal of Modern Mathematical Sciences **ISSN: 2166-286X**

Journal homepage: www.ModernScientificPress.com/Journals/ijmms.aspx *Florida, USA*

Article

New Results for a Boundary Value Problem for Differential Equations of Arbitrary Order

Mohamed Houas [1], Zoubir Dahmani [2], *, Maamar Benbachir [1]

[1] *Department of Mathematics, University of Khemis Miliana, Algeria*

[2] *LPAM, Faculty SEI, UMAB Mostaganem, Algeria*

* Author to whom correspondence should be addressed; Email: zzdahmani@yahoo.fr

Article history: Received 1 May 2013, Received in revised form 7 July 2013, Accepted 10 July 2013, Published 12 July 2013.

Abstract: In this paper, we study the existence and uniqueness of solutions for a nonlinear fractional boundary value problem. New results are established using a Banach contraction principle and Schaefer's fixed point theorem. An illustrative example is also presented.

Keywords: Caputo derivative; Banach fixed point; fractional differential equation; existence; uniqueness.

Mathematics Subject Classification (2010): 34A08, 34B25.

1. Introduction

The theory of differential equations of fractional order has been shown to be very useful in the study of models of many phenomena in various fields of science and engineering, such as electrochemistry, physics, chemistry, visco-elasticity, control, image and signal processing, biophysics. For more details, we refer the reader to [2, 4, 7, 8, 9, 13, 14, 15] and references therein. Recently, there has been a significant progress in the investigation of these equations (see [5, 6, 18]). More recently, some basic theory for the initial boundary value problems of fractional differential equations has been discussed in [1, 10, 11, 13, 19]. Moreover, existence and uniqueness of solutions to boundary value problems for fractional differential equations had attracted the attention of many authors, see for example, [3, 5, 6, 12, 13, 17] and the references therein. Motivated by the problem (1) in [12], this paper deals with the existence and uniqueness of solutions for the following problem:

$$\begin{aligned}&D^{\alpha}x(t)+f\left(t,x(t),D^{\beta}x(t)\right)=0,t\in J,\\&x(0)=0,x(1)-\lambda_1 x(\eta)=0,\\&x^{''}(0)=0,x^{''}(1)-\lambda_2 x^{''}(\xi)=0,\end{aligned}\tag{1.1}$$

where $3<\alpha\leq 4,\beta\leq\alpha-1,0<\eta,\xi<1,$ an D^{α},D^{β} are the Caputo fractional derivatives, $J=[0,1],\lambda_1,\lambda_2$ are real constants with $\lambda_1\eta\neq 1,\lambda_2\xi\neq 1$ and f continuous function on $J\times\Re^2$. The paper is organized as follows. In section 2, we present some preliminaries and lemmas. Section 3 is devoted to the existence of solution of (1.1). In section 4, we will give an example to illustrate our main results.

2. Preliminaries

Definition 2.1. The Riemann-Liouville fractional integral operator of order $\alpha\geq 0$, for a continuous function f on $[0,\infty[$ is defined as:

$$J^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}f(\tau)d\tau,\alpha>0,\tag{2.1}$$

$$J^0 f(t)=f(t),$$

where $\Gamma(\alpha):=\int_0^{\infty}e^{-u}u^{\alpha-1}du.$

Definition 2.2. The fractional derivative of $f\in C^n([0,\infty[)$ in the sense of Caputo is defined as:

$$D^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t(t-\tau)^{n-\alpha-1}f^{(n)}(\tau)d\tau,n-1<\alpha,n\in N^{*}.\tag{2.2}$$

For more details, we refer the reader to [15, 16].

Let us now introduce the following Banach space $X=\{x\ :\ x\in C(J),D^{\beta}x\in C(J)\}$, endowed with the norm $\|x\|_X=\|x\|+\|D^{\beta}x\|;\|x\|=\sup_{t\in J}|x(t)|,\|D^{\beta}x\|=\sup_{t\in J}|D^{\beta}x(t)|.$

We give the following lemmas [9]:

Lemma 2.3. For $\alpha>0,$ the general solution of the fractional differential equation $D^{\alpha}x(t)=0$ is given by

$$x(t)=c_0+c_1t+c_2t^2+\ldots+c_{n-1}t^{n-1},\tag{2.3}$$

where $c_i\in\Re,i=0,1,2,..,n-1,n=[\alpha]+1.$

Lemma 2.4. Let $\alpha > 0.$ Then

$$J^{\alpha} D^{\alpha} x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1}, \tag{2.4}$$

for some $c_i \in \mathrm{R}, i = 0,1,2,\ldots,n-1, n = [\alpha]+1.$

We give also the following result:

Lemma 2.5. Let $g \in C(J),$ the solution of the equation

$$D^{\alpha} x(t) + g(t) = 0, t \in J, 3 < \alpha \leq 4, \tag{2.5}$$

such that

$$x(0) = 0, x(1) - \lambda_1 x(\eta) = 0, \tag{2.6}$$

$$x''(0) = 0, x''(1) - \lambda_2 x''(\xi) = 0.$$

is given by:

$$\begin{aligned} x(t) = & -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(s, x(s), D^{\beta} x(s)\right) ds \\ & + \frac{\lambda_1 t}{(\lambda_1 \eta - 1)\Gamma(\alpha)} \int_0^{\eta} (\eta - s)^{\alpha-1} f\left(s, x(s), D^{\beta} x(s)\right) ds \\ & - \frac{1}{(\lambda_1 \eta - 1)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f\left(s, x(s), D^{\beta} x(s)\right) ds \\ & + \frac{\left(\lambda_2 - \lambda_2 \lambda_1 \eta^3\right) t + \left(\lambda_2 \lambda_1 \eta - \lambda_2\right) t^3}{6(\lambda_1 \eta - 1)(\lambda_2 \xi - 1)\Gamma(\alpha - 2)} \int_0^{\xi} (\xi - s)^{\alpha-3} f\left(s, x(s), D^{\beta} x(s)\right) ds \\ & - \frac{\left(1 - \lambda_1 \eta^3\right) t + (\lambda_1 \eta - 1) t^3}{6(\lambda_1 \eta - 1)(\lambda_2 \xi - 1)\Gamma(\alpha - 2)} \int_0^1 (1-s)^{\alpha-3} f\left(s, x(s), D^{\beta} x(s)\right) ds. \end{aligned} \tag{2.7}$$

