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THESIS

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Fractional Differential Equations and Travelling Waves

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List of publications

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• M. Rakah, Y. Gouari, R.W. Ibrahim, Z. Dahmani, H. Kahtan, *Unique solutions, stability and travelling waves for some generalized fractional differential problems.* Applied Mathematics in Science and Engineering, **23(1)** (2023).

• M. Rakah, Z. Dahmani, Y. Gouari, *A Fractional BVP Problem and Some Travelling Waves.* Int. J. Open Problems Compt. Math., **16(4)** (2023).

• Y. Gouari, M. Rakah, Z. Dahmani, *A sequential differetial problem with caputo and riemann liouville derivatives involving convergent series*, Adv. Theory Nonlinear Anal. Appl., **7**(2), 319-335, (2023).

• I. Jebril, Y. Gouari, M. Rakah, Z. Dahmani,*Solvability For a Class of FDEs With Some* $(e1, e2,)$*-nonlocal anti periodic conditions and Another Class of Kdv Burger equation type*, Nonlinear Functional Analysis and Applications Vol. 28, No. 4, 1017-1034, (2023).

• K. Bensassa, Z. Dahmani, M. Rakah, M. Z. Sarikaya, *Beam deflection coupled systems of fractional differential equations: existence of solutions, Ulam-Hyers stability and travelling waves.*Analysis and Mathematical Physics, 14(2), (2024).

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Abstract

In this thesis, we will study certain classes of differential equations of fractional order in the sense of Caputo and some other classes in the conformable fractional sense of Khalil. We use the theory of fixed points on Banach spaces. We use also the theory of nonlinear operators and the theory of inequalities to study the existence, uniqueness and stability in the sense of Ulam-Hyers. We illustrate the main results with several academic applications. We will also try as far as possible to deal with problems inspired by physics. We devote a final part of our thesis project to physical applications, we will be interested to study some important classes of EDF solutions that are called traveling wave solutions". Such classes of solutions are very important in applications because they can be used to model the spread of epidemics. We use numerical methods, such as: Tanh Method to obtain and calculate these important classes of solutions.

Keywords: Caputo derivative, Conformable derivative, Riemann-Liouville, Fixed point, Existence, Uniqueness, Stability Ulam-Hyers, Traveling waves.

Arabic abstract

في هذه الأطروحة سوف نقوم بدراسة فئات معينة من المعادلات التفاضلية ذات الرتبة الكسرية بمعنى كابوتو وبعض الفئات الأخرى بمعنى الكسر المطابق لخليل. نستخدم نظرية النقطة الثابتة على فضاء باناخ. كما نستخدم نظرية العوامل غير الخطية ونظرية المتباينات لدراسة الوجود والوحدانية والاستقرار بمعنى أولام هايرز. ونوضح النتائج الرئيسية مع العديد من التطبيقات الأكاديمية. وسنحاول أيضًا قدر الإمكان التعامل مع المشكلات المستوحاة من الفيزياء. كما خصصنا الجزء الأخير من مشروع أطروحتنا للتطبيقات الفيزيائية، وسنكون مهتمين بدراسة بعض الفئات المهمة من حلول المعادلات الموجية والتي تسمى حلول الموجات المتنقلة" . تعتبر هذه الفئات من الحلول مهمة جدًا في التطبيقات لأنه يمكن استخدامها لنمذجة انتشار الأوبئة، ونستخدم الطرق العددية مثل: "طريقة دالة الظل الزائدي" للحصول على هذه الفئات المهمة من الحلول وحسابها.

<u>الكلمات المفتاحية:</u> مشتق كابوتو، المشتق المطابق، تكامل ريمان–ليوفيل، النقطة الثابتة، الوجود، الوحدانية، الاستقرار أولام–هايرز، الموجات المتنقلة.

Résumé

Dans cette thèse, nous étudions certaines classes d'équations différentielles d'ordre fractionnaire au sens de Caputo et d'autres au sens fractionnaire conforme de Khalil. Nous utilisons la théorie des points fixes appliquée sur les espaces de Banach, nous utilisons aussi la théorie des opérateurs non linéaires ainsi que la théorie des estimations pour traiter les questions d'existence, d'unicité et la stabilité des solutions au sens d'Ulam-Hyers. Nous illustrons les résultats obtenus par des applications académiques et nous essayons, lorsque si possible, de présenter un cadre théroqie pour traiter des problèmes inspirs de la physique. Nous consacrons une autre partie de notre thèse des applications physiques, nous nous intéressons à des classes très importantes des solutions des EDFs á savoir " les travelling wave solutions "; les solutions ondes progressives. Ces classes de solutions sont très importantes en application car elles peuvent être utilisées à modéliser la propagation des pidémies. Nous utilisons des méthodes numériques, comme par exemple: Tanh Methode pour déterminer et calculer ces classes de solutions.

Mots clés: Dérivés au sens de Caputo, Dérivés au sens conforme de Khalil, Riemann-Liouville, Point fixe, Existence, Unicité, Stabilité Ulam-Hyers, Ondes progressives.

Contents

General Introduction

Applied mathematics as it is known is the application of mathematical knowledge for the formal purposes of problem solving that comes from various applications, such as physical, biological, engineering, social sciences and other areas of science. Their solutions require knowledge of various branches of mathematics, such as analysis, differential equations, stochastics, optimization and graph theory...., using analytical and numerical methods.

In recent years, fractional calculus has successfully attracted many researchers in various fields of science and engineering. One of the major advantages of fractional arithmetic is that fractional derivatives provide a superior approach to describing the memory and genetic properties of various substances and processes [3],[6],[12].

Fractional derivation theory is a subject almost as old as classical calculus as we know it today. These origins date back to the end of the $17th$ century, the time when Newton and Leibniz developed the foundations of differential and integral calculus [34]. The first question that led to fractional calculus was: Can the integer derivative $\frac{d^n f}{dx^n}$ be extended to make sense when n is a fraction? Later the question became: can n be any number: Fractional, irrational or complex? Because the latter question was answered in the affirmative, fractional calculus became a misnomer and might better be called fractional order integration and differentiation. Then the goal of fractional calculus is to generalize traditional derivatives to non-integer orders. As is well known, many dynamic systems are best characterized by a fractional order dynamic model, generally based on the notion of differentiation or integration of non-integer order. The study of fractional order systems is more difficult than for their integer order counterparts.

The theory of fractional differential equations has emerged as an interesting area to explore in recent years. Note that several phenomena of physics and engineering sciences can very well be modeled using differential equations of standard order as well as of non-integer order.

In recent years, numerous results have been established concerning the existence and uniqueness of the solution of fractional differential equations using fixed point theory (see [8],[9],[12],[15]). The concept of Ulam-Hyers stability means that we are not looking for the exact solution for a stable Ulam-Hyers system but that it is necessary to find a function that satisfies an appropriate approximation inequality. This approach can ensure that there is a close exact solution useful in many applications (see [19],[21]).

Another type of non classical derivative is the so called conformable derivative that was introduced by Khalil et al. in [20]. This interesting fractional derivative is based on a limit form as in the classical derivative and has similar properties than the classical one. The new conformable fractional derivative is now knowing a great interest and is the subject of several articles concerning boundary value problems, (see [11],[31],[32],[33].)

Motived by works on non classical derivatives, the thesis has two objectives, the first one is the study of existence, uniqueness and stability of solutions for systems of fractional differential equations in the sense of Caputo with boundary conditions , to this end, some fixed point theorems are used. The second objective is to study traveling wave solutions of conformable time-fractional differential equations. The thesis consists of four chapters.

- ***Chapter one:*** In this chapter, we present some of the fundamental notions of fractional calculus like special functions, integral operators and derivative operators and their properties as well. We also find it useful to mention some of the basics of analysis and topology as a helpful mean to understand functional analysis theorems and how they work.

- ***Chapter two:*** This chapter is study of the existence and uniqueness of solutions of new nonlinear fractional differential equations with integral boundary conditions, using fixed point techniques. We conclude the chapter by providing some illustrative examples in order to show the validity of the results.

- ***Chapter three:*** This chapter is concerned with a new alpha-beta-point non-linear boundary value problem that involves both Caputo derivatives and RL integrals, such that its limiting-case on the parameters alpha and beta product a fourth order problem that arises in bridge design. We begin first by presenting a uniqueness of solution result for the problem. Then, the Ulam-Hyers stability is discussed. An example is then discussed.

- ***Chapter four:*** In this chapter, we apply the tanh method and exp-funtion method for finding travelling waves for a nonlinear problem that involves Khalil derivatives in time and space. Some graphs are plotted for the obtained travelling waves.

CHAPTER 1

Notions of Fractional Calculation

In this chapter we recall some definitions, notions, properties and results on the different approaches of fractional derivation and some results which will be useful in the rest of this thesis.

1.1 Gamma Function

In this section, we present the Gamma function, which will be used in the other chapters. This function plays a very important role in the theory of fractional calculus.

Definition 1.1 *[22][29] The Gamma function noted* Γ *is defined by the following integral:*

$$\Gamma(\psi) = \int_0^{+\infty} e^{-t} t^{\psi-1} dt, \psi \in \mathbb{R}_+,$$

with $\Gamma(1) = 1, \Gamma(0^+) = +\infty$.

1.1.1 Some Properties of The Gamma Function

1) $\Gamma(\psi + 1) = \psi\Gamma(\psi)$.

2) $\Gamma(m + 1) = m!, m \in \mathbb{Z}$.

Proof.

1) Using integration by part, we have

$$\Gamma(\psi + 1) = \int_0^{+\infty} e^{-t} t^{\psi} dt = \left[-t^{\psi} e^{-t}\right]_0^{+\infty} + \int_0^{+\infty} \psi e^{-t} t^{\psi-1} dt$$

$$= \psi \int_0^{+\infty} e^{-t} t^{\psi-1} dt = \psi \Gamma(\psi).$$

2) Using the property 1), we will have

$$\begin{aligned}
\Gamma(2) &= 1\Gamma(1) = 1!, \\
\Gamma(3) &= 2\Gamma(2) = 2 \times 1! = 2!, \\
\Gamma(4) &= 3\Gamma(3) = 3 \times 2! = 3!, \\
& ... \\
& ... \\
& ... \\
\Gamma(m+1) &= m\Gamma(m) = m(m-1)! = m!.
\end{aligned}$$

Proposition 1.1 *The Gamma function is well defined on* $\mathbb{R}^+$.

Proof. We can write $\Gamma(\psi)$ under the form

$$\Gamma(\psi) := \int_0^{+\infty} e^{-t} t^{\psi-1} dt = \int_0^1 e^{-t} t^{\psi-1} dt + \int_1^{+\infty} e^{-t} t^{\psi-1} dt,$$

we put $M_1 = \int_0^1 e^{-t} t^{\psi-1} dt$ and $M_2 = \int_1^{+\infty} e^{-t} t^{\psi-1} dt$.

We have

$$M_1 = \int_0^1 e^{-t} t^{\psi-1} dt < \int_0^1 t^{\psi-1} dt = \frac{1}{\psi},$$

from where M_1 is convergent for $0 < \psi \leq 1$. Let us study the convergence of M_2. We have

$$\frac{t^{\psi-1}}{e^{-\frac{t}{2}}} \leq 1 \text{ because } \lim_{t \longrightarrow +\infty} \frac{t^{\psi-1}}{e^{-\frac{t}{2}}} = 0.$$

Then

$$M_2 = \int_1^{+\infty} e^{-t} t^{\psi-1} dt < \int_1^{+\infty} e^{-\frac{t}{2}} dt = 2e^{-\frac{1}{2}}.$$

Finally the Gamma function is defined for every $\psi > 0$.

1.1.2 Beta Function

Definition 1.2 *[29],[22] Let $\psi, \psi^* \in \mathbb{R}_+^*$. The beta function of Euler is the function defined by*

$$B(\psi, \psi^*) = \int_0^1 t^{\psi-1}(1-t)^{\psi^*-1}\, dt.$$

⋆ Some of the properties of the beta function of Euler is that it can be written as follow ([29])

$$B(\psi, \psi^*) = \frac{\Gamma(\psi)\Gamma(\psi^*)}{\Gamma(\psi+\psi^*)}.$$

⋆ Another property to this function that is related to the previous property is that ([29])

$$B(\psi, \psi^*) = B(\psi^*, \psi), \quad (\psi > 0, \psi^* > 0).$$

1.2 Fractional Integrals

There are several mathematical definitions of fractional order integration, in this section we will introduce the fractional integration operators that are more used in our work and their definitions and characteristics.

1.2.1 Riemann-Liouville Integral

Fractional order integration is a generalization of the notion of integer order integration. The definition of fractional integration in the Riemann-Liouville sense is based on the Cauchy formula which calculates the repeated integral n times of a causal function g,

$$(I_a^n g)(t) = \int_a^t \frac{(t-s)^{n-1} g(s)}{(n-1)!} ds.$$

Definition 1.3 *[27] Let g be a continuous function on $[a,b]$. The integral of g of order α with the approach of Riemann-Liouville is:*

$$(I_a^\alpha g)(t) = \int_a^t \frac{(t-s)^{\alpha-1} g(s)}{\Gamma(\alpha)} ds, \ t \in [a,b]. \tag{1.1}$$

Remark 1.4 *In the case* $\alpha = 0$, *the fractional integral* I^0 *is interpreted as an identity operator.*

Example 1.5 *Let* $\alpha > 0$ *and* $g(t) = \lambda$

$$\begin{aligned} I_a^{\alpha}(\lambda) &= \int_a^t \frac{(t-s)^{\alpha-1}\lambda}{\Gamma(\alpha)} ds, \\ &= \frac{\lambda}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} ds, \\ &= \frac{\lambda}{\Gamma(\alpha)} \left[\frac{(t-s)^{\alpha}}{-\alpha} \right]_{s=a}^{s=t}, \\ &= \frac{\lambda(t-a)^{\alpha}}{\alpha\Gamma(\alpha)}. \end{aligned}$$

Proposition 1.2 *Let* $g \in C\left([a,b]\right)$. *Then, we have*

$$I_a^{\alpha}\left(I_a^{\beta} f\left(t\right)\right) = I_a^{\alpha+\beta} f(t),\ \ \alpha > 0, \beta > 0.$$

Proof

By definition, we have

$$\begin{aligned} I_a^{\alpha}\left(I_a^{\beta} f\right)(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(I_a^{\beta} f\right)(s) ds \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t (t-s)^{\alpha-1} \left(\int_0^s (s-\tau)^{\beta-1} f(\tau) d\tau \right) ds \qquad (1.2) \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t f(\tau) \left(\int_{\tau}^t (t-s)^{\alpha-1}(s-\tau)^{\beta-1} ds \right) d\tau. \end{aligned}$$

We put

$$y = \frac{s-\tau}{t-\tau},$$

so we can write

$$\begin{aligned} \int_{\tau}^t (t-s)^{\alpha-1}(s-\tau)^{\beta-1} ds &= (t-\tau)^{\alpha+\beta-1} \int_0^1 (1-x)^{\alpha-1} x^{\beta-1} dx \\ &= (t-\tau)^{\alpha+\beta-1} B(\alpha,\beta) \qquad (1.3) \\ &= (t-\tau)^{\alpha+\beta-1} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \end{aligned}$$

Replacing (1.3) in (1.2), we will have

$$I_a^{\alpha}\left(I_a^{\beta} f\right)(t) = \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-\tau)^{\alpha+\beta-1} f(\tau) d\tau = I_a^{\alpha+\beta} f(t).$$

■

1.2.2 Conformable Integral

Definition 1.6 *[1] [20] The conformable fractional integral of a function $f:(0,\infty)\to\mathbb{R}$ of order α is defined as*

$$J_\alpha f(t)=\int_0^t \tau^{\alpha-1}f(\tau)\,d\tau,\qquad 0<\alpha\leq 1.$$

1.3 Fractional Derivatives

There are several definitions of fractional derivatives, unfortunately they are not all equivalent. In this part we present the definitions of Riemann-Liouville, Liouville, Caputo as well as Al-khalil which are the most used.