Proof. For $c_i \in \mathrm{R}, i = 0,1,2,3,$ and by lemmas 3, 4, the general solution of (2.5) is given by

$$x(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) ds - c_0 - c_1 t - c_2 t^2 - c_3 t^3. \tag{2.8}$$

Using the boundary condition (2.6) we have $c_0 = c_2 = 0,$ and

$$\begin{aligned} c_1 = & -\frac{\lambda_1}{(\lambda_1 \eta - 1)\Gamma(\alpha)} \int_0^{\eta} (\eta - s)^{\alpha-1} g(s) ds \\ & + \frac{1}{(\lambda_1 \eta - 1)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} g(s) ds \\ & - \frac{\lambda_2 \left(1 - \lambda_1 \eta^3\right)}{6(\lambda_1 \eta - 1)(\lambda_2 \xi - 1)\Gamma(\alpha - 2)} \int_0^{\xi} (\xi - s)^{\alpha-3} g(s) ds \\ & + \frac{(1 - \lambda_1 \eta)}{6(\lambda_1 \eta - 1)(\lambda_2 \xi - 1)\Gamma(\alpha - 2)} \int_0^1 (1-s)^{\alpha-3} g(s) ds, \end{aligned} \tag{2.9}$$

and

$$c_3 = -\frac{\lambda_2}{6(\lambda_2\xi - 1)\Gamma(\alpha-2)}\int_0^{\xi}(\xi - s)^{\alpha-3}g(s)ds \qquad (2.10)$$
$$+\frac{1}{6(\lambda_2\xi - 1)\Gamma(\alpha-2)}\int_0^{1}(1 - s)^{\alpha-3}g(s)ds.$$

Substituting the value of c_1 and c_3 in (2.8) we obtain the desired quantity in Lemma 2.5.

3. Main Results

Let us introduce the following notations

$$L_1 = \frac{|\lambda_1\eta - 1| + |\lambda_1|\eta^{\alpha} + 1}{|\lambda_1\eta - 1|\Gamma(\alpha+1)}$$
$$+\frac{\left(|\lambda_2 - \lambda_1\lambda_2\eta^3| + |\lambda_1\lambda_2\eta - \lambda_2|\right)\xi^{\alpha-2} + |1 - \lambda_1\eta^3| + |\lambda_1\eta - 1|}{6|\lambda_1\eta - 1||\lambda_2\xi - 1|\Gamma(\alpha-1)}, \qquad (3.1)$$
$$L_2 = \frac{1}{\Gamma(\alpha-\beta+1)} + \frac{|\lambda_1|\eta^{\alpha} + 1}{|\lambda_1\eta - 1|\Gamma(\alpha+1)\Gamma(2-\beta)}$$
$$+\frac{|\lambda_2 - \lambda_2\lambda_1\eta^3|\xi^{\alpha-2} + |1 - \lambda_1\eta^3|}{6|\lambda_1\eta - 1||\lambda_2\xi - 1|\Gamma(\alpha-1)\Gamma(2-\beta)} + \frac{|\lambda_2\lambda_1\eta - \lambda_2|\xi^{\alpha-2} + |\lambda_1\eta - 1|}{|\lambda_1\eta - 1||\lambda_2\xi - 1|\Gamma(\alpha-1)\Gamma(4-\beta)},$$

and list the following hypotheses:

(H1): The function $f : J \times \Re^2 \to \Re$ is continuous.

(H2): There exist a, b non negative continuous functions on J, such that for all $t \in J, (x, y), (x_1, y_1) \in \Re^2,$ we have

$$|f(t, x, y) - f(t, x_1, y_1)| \le a(t)|x - x_1| + b(t)|y - y_1|, \qquad (3.2)$$

where, $\omega = \sup_{t\in J}|a(t)|,$ and $\varpi = \sup_{t\in J}|b(t)|.$

(H3): There exists $M > 0$ such that

$$|f(t, x, y)| \le M; t \in J, x, y \in \Re. \qquad (3.3)$$

Our first result is based on the Banach contraction principle. We have:

Theorem 3.1. Assume that the hypothesis (H1) and (H2) are valid.

If

$$(L_1 + L_2)(\omega + \varpi) < 1, \qquad (3.4)$$

then the problem (1.1) has a unique solution on J.

Proof. Consider the operator $\phi : X \to X$ defined by:

$$\begin{aligned}\phi x(t) = &-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\left(s, x(s), D^\beta x(s)\right)ds \\ &+\frac{\lambda_1 t}{(\lambda_1\eta-1)\Gamma(\alpha)}\int_0^\eta (\eta-s)^{\alpha-1} f\left(s, x(s), D^\beta x(s)\right)ds \\ &-\frac{1}{(\lambda_1\eta-1)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} f\left(s, x(s), D^\beta x(s)\right)ds \\ &+\frac{\left(\lambda_2-\lambda_2\lambda_1\eta^3\right)t+(\lambda_2\lambda_1\eta-\lambda_2)t^3}{6(\lambda_1\eta-1)(\lambda_2\xi-1)\Gamma(\alpha-2)}\int_0^\xi (\xi-s)^{\alpha-3} f\left(s, x(s), D^\beta x(s)\right)ds \\ &-\frac{\left(1-\lambda_1\eta^3\right)t+(\lambda_1\eta-1)t^3}{6(\lambda_1\eta-1)(\lambda_2\xi-1)\Gamma(\alpha-2)}\int_0^1 (1-s)^{\alpha-3} f\left(s, x(s), D^\beta x(s)\right)ds.\end{aligned} \tag{3.5}$$

We shall prove that ϕ is a contraction:

$$\begin{aligned}|\phi x(t)-\phi y(t)| \le &\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|f\left(s, x(s), D^\beta x(s)\right)-f\left(s, y(s), D^\beta y(s)\right)\right|ds \\ &+\frac{|\lambda_1|t}{|\lambda_1\eta-1|\Gamma(\alpha)}\int_0^\eta (\eta-s)^{\alpha-1}\left|f\left(s, x(s), D^\beta x(s)\right)-f\left(s, y(s), D^\beta y(s)\right)\right|ds \\ &+\frac{t}{|\lambda_1\eta-1|\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\left|f\left(s, x(s), D^\beta x(s)\right)-f\left(s, y(s), D^\beta y(s)\right)\right|ds \\ &+\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|t+|\lambda_2\lambda_1\eta-\lambda_2|t^3}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)}\int_0^\xi (\xi-s)^{\alpha-3}\left|f\left(s, x(s), D^\beta x(s)\right)-f\left(s, y(s), D^\beta y(s)\right)\right|ds \\ &+\frac{\left|1-\lambda_1\eta^3\right|t+|\lambda_1\eta-1|t^3}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)}\int_0^1 (1-s)^{\alpha-3}\left|f\left(s, x(s), D^\beta x(s)\right)-f\left(s, y(s), D^\beta y(s)\right)\right|ds.\end{aligned} \tag{3.6}$$

Using the (H2), we can write:

$$\begin{aligned}|\phi x(t)-\phi y(t)| \le &\frac{\omega\|x-y\|+\varpi\left\|D^\beta x-D^\beta y\right\|}{\Gamma(\alpha+1)} \\ &+\frac{|\lambda_1|\eta^\alpha\left(\omega\|x-y\|+\varpi\left\|D^\beta x-D^\beta y\right\|\right)}{|\lambda_1\eta-1|\Gamma(\alpha+1)}+\frac{\omega\|x-y\|+\varpi\left\|D^\beta x-D^\beta y\right\|}{|\lambda_1\eta-1|\Gamma(\alpha+1)} \\ &+\frac{\left(\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|+|\lambda_2\lambda_1\eta-\lambda_2|\right)\xi^{\alpha-2}\left(\omega\|x-y\|+\varpi\left\|D^\beta x-D^\beta y\right\|\right)}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)} \\ &+\frac{\left(\left|1-\lambda_1\eta^3\right|+|\lambda_1\eta-1|\right)\left(\omega\|x-y\|+\varpi\left\|D^\beta x-D^\beta y\right\|\right)}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}\end{aligned} \tag{3.7}$$

$$\leq \frac{\left(\left|\lambda_1\eta-1\right|+\left|\lambda_1\right|\eta^{\alpha}+1\right)(\omega+\varpi)\left(\|x-y\|+\left\|D^{\beta}x-D^{\beta}y\right\|\right)}{\left|\lambda_1\eta-1\right|\Gamma(\alpha+1)}$$

$$\frac{\left[\left(\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|+\left|\lambda_2\lambda_1\eta-\lambda_2\right|\right)\xi^{\alpha-2}+\left|1-\lambda_1\eta^3\right|+\left|\lambda_1\eta-1\right|\right](\omega+\varpi)\left(\|x-y\|+\left\|D^{\beta}x-D^{\beta}y\right\|\right)}{6\left|\lambda_1\eta-1\right|\left|\lambda_2\xi-1\right|\Gamma(\alpha-1)}.$$

Consequently we obtain,

$$\left|\phi x(t)-\phi y(t)\right| \leq L_1(\omega+\varpi)\left(\|x-y\|+\left\|D^{\beta}x-D^{\beta}y\right\|\right). \qquad (3.8)$$

Hence, we have

$$\left\|\phi(x)-\phi(y)\right\| \leq L_1(\omega+\varpi)\left(\|x-y\|+\left\|D^{\beta}x-D^{\beta}y\right\|\right). \qquad (3.9)$$

On the other hand,

$$\left|D^{\beta}\phi x(t)-D^{\beta}\phi y(t)\right| \leq \frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}\left|f\left(s,x(s),D^{\beta}x(s)\right)-f\left(s,y(s),D^{\beta}y(s)\right)\right|ds$$

$$+\frac{\left|\lambda_1\right|t^{1-\beta}}{\left|\lambda_1\eta-1\right|\Gamma(\alpha)\Gamma(2-\beta)}\int_0^{\eta}(\eta-s)^{\alpha-1}\left|f\left(s,x(s),D^{\beta}x(s)\right)-f\left(s,y(s),D^{\beta}y(s)\right)\right|ds$$

$$+\frac{t^{1-\beta}}{\left|(\lambda_1\eta-1)\right|\Gamma(\alpha)\Gamma(2-\beta)}\int_0^1(1-s)^{\alpha-1}\left|f\left(s,x(s),D^{\beta}x(s)\right)-f\left(s,y(s),D^{\beta}y(s)\right)\right|ds \qquad (3.10)$$

$$+\frac{1}{\Gamma(\alpha-2)}\left(\begin{array}{c}\dfrac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|t^{1-\beta}}{6\left|\lambda_1\eta-1\right|\left|\lambda_2\xi-1\right|\Gamma(2-\beta)}\\ +\dfrac{\left|\lambda_2\lambda_1\eta-\lambda_2\right|t^{3-\beta}}{\left|\lambda_1\eta-1\right|\left|\lambda_2\xi-1\right|\Gamma(4-\beta)}\end{array}\right)\int_0^{\xi}(\xi-s)^{\alpha-3}\left|f\left(s,x(s),D^{\beta}x(s)\right)-f\left(s,y(s),D^{\beta}y(s)\right)\right|ds$$

$$+\frac{1}{\Gamma(\alpha-2)}\left(\begin{array}{c}\dfrac{\left|1-\lambda_1\eta^3\right|t^{1-\beta}}{6\left|\lambda_1\eta-1\right|\left|\lambda_2\xi-1\right|\Gamma(2-\beta)}\\ +\dfrac{\left|\lambda_1\eta-1\right|t^{3-\beta}}{\left|\lambda_1\eta-1\right|\left|\lambda_2\xi-1\right|\Gamma(4-\beta)}\end{array}\right)\int_0^1(1-s)^{\alpha-3}\left|f\left(s,x(s),D^{\beta}x(s)\right)-f\left(s,y(s),D^{\beta}y(s)\right)\right|ds.$$