1.3.1 Riemann-Liouville Derivative

Definition 1.7 *[27] Let $f\in L^1([a,b]),\alpha>0,$ and $n=[\alpha]$. The fractional derivative of Riemann-Liouville of order α is given by*

$$\begin{aligned}\forall t\in[a,b],\left({}^{RL}D_a^\alpha f\right)(t) &= \left(\frac{d}{dt}\right)^n\left(I_a^{n-\alpha}f\right)(t)\\ &= \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n\int_a^t(t-\psi)^{n-\alpha-1}f(\psi)\,d\psi.\end{aligned}$$

Remark 1.8

If $\alpha=n\in\mathbb{N}_+$, then

$$\left({}^{RL}D_a^\alpha f\right)(t) = f^{(n)}(t),$$

where $f^{(n)}$ is the standard derivative of order n of the function f.

Example 1.9

Let f be the function defined by $f(t)=t^\lambda, t\in[0,b], b>0, \lambda>-1$ and $n-1<\alpha\leq n, n\in\mathbb{N}^*$.
So, we have

$$\left({}^{RL}D_0^\alpha f\right)(x) \;:\; = \left(\frac{d}{dx}\right)^n\left(\frac{1}{\Gamma(m-\alpha)}\int_0^x(x-s)^{n-\alpha-1}s^\lambda ds\right)$$

$$\begin{aligned} &= \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n\left(x^{n-\alpha+\lambda}\int_0^1(1-u)^{n-\alpha-1}u^{\lambda}du\right) \\ &= \frac{1}{\Gamma(m-\alpha)}B(\lambda+1,n-\alpha)\left(\frac{d}{dx}\right)^n x^{n-\alpha+\lambda}. \end{aligned}$$

Since

$$\left(\frac{d}{dx}\right)^n x^p = p(p-1)\ldots(p-n+1)x^{p-n} = \frac{\Gamma(p+1)}{\Gamma(p-n+1)}x^{p-n},$$

for any $p \in \mathbb{R}\setminus\{-1,-2,-3,\ldots\}$.

Therefore, we obtain

$$\begin{aligned} \left({}^{RL}D_0^{\alpha}f\right)(x) &= \frac{1}{\Gamma(n-\alpha)} \times \frac{\Gamma(\lambda+1)\Gamma(n-\alpha)}{\Gamma(\lambda+1+n-\alpha)} \times \frac{\Gamma(n-\alpha+\lambda+1)}{\Gamma(n-\alpha+\lambda-m+1)}x^{\lambda-\alpha} \\ &= \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)}x^{\lambda-\alpha}. \end{aligned}$$

Remark 1.10

As a special case, if $\lambda = 0$, then we get

$$\begin{aligned} {}^{RL}D_0^{\alpha}1 &= \frac{x^{-\alpha}}{\Gamma(1-\alpha)},\ \forall\alpha \in \mathbb{R}^+\setminus\{0,1,2,3,\ldots\}. \\ {}^{RL}D_0^{\alpha}1 &= 0, \forall\alpha \in \mathbb{Z}_+. \end{aligned}$$

Remark 1.11 *The fractional derivative in the sense of Riemann-Liouville of a constant function is not zero.*

Proposition 1.3 *Let $\alpha, \beta > 0$ such as $n-1 < \alpha \le n$ and $m-1 < \beta \le m, n, m \in \mathbb{N}^*$. If $\alpha > \beta > 0$, then for $f \in L^1([a,b])$, we have*

$$\left({}^{RL}D^{\beta}I_a^{\alpha}f\right)(t) = I_a^{\alpha-\beta}f(t).$$

Proposition 1.4 *Let $\alpha > 0$ such as $n-1 < \alpha \le n, n \in \mathbb{N}^*$. For $f \in L^1([a,b])$, we have*

$$\left({}^{RL}D^{\alpha}I_a^{\alpha}f\right)(t) = f(t).$$

Proposition 1.5 *Let $n-1 < \alpha \le n, n \in \mathbb{N}^*, m \in \mathbb{N}^*$ and $f \in L^1([a,b])$. If the fractional derivatives $\left({}^{RL}D^{\alpha}f\right)(t)$ and $\left(D^{\alpha+m}f\right)(t)$ exist, then we have*

$$\left({}^{RL}D^{m}\left({}^{RL}D^{\alpha}f\right)\right)(t) = \left({}^{RL}D^{\alpha+m}f\right)(t).$$

1.3.2 Caputo Derivative

Definition 1.12 *The Caputo fractional derivative of order $\alpha \in \mathbb{R}$ $(\alpha > 0)$ of a function $f \in C^n([a,b])$ is defined by*

$$D_a^\alpha f(t) := I_a^{n-\alpha} f^{(n)}(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s)\, ds, \ n \in \mathbb{N}^*, n-1 < \alpha < n, t > a.$$

Remark 1.13

1) In particular, when $0 < \alpha < 1$ and $f \in C([a,b])$, then

$$D_a^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-s)^{-\alpha} f'(s)\, ds = I_a^{1-\alpha} f'(t).$$

2) If $\alpha \in \mathbb{N}$, then we have

$$D_a^\alpha f(t) = f^{(n)}(t).$$

Example 1.14

Let $f(t) = C, t \in [a,b]$, the constant function, then we have

$$D^\alpha f(t) = 0 \text{ but } {}^{RL}D^\alpha f(t) \neq 0.$$

Proposition 1.6 *Let f and g be two functions such that $D^\alpha f(t), D^\alpha g(t)$ exist. Then the Caputo fractional derivation is a linear operator:*

$$D_a^\alpha (\lambda f + \gamma g)(t) = \lambda D_a^\alpha f(t) + \gamma D_a^\alpha g(t), \ \forall \lambda, \gamma \in \mathbb{R}.$$

Theorem 1.15 *Let $\alpha > 0$ with $n-1 < \alpha < n, n \in \mathbb{N}^*$, and let f be a function such that $D_a^\alpha f(t)$ et ${}^{RL}D_a^\alpha f(t)$ exist. Then, we have:*

$$D_a^\alpha f(t) = {}^{RL}D_a^\alpha f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(k-\alpha+1)} (t-a)^{k-\alpha}.$$

1.3.3 Conformable Derivative

Recently, Khalil et al. gave a new definition of integral and derivative of non-integer order [20]. This new definition is used as a limit form as in the case of the classical derivative. They proved the product rule, the fractional Rolle theorem and the mean value theorem. Later, this theory is developed by Abdeljawad [1] who gave definitions of the left and right conformable derivatives of higher order, integration by part formulas, chain rule, Taylor power series representation.

Definition 1.16 *[1],[20] Given a function* $f : (0, \infty) \to \mathbb{R}$.

Then, the conformable fractional derivative of order α *is defined by*

$$T_\alpha(f)(t) = \frac{\partial^\alpha f(t,x)}{\partial t^\alpha} = \lim_{\varepsilon \to 0} \left(\frac{f\left(t + \varepsilon t^{1-\alpha}\right) - f(t)}{\varepsilon} \right), \quad t > 0, \quad 0 < \alpha \leq 1.$$

$\diamondsuit$ **Some Properties of The Conformable Derivative**

Several properties of conformable fractional derivative definition are as follows [1],[20]:

Let $f, g : [0, \infty) \to \mathbb{R}$ two functions α-differentiables, with $\alpha \in (0, 1]$. Then:

1) $T_\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g)$, for all $a, b \in \mathbb{R}$.

2) $T_\alpha(C) = 0$.

3) $T_\alpha\left(t^b\right) = bt^{b-\alpha}$, for all $b \in \mathbb{R}$.

4) $T_\alpha(fg) = gT_\alpha(f) + fT_\alpha(g)$.

5) $T_\alpha(f)(t) = t^{1-\alpha}\frac{df}{dt}(t)$.

1.4 Auxiliary Results

In this section, we provide some lemmas of fractional derivatives, witch will play major roles in our analysis, see [3],[12].

Lemma 1.7 *The general solution of the equation* $D_a^{\alpha} u(t) = 0, t \in [a, b]$ *can be given by:*

$$u(t) = \sum_{i=0}^{n-1} \lambda_i (t-a)^i, t \in [a, b],$$

such that $\lambda_i \in \mathbb{R}, i = 0, 1, 2, .., n-1,\ n = [\alpha] + 1.$

Lemma 1.8 *Let* $\alpha > 0$. *Then, it yields that*

$$I^{\alpha}(D^{\alpha} u(t)) = u(t) + \sum_{i=0}^{n-1} \lambda_i (t-a)^i, t \in [a, b],$$

for $\lambda_i \in \mathbb{R}, i = 0, 1, 2, ..., n-1,\ n = [\alpha] + 1.$

1.5 Fixed Point Theory

In this chapter, we are concerned with some important notions on fixed point theory. Some integral inequalities are shown to the reader in order to be used in the two last chapters [14],[23],[35].

1.5.1 Essential Concepts

The Banach contraction principle is the most elementary result which ensures the uniqueness of a fixed point. This theorem is essentially based on the following definitions:

Definition 1.17 *Let* V *a normed vector space, of norm* $\|\cdot\|_V$ *et* $(u_n)_n, n \in \mathbb{N}$, *a sequence of* V. *We say that* $(u_n)_n$ *is a Cauchy sequence if*

$$\forall \varepsilon > 0, \quad \exists N_{\varepsilon} \geq 0, \forall n \geq N_{\varepsilon}, \forall q \geq N_{\varepsilon}, \|u_q - u_n\|_V \leq \varepsilon.$$

Definition 1.18 *We say that the norm vector space* S *is complete for the norm* $\|\cdot\|_V$ *if every Cauchy sequence (for this norm) is convergent (for this norm). Such a space is also called Banach space.*

Definition 1.19 *Let* B *be a Banach space with the norm* $\|\cdot\|_B$ *and* Z *be a map from* B *to* B, *we call a fixed point of* Z *any point* u *such that:*

$$Zu = u.$$

Definition 1.20 *Let V be a norm vector space, with norm $\|\cdot\|_V$. A function f from V to V is called Lipschitzian with constant $\lambda \geq 0$ if it satisfies:*

$$\forall u_1, u_2 \in V, \|f(u_1) - f(u_2)\|_V \leq \lambda \|u_1 - u_2\|_V.$$

Definition 1.21 *The Lipschitzian function f is called a contraction if $\lambda \in]0,1[$.*

1.5.2 Banach Contraction Principle

Theorem 1.22 *[22] [23] Let be B a Banach space and $Z : B \longrightarrow B$ be a contracting application. Then Z has a unique fixed point.*

We propose to the reader also the following theorem:

Theorem 1.23 *Let Z be an application on a Banach space B, such as Z^M is contraction on B for a positive integer M. So Z admits a unique fixed point.*

1.5.3 Schaefer Fixed Point Theorem

Theorem 1.24 *[22] [23] Let B be a Banach space and $Z : B \to B$ be a completely continuous operator. If the set:*

$$\Omega := \{u \in B : u = \lambda Z u, \lambda \in]0,1[\}$$

is bounded, hence Z has at least one fixed point.

1.6 Travelling Waves

It is now interesting to give some definitions of certain concepts of the mathematical theory of waves.

1.6.1 Wave Equation

The physical definition of a wave is movement up and down or back and forth. The wave is also a disturbance that transmits energy from one place to another. The main characteristic of the traveling wave is that the disturbance finds itself identical to itself after a duration T (temporal period of propagation) and at a distance of X (spatial period of propagation or wavelength). For this, the propagation medium must have an infinite extension or, at least, a very large size compared to that of the wavelength. The simplest wave propagation equation is given by

$$u_{tt} = c^2 u_{xx},$$

where $u(x,t)$ represents the amplitude of the wave, and c is the speed of the wave. This equation has the general d'Alembert solution

$$u(x,t) = f(x-ct) + g(x+ct),$$

where f and g are arbitrary functions which represent right and left propagating waves respectively.

1.6.2 Traveling Wave Solution

The traveling wave solution is a permanent shape solution moving with a constant speed c. Traveling wave solutions are generally obtained by reducing Nonlinear Partial Differential Equations (NPDE) into associated ordinary differential equations (EDOs), which are solved by several suitable methods. This is mainly handled by using $u(x,t) = u(\xi)$, $\xi = x - ct$.

CHAPTER 2

An Analytical Study for a Class of Fractional Differential Equations

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2.1 Introduction

Fractional differential equations have wide-ranging applications in diverse areas of science and engineering, such as physics, biology, biotechnology, medicine, control theory, and signal and image processing. The substantial number of important mathematical and physical publications, as well as numerous outstanding monographs, underscores the significant interest in this field. For more detailed information, please consult, for example,[6] [8]. The theory on existence and uniqueness of solutions of nonlinear fractional differential equations has attracted the attention of many authors. Fixed point theorems contribute with a substantial and great role in the study of the uniqueness and existence. For some recent results, we refer the interested reader to [9][15][22].

The aim of this chapter to study the following fractional differential problem:

$$\begin{cases} D^{\alpha}D^{\beta}D^{\gamma}u(t) = \dfrac{a_1 f(t,u(t),D^{\gamma}u(t)) + a_2 g(t,D^{\gamma}u(t),D^{\gamma}(D^{\rho}u(t)) + a_3 h(t,u(t))}{K(u(t))}, \\ u(0)+u(1) = \int_0^{\eta} bu(s)ds, 0<\eta<1, \\ D^{\gamma}u(0)+D^{\gamma}u(1)=0, \\ D^{\mu}D^{\mu}u(0)+D^{\mu}D^{\mu}u(1)=0, \\ t\in[0,1], \\ 0<\alpha,\beta,\gamma,\rho,\mu\leq 1, \\ a_1,a_2,a_3,b\in\mathbb{R}, \end{cases} \tag{2.1}$$

where $D^{\alpha}, D^{\beta}, D^{\gamma}, D^{\mu}, D^{\rho}$ are the Caputo fractional, with $0<\alpha,\beta,\gamma,\rho\leq 1$,$\alpha+\beta\notin]0,1), \beta+\gamma\notin]0,1), \gamma+\rho\notin]0,1), J:=[0,1]$ and $f,g: I\times\mathbb{R}^2\to\mathbb{R}$, $h: I\times\mathbb{R}\to\mathbb{R}$, and $K:\mathbb{R}\to\mathbb{R}_+^*$ are given functions.

We prove an uniqueness result, then we discuss an illustrative example.

2.2 The Integral Representation

Since the system we have is not linear, we consider what is called "integral representation" or "integral solution" of the given problem.

The following lemma is an auxiliary result that highlights the integral representation of the system (2.1) which is very important for the main results.

Lemma 2.1 *Let G be any continuous function over J. Then, writing*

$$\begin{cases} D^{\alpha}D^{\beta}D^{\gamma}u(t) = \frac{G(t)}{K(u(t))}, \\ u(0)+u(1) = \int_0^{\eta} bu(s)ds, 0<\eta<1, \\ D^{\gamma}u(0)+D^{\gamma}u(1)=0, \\ D^{\mu}D^{\mu}u(0)+D^{\mu}D^{\mu}u(1)=0, \end{cases} \tag{2.2}$$

is geven by:

$$\begin{aligned} u(t) &= I^{\alpha+\beta+\gamma}\Big(\frac{G(t)}{K(u(t))}\Big) - \theta^{-1}I^{\alpha+\beta+\gamma}\Big(\frac{G(1)}{K(u(1))}\Big) + b\theta^{-1}I^{\alpha+\beta+\gamma+1}\Big(\frac{G(\eta)}{K(u(\eta))}\Big) \\ &+ \Lambda_1\Big[\frac{\theta t^{\gamma}-1}{z_5} + \frac{2z_2-2\theta z_2 t^{\beta+\gamma}}{z_4} + \frac{b\eta^{\gamma+1}}{z_6} - \frac{2bz_2\eta^{\beta+\gamma+1}}{z_7}\Big] I^{\alpha+\beta+\gamma-2\mu}\Big(\frac{G(1)}{K(u(1))}\Big) \\ &+ \Lambda_2\Big[\frac{\theta z_1 t^{\beta+\gamma}-z_1}{z_4} + \frac{z_3-\theta z_3 t^{\gamma}}{z_5} + \frac{bz_1\eta^{\beta+\gamma+1}}{z_7} - \frac{bz_3\eta^{\gamma+1}}{z_6}\Big] I^{\alpha+\beta}\Big(\frac{G(1)}{K(u(1))}\Big), \end{aligned} \tag{2.3}$$

such that one takes into account:

$$
\begin{aligned}
&z_1 = \Gamma(\beta + \gamma - 2\mu + 1),\\
&z_2 = \Gamma(\beta + 1),\\
&z_3 = \Gamma(\gamma - 2\mu + 1),\\
&z_4 = \Gamma(\beta + \gamma + 1),\\
&z_5 = \Gamma(\gamma + 1),\\
&z_6 = \Gamma(\gamma + 2),\\
&z_7 = \Gamma(\beta + \gamma + 2),\\
&2z_2z_3 - z_1 \neq 0,\\
&2 - b\eta = \theta \neq 0,\\
&\Lambda_1 = \frac{z_1z_3}{\theta[2z_2z_3 - z_1]},\\
&\Lambda_2 = \frac{z_2}{\theta[2z_2z_3 - z_1]}.
\end{aligned}
\tag{2.4}
$$

Proof The reader who is familiarized with fractional calculus theory can see that the two identities:

$$D^\beta D^\gamma u(t)) = I^\alpha\Big(\frac{G(t)}{K(u(t))}\Big) - c_0,$$

and

$$D^\gamma u(t) = I^{\alpha+\beta}\Big(\frac{G(t)}{K(u(t))}\Big) - c_0 I^\beta(1) - c_1,$$

are valid.

So, it yields that

$$u(t) = I^{\alpha+\beta+\gamma}\Big(\frac{G(t)}{K(u(t))}\Big) - c_0\frac{t^{\beta+\gamma}}{\Gamma(\beta+\gamma+1)} - c_1\frac{t^\gamma}{\Gamma(\gamma+1)} - c_2. \tag{2.5}$$

Therefore,

$$\begin{cases} -2c_2 + I^{\alpha+\beta+\gamma}\Big(\dfrac{G(1)}{K(u(1))}\Big) - \dfrac{c_0}{\Gamma(\beta+\gamma+1)} - \dfrac{c_1}{\Gamma(\gamma+1)} = \displaystyle\int_0^\eta bu(s)ds, \\ \\ -2c_1 + I^{\alpha+\beta}\Big(\dfrac{G(1)}{K(u(1))}\Big) - \dfrac{c_0}{\Gamma(\beta+1)} = 0, \\ \\ I^{\alpha+\beta+\gamma-2\mu}\Big(\dfrac{G(1)}{K(u(1))}\Big) - \dfrac{c_0}{\Gamma(\beta+\gamma-2\mu+1)} - \dfrac{c_1}{\Gamma(\gamma-2\mu+1)} = 0. \end{cases}$$

We solve the above system to write

$$\begin{aligned}
c_0 &= \frac{2\Gamma(\beta+\gamma-2\mu+1)\Gamma(\beta+1)\Gamma(\gamma-2\mu+1)}{2\Gamma(\beta+1)\Gamma(\gamma-2\mu+1)-\Gamma(\beta+\gamma-2\mu+1)} I^{\alpha+\beta+\gamma-2\mu}\Big(\frac{G(1)}{K(u(1))}\Big) \\
&- \frac{\Gamma(\beta+\gamma-2\mu+1)\Gamma(\beta+1)}{2\Gamma(\beta+1)\Gamma(\gamma-2\mu+1)-\Gamma(\beta+\gamma-2\mu+1)} I^{\alpha+\beta}\Big(\frac{G(1)}{K(u(1))}\Big), \\
c_1 &= -\frac{\Gamma(\beta+\gamma-2\mu+1)\Gamma(\gamma-2\mu+1)}{2\Gamma(\beta+1)\Gamma(\gamma-2\mu+1)-\Gamma(\beta+\gamma-2\mu+1)} I^{\alpha+\beta+\gamma-2\mu}\Big(\frac{G(1)}{K(u(1))}\Big) \\
&+ \frac{\Gamma(\beta+1)\Gamma(\gamma-2\mu+1)}{2\Gamma(\beta+1)\Gamma(\gamma-2\mu+1)-\Gamma(\beta+\gamma-2\mu+1)} I^{\alpha+\beta}\Big(\frac{G(1)}{K(u(1))}\Big), \\
c_2 &= \frac{1}{2-b\eta} I^{\alpha+\beta+\gamma}\Big[\frac{G(1)}{K(u(1))}\Big] - \frac{b}{2-b\eta} I^{\alpha+\beta+\gamma+1}\Big(\frac{G(\eta)}{K(u(\eta))}\Big) \\
&+ \frac{\Gamma(\beta+\gamma-2\mu+1)\Gamma(\gamma-2\mu+1)}{2\Gamma(\beta+1)\Gamma(\gamma-2\mu+1)-\Gamma(\beta+\gamma-2\mu+1)} \Big[\frac{1}{(2-b\eta)\Gamma(\gamma+1)} - \frac{2\Gamma(\beta+1)}{(2-b\eta)\Gamma(\beta+\gamma+1)} \\
&+ \frac{2b\Gamma(\beta+1)\eta^{\beta+\gamma+1}}{(2-b\eta)\Gamma(\beta+\gamma+2)} - \frac{b\eta^{\gamma+1}}{(2-b\eta)\Gamma(\gamma+2)}\Big] I^{\alpha+\beta+\gamma-2\mu}\Big(\frac{G(1)}{K(u(1))}\Big) \\
&+ \frac{\Gamma(\beta+1)}{2\Gamma(\beta+1)\Gamma(\gamma-2\mu+1)-\Gamma(\beta+\gamma-2\mu+1)} \Big[\frac{\Gamma(\beta+\gamma-2\mu+1)}{(2-b\eta)\Gamma(\beta+\gamma+1)} - \frac{\Gamma(\gamma-2\mu+1)}{(2-b\eta)\Gamma(\gamma+1)} \\
&+ \frac{b\Gamma(\beta+\gamma-2\mu+1)\eta^{\beta+\gamma+1}}{(2-b\eta)\Gamma(\beta+\gamma+2)}\Big] I^{\alpha+\beta}\Big(\frac{G(1)}{K(u(1))}\Big).
\end{aligned}$$

Inserting c_0, c_1 and c_2 in (2.5), we end the proof. ■

We will use fixed point theory to study the above problem. Hence, we need the space:

$$X := \{u \in C(J,\mathbb{R}), D^\gamma u \in C(J,\mathbb{R}), D^\gamma D^\rho u \in C(J,\mathbb{R})\},$$

and

$$\|u\|_X = \|u\|_\infty + \|D^\gamma u\|_\infty + \|D^\gamma D^\rho u\|_\infty,$$

where,

$$\|u\|_\infty = \sup_{t\in J}|u(t)|\ , \|D^\gamma u\|_\infty = \sup_{t\in J}|D^\gamma u(t)|, \|D^\gamma D^\rho u\|_\infty = \sup_{t\in J}|D^\gamma D^\rho u(t)|.$$

Then, we consider the application $Z : X \to X$,

$$\begin{aligned}
Hu(t) &= z_8^{-1}\int_0^t (t-\tau)^{\alpha+\beta+\gamma-1}\left[\frac{a_1 f(\tau,u(\tau),D^\gamma u(\tau)) + a_2 g(\tau,D^\gamma u(\tau),D^\gamma D^\rho u(\tau))}{K(u(\tau))}\right.\\
&+ \left.\frac{a_3 h(\tau,u(\tau))}{K(u(\tau))}\right]d\tau - (\theta z_8)^{-1}\int_0^1 (t-\tau)^{\alpha+\beta+\gamma-1}\left[\frac{a_1 f(\tau,u(\tau),D^\gamma u(\tau))}{K(u(\tau))}\right.\\
&+ \left.\frac{a_2 g(\tau,D^\gamma u(\tau),D^\gamma D^\rho u(\tau)) + a_3 h(\tau,u(\tau))}{K(u(\tau))}\right]d\tau - b(\theta z_9)^{-1}\int_0^\eta (t-\tau)^{\alpha+\beta+\gamma}\\
&\times \left[\frac{a_1 f(\tau,u(\tau),D^\gamma u(\tau)) + a_2 g(\tau,D^\gamma u(\tau),D^\gamma D^\rho u(\tau)) + a_3 h(\tau,u(\tau))}{K(u(\tau))}\right]d\tau\\
&+ \Lambda_1 z_{10}^{-1}\left[\frac{\theta t^\gamma - 1}{z_5} + \frac{2z_2 - 2\theta z_2 t^{\beta+\gamma}}{z_4} + \frac{b\eta^{\gamma+1}}{z_6} - \frac{2bz_2\eta^{\beta+\gamma+1}}{z_7}\right]\\
&\times \int_0^1 (t-\tau)^{\alpha+\beta+\gamma-2\mu-1}\left[\frac{a_1 f(\tau,u(\tau),D^\gamma u(\tau)) a_2 g(\tau,D^\gamma u(\tau),D^\gamma D^\rho u(\tau))}{K(u(\tau))}\right.\\
&+ \left.\frac{a_3 h(\tau,u(\tau))}{K(u(\tau))}\right]d\tau + \Lambda_2 z_{11}^{-1}\left[\frac{\theta z_1 t^{\beta+\gamma} - z_1}{z_4} + \frac{z_3 - \theta z_3 t^\gamma}{z_5} + \frac{bz_1\eta^{\beta+\gamma+1}}{z_7} - \frac{bz_3\eta^{\gamma+1}}{z_6}\right]\\
&\times \int_0^1 (t-\tau)^{\alpha+\beta-1}\left[\frac{a_1 f(\tau,u(\tau),D^\gamma u(\tau)) + a_2 g(\tau,D^\gamma u(\tau),D^\gamma D^\rho u(\tau)) + a_3 h(\tau,u(\tau))}{K(u(\tau))}\right]d\tau,
\end{aligned}$$

where

$$\begin{aligned}
z_8 &= \Gamma(\alpha+\beta+\gamma),\\
z_9 &= \Gamma(\alpha+\beta+\gamma+1),\\
z_{10} &= \Gamma(\alpha+\beta+\gamma-2\mu),\\
z_{11} &= \Gamma(\alpha+\beta),\\
z_{12} &= \Gamma(\alpha+\beta+\gamma+2),\\
z_{13} &= \Gamma(\alpha+\beta+\gamma-2\mu+1),\\
z_{14} &= \Gamma(\alpha+\beta+1),\\
z_{15} &= \Gamma(\alpha+\beta-\rho),\\
z_{16} &= \Gamma(\beta-\rho+1),\\
z_{17} &= \Gamma(\alpha+\beta-\rho+1).
\end{aligned} \tag{2.6}$$

2.3 Uniqueness of Solutions

We are concerned with studying the uniqueness of one solution for (2.1). So, we consider the following hypotheses that can be seen as sufficient conditions, they can be changed by some other conditions (of Caratheodory type, for instance). But for our case, all what is needed is only the establishment of "some" conditions that allow us to assure the uniqueness of solutions. Therefore, we put:

$(A1)$: The functions f, g defined on $J \times \mathbb{R}^2$ and h defined on $J \times \mathbb{R}$ are continuous.