By (H2), we obtain

$$\begin{aligned}\left|D^{\beta}\phi x(t)-D^{\beta}\phi y(t)\right| \le{} & \frac{\omega\|x-y\|+\varpi\left\|D^{\beta}x-D^{\beta}y\right\|}{\Gamma(\alpha-\beta+1)} \\ & +\frac{|\lambda_1|\eta^{\alpha}\left(\omega\|x-y\|+\varpi\left\|D^{\beta}x-D^{\beta}y\right\|\right)}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\beta)}+\frac{\omega\|x-y\|+\varpi\left\|D^{\beta}x-D^{\beta}y\right\|}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\beta)} \\ & +\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}\left(\omega\|x-y\|+\varpi\left\|D^{\beta}x-D^{\beta}y\right\|\right)}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\beta)} \qquad (3.11) \\ & +\frac{|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}\left(\omega\|x-y\|+\varpi\left\|D^{\beta}x-D^{\beta}y\right\|\right)}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\beta)} \\ & +\frac{\left|1-\lambda_1\eta^3\right|\left(\omega\|x-y\|+\varpi\left\|D^{\beta}x-D^{\beta}y\right\|\right)}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\beta)}+\frac{|\lambda_1\eta-1|\left(\omega\|x-y\|+\varpi\left\|D^{\beta}x-D^{\beta}y\right\|\right)}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\beta)}.\end{aligned}$$

Hence,

$$\begin{aligned}\left|D^{\beta}\phi x(t)-D^{\beta}\phi y(t)\right| \le{} & \left[\frac{\omega+\varpi}{\Gamma(\alpha-\beta+1)}+\frac{\left(|\lambda_1|\eta^{\alpha}+1\right)(\omega+\varpi)}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\beta)}\right]\left(\|x-y\|+\left\|D^{\beta}x-D^{\beta}y\right\|\right) \\ & +\left[\begin{array}{c}\dfrac{\left(\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}+\left|1-\lambda_1\eta^3\right|\right)(\omega+\varpi)}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\beta)} \\ +\dfrac{\left(|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}+|\lambda_1\eta-1|\right)(\omega+\varpi)}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\beta)}\end{array}\right]\left(\|x-y\|+\left\|D^{\beta}x-D^{\beta}y\right\|\right). \qquad (3.12)\end{aligned}$$

Therefore,

$$\left|D^{\beta}\phi x(t)-D^{\beta}\phi y(t)\right| \le L_2(\omega+\varpi)\left(\|x-y\|+\left\|D^{\beta}x-D^{\beta}y\right\|\right), \qquad (3.13)$$

which implies,

$$\left\|D^{\beta}\phi(x)-D^{\beta}\phi(y)\right\| \le L_2(\omega+\varpi)\left(\|x-y\|+\left\|D^{\beta}x-D^{\beta}y\right\|\right). \qquad (3.14)$$

It follows from (3.9) and (3.14) that

$$\|\phi(x)-\phi(y)\|_X \le (L_1+L_2)(\omega+\varpi)\left(\|x-y\|+\left\|D^{\beta}x-D^{\beta}y\right\|\right). \qquad (3.15)$$

Thanks to (3.4) we deduce that ϕ is a contraction. As a consequence of Banach contraction principle, the problem (1.1) has a unique solution on J.

The second result is the following:

Theorem 3.2. Suppose that (H1) and (H3) hold. Then, the problem (1.1) has at least one solution on J.

Proof. We use Scheafer's fixed point theorem to prove that ϕ has at least a fixed point on X. The proof will be given in the following steps:

Step1: Thanks to (H1) we conclude that the operator ϕ is continuous.

Step2: The operator ϕ maps bounded sets into bounded sets in X: For $\delta > 0$, we take $x \in B_\delta = \{x \in X; \|x\|_X \le \delta\}$.

For $x \in B_\delta$, and for each $t \in J$, we have:

$$
\begin{aligned}
|\phi x(t)| \le{}& \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|f\left(s, x(s), D^\beta x(s)\right)\right| ds \\
&+ \frac{|\lambda_1| t}{|\lambda_1\eta - 1|\Gamma(\alpha)}\int_0^\eta (\eta-s)^{\alpha-1}\left|f\left(s, x(s), D^\beta x(s)\right)\right| ds \\
&+ \frac{t}{|\lambda_1\eta - 1|\Gamma(\alpha)}\int_0^\eta (\eta-s)^{\alpha-1}\left|f\left(s, x(s), D^\beta x(s)\right)\right| ds \\
&+ \frac{\left|\lambda_2 - \lambda_2\lambda_1\eta^3\right| t + |\lambda_2\lambda_1\eta - \lambda_2| t^3}{6|\lambda_1\eta - 1||\lambda_2\xi - 1|\Gamma(\alpha-2)}\int_0^\xi (\xi-s)^{\alpha-3}\left|f\left(s, x(s), D^\beta x(s)\right)\right| ds \\
&+ \frac{\left|1 - \lambda_1\eta^3\right| t + |\lambda_1\eta - 1| t^3}{6|\lambda_1\eta - 1||\lambda_2\xi - 1|\Gamma(\alpha-2)}\int_0^1 (1-s)^{\alpha-3}\left|f\left(s, x(s), D^\beta x(s)\right)\right| ds.
\end{aligned}
\tag{3.16}
$$

Using (H3), we can write

$$
\begin{aligned}
|\phi x(t)| \le{}& \frac{M\left(|\lambda_1\eta - 1||\lambda_1|\eta^\alpha + 1\right)}{|\lambda_1\eta - 1|\Gamma(\alpha+1)} \\
&+ \frac{M\left[\left(\left|\lambda_2 - \lambda_2\lambda_1\eta^3\right| + |\lambda_2\lambda_1\eta - \lambda_2|\right)\xi^{\alpha-2} + \left(\left|1 - \lambda_1\eta^3\right| + |\lambda_1\eta - 1|\right)\right]}{6|\lambda_1\eta - 1||\lambda_2\xi - 1|\Gamma(\alpha-1)} \\
\le{}& M\left(\frac{|\lambda_1\eta - 1||\lambda_1|\eta^\alpha + 1}{|\lambda_1\eta - 1|\Gamma(\alpha+1)} + \frac{\left(\left|\lambda_2 - \lambda_2\lambda_1\eta^3\right| + |\lambda_2\lambda_1\eta - \lambda_2|\right)\xi^{\alpha-2} + \left|1 - \lambda_1\eta^3\right| + |\lambda_1\eta - 1|}{6|\lambda_1\eta - 1||\lambda_2\xi - 1|\Gamma(\alpha-1)}\right).
\end{aligned}
\tag{3.17}
$$

Thus,

$$
\|\phi(x)\| \le ML_1. \tag{3.18}
$$

On the other hand,

$$\begin{aligned}\left|D^{\beta}\phi x(t)\right| \leq {} & \frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}\left|f\left(s,x(s),D^{\beta}x(s)\right)\right|ds \\ & +\frac{|\lambda_1|t^{1-\beta}}{|\lambda_1\eta-1|\Gamma(\alpha)\Gamma(2-\beta)}\int_0^{\eta}(\eta-s)^{\alpha-1}\left|f\left(s,x(s),D^{\beta}x(s)\right)\right|ds \\ & +\frac{t^{1-\beta}}{|(\lambda_1\eta-1)|\Gamma(\alpha)\Gamma(2-\beta)}\int_0^1(1-s)^{\alpha-1}\left|f\left(s,x(s),D^{\beta}x(s)\right)\right|ds \\ & +\frac{1}{\Gamma(\alpha-2)}\left(\begin{array}{l}\dfrac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|t^{1-\beta}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(2-\beta)} \\ +\dfrac{|\lambda_2\lambda_1\eta-\lambda_2|t^{3-\beta}}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(4-\beta)}\end{array}\right)\int_0^{\xi}(\xi-s)^{\alpha-3}\left|f\left(s,x(s),D^{\beta}x(s)\right)\right|ds \qquad (3.19) \\ & +\frac{1}{\Gamma(\alpha-2)}\left(\begin{array}{l}\dfrac{\left|1-\lambda_1\eta^3\right|t^{1-\beta}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(2-\beta)} \\ +\dfrac{|\lambda_1\eta-1|t^{3-\beta}}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(4-\beta)}\end{array}\right)\int_0^{1}(1-s)^{\alpha-3}\left|f\left(s,x(s),D^{\beta}x(s)\right)\right|ds.\end{aligned}$$

By (H3), we obtain

$$\begin{aligned}\left|D^{\beta}\phi x(t)\right| \leq {} & M\left[\frac{1}{\Gamma(\alpha-\beta+1)}+\frac{|\lambda_1|\eta^{\alpha}+1}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\beta)}\right] \\ & +M\left[\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}+\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\beta)}+\frac{|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}+|\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\beta)}\right] \qquad (3.20) \\ \leq {} & M\left[\begin{array}{l}\dfrac{1}{\Gamma(\alpha-\beta+1)}+\dfrac{|\lambda_1|\eta^{\alpha}+1}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\beta)}+\dfrac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}+\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\beta)} \\ +\dfrac{|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}+|\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\beta)}\end{array}\right].\end{aligned}$$

Consequently,

$$\left\|D^{\beta}\phi(x)\right\| \leq ML_2. \qquad (3.21)$$

Thanks to (3.18) and (3.21), yields

$$\|\phi(x)\|_X \leq M\left(L_1+L_2\right)<\infty. \qquad (3.22)$$

Step3: The operator ϕ is equicontinuous on J: Let us take $x\in B_{\delta}, t_1, t_2\in J, t_1<t_2$. We have:

$$
\begin{aligned}
\left|\phi x(t_2)-\phi x(t_1)\right| \le{}& \frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left((t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}\right)\left|f\left(s,x(s),D^{\beta}x(s)\right)\right|ds \\
&+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\left|f\left(s,x(s),D^{\beta}x(s)\right)\right|ds \\
&+\frac{|\lambda_1|(t_2-t_1)}{|\lambda_1\eta-1|\Gamma(\alpha)}\int_0^{\eta}(\eta-s)^{\alpha-1}\left|f\left(s,x(s),D^{\beta}x(s)\right)\right|ds \\
&+\frac{(t_1-t_2)}{|\lambda_1\eta-1|\Gamma(\alpha)}\int_0^{1}(1-s)^{\alpha-1}\left|f\left(s,x(s),D^{\beta}x(s)\right)\right|ds \\
&+\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|(t_2-t_1)+|\lambda_2\lambda_1\eta-\lambda_2|\left(t_2^3-t_1^3\right)}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)}\int_0^{\xi}(\xi-s)^{\alpha-3}\left|f\left(s,x(s),D^{\beta}x(s)\right)\right|ds \\
&+\frac{\left|1-\lambda_1\eta^3\right|(t_1-t_2)+|\lambda_1\eta-1|\left(t_1^3-t_2^3\right)}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)}\int_0^{1}(1-s)^{\alpha-3}\left|f\left(s,x(s),D^{\beta}x(s)\right)\right|ds \\
&\le \frac{M\left(t_1^{\alpha}-t_2^{\alpha}+2\left(t_2^{\alpha}-t_1^{\alpha}\right)\right)}{\Gamma(\alpha+1)}+\frac{M|\lambda_1|\eta^{\alpha}(t_2-t_1)}{|\lambda_1\eta-1|\Gamma(\alpha+1)}+\frac{M(t_1-t_2)}{|\lambda_1\eta-1|\Gamma(\alpha+1)} \\
&\quad+\frac{M\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}(t_2-t_1)+M|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}\left(t_2^3-t_1^3\right)}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)} \\
&\quad+\frac{M\left|1-\lambda_1\eta^3\right|(t_1-t_2)+M|\lambda_1\eta-1|\left(t_1^3-t_2^3\right)}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}.
\end{aligned}
\tag{3.23}
$$

Thus,

$$
\begin{aligned}
\left|\phi x(t_2)-\phi x(t_1)\right| \le{}& M\left[\frac{|\lambda_1\eta-1|+|\lambda_1|\eta^{\alpha}}{|\lambda_1\eta-1|\Gamma(\alpha+1)}+\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}\right](t_2-t_1) \\
&+M\left[\frac{1}{|\lambda_1\eta-1|\Gamma(\alpha+1)}+\frac{\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}\right](t_1-t_2) \\
&+M\left[\frac{|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}\right]\left(t_2^3-t_1^3\right)+\frac{M|\lambda_1\eta-1|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}\left(t_1^3-t_2^3\right) \\
&+\frac{M}{\Gamma(\alpha+1)}\left(t_1^{\alpha}-t_2^{\alpha}\right)+\frac{2M}{\Gamma(\alpha+1)}(t_2-t_1)^{\alpha}.
\end{aligned}
\tag{3.24}
$$