$(A2)$: There are non-negative continuous functions $\phi_1(t), \phi_2(t), \psi_1(t), \psi_2(t)$, $\delta(t)$, such that for any $t \in J$, $u_i, v_i \in \mathbb{R}$,

$$\begin{aligned}
|f(t,u_1,u_2) - f(t,v_1,v_2)| &\leq \sum_{i=1}^{2} \phi_i(t)|u_i - v_i|,\\
|g(t,u_1,u_2) - g(t,v_1,v_2)| &\leq \sum_{i=1}^{2} \psi_i(t)|u_i - v_i|,
\end{aligned}$$

and for any $t \in J$, $u, v \in \mathbb{R}$,

$$|h(t,u) - h(t,v)| \leq \delta(t)|u - v|.$$

It is to note that we take:

$$\|\phi\| = Max(\sup_{t\in J}|\phi_1(t)|, \sup_{t\in J}|\phi_2(t)|),$$
$$\|\psi\| = Max(\sup_{t\in J}|\psi_1(t)|, \sup_{t\in J}|\psi_2(t)|),$$
$$\|\delta\| = \sup_{t\in I}|\delta(t)|.$$

$(A3)$: The function K satisfies: $m \leq K(u) \leq M$, with $m, M \in \mathbb{R}_+^*$, for any $u \in \mathbb{R}$. Also we need to consider the quantities to facilitate the fastidious calculations:

$$\begin{aligned}
\Delta_1 &= \Omega\Bigg[|(|\theta|z_9)^{-1} + z_9^{-1}| + |b(\theta z_{12})^{-1}| + |\Lambda_1 z_{13}^{-1}| \\
&\times \left(\frac{|\theta|+1}{z_5} + \frac{2z_2 + 2|\theta|z_2}{z_4} + \frac{|b|\eta^{\gamma+1}}{z_6} + \frac{2|b|z_2\eta^{\beta+\gamma+1}}{z_7}\right) \\
&+ |\Lambda_2 z_{14}^{-1}|\left(\frac{|\theta|z_1 + z_1}{z_4} + \frac{z_3 + |\theta|z_3}{z_5} + \frac{|b|z_1\eta^{\beta+\gamma+1}}{z_7} + \frac{|b|z_3\eta^{\gamma+1}}{z_6}\right)\Bigg], \\
\Delta_2 &= \Omega\left[z_{14} + |\theta\Lambda_1 z_{13}^{-1}| + |\theta\Lambda_2 z_{14}^{-1}|\left(\frac{z_1}{z_2} + z_3\right)\right], \\
\Delta_3 &= \Omega\left[z_{17}^{-1} + |2\theta\Lambda_1 z_2 z_{13}^{-1} z_{16}^{-1}| + |\theta\Lambda_2 z_1 z_{14}^{-1} z_{16}^{-1}|\right].
\end{aligned}$$

With

$$\Omega = \frac{2|a_1|\|\phi\| + 2|a_2|\|\psi\| + |a_3|\|\delta\|}{m}.$$

Now, we pass to prove the main result.

Theorem 2.1 *Assume that the conditions (A_2) and (A_3) are valid and suppose that the constant satisfies $\Delta : \Delta = \sum_{i=1}^{3}\Delta_i < 1$. Then, problem (2.1) has a unique solution over J.*

Proof We can use Banach principle to prove this result. For $(u_1, u_2) \in B^2$, we can write

$$\begin{aligned} \|Zu_1 - Zu_2\|_\infty &\leq \Omega\Bigg[|(|\theta|z_9)^{-1} + z_9^{-1}| + |b(\theta z_{12})^{-1}| + |\Lambda_1 z_{13}^{-1}| \\ &\times\left(\frac{|\theta|+1}{z_5} + \frac{2z_2 + 2|\theta|z_2}{z_4} + \frac{|b|\eta^{\gamma+1}}{z_6} + \frac{2|b|z_2\eta^{\beta+\gamma+1}}{z_7}\right) \\ &+|\Lambda_2 z_{14}^{-1}|\left(\frac{|\theta|z_1 + z_1}{z_4} + \frac{z_3 + |\theta|z_3}{z_5} + \frac{|b|z_1\eta^{\beta+\gamma+1}}{z_7}\right. \\ &\left.+\frac{|b|z_3\eta^{\gamma+1}}{z_6}\right)\Bigg]\|u_1 - u_2\|_X. \end{aligned} \tag{2.7}$$

We have also

$$\begin{aligned} D^\gamma Zu(t) &= z_{11}^{-1}\int_0^t (t-\tau)^{\alpha+\beta-1}\Bigg[\frac{a_1 f(\tau, u(\tau), D^\gamma u(\tau)) + a_2 g(\tau, D^\gamma u(\tau), D^\gamma D^\rho u(\tau))}{K(u(\tau))} \\ &+ \frac{a_3 h(\tau, u(\tau))}{K(u(\tau))}\Bigg]d\tau + \Lambda_1\theta\left(1 - 2t^\beta\right) z_{10}^{-1}\int_0^1 (t-\tau)^{\alpha+\beta+\gamma-2\mu-1} \\ &\times \left[\frac{a_1 f(\tau, u(\tau), D^\gamma u(\tau)) + a_2 g(\tau, D^\gamma u(\tau), D^\gamma D^\rho u(\tau)) + a_3 h(\tau, u(\tau))}{K(u(\tau))}\right]d\tau \\ &+ \Lambda_2\theta\left(\frac{z_1}{z_2}t^\beta - z_3\right)\frac{1}{\Gamma(\alpha+\beta)}\int_0^1 (t-\tau)^{\alpha+\beta-1}\Bigg[\frac{a_1 f(\tau, u(\tau), D^\gamma u(\tau))}{K(u(\tau))} \\ &+ \left[\frac{a_2 g(\tau, D^\gamma u(\tau), D^\gamma D^\rho u(\tau)) + a_3 h(\tau, u(\tau))}{K(u(\tau))}\right]d\tau, \end{aligned}$$

As before, we have

$$\|D^\gamma Zu_1 - D^\gamma Zu_2\|_\infty \leq \Omega\left[z_{14} + |\theta\Lambda_1 z_{13}^{-1}| + |\theta\Lambda_2 z_{14}^{-1}|\left(\frac{z_1}{z_2} + z_3\right)\right]\|u_1 - u_2\|_X. \tag{2.8}$$

Also, taking into account that

$$
\begin{aligned}
D^{\gamma}D^{\rho}Hu(t) &= z_{15}^{-1}\int_0^t (t-\tau)^{\alpha+\beta-\rho-1}\Bigg[\frac{a_1 f(\tau,u(\tau),D^{\gamma}u(\tau)) = a_2 g(\tau,D^{\gamma}u(\tau),D^{\gamma}D^{\rho}u(\tau))}{K(u(\tau))} \\
&+ \frac{a_3 h(\tau,u(\tau))}{K(u(\tau))}\Bigg]d\tau + \theta\Lambda_1\left(-2z_2 z_{16}^{-1}t^{\beta-\rho}\right) z_{10}^{-1}\int_0^1 (t-\tau)^{\alpha+\beta+\gamma-2\mu-1} \\
&\times \left[\frac{a_1 f(\tau,u(\tau),D^{\gamma}u(\tau)) a_2 g(\tau,D^{\gamma}u(\tau),D^{\gamma}D^{\rho}u(\tau)) + a_3 h(\tau,u(\tau))}{K(u(\tau))}\right]d\tau \\
&+ \theta\Lambda_1 z_{11}^{-1}\left(z_1 z_{16}^{-1}t^{\beta-\rho}\right)\int_0^1 (t-\tau)^{\alpha+\beta-1}\Bigg[\frac{a_1 f(\tau,u(\tau),D^{\gamma}u(\tau))}{K(u(\tau))} \\
&+ \frac{a_2 g(\tau,D^{\gamma}u(\tau),D^{\gamma}D^{\rho}u(\tau)) + a_3 h(\tau,u(\tau))}{K(u(\tau))}\Bigg]d\tau.
\end{aligned}
$$

So, the reader can observe that

$$
\|D^{\gamma}D^{\rho}Zu_1 - D^{\gamma}D^{\rho}Zu_2\|_{\infty} \quad \le \Omega\left[z_{17}^{-1} + |2\theta\Lambda_1 z_2 z_{13}^{-1} z_{16}^{-1}| + |\theta\Lambda_2 z_1 z_{14}^{-1} z_{16}^{-1}|\right]\|u_1-u_2\|_X. \tag{2.9}
$$

Now, thanks to (2.7), (2.8) and (2.9), we find that

$$
\begin{aligned}
\|Zu_1 - Zu_2\|_{\infty} &\le \Delta_1\|u_1-u_2\|_X, \\
\|D^{\gamma}Zu_1 - D^{\gamma}Zu_2\|_{\infty} &\le \Delta_2\|u_1-u_2\|_X, \\
\|D^{\gamma}D^{\rho}Zu_1 - D^{\gamma}D^{\rho}Zu_2\|_{\infty} &\le \Delta_3\|u_1-u_2\|_X.
\end{aligned}
$$

Finally, we get

$$
\|Zu_1 - Zu_2\|_X \le (\sum_{i=1}^{3}\Delta_i)\|u_1-u_2\|_X.
$$

We deduce that Z is contractive. We conclude that Z has a unique fixed point which is the solution of (2.1). ■

2.4 An Illustrative Example

As illustrative example for the first part of our results, we consider the following problem:

$$\begin{cases} D^{\frac{17}{20}} D^{\frac{9}{10}} D^{\frac{13}{20}} u(t) = \dfrac{2f(t, u(t), D^{\frac{13}{20}} u(t)) + \frac{1}{3} g(t, u(t), D^{\frac{11}{20}} D^{\frac{6}{10}} u(t)) + \frac{1}{2} h(t, u(t))}{K(u)}, \\ u(0) + u(1) = 5 \int_0^{0.6} u(s)ds, \\ D^{\frac{13}{20}} u(0) + D^{\frac{13}{20}} u(1) = 0, \\ D^{\frac{11}{20}} D^{\frac{11}{20}} u(0) + D^{\frac{11}{20}} D^{\frac{11}{20}} u(1) = 0, \end{cases} \tag{2.10}$$

with the conditions:

$$f(t, u, v) = \frac{u^2}{2(t^3 + 115)(u^2 + 1)} + \frac{|v|}{(200 + t)(1 + |v|)},$$

$$g(t, u, v) = \frac{4|u|}{(e^t + 25)^2 - 25} + \frac{cosv}{320 + t^2},$$

$$h(t, u) = \frac{3|u|}{15(t^3 + 70)(1 + |u|)},$$

$$k(u) = \frac{1 + 2u^2}{10(1 + u^2)},$$

and

$$\alpha = \tfrac{17}{20},\ \beta = \tfrac{9}{10},\ \gamma = \tfrac{13}{20},\ \rho = \tfrac{6}{10},\ b = \tfrac{3}{10},\ \mu = \tfrac{11}{20},\ \eta = 0.6,\ m = 0.1,$$

$$\Delta_1 = 0.5439, \quad \Delta_2 = 0.0039, \quad \Delta_3 = 0.0102,$$

$$\Delta = \Delta_1 + \Delta_2 + \Delta_3 = 0.5580.$$

The conditions of the first main result hold. Therefore, problem (2.10) has a unique solution on J.

CHAPTER 3

A Fractional Boundary Value Problem

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3.1 Presentation

Fractional differential equations theory is attracting more popularity and increasing importance, due to its numerous applications in various areas, such as optics, medicine, statistical physics, automatics, and control theory, see [6] [8]. In this context, many authors have been interested in studying the question of the existence, uniqueness and stability of solutions for certain types of such equations. We refer the interested reader to [22][9][15]. Let us now, on the other hand, recall some papers that have motivated the present work. We begin by the references [36] where N. Urus et al. have studied the existence of solutions for a fourth order four-point non-linear boundary value problem (BVP) which arises in bridge design:

$$\begin{cases} -y^{(4)}(s) - \lambda y''(s) = f(s, y(s)), s \in (0,1), \\ y(0) = 0, \\ y''(0) = 0, \\ y(1) = \delta_1 y(\eta_1) + \delta_2 y(\eta_2), \\ y''(1) = \delta_1 y''(\eta_1) + \delta_2 y''(\eta_2), \end{cases}$$

where $f \in C([0,1] \times \mathbb{R}, \mathbb{R}), \delta_1, \delta_2 > 0, 0 < \eta_1 \leq \eta_2 < 1, \lambda = \eta_1 + \eta_2$, η_1 and η_2 are real constants.

Also in [4], the authors have been concerned with a suitable fractional presentation for a simple Jerk circuit that allows us to study chaotic dynamics which was modeled by the problem:

$$\begin{cases} D^\alpha \left(D^2 + \lambda^2 D^\alpha\right) y(t) = f\left(t, y(t), D^\alpha y(t)\right), \quad t \in [0,T], \quad T > 0, \\ y(0) = 0, \\ D^{1-\alpha} D^\alpha y(0) = 0, \\ y(T) = \beta J^\gamma y(\eta), \quad 0 < \eta \leq T, \end{cases}$$

where D^α is Caputo fractional derivative of order $\alpha \in [0,1], J^\gamma$ is the Riemann-Liouville fractional integral order $\gamma \in [0, \infty[$, $\lambda \in \mathbb{R}_+$, and $\beta \in \mathbb{R}$.