We have also,

$$
\begin{aligned}
\left|D^{\beta}\phi x(t_2)-D^{\beta}\phi x(t_1)\right| \le{}& \frac{1}{\Gamma(\alpha-\beta)}\int_0^{t_1}\left((t_1-s)^{\alpha-\beta-1}-(t_2-s)^{\alpha-\beta-1}\right)\left|f\left(s,x(s),D^{\beta}x(s)\right)\right|ds \\
&+\frac{1}{\Gamma(\alpha-\beta)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-\beta-1}\left|f\left(s,x(s),D^{\beta}x(s)\right)\right|ds \\
&+\frac{|\lambda_1|\left(t_2^{1-\beta}-t_1^{1-\beta}\right)}{|\lambda_1\eta-1|\Gamma(\alpha)\Gamma(2-\beta)}\int_0^{\eta}(\eta-s)^{\alpha-1}\left|f\left(s,x(s),D^{\beta}x(s)\right)\right|ds \qquad (3.25)\\
&+\frac{\left(t_1^{1-\beta}-t_2^{1-\beta}\right)}{|\lambda_1\eta-1|\Gamma(\alpha)\Gamma(2-\beta)}\int_0^{1}(1-s)^{\alpha-1}\left|f\left(s,x(s),D^{\beta}x(s)\right)\right|ds \\
&+\left[\begin{aligned}&\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\left(t_2^{1-\beta}-t_1^{1-\beta}\right)}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(2-\beta)}\\ &+\frac{|\lambda_2\lambda_1\eta-\lambda_2|\left(t_2^{3-\beta}-t_1^{3-\beta}\right)}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(4-\beta)}\end{aligned}\right]\int_0^{\xi}(\xi-s)^{\alpha-3}\left|f\left(s,x(s),D^{\beta}x(s)\right)\right|ds \\
&+\left[\begin{aligned}&\frac{\left|1-\lambda_1\eta^3\right|\left(t_1^{1-\beta}-t_2^{1-\beta}\right)}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(2-\beta)}\\ &+\frac{|\lambda_1\eta-1|\left(t_1^{3-\beta}-t_2^{3-\beta}\right)}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(4-\beta)}\end{aligned}\right]\int_0^{1}(1-s)^{\alpha-3}\left|f\left(s,x(s),D^{\beta}x(s)\right)\right|ds.
\end{aligned}
$$

By (H3), we have:

$$
\begin{aligned}
\left|D^{\beta}\phi x(t_2)-D^{\beta}\phi x(t_1)\right| \le{}& \frac{M}{\Gamma(\alpha-\beta+1)}\left(t_1^{\alpha-\beta}-t_2^{\alpha-\beta}+2(t_2-t_1)^{\alpha-\beta}\right) \\
&+M\left[\begin{aligned}&\frac{|\lambda_1|\eta^{\alpha}+1}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\beta)}\\ &+\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\beta)}\end{aligned}\right]\left(t_2^{1-\beta}-t_1^{1-\beta}\right) \\
&+M\left[\begin{aligned}&\frac{1}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\beta)}\\ &+\frac{\left|1-\lambda_1\eta^3\right|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\beta)}\end{aligned}\right]\left(t_1^{1-\beta}-t_2^{1-\beta}\right) \qquad (3.26)\\
&+\frac{M|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\beta)}\left(t_2^{3-\beta}-t_1^{3-\beta}\right) \\
&+\frac{M|\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\beta)}\left(t_1^{3-\beta}-t_2^{3-\beta}\right).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\|\phi x(t_2)-\phi x(t_1)\|_X \le{}& M\left[\frac{|\lambda_1\eta-1|+|\lambda_1|\eta^{\alpha}}{|\lambda_1\eta-1|\Gamma(\alpha+1)}+\frac{|\lambda_2-\lambda_2\lambda_1\eta^3|\xi^{\alpha-2}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}\right](t_2-t_1)\\
&+M\left[\frac{1}{|\lambda_1\eta-1|\Gamma(\alpha+1)}+\frac{|1-\lambda_1\eta^3|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}\right](t_1-t_2)\\
&+M\left[\frac{|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}\right](t_2^3-t_1^3)\\
&+\frac{M|\lambda_1\eta-1|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}(t_1^3-t_2^3)\\
&+\frac{M}{\Gamma(\alpha+1)}\left(t_1^{\alpha}-t_2^{\alpha}+2(t_2-t_1)^{\alpha}\right)\\
&+\frac{M}{\Gamma(\alpha-\beta+1)}\left(t_1^{\alpha-\beta}-t_2^{\alpha-\beta}+2(t_2-t_1)^{\alpha-\beta}\right)\\
&+M\left[\frac{|\lambda_1|\eta^{\alpha}+1}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\beta)}+\frac{|\lambda_2-\lambda_2\lambda_1\eta^3|\xi^{\alpha-2}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\beta)}\right]\left(t_2^{1-\beta}-t_1^{1-\beta}\right)\\
&+M\left[\frac{1}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\beta)}+\frac{|1-\lambda_1\eta^3|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\beta)}\right]\left(t_1^{1-\beta}-t_2^{1-\beta}\right)\\
&+\frac{M|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\beta)}\left(t_2^{3-\beta}-t_1^{3-\beta}\right)\\
&+\frac{M|\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\beta)}\left(t_1^{3-\beta}-t_2^{3-\beta}\right),
\end{aligned}
\tag{3.27}
$$

which implies $\|\phi x(t_2)-\phi x(t_1)\|_X \to 0$ as $t_2 \to t_1$. By Arzela-Ascoli theorem, we conclude that ϕ is completely continuous operator.