Very recently, A. Abdelnebi and Z. Dahmani [2] have studied the existence, uniqueness, and stability of solutions for the following Van der Pol-Duffing jerk equation:

$$\begin{cases} D^{\alpha}\left(D^{2-\beta}+\lambda D^{\alpha}\right)x(t)+k_1 f_1\left(t,x(t),D^{\alpha}x(t)\right)+k_2 f_2\left(t,x(t),J^{p}x(t)\right)=h(t), \\ x(1)=0, \quad D^{1-(\alpha-\beta)}D^{\alpha-\beta}x(1)=A^{*}\in\mathbb{R}, \quad x(T)=0, \\ 0\leq\beta<\alpha\leq 1, \quad 0\leq\alpha+\beta<1, \quad 0<p, \quad t\in I, \end{cases}$$

where $D^{\alpha}, D^{2-\beta}$, are the Caputo-Hadamard fractional derivatives, J^{p} is the Hadamard fractional integral, $I=[1,T], k_1, k_2$ are real constants, and the functions f_1, f_2 and h are continuous.

The aim in this chapter to study the following fourth order four-point non-linear boundary value problem (BVP) which arises in bridge design fractional differential problem:

$$\begin{cases} -D^{\beta}(D^{\alpha}+\lambda)y(t)=m(t,y(t),D^{\vartheta}y(t))+n(t,y(t),I^{p}y(t))+r(t,y(t))+l(t), t\in J, \\ y(0)=0, \\ D^{\alpha}y(0)=0, \\ y(1)=ay(\xi_1)+by(\xi_2), \\ D^{\alpha}y(1)=a^{'}D^{\alpha}y(\xi_1)+b^{'}D^{\alpha}y(\xi_2), \end{cases} \tag{3.1}$$

under the fact that $J:=[0,1]$, $a,a^{'},b,b^{'}\in\mathbb{R}, 0<\xi_1\leq\xi_2<1, \lambda=\xi_1+\xi_2$, where ξ_1 and ξ_2 are the real constants, $0<\vartheta\leq 1$, $1<\alpha,\beta\leq 2$, and m,n,r, and l are some functions that will be specified later, D^{α}, D^{β} and D^{ϑ} are the derivatives in the sense of Caputo.

3.2 Some Auxiliary Results

Lemma 3.1 *Let G be a Banach space and $Z:G\longrightarrow G$ is an application that is contractible. Then, Z has exactly one fixed point in G.*

Let us now present then prove another lemma.

Lemma 3.2 *We consider $K \in C(J)$. Then the BVP*

$$\begin{cases} -D^{\beta}(D^{\alpha}+\lambda)y(t) = K(t), t \in [0,1], \\ y(0) = 0, \\ D^{\alpha}y(0) = 0, \\ y(1) = ay(\xi_1) + by(\xi_2), \\ D^{\alpha}y(1) = a^{'}D^{\alpha}y(\xi_1) + b^{'}D^{\alpha}y(\xi_2), \\ 0 < \alpha, \beta \leq 2, \end{cases} \tag{3.2}$$

is geven by:

$$\begin{aligned} y(t) &= -I^{\alpha+\beta}K(t) + \lambda I^{\alpha}y(t) - \Delta_1^{-1}\Big[I^{\beta}K(1) - \lambda y(1) - a^{'}I^{\beta}K(\xi_1) + a^{'}\lambda y(\xi_1) - b^{'}I^{\beta}K(\xi_2) \\ &+ b^{'}\lambda y(\xi_2)\Big]\frac{t^{\alpha}}{\Gamma(\alpha+1)} - \Delta_3^{-1}\Big[I^{\alpha+\beta}K(1) - \lambda I^{\alpha}y(1) - aI^{\alpha+\beta}K(\xi_1) + a\lambda I^{\alpha}y(\xi_1) \\ &- bI^{\alpha+\beta}K(\xi_2) + b\lambda I^{\alpha}y(\xi_2) - \Delta_2\Delta_1^{-1}\Big(I^{\beta}K(1) - \lambda y(1) - a^{'}I^{\beta}K(\xi_1) + a^{'}\lambda y(\xi_1) \\ &- b^{'}I^{\beta}K(\xi_2) + b^{'}\lambda y(\xi_2)\Big)\Big]t, \end{aligned}$$

such that

$$\Delta_1 = a^{'}\xi_1 + b^{'}\xi_2 - 1,$$

$$\Delta_2 = \frac{a\xi_1^{\alpha+1} + b\xi_2^{\alpha+1} - 1}{\Gamma(\alpha+2)},$$

$$\Delta_3 = a\xi_1 + b\xi_2 - 1,$$

and,

$$\Delta_1\Delta_3 \neq 0.$$

Proof

Thanks to Lemma 1.8, it yields

$$D^{\alpha}y(t)+\lambda y(t)=I^{\beta}K(t)+c_0t+c_1.$$

Therefore,

$$y(t)=-I^{\alpha+\beta}K(t)+\lambda I^{\alpha}y(t)-c_0\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}-c_1\frac{t^{\alpha}}{\Gamma(\alpha+1)}-c_2t-c_3. \tag{3.3}$$

The conditions

$$y(0)=0, \qquad D^{\alpha}y(0)=0,$$

allow us to write

$$c_1=c_3=0,$$

and the conditions

$$y(1)=ay(\xi_1)+by(\xi_2),$$

and

$$D^{\alpha}y(1)=a^{'}D^{\alpha}y(\xi_1)+b^{'}D^{\alpha}y(\xi_2),$$

allow us to get

$$\begin{aligned}
c_0 &= \Delta_1^{-1}\left[I^{\beta}K(1)-\lambda y(1)-a^{'}I^{\beta}K(\xi_1)+a^{'}\lambda y(\xi_1)-b^{'}I^{\beta}K(\xi_2)+b^{'}\lambda y(\xi_2)\right],\\
c_2 &= \Delta_3^{-1}\Bigg[I^{\alpha+\beta}K(1)-\lambda I^{\alpha}y(1)-aI^{\alpha+\beta}K(\xi_1)+a\lambda I^{\alpha}y(\xi_1)-bI^{\alpha+\beta}K(\xi_2)+b\lambda I^{\alpha}y(\xi_2)\\
&- \Delta_2\Delta_1^{-1}\left(I^{\beta}K(1)-\lambda y(1)-a^{'}I^{\beta}K(\xi_1)+a^{'}\lambda y(\xi_1)-b^{'}I^{\beta}K(\xi_2)+b^{'}\lambda y(\xi_2)\right)\Bigg].
\end{aligned}$$

The proof is thus achieved. ■

3.3 Main Results

Let consider the space

$$N:=\{y\in C(J,\mathbb{R}),D^{\vartheta}y\in C(J,\mathbb{R})\},$$

and the norm:

$$\|y\|_N = \|y\|_\infty \ + \|D^\vartheta y\|_\infty,$$

where,

$$\|y\|_\infty = \sup_{t\in J} |y(t)| \ , \|D^\vartheta y\|_\infty = \sup_{t\in J} |D^\vartheta y(t)|.$$

Then, we take the nonlinear operator $Z : N \to N$, where:

$$\begin{aligned} Zy(t) &= -I^{\alpha+\beta}K_y^*(t) + \lambda I^\alpha y(t) - \Delta_1^{-1}\Bigg[I^\beta K_y^*(1) - \lambda y(1) - a' I^\beta K_y^*(\xi_1) + a'\lambda y(\xi_1) \\ &- b' I^\beta K_y^*(\xi_2) + b'\lambda y(\xi_2)\Bigg]\frac{t^\alpha}{\Gamma(\alpha+1)} - \Delta_3^{-1}\Bigg[I^{\alpha+\beta}K_y^*(1) - \lambda I^\alpha y(1) - aI^{\alpha+\beta}K_y^*(\xi_1) \\ &+ a\lambda I^\alpha y(\xi_1) - bI^{\alpha+\beta}K_y^*(\xi_2) + b\lambda I^\alpha y(\xi_2) - \Delta_2\Delta_1^{-1}\Bigg(I^\beta K_y^*(1) - \lambda y(1) - a' I^\beta K_y^*(\xi_1) \\ &+ a'\lambda y(\xi_1) - b' I^\beta K_y^*(\xi_2) + b'\lambda y(\xi_2)\Bigg)\Bigg]t, \end{aligned}$$

where

$$K_y^*(t) = m(t, y(t), D^\vartheta y(t)) + n(t, y(t), I^p y(t)) + r(t, y(t)) + l(t).$$

The above notions are introduced in order to transform our problem into a one of fixed point.

The following hypotheses are only sufficient; one can use other conditions of Caratheodory functions to obtain the same results.

$(\Phi 1)$: The introduced functions are continuous.

$(\Phi 2)$: There exist nonnegative constants $\chi_{m1}, \chi_{m2}, \chi_{n1}, \chi_{n2}, Q$, such that for any t $\in J$, $y_i, y_i^* \in \mathbb{R}$,

$$|m(t, y_1, y_2) - m(t, {y_1}^*, {y_2}^*)| \ \leq \ \sum_{i=1}^{2} \chi_{mi}|y_i - {y_i}^*|,$$

$$|n(t, y_1, y_2) - n(t, {y_1}^*, {y_2}^*)| \ \leq \ \sum_{i=1}^{2} \chi_{ni}|y_i - {y_i}^*|.$$

and for any t $\in J$, $y, y' \in \mathbb{R}$,

$$|r(t, y) - r(t, y')| \ \leq \ Q|y - y'|.$$

We put in passage:

$$X := Max(\chi_{m1}, \chi_{m2}), X^* := Max(\chi_{n1}, \chi_{n2}).$$

Also, the quantities

$$\theta_1 = \frac{1}{\Gamma(\alpha+\beta+1)} + |\Delta_3^{-1}| \times \frac{1+|a|+|b|}{\Gamma(\alpha+\beta+1)} + \left(\frac{|\Delta_1^{-1}|}{\Gamma(\alpha+1)} + |\Delta_1^{-1}\Delta_2\Delta_3^{-1}|\right) \times \frac{1+|a'|+|b'|}{\Gamma(\beta+1)},$$

$$\varphi_1 = \frac{1}{\Gamma(\alpha+1)} + |\Delta_3^{-1}| \times \Big(1+|a|+|b|\Big) + \left(\frac{|\Delta_1^{-1}|}{\Gamma(\alpha+1)} + |\Delta_1^{-1}\Delta_2\Delta_3^{-1}|\right)\Big(1+|a'|+|b'|\Big),$$

$$\theta_2 = \frac{1}{\Gamma(\alpha+\beta-\vartheta+1)} + \frac{|\Delta_3^{-1}|}{\Gamma(2-\vartheta)} \times \frac{1+|a|+|b|}{\Gamma(\alpha+\beta+1)} + \left(\frac{|\Delta_1^{-1}|}{\Gamma(\alpha-\vartheta+1)} + |\Delta_1^{-1}\Delta_2\Delta_3^{-1}|\right)$$

$$\times \frac{1+|a'|+|b'|}{\Gamma(\beta+1)},$$

and

$$\varphi_2 = \frac{1}{\Gamma(\alpha+1)} + \frac{|\Delta_3^{-1}|}{\Gamma(2-\vartheta)} \times \Big(1+|a|+|b|\Big) + \left(\frac{|\Delta_1^{-1}|}{\Gamma(\alpha-\vartheta+1)} + |\Delta_1^{-1}\Delta_2\Delta_3^{-1}|\right)$$

$$\times \Big(1+|a'|+|b'|\Big),$$

are to be considered in this paper

3.3.1 A Main Result for Unique Solutions

We propose to establish the following result:

Theorem 3.1 *Assume that* $(\Phi 1), (\Phi 2)$ *are satisfied. Then, (3.1) has a unique solution if* $\Sigma(\theta_1+\theta_2) < 1 - \lambda(\varphi_1+\varphi_2)$*;* $\Sigma = Q + 2X + X^* + \dfrac{X^*}{\Gamma(p+1)}$.

Proof

Let us take $(y,y^{'})\in N^2$, we can write

$$\begin{aligned}
\|Zy-Zy^{'}\|_\infty &\leq \Sigma\Bigg[\frac{1}{\Gamma(\alpha+\beta+1)}+|\Delta_3^{-1}|\times\frac{1+|a|+|b|}{\Gamma(\alpha+\beta+1)}+\Bigg(\frac{|\Delta_1^{-1}|}{\Gamma(\alpha+1)}+|\Delta_1^{-1}\Delta_2\Delta_3^{-1}|\Bigg)\\
&\times \frac{1+|a'|+|b'|}{\Gamma(\beta+1)}\Bigg]\|y-y^{'}\|_N+\lambda\Bigg[\frac{1}{\Gamma(\alpha+1)}+|\Delta_3^{-1}|\times\Big(1+|a|+|b|\Big)\\
&+ \Bigg(\frac{|\Delta_1^{-1}|}{\Gamma(\alpha+1)}+|\Delta_1^{-1}\Delta_2\Delta_3^{-1}|\Bigg)\Big(1+|a'|+|b'|\Big)\Bigg]\|y-y^{'}\|_N.
\end{aligned}$$

The reader can see also that

$$\begin{aligned}
D^\vartheta Zy(t) &= -I^{\alpha+\beta-\vartheta}K_y^*(t)+\lambda I^{\alpha-\vartheta}y(t)-\Delta_1^{-1}\Bigg[I^\beta K_y^*(1)-\lambda y(1)-a^{'}I^\beta K_y^*(\xi_1)+a^{'}\lambda y(\xi_1)\\
&- b^{'}I^\beta K_y^*(\xi_2)+b^{'}\lambda y(\xi_2)\Bigg]\frac{t^{\alpha-\vartheta}}{\Gamma(\alpha-\vartheta+1)}-\Delta_3^{-1}\Bigg[I^{\alpha+\beta}K_y^*(1)-\lambda I^\alpha y(1)-aI^{\alpha+\beta}K_y^*(\xi_1)\\
&+ a\lambda I^\alpha y(\xi_1)-bI^{\alpha+\beta}K_y^*(\xi_2)+b\lambda I^\alpha y(\xi_2)-\Delta_2\Delta_1^{-1}\Bigg(I^\beta K_y^*(1)-\lambda y(1)-a^{'}I^\beta K_y^*(\xi_1)\\
&+ a^{'}\lambda y(\xi_1)-b^{'}I^\beta K_y^*(\xi_2)+b^{'}\lambda y(\xi_2)\Bigg)\Bigg]\frac{t^{1-\vartheta}}{\Gamma(2-\vartheta)},
\end{aligned}$$

and

$$
\begin{aligned}
\|D^{\vartheta}Zy - D^{\vartheta}Zy^{'}\|_{\infty} \quad &\leq \quad \Sigma\Bigg[\frac{1}{\Gamma(\alpha+\beta-\vartheta+1)} + \frac{|\Delta_3^{-1}|}{\Gamma(2-\vartheta)} \times \frac{1+|a|+|b|}{\Gamma(\alpha+\beta+1)} + \Bigg(\frac{|\Delta_1^{-1}|}{\Gamma(\alpha-\vartheta+1)} \\
&+ \quad |\Delta_1^{-1}\Delta_2\Delta_3^{-1}|\Bigg) \times \frac{1+|a'|+|b'|}{\Gamma(\beta+1)}\Bigg]\|y-y^{'}\|_N + \lambda\Bigg[\frac{1}{\Gamma(\alpha+1)} \\
&+ \quad \frac{|\Delta_3^{-1}|}{\Gamma(2-\vartheta)} \times \Big(1+|a|+|b|\Big) + \Bigg(\frac{|\Delta_1^{-1}|}{\Gamma(\alpha-\vartheta+1)} + |\Delta_1^{-1}\Delta_2\Delta_3^{-1}|\Bigg) \\
&\times \quad \Big(1+|a'|+|b'|\Big)\Bigg]\|y-y^{'}\|_N.
\end{aligned}
$$

Therefore,

$$
\|Zy - Zy^{'}\|_N \quad \leq \quad \Bigg[\Sigma(\theta_1+\theta_2) + \lambda(\varphi_1+\varphi_2)\Bigg]\|y-y^{'}\|_N.
$$

■

We conclude that Z is contraction. As a consequence of Banach fixed point theorem, we deduce that Z has a unique fixed point which is a solution of (3.1).

Example 3.2 *We consider the following problem:*

$$
\begin{cases}
-D^{\frac{4}{3}}(D^{\frac{3}{2}} + \frac{1}{20})y(t) = \frac{1}{100}\Big(e^{t-4}\sin(y(t)) + \frac{1}{e^{t+2}}D^{\frac{1}{2}}y(t) + \frac{\sin(t+1)}{2}\Big) \\
+10^{-1}\Big(\frac{1}{25}y(t) + \frac{\cos(3+t^2)}{\pi(10+t)} + \frac{1}{30}I^{\frac{1}{2}}y(t)\Big) \\
+\frac{1}{40}\Big(\frac{1}{\pi^2}y(t) + ln(t+1)\Big) + \frac{3}{2}t, \\
y(0) = 0, \\
D^{\frac{3}{2}}y(0) = 0, \\
y(1) = \frac{1}{10}y(\frac{7}{200}) + \frac{1}{5}y(\frac{3}{200}), \\
D^{\frac{3}{2}}y(1) = \frac{1}{5}D^{\alpha}y(\frac{7}{200}) + \frac{3}{10}D^{\frac{3}{2}}y(\frac{3}{200}),
\end{cases}
$$

with

$$m(t,y_1,y_2) = \frac{1}{100}\Big(e^{t-4}\sin(y(t)) + \frac{1}{e^{t+2}}D^{\frac{1}{2}}y(t) + \frac{\sin(t+1)}{2}\Big),$$
$$n(t,y_1,y_2) = 10^{-1}\Big(\frac{1}{25}y(t) + \frac{\cos(3+t^2)}{\pi(10+t)} + \frac{1}{30}I^{\frac{1}{2}}y(t)\Big),$$
$$r(t,y) = \frac{1}{40}\Big(\frac{1}{\pi^2}y(t) + ln(t+1)\Big),$$
$$l(t) = \frac{3}{2}t,$$
$$\Sigma = 0.0138, \quad \theta_1 = 3.4010, \quad \theta_2 = 5.0352, \quad \lambda = \frac{1}{20}, \quad \varphi_1 = 5.3492, \quad \varphi_2 = 5.8544.$$

So, we see that

$$\Sigma(\theta_1+\theta_2) < 1 - \lambda(\varphi_1+\varphi_2).$$

The conditions of Theorem 3.1 hold. Therefore, the problem has a unique solution over J.

3.3.2 A Main Result for Stability of Solutions

Definition 3.3 *The equation* (3.1) *has the Ulam Hyers stability if there is a* $\Lambda > 0$*; for each* $\varpi > 0, t \in [0,1]$ *and for each* $y \in N$ *solution of*

$$\Big|D^{\beta}(D^{\alpha}+\lambda)y(t) + m(t,y(t),D^{\vartheta}y(t)) + n(t,y(t),I^{p}y(t)) + r(t,y(t)) + l(t)\Big| \le \varpi, \tag{3.4}$$

there is certainly $y^* \in N$ *a solution of* (3.1)*;*

$$\|y-y^*\|_N \le \Lambda\varpi.$$

Definition 3.4 *The equation* (3.1) *has the Ulam Hyers stability in the generalized sense if there is* $\rho \in C(\mathbb{R}^+,\mathbb{R}^+)$*;* $\rho(0) = 0$*; for each* $\varpi > 0$*, and for any* $y \in N$ *solution of* (3.4)*, there is a solution* $y^* \in N$ *of (3.1);*

$$\|y-y^*\|_N < \rho(\varpi).$$

We prove the following theorem

Theorem 3.5 *Under the conditions of Theorem* 3.1*, problem* (3.1) *is Ulam Hyers stable.*

Proof: Let $y \in N$ be a solution of (3.4), and let $y^* \in N$ be the unique solution of (3.1). We have:

$$\Bigg| y(t) + I^{\alpha+\beta} K_y^*(t) - \lambda I^{\alpha} y(t) + \Delta_1^{-1} \Bigg[I^{\beta} K_y^*(1) - \lambda y(1) - a' I^{\beta} K_y^*(\xi_1) + a' \lambda y(\xi_1) - b' I^{\beta} K_y^*(\xi_2)$$

$$+ b' \lambda y(\xi_2) \Bigg] \times \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \Delta_3^{-1} \Bigg[I^{\alpha+\beta} K_y^*(1) - \lambda I^{\alpha} y(1) - a I^{\alpha+\beta} K_y^*(\xi_1) + a \lambda I^{\alpha} y(\xi_1) - b I^{\alpha+\beta} K_y^*(\xi_2)$$

$$+ b \lambda I^{\alpha} y(\xi_2) - \Delta_2 \Delta_1^{-1} \Bigg(I^{\beta} K_y^*(1) - \lambda y(1) - a' I^{\beta} K_y^*(\xi_1) + a' \lambda y(\xi_1) - b' I^{\beta} K_y^*(\xi_2) + b' \lambda y(\xi_2) \Bigg) \Bigg] t \Bigg|$$

$$\leq \frac{\varpi}{\Gamma(\alpha+\beta+1)}. \tag{3.5}$$

Using (3.4) and (3.5), we can write

$$\|y - y^*\|_{\infty} \leq \frac{\varpi}{\Gamma(\alpha+\beta+1)} + \Sigma \Bigg[\frac{1}{\Gamma(\alpha+\beta+1)} + |\Delta_3^{-1}| \times \frac{1+|a|+|b|}{\Gamma(\alpha+\beta+1)} + \Bigg(\frac{|\Delta_1^{-1}|}{\Gamma(\alpha+1)}$$

$$+ |\Delta_1^{-1} \Delta_2 \Delta_3^{-1}| \Bigg) \times \frac{1+|a'|+|b'|}{\Gamma(\beta+1)} \Bigg] \|y - y^*\|_N + \lambda \Bigg[\frac{1}{\Gamma(\alpha+1)} + |\Delta_3^{-1}| \times \Big(1+|a|+|b| \Big)$$

$$+ \Bigg(\frac{|\Delta_1^{-1}|}{\Gamma(\alpha+1)} + |\Delta_1^{-1} \Delta_2 \Delta_3^{-1}| \Bigg) \Big(1+|a'|+|b'| \Big) \Bigg] \|y - y^*\|_N. \tag{3.6}$$

Hence,

$$\|y - y^*\|_{\infty} \leq \frac{\varpi}{\Gamma(\alpha+\beta+1)} + (\Sigma\theta_1 + \lambda\varphi_1) \|y - y^*\|_N.$$

Also, the reader can observe that

$$\|D^{\vartheta}(y - y^*)\|_{\infty} \leq \frac{\varpi}{\Gamma(\alpha+\beta-\vartheta+1)} + (\Sigma\theta_2 + \lambda\varphi_2) \|y - y^*\|_N.$$

Thus,

$$\|y-y^*\|_N \leq \left(\frac{\varpi}{\Gamma(\alpha+\beta+1)}+\frac{\varpi}{\Gamma(\alpha+\beta-\vartheta+1)}\right)+\Bigg[\Sigma(\theta_1+\theta_2)+\lambda(\varphi_1+\varphi_2)\Bigg]\|y-y^*\|_N.$$

$$\|y-y^*\|_N \leq \frac{\dfrac{\varpi}{\Gamma(\alpha+\beta+1)}+\dfrac{\varpi}{\Gamma(\alpha+\beta-\vartheta+1)}}{1-\Bigg[\Sigma(\theta_1+\theta_2)+\lambda(\varphi_1+\varphi_2)\Bigg]}.$$

So,

$$\|y-y^*\|_N \leq \Lambda\varpi.$$

Consequently, (3.1) has the Ulam Hyers stability.

Remark 3.6 *The case* $\rho(\varpi)=\Lambda\varpi$ *allows us to guarantee the generalised Ulam Hyers stability for* (3.1)*.*

CHAPTER 4

Traveling Waves

4.1 Motivation

The travelling wave is one that advances in a particular direction, with an additional property of retaining a fixed shape. Travelling waves are associated with having a constant velocity throughout their propagation. Such waves are observed in many areas of science.The phenomenon of waves is observed in interaction, convection and natural propagation. Recently, new exact travelling wave solutions may help to find new phenomena. So the most important thing is to seek exact solutions of nonlinear partial differential equation. There are many method to look for exact travelling solutions such as the first integral method [24], the exp-function method [16], the (G'/G) expansion method [39], and the tanh method [25, 37].

4.2 How to Obtain Travelling Waves

We consider the equation

$$M\left(u, T_t^{\alpha}u, T_x^{\beta}u, T_t^{2\alpha}u, T_t^{\alpha}(T_x^{\beta}u), T_x^{2\beta}u, ...\right) = 0, \tag{4.1}$$

where $T_t^{\alpha}u$ is the fractional conformable derivative of u of order $\alpha, 0 < \alpha \leq 1$. Introducing the new wave variable

$$\xi = \tfrac{k}{\alpha}t^{\alpha} + \tfrac{\omega}{\beta}x^{\beta}, \tag{4.2}$$

so, (4.1) can be transformed to the equation:

$$M^{\star} = \left(U, U', U'', U''', ...\right) = 0. \tag{4.3}$$

4.2.1 Tanh Method

The tanh method is one of most direct method for finding solutions of nonlinear diffusion equations. This method has been presented by Malfliet [25, 26] and by also by Wazwaz [37] for the computation of exact traveling wave solutions. Its ida is to express the solution of the nonlinear differential equation as a polynomial and it is based on "the balance principle".

The Tanh method is based on the fact that travelling wave solutions can be expressed as follows:

$$u(x,t) = U(\xi) = P(\Upsilon) = \sum_{i=0}^{m} a_i \Upsilon^i, \tag{4.4}$$

where

$$\Upsilon = tanh(\xi). \tag{4.5}$$

leads to the change of derivatives:

$$\begin{aligned}
\frac{d}{d\xi} &= \left(1-\Upsilon^2\right)\frac{d}{d\Upsilon},\\
\frac{d^2}{d\xi^2} &= -2\Upsilon\left(1-\Upsilon^2\right)\frac{d}{d\Upsilon} + \left(1-\Upsilon^2\right)^2\frac{d^2}{d\Upsilon^2},\\
\frac{d^3}{d\xi^3} &= 2\left(1-\Upsilon^2\right)\left(3\Upsilon^2-1\right)\frac{d}{d\Upsilon} - 6\Upsilon\left(1-\Upsilon^2\right)^2\frac{d^2}{d\Upsilon^2} + \left(1-\Upsilon^2\right)^3\frac{d^3}{d\Upsilon^3},\\
\frac{d^4}{d\xi^4} &= -8\Upsilon\left(1-\Upsilon^2\right)\left(3\Upsilon^2-2\right)\frac{d}{d\Upsilon} + 4\left(1-\Upsilon^2\right)^2\left(9\Upsilon^2-2\right)\frac{d^2}{d\Upsilon^2}\\
&\quad - 12\Upsilon\left(1-\Upsilon^2\right)^3\frac{d^3}{d\Upsilon^3} + \left(1-\Upsilon^2\right)^4\frac{d^4}{d\Upsilon^4}.
\end{aligned} \tag{4.6}$$

We recall that m is a positive integer determined by the balancing procedure in the resulting nonlinear ODE in P. Thus, we have an algebraic system of equations from which the constants $k, \omega, a_i (i = 0, \cdots, m)$ are obtained and determine the function U, hence we get the exact solutions of (4.1).

4.2.2 Exp-Function Method

The method is the most recent analytical approach that has been used for solving nonlinear differential problems in physics. Also, it seems that the Exp-function method is more effective and simple than other methods and a lot of solutions can be obtained by using a simple computer program. The EFM method was first proposed in 2006 [17], also it has been applied by and Navickas et al. in [28], and later in 2012, by Ebaid [13]. However, its applications to fractional calculus are rare.

The EFM method is based on the fact that travelling wave solutions (as in [16]) can be expressed as follows:

$$U(\zeta) = \frac{\sum_{n=-c}^{d} a_n \exp(n\zeta)}{\sum_{m=-p}^{q} b_m \exp(m\zeta)}. \tag{4.7}$$

We suppose that the solution of (4.3) can be expressed as:

$$U(\zeta) = \frac{a_d \exp(d\zeta)+...+a_{-c}\exp(-c\zeta)}{b_q \exp(q\zeta)+...+b_{-p}\exp(-p\zeta)}, \tag{4.8}$$

we can write (4.8) in the following form

$$U(\zeta) = \frac{a_c \exp(c\zeta)+...+a_{-d}\exp(-d\zeta)}{b_p \exp(p\zeta)+...+b_{-q}\exp(-q\zeta)}. \tag{4.9}$$

To determine the value of c and p, we balance the linear term of highest order of (4.3) with the highest order nonlinear term. Similarly, to determine the value of d and q, we balance the linear term of lowest order of (4.3) with lowest order nonlinear term. After this separated algebraic equation, we can find a_n and b_m constants.