Step4: We show that the set Φ defined by:

$$\Phi=\{x\in X, x=\mu\phi(x), 0<\mu<1\}, \tag{3.28}$$

is bounded:

Let $x\in\Phi$, then $x=\mu\phi(x)$, for some $0<\mu<1$. Thus, for each $t\in J$, we have:

$$\begin{aligned}\frac{1}{\mu}\left|x(t)\right| \le{}& \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|f\left(s,x(s),D^{\beta}x(s)\right)\right|ds\\ &+\frac{|\lambda_1|\,t}{|\lambda_1\eta-1|\,\Gamma(\alpha)}\int_0^{\eta}(\eta-s)^{\alpha-1}\left|f\left(s,x(s),D^{\beta}x(s)\right)\right|ds\\ &+\frac{t}{|\lambda_1\eta-1|\,\Gamma(\alpha)}\int_0^{\eta}(\eta-s)^{\alpha-1}\left|f\left(s,x(s),D^{\beta}x(s)\right)\right|ds\\ &+\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|t+\left|\lambda_2\lambda_1\eta-\lambda_2\right|t^3}{6|\lambda_1\eta-1||\lambda_2\xi-1|\,\Gamma(\alpha-2)}\int_0^{\xi}(\xi-s)^{\alpha-3}\left|f\left(s,x(s),D^{\beta}x(s)\right)\right|ds\\ &+\frac{\left|1-\lambda_1\eta^3\right|t+|\lambda_1\eta-1|\,t^3}{6|\lambda_1\eta-1||\lambda_2\xi-1|\,\Gamma(\alpha-2)}\int_0^{1}(1-s)^{\alpha-3}\left|f\left(s,x(s),D^{\beta}x(s)\right)\right|ds.\end{aligned} \tag{3.29}$$

Thanks to (H3), we can write

$$\begin{aligned}\frac{1}{\mu}\left|x(t)\right| \le{}& \frac{M\left(|\lambda_1\eta-1|+|\lambda_1|\eta^{\alpha}+1\right)}{|\lambda_1\eta-1|\,\Gamma(\alpha+1)}\\ &\frac{M\left[\left(\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|+\left|\lambda_2\lambda_1\eta-\lambda_2\right|\right)\xi^{\alpha-2}+\left|1-\lambda_1\eta^3\right|+|\lambda_1\eta-1|\right]}{6|\lambda_1\eta-1||\lambda_2\xi-1|\,\Gamma(\alpha-1)}.\end{aligned} \tag{3.30}$$

Therefore,

$$\left|x(t)\right| \le \mu M\left[\begin{array}{l}\dfrac{\left(|\lambda_1\eta-1|+|\lambda_1|\eta^{\alpha}+1\right)}{|\lambda_1\eta-1|\,\Gamma(\alpha+1)}\\ +\dfrac{\left(\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|+\left|\lambda_2\lambda_1\eta-\lambda_2\right|\right)\xi^{\alpha-2}+\left|1-\lambda_1\eta^3\right|+|\lambda_1\eta-1|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\,\Gamma(\alpha-1)}\end{array}\right]. \tag{3.31}$$

Hence,

$$\|x\| \le \mu M L_1. \tag{3.32}$$

On the other hand,

$$
\begin{aligned}
\frac{1}{\mu}\left|D^{\beta}x(t)\right| \le{}& \frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}\left|f\left(s,x(s),D^{\beta}x(s)\right)\right|ds \\
&+\frac{|\lambda_1|\,t^{1-\beta}}{|\lambda_1\eta-1|\,\Gamma(2-\beta)\Gamma(\alpha)}\int_0^{\eta}(\eta-s)^{\alpha-1}\left|f\left(s,x(s),D^{\beta}x(s)\right)\right|ds \qquad (3.33)\\
&+\frac{t^{1-\beta}}{|\lambda_1\eta-1|\,\Gamma(2-\beta)\Gamma(\alpha)}\int_0^{\eta}(\eta-s)^{\alpha-1}\left|f\left(s,x(s),D^{\beta}x(s)\right)\right|ds \\
&+\left[\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|t^{1-\beta}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\,\Gamma(2-\beta)\Gamma(\alpha-2)}+\frac{|\lambda_2\lambda_1\eta-\lambda_2|\,t^{3-\beta}}{|\lambda_1\eta-1||\lambda_2\xi-1|\,\Gamma(\alpha-2)\Gamma(4-\beta)}\right]\int_0^{\xi}(\xi-s)^{\alpha-3}\left|f\left(s,x(s),D^{\beta}x(s)\right)\right|ds \\
&+\left[\frac{\left|1-\lambda_1\eta^3\right|t^{1-\beta}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\,\Gamma(2-\beta)\Gamma(\alpha-2)}+\frac{|\lambda_1\eta-1|\,t^{3-\beta}}{|\lambda_1\eta-1||\lambda_2\xi-1|\,\Gamma(4-\beta)\Gamma(\alpha-2)}\right]\int_0^{1}(1-s)^{\alpha-3}\left|f\left(s,x(s),D^{\beta}x(s)\right)\right|ds.
\end{aligned}
$$

By (H3), we get

$$
\begin{aligned}
\frac{1}{\mu}\left|D^{\beta}x(t)\right| \le{}& M\left[\frac{1}{\Gamma(\alpha-\beta+1)}+\frac{|\lambda_1|\eta^{\alpha}+1}{|\lambda_1\eta-1|\,\Gamma(\alpha+1)\Gamma(2-\beta)}\right] \qquad (3.34)\\
&+M\left[\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}+\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\,\Gamma(\alpha-1)\Gamma(2-\beta)}+\frac{|\lambda_2\lambda_1\eta-\lambda_2|\,\xi^{\alpha-2}+|\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\,\Gamma(\alpha-1)\Gamma(4-\beta)}\right].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\left|D^{\beta}x(t)\right| \le{}& \mu M\left[\frac{1}{\Gamma(\alpha-\beta+1)}+\frac{|\lambda_1|\eta^{\alpha}+1}{|\lambda_1\eta-1|\,\Gamma(\alpha+1)\Gamma(2-\beta)}\right] \qquad (3.35)\\
&+\mu M\left[\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}+\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\,\Gamma(\alpha-1)\Gamma(2-\beta)}+\frac{|\lambda_2\lambda_1\eta-\lambda_2|\,\xi^{\alpha-2}+|\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\,\Gamma(\alpha-1)\Gamma(4-\beta)}\right].
\end{aligned}
$$

$$\left\|D^{\beta}x\right\| \le \mu M L_2. \qquad (3.36)$$

Thus, from (3.32) and (3.36) we obtain

$$\|x\|_X \le \mu M\left(L_1+L_2\right). \qquad (3.37)$$

Hence,

$$\|\phi(x)\|_X < \infty. \qquad (3.38)$$

This shows that Φ is bounded.

As consequence of Schaefer's fixed point theorem, the problem (1.1) has at least one solution on $[0,1]$.