4.3 Applications

4.3.1 Travelling Waves for the fractional Beam Equation

In this section, we are interested in using the tanh method to solve the conformable problem:

$$T_t^{2\alpha}u + T_x(G(u)T_x^{3\beta}u) + T_x(H(u)T_x^{\beta}u) = F(u), \tag{4.10}$$

To demonstrate the power of the tanh method, some of well known nonlinear equations will be examined.

Example 1 [31]

The oscillations and motion of waves of the elastic beams on elastic foundation scan be described by means of the following equation [18] ($G(u) = 1$, $H(u) = 0$):

$$T_t^{2\alpha} u\,(x,t) + T_x^{4\beta} u\,(x,t) = (u\,(x,t))^3 - cu\,(x,t)\,. \tag{4.11}$$

Using (4.2), to change (4.11) into the following nonlinear ODE

$$k^2 U_{\xi\xi} + \omega^4 U_{\xi\xi\xi\xi} = U^3 - cU, \tag{4.12}$$

Substituting (4.4) and (4.6) into (4.12),we can get

$$\begin{aligned}(k^2 + c_1\omega^2)\Big[-2\Upsilon(1-\Upsilon^2)\frac{dP}{d\Upsilon} + (1-\Upsilon^2)^2\frac{d^2P}{d\Upsilon^2}\Big] + \omega^4\Big[-8\Upsilon(1-\Upsilon^2)(3\Upsilon^2-2)\frac{dP}{d\Upsilon}\\ +4(1-\Upsilon^2)^2(9\Upsilon^2-2)\frac{d^2P}{d\Upsilon^2} - 12\Upsilon(1-\Upsilon^2)^3\frac{d^3P}{d\Upsilon^3} + (1-\Upsilon^2)^4\frac{d^4P}{dY^4}\Big] = P^3 - cP.\end{aligned} \tag{4.13}$$

To determine the parameter m we usually balance $\Upsilon^8 \frac{d^4P}{d\Upsilon^4}$ with P^3. This in turn gives

$$8 + m - 4 = 3m$$

so that $m = 2$. This gives the solution in the form

$$P(\Upsilon) = a_0 + a_1\Upsilon + a_2\Upsilon^2. \tag{4.14}$$

Substituting (4.14) into (4.13), we can get

$$\begin{aligned}(k^2 + c_1\omega^2)(1-\Upsilon^2)\Big[-2\Upsilon(a_1 + 2a_2\Upsilon) + 2a_2(1-\Upsilon^2)\Big] + \omega^4(1-\Upsilon^2)\Big[-8\Upsilon(3\Upsilon^2-2)(a_1+2a_2\Upsilon)\\ +8a_2(1-\Upsilon^2)(9\Upsilon^2-2)\Big] - (a_0 + a_1\Upsilon + a_2\Upsilon^2)^3 + c(a_0 + a_1\Upsilon + a_2\Upsilon^2) = 0.\end{aligned} \tag{4.15}$$

Then, we have the system:

$$\begin{cases}\Upsilon^0 : -16\omega^4 a_2 + 2k^2 a_2 - a_0^3 + ca_0 = 0,\\ \Upsilon^1 : 16\omega^4 a_1 - 2k^2 a_1 - 3a_0^2 a_1 + ca_1 = 0,\\ \Upsilon^2 : 136\omega^4 a_2 - 8k^2 a_2 - 3a_2 a_0^2 - 3a_1^2 a_0 + ca_2 = 0,\\ \Upsilon^3 : -40\omega^4 a_1 + 2k^2 a_1 - 6a_0 a_1 a_2 - a_1^3 = 0,\\ \Upsilon^4 : -240\omega^4 a_2 + 6k^2 a_2 - 3a_0 a_2^2 - 3a_1^2 a_2 = 0,\\ \Upsilon^5 : 24\omega^4 a_1 - 3a_1 a_2^2 = 0,\\ \Upsilon^6 : 120\omega^4 a_2 - a_2^3 = 0.\end{cases}$$

We solve the algebraic system with the aid of Maple. We obtain traveling wave solutions of (4.11) as follows:

Case 1.

$$a_0 = \frac{\sqrt{30c}}{4}, a_1 = 0, a_2 = \frac{\sqrt{30c}}{4},$$

$$u(x,t) = \frac{\sqrt{30c}}{4} + \frac{\sqrt{30c}}{4}\tanh^2(\xi). \tag{4.16}$$

Case 2.

$$a_0 = \frac{\sqrt{30c}}{4}, a_1 = 0, a_2 = -\frac{\sqrt{30c}}{4},$$

$$u(x,t) = \frac{\sqrt{30c}}{4} - \frac{\sqrt{30c}}{4}\tanh^2(\xi). \tag{4.17}$$

Case 3.

$$a_0 = -\frac{\sqrt{30c}}{4}, a_1 = 0, a_2 = -\frac{\sqrt{30c}}{4},$$

$$u(x,t) = -\frac{\sqrt{30c}}{4} - \frac{\sqrt{30c}}{4}\tanh^2(\xi). \tag{4.18}$$

Case 4.

$$a_0 = -\frac{\sqrt{30c}}{4}, a_1 = 0, a_2 = \frac{\sqrt{30c}}{4},$$

$$u(x,t) = -\frac{\sqrt{30c}}{4} + \frac{\sqrt{30c}}{4}\tanh^2(\xi). \tag{4.19}$$

Case 5.

$$a_0 = -\frac{\sqrt{30c}}{4}, a_1 = 0, a_2 = \frac{\sqrt{30c}}{4},$$

$$u(x,t) = \sqrt{c}. \tag{4.20}$$

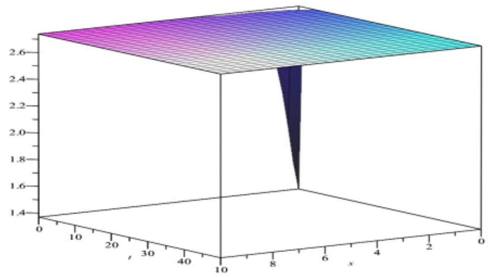


Figure 4.1: *3D plot of traveling wave solution (case 1.) of (4.16) sketched within the intervals* $0 \leq x \leq 10$ *and* $0 \leq t \leq 50$.

Example 2 [31]

We now consider the nonlinear beam equation [38] ($G(u) = 1$, $H(u) = 1$):

$$T_t^{2\alpha} u(x,t) + T_x^{4\beta} u(x,t) + c_1 T_x^{2\beta} u(x,t) + c_2 u(x,t) + (u(x,t))^2 = 0. \tag{4.21}$$

Using (4.2), to change (4.21) into the following nonlinear ODE

$$(k^2 + c_1\omega^2) U_{\xi\xi} + \omega^4 U_{\xi\xi\xi\xi} + c_2 U + U^2 = 0, \tag{4.22}$$

Substituting (4.4) and (4.6) into (4.22),we can get

$$(k^2 + c_1\omega^2)\Big[-2\Upsilon(1-\Upsilon^2)\frac{dP}{dY} + (1-\Upsilon^2)^2\frac{d^2P}{d\Upsilon^2}\Big] + \omega^4\Big[-8\Upsilon(1-\Upsilon^2)(3\Upsilon^2-2)\frac{dP}{d\Upsilon} + 4(1-\Upsilon^2)^2(9\Upsilon^2-2)\frac{d^2P}{d\Upsilon^2} - 12\Upsilon(1-\Upsilon^2)^3\frac{d^3P}{d\Upsilon^3} + (1-\Upsilon^2)^4\frac{d^4\Upsilon}{d\Upsilon^4}\Big] + c_2 P + P^2 = 0. \tag{4.23}$$

To determine the parameter m we usually balance $\Upsilon^8 \frac{d^4P}{d\Upsilon^4}$ with P^2. This in turn gives

$$8 + m - 4 = 2m$$

so that $m = 4$. This gives the solution in the form

$$P(\Upsilon) = a_0 + a_1\Upsilon + a_2\Upsilon^2 + a_3\Upsilon^3 + a_4\Upsilon^4. \tag{4.24}$$

Substituting (4.24) into (4.23), we can get

$$(k^2 + c_1\omega^2)(1 - \Upsilon^2)\Big[- 2\Upsilon(a_1 + 2a_2\Upsilon + 3a_3\Upsilon^2 + 4a_4\Upsilon^3) + (2a_2 + 6a_3\Upsilon + 12a_4\Upsilon^2)(1 - \Upsilon^2)\Big]$$
$$+ \omega^4(1 - \Upsilon^2)\Big[- 8\Upsilon(3\Upsilon^2 - 2)(a_1 + 2a_2\Upsilon + 3a_3\Upsilon^2 + 4a_4\Upsilon^3) + 4(2a_2 + 6a_3\Upsilon + 12a_4\Upsilon^2)$$
$$\times (1 - \Upsilon^2)(9\Upsilon^2 - 2) - 12\Upsilon(1 - \Upsilon^2)^2(6a_3 + 24a_4\Upsilon) + (1 - \Upsilon^2)^3(24a_4)\Big]$$
$$+ c_2(a_0 + a_1\Upsilon + a_2\Upsilon^2 + a_3\Upsilon^3 + a_4\Upsilon^4) + (a_0 + a_1\Upsilon + a_2\Upsilon^2 + a_3\Upsilon^3 + a_4\Upsilon^4)^2 = 0. \tag{4.25}$$

We solve the algebraic system with the aid of Maple. We obtain traveling wave solutions of (4.21) as follows:

Case 1.

$$a_0 = \frac{c_2}{2}, a_1 = 0, a_2 = -\frac{3c_2}{2}, a_3 = 0, a_4 = 0,$$

$$u(x,t) = \frac{c_2}{2} - \frac{3c_2}{2}\tanh^2(\xi). \tag{4.26}$$

Case 2.

$$a_0 = -\frac{3c_2}{2}, a_1 = 0, a_2 = \frac{3c_2}{2}, a_3 = 0, a_4 = 0,$$

$$u(x,t) = -\frac{3c_2}{2} + \frac{3c_2}{2}\tanh^2(\xi). \tag{4.27}$$

Case 3.

$$a_0 = -c_2, a_1 = 0, a_2 = 0, a_3 = 0, a_4 = 0,$$

$$u(x,t) = -c_2. \tag{4.28}$$

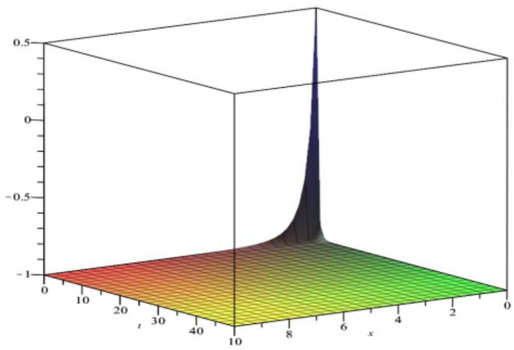


Figure 4.2: *3D plot of traveling wave solution (case 1.) of (4.26) sketched within the intervals* $0 \leq x \leq 10$ *and* $0 \leq t \leq 50$.

Example 3 [33]

In the third example we will bring to bear the Tanh method to find exact solutions, and then the solitary wave solutions of Beam equation in the form [5]:

$$T_t^{2\alpha}u\,(x,t) + T_x^{4\beta}u\,(x,t) + (n_1 + n_2 u)u = 0. \tag{4.29}$$

Using (4.2) to change (4.29) into the following nonlinear ODE

$$k^2 U_{\xi\xi} + \omega^4 U_{\xi\xi\xi\xi} + (n_1 + n_2 U)U = 0.$$

Then, we have

$$k^2 U_{\xi\xi} + \omega^4 U_{\xi\xi\xi\xi} + n_1 U + n_2 U^2 = 0. \tag{4.30}$$

Substituting (4.4) and (4.6) into (4.30), we can get

$$\begin{aligned} &-2k^2\Upsilon(1-\Upsilon^2)\frac{dP}{d\Upsilon} + k^2(1-\Upsilon^2)^2\frac{d^2P}{d\Upsilon^2} - 8\omega^4\Upsilon(1-\Upsilon^2)(3\Upsilon^2-2)\frac{dP}{d\Upsilon} + 4\omega^4(1-\Upsilon^2)^2 \\ &\times(9\Upsilon^2-2)\frac{d^2P}{d\Upsilon^2} - 12\omega^4\Upsilon(1-\Upsilon^2)^3\frac{d^3P}{d\Upsilon^3} + (1-\Upsilon^2)^4\frac{d^4P}{d\Upsilon^4} + n_1P + n_2P^2 = 0. \end{aligned} \tag{4.31}$$

To determine the parameter m we usually balance $\Upsilon^8\frac{d^4P}{d\Upsilon^4}$ with P^2. This in turn gives

$$8 + m - 4 = 2m$$

so that $m = 4$. This gives the solution in the form

$$P(\Upsilon) = b_0 + b_1\Upsilon + b_2\Upsilon^2 + b_3\Upsilon^3 + b_4\Upsilon^4. \tag{4.32}$$

Substituting (4.32) into (4.31), we can get

$$\begin{aligned}
&-2k^2\Upsilon(1-\Upsilon^2)(b_1 + 2b_2\Upsilon + 3b_3\Upsilon^2 + 4b_4\Upsilon^3) + k^2(1-\Upsilon^2)^2(2b_2 + 6b_3\Upsilon + 12b_4\Upsilon^2)\\
&-8\omega^4\Upsilon(1-\Upsilon^2)(3\Upsilon^2-2)(b_1 + 2b_2\Upsilon + 3b_3\Upsilon^2 + 4b_4\Upsilon^3) + 4\omega^4(1-\Upsilon^2)^2(9\Upsilon^2-2)\\
&\times(2b_2 + 6b_3\Upsilon + 12b_4\Upsilon^2) - 12\omega^4\Upsilon(1-\Upsilon^2)^3(6b_3 + 24b_4\Upsilon) + 24b_4(1-\Upsilon^2)^4\\
&+n_1(b_0 + b_1\Upsilon + b_2\Upsilon^2 + b_3\Upsilon^3 + b_4\Upsilon^4) + n_2(b_0 + b_1\Upsilon + b_2\Upsilon^2 + b_3\Upsilon^3 + b_4\Upsilon^4)^2 = 0.
\end{aligned} \tag{4.33}$$

Then, we have the system:

$$\begin{cases}
\Upsilon^0 : -16\omega^4 b_2 + 24\omega^4 b_4 + +2k^2 b_2 + b_0^2 n_2 + b_0 n_1 = 0,\\
\Upsilon^1 : 16\omega^4 b_1 - 120\omega^4 b_3 - 2k^2 b_1 + 6b_3 k^2 + 2b_0 b_1 n_2 + b_1 n_1 = 0,\\
\Upsilon^2 : 136\omega^4 b_2 - 480\omega^4 b_4 - 8k^2 b_2 + 12k^2 b_4 + 2b_0 b_2 n_2 + b_1^2 n_2 + b_2 n_1 = 0,\\
\Upsilon^3 : -40\omega^4 b_1 + 576\omega^4 b_3 + 2k^2 b_1 - 18k^2 b_3 + 2b_0 b_3 n_2 + 2b_1 b_2 n_2 + b_3 n_1 = 0,\\
\Upsilon^4 : -240\omega^4 b_2 + 1696\omega^4 b_4 + 6k^2 b_2 - 32k^2 b_4 + 2b_0 b_4 n_2 + 2b_1 b_3 n_2 + b_2^2 n_2 + b_4 n_1 = 0,\\
\Upsilon^5 : 24\omega^4 b_1 - 816\omega^4 b_3 + 12k^2 b_3 + 2b_1 b_4 n_2 + 2b_2 b_3 n_2 = 0,\\
\Upsilon^6 : 120\omega^4 b_2 - 2080\omega^4 b_4 + 20k^2 b_4 + 2b_2 b_4 n_2 + b_3^2 n_2 = 0,\\
\Upsilon^7 : 360\omega^4 b_3 + 2b_3 b_4 n_2 = 0,\\
\Upsilon^8 : 840\omega^4 b_4 + b_4^2 n_2 = 0.
\end{cases} \tag{4.34}$$

We solve the algebraic system with the aid of Maple. We obtain traveling wave solutions of (4.29) as follows:

Case 1:

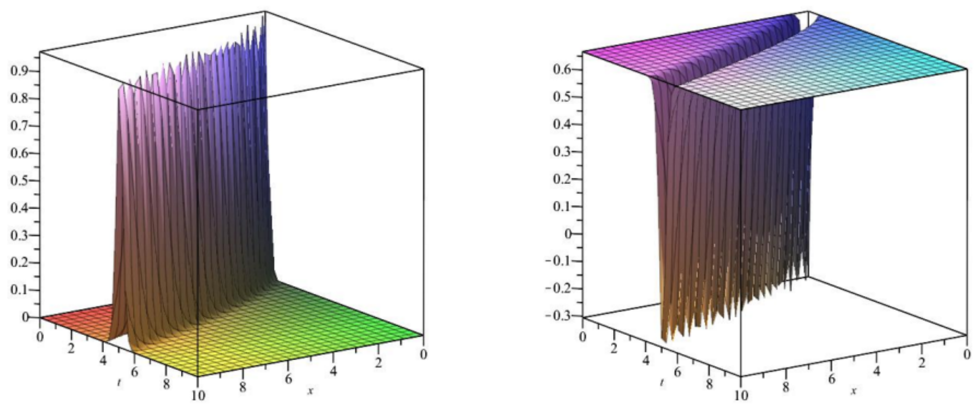
$$b_0 = -\frac{35n_1}{24n_2}, b_1 = 0, b_2 = \frac{35n_1}{12n_2}, b_3 = 0, b_4 = -\frac{35n_1}{24n_2},$$

$$u(x,t) = -\frac{35n_1}{24n_2} + \frac{35n_1}{12n_2}\tanh^2(\xi) - \frac{35n_1}{24n_2}\tanh^4(\xi). \tag{4.35}$$

Case 2:

$$b_0 = \frac{11n_1}{24n_2}, b_1 = 0, b_2 = -\frac{35n_1}{12n_2}, b_3 = 0, b_4 = \frac{35n_1}{24n_2},$$

$$u(x,t) = \frac{11n_1}{24n_2} - \frac{35n_1}{12n_2}\tanh^2(\xi) + \frac{35n_1}{24n_2}\tanh^4(\xi). \tag{4.36}$$



(a) Plots of solution (4.35) **(b)** Plots of solution (4.36)

Figure 4.3: *Plots of solution with* $0 \le x \le 10$, $0 \le t \le 30$ and $n_1 = -2, n_2 = 3, \alpha = \frac{9}{10}, \beta = \frac{7}{10}$.

4.3.2 Travelling Waves for fractional Kdv-Burger Equation

Let us consider the following problem [33]

$$T_t^{\alpha} u + \nu(u T_x^{\beta} u) + \eta T_x^{2\beta} u + \mu T_x^{3\beta} u = 0, \tag{4.37}$$

where, $T_x^{\beta}, T_t^{\alpha}$ are the conformable fractional derivative, with $0 < \alpha, \beta \le 1$.

Using (4.2), to change (4.37) into the following nonlinear ODE

$$kU_{\xi} + \nu\omega U U_{\xi} + \eta\omega^2 U_{\xi\xi} + \mu\omega^3 U_{\xi\xi\xi} = 0.$$

Integrating the above equation, we have

$$kU + \tfrac{\nu\omega}{2} U^2 + \eta\omega^2 U_{\xi} + \mu\omega^3 U_{\xi\xi} = 0. \tag{4.38}$$

Substituting (4.4) and (4.6) into (4.38), we can get

$$kP+\frac{\nu\omega}{2}P^2+\eta\omega^2\left(1-\Upsilon^2\right)\frac{dP}{d\Upsilon}-2\mu\omega^3\Upsilon\left(1-\Upsilon^2\right)\frac{dF}{d\psi}+\mu\omega^3\left(1-\Upsilon^2\right)^2\frac{d^2P}{d\Upsilon^2}=0. \quad (4.39)$$

To determine the parameter m we usually balance $\Upsilon^4\frac{d^2P}{d\Upsilon^2}$ with P^2. This in turn gives

$$4+m-2=2m$$

so that $m=2$. This gives the solution in the form

$$P(\Upsilon)=a_0+a_1\Upsilon+a_2\Upsilon^2. \quad (4.40)$$

Substituting (4.40) into (4.39), we can get

$$\begin{aligned}&k(a_0+a_1\Upsilon+a_2\Upsilon^2)+\frac{\nu\omega}{2}(a_0+a_1\Upsilon+a_2\Upsilon^2)^2+\eta\omega^2(1-\Upsilon^2)(a_1+2a_2\Upsilon)\\&-2\mu\omega^3\Upsilon\left(1-\Upsilon^2\right)(a_1+2a_2\Upsilon)+2a_2\mu\omega^3\left(1-\Upsilon^2\right)^2\Big]=0,\end{aligned} \quad (4.41)$$

Then, we have the system:

$$\begin{cases}\Upsilon^0: ka_0+\frac{1}{2}\nu\omega a_0^2+\eta\omega^2a_1+2\mu\omega^3a_1=0,\\ \Upsilon^1: -2\mu\omega^3a_1+2\eta\omega^2a_2+\nu\omega a_0a_1+ka_1=0,\\ \Upsilon^2: ka_2+\nu\omega a_0a_2+\frac{1}{2}\nu\omega a_1^2-\eta\omega^2a_1-4\mu\omega^3a_1-4\mu\omega^3a_2=0,\\ \Upsilon^3: 2\mu\omega^3a_1-2\eta\omega^2a_2+\nu\omega a_1a_2=0,\\ \Upsilon^4: \frac{1}{2}\nu\omega a_2^2+2\mu\omega^3a_1+4\mu\omega^3a_2=0.\end{cases}$$

We solve the algebraic system with the aid of Maple. We obtain traveling wave solutions of (4.37) as follows:

Case 1:

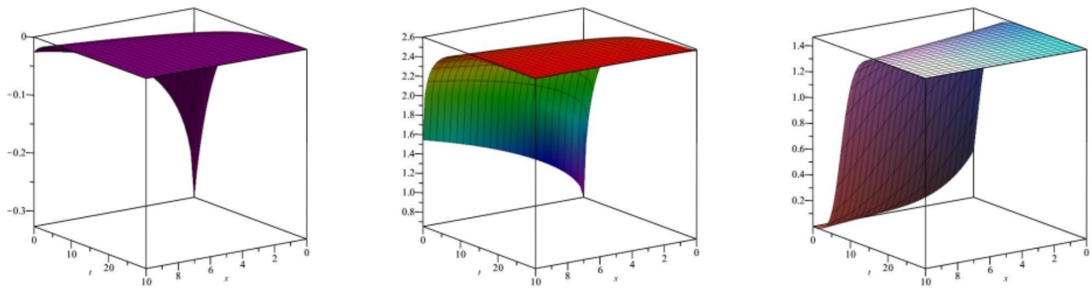
$$a_0=-\frac{4\eta^2}{49\mu\nu}, a_1=-\frac{8\eta^2}{49\mu\nu}, a_2=-\frac{4\eta^2}{49\mu\nu}, k=-\frac{4\eta^3}{343\mu^2}, w=-\frac{\eta}{14\mu},$$

$$u(x,t)=-\frac{8\eta^2}{49\mu\nu}-\frac{8\eta^2}{49\mu\nu}\tanh(\xi)-\frac{4\eta^2}{49\mu\nu}\tanh^2(\xi). \quad (4.42)$$

Case 2:

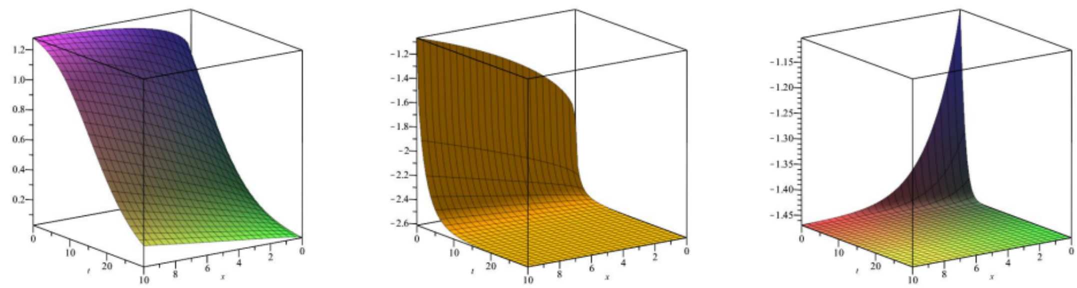
$$a_0=\frac{12\eta^2}{49\mu\nu}, a_1=-\frac{8\eta^2}{49\mu\nu}, a_2=-\frac{4\eta^2}{49\mu\nu}, k=\frac{4\eta^3}{343\mu^2}, w=-\frac{\eta}{14\mu},$$

$$u(x,t)=\frac{12\eta^2}{49\mu\nu}-\frac{8\eta^2}{49\mu\nu}\tanh(\xi)-\frac{4\eta^2}{49\mu\nu}\tanh^2(\xi). \quad (4.43)$$



(a) $\nu = 1, \mu = 1,$ $\eta = 2, \alpha = \frac{9}{10}, \beta = \frac{1}{2}.$ **(b)** $\nu = -2, \mu = 9,$ $\eta = -12, \alpha = \frac{1}{2}, \beta = \frac{1}{2}.$ **(c)** $\nu = 2, \mu = -1,$ $\eta = -3, \alpha = \frac{95}{100}, \beta = \frac{9}{10}$

Figure 4.4: Plots of solution (4.42) with $0 \le x \le 10$ and $0 \le t \le 30$.



(a) $\nu = 1, \mu = 1,$ $\eta = 2, \alpha = \frac{9}{10}, \beta = \frac{1}{2}.$ **(b)** $\nu = -2, \mu = 9,$ $\eta = -12, \alpha = \frac{1}{2}, \beta = \frac{1}{2}.$ **(c)** $\nu = 2, \mu = -1,$ $\eta = -3, \alpha = \frac{95}{100}, \beta = \frac{9}{10}$

Figure 4.5: Plots of solution (4.43) with $0 \le x \le 10$ and $0 \le t \le 30$.

4.3.3 Travelling Waves for Ostrovsky Type Equation

The aim of this subsection is use the Exp-function method to study the following problem: [30]

$$\left(T_x^{\beta} u\right)\left(T_x^{\beta}\left(T_t^{\alpha} u\right)\right) - (u)\left(T_x^{2\beta}\left(T_t^{\alpha} u\right)\right) = f\left(u, T_t^{\alpha} u\right), \tag{4.44}$$

where $T_x^{\beta}, T_t^{\alpha}$ are the conformable fractional derivative, with $0 < \alpha, \beta \le 1$ and $f : \mathbb{R}^2 \to \mathbb{R}$ is a given function.

Note, when $\alpha = \beta = 1$ and $f(u, v) = u^2 v$, so problem (4.45) is transformed into the

Ostrovsky equation:

$$(u_x)(u_{xt}) - (u)(u_{xxt}) = (u^2)(u_t).$$

Example 1

We consider the following problem:

$$\left(T_x^{\beta} u\right)\left(T_x^{\beta}\left(T_t^{\alpha} u\right)\right) - (u)\left(T_x^{2\beta}\left(T_t^{\alpha} u\right)\right) = u^2 T_t^{\alpha} u. \tag{4.45}$$

Substituting (4.2) into (4.45), so

$$(U_\xi)(U_{\xi\xi}) - UU_{\xi\xi\xi} = \tfrac{1}{\omega^2} U^2 U_\xi. \tag{4.46}$$

Therefore,

$$(U_\xi)^2 - UU_{\xi\xi} - \tfrac{1}{3\omega^2} U^3 = 0. \tag{4.47}$$

Now, we can set $p = c = q = d = 1$, then (4.8) is transformed to

$$U(\xi) = \tfrac{a_1 \exp(\xi) + a_0 + a_{-1} \exp(-\xi)}{b_1 \exp(\xi) + b_0 + b_{-1} \exp(-\xi)}. \tag{4.48}$$

Putting (4.48) into (4.47) so this yields a set of algebraic equations for $a_0, a_1, a_{-1}, b_0, b_1, b_{-1}, k$ and ω. We solve the algebraic system with the aid of Maple. We