4. Example

Consider the following fractional problem:

$$
\begin{gathered}
D^{\frac{7}{2}}x(t)+\frac{\sqrt{\pi}e^{-2\pi t}\cos(\pi t)\left(x(t)+D^{\frac{5}{2}}x(t)\right)}{\left(7\pi+15e^{t}\right)^{2}\left(2+x+D^{\frac{5}{2}}x(t)\right)}=0,t\in[0,1],\\
x(0)=0,x(1)-\frac{2}{3}x\left(\frac{1}{5}\right)=0, \qquad (4.1)\\
x''(0)=0,x''(1)-\frac{1}{2}x''\left(\frac{1}{4}\right)=0.
\end{gathered}
$$

We have

$$
f(t,x,y)=\frac{\sqrt{\pi}e^{-2\pi t}\cos(\pi t)(x+y)}{\left(7\pi+15e^{t}\right)^{2}(2+x+y)},t\in J\ :=[0,1],x,y\in[0,\infty).
$$

Let $x,y\in[0,\infty)$ and $t\in J$. Then we have:

$$
\begin{aligned}
|f(t,x,y)-f(t,x_1,y_1)|&=\frac{\sqrt{\pi}e^{-2\pi t}|\cos\pi t|}{\left(7\pi+15e^{t}\right)^{2}}\left|\frac{x+y}{2+x+y}-\frac{x_1+y_1}{2+x_1+y_1}\right|\\
&\le\frac{\sqrt{\pi}e^{-2\pi t}\left(|x-x_1|+|y-y_1|\right)}{\left(7\pi+15e^{t}\right)^{2}(2+x+y)(2+x_1+y_1)}\\
&\le\frac{\sqrt{\pi}e^{-2\pi t}}{\left(7\pi+15e^{t}\right)^{2}}|x-x_1|+\frac{\sqrt{\pi}e^{-2\pi t}}{\left(7\pi+15e^{t}\right)^{2}}|y-y_1|.
\end{aligned}
$$

So we can take

$$
a(t)=\frac{\sqrt{\pi}e^{-2\pi t}}{\left(7\pi+15e^{t}\right)^{2}},b(t)=\frac{\sqrt{\pi}e^{-2\pi t}}{\left(7\pi+15e^{t}\right)^{2}}.
$$

Then,

$$
\omega=\sup_{t\in[0,1]}a(t),\varpi=\sup_{t\in[0,1]}b(t).
$$

We have also

$$
(L_1+L_2)(\omega+\varpi)=\frac{4,886\sqrt{\pi}}{(7\pi+15)^{2}}=0,0063311<1.
$$

Hence by Theorem (3.1) the boundary value problem (1.1) has a unique solution on $[0,1]$.

References

1] A. Bashir. N. Juan, .J. A. Aalsaedi, Existence and uniqueness of solutions for nonlinear fractional differential equations with non-separated type integral boundary conditions, *Acta Mathematica Scientia*, 31B(6) (**2011**): 2122-2130.

[2] M.E. Bengrine, Z. Dahmani, Boundary value problems for fractional differential equations, *Int. J. Open problems compt. Math*, 5(4)(**2012**): 09 pages.

[3] D. Delbosco, L. Rodino, Existence and uniqueness for a nonlinear fractional differential equation, *J. Math. Anal. Appl*, 204(3-4)(**1996**): 429-440.

[4] K. Diethelm, N.J. Ford, Analysis of fractional differential equations, *J. Math. Anal. Appl*, 265(2)(**2002**): 229-248,

[5] K. Diethelm, G. Walz, Numerical solution of fraction order differential equations by extrapolation, *Numer. Algorithms*, 16(3)(**1998**): 231-253.

[6] A.M.A. El-Sayed, Nonlinear functional differential equations of arbitrary orders, *Nonlinear Anal*, 33(2)(**1998**): 181-186.

[7] M. Houas, Z. Dahmani, New fractional results for a boundary value problem with Caputo derivative, *Int. J. Open problems compt. Math*, *Accepted* (2013): 13 pages.

[8] R. Hilfer, *Applications of fractional calculus in physics*, World Scientific Publishing Co., Inc., River Edge, NJ, (**2000**).

[9] A.A. Kilbas, S.A. Marzan, Nonlinear differential equation with the Caputo fraction derivative in the space of continuously differentiable functions, *Differ. Equ*, 41(1)(**2005**): 84-89.

[10] V. Lakshmikantham, A.S. Vatsala, Basic theory of fractional differential equations, *Nonlinear Anal*. 69(8)(**2008**): 2677-2682.

[11] C. Li, X. Luo, Y. Zhou, Existence of positive solutions of the boundary value problem for nonlinear fractional differential equations, *Comput. Math. Appl*, 59(**2010**): 1363-1375.

[12] D.X. Ma and X.Z. Yang, Upper and lower solution method for fourth-order four-point boundary value problems, *Journal of Computational and Applied Mathematics*, 223(**2009**): 543-551.

[13] F. Mainardi, Fractional calculus some basic problem in continuum and statistical mechanics Fractals and fractional calculus in continuum mechanics, Springer, *Vienna*, (**1997**).

[14] S.M. Momani, S.B. Hadid, Some comparison resultats for integro-fractional differential inequalities, *J. Fract. Calc*, 24(**2003**): 37-44.

[15] S.K. Ntouyas, Existence results for first order boundary value problems for fractional differential equations and inclusions with fractional integral boundary conditions, *Journal of Fractional Calculus and Applications*, 3(9)(**2012**): 1-14.

[16] I. Podlubny, I. Petras, B.M. Vinagre, P. O'leary, L. Dorcak, Analogue realizations of fractional-order controllers Fractional order calculus and its applications, *Nonlinear Dynam*, 29(4)(**2002**): 281-296.

[17] M.U. Rehman, R.A Khan and N.A Asif, Three point boundary value problems for nonlinear fractional differential equations, *Acta Mathematica Scientia*, 31B(4) (**2011**): 1337-1346.

[18] G. Wang, R. P. Agarwal, and A. Cabada, Existence results and monotone iterative technique for systems of nonlinear fractional differential equations, *Applied Mathematics Letters*, 25(**2012**): 1019-1024.

[19] J. Wang, H. Xiang, and Z. Liu, Positive solution to nonzero boundary values problem for a coupled system of nonlinear fractional differential equations, *International Journal of Differential Equations*, *Article* ID 186928(**2010**): 12 pages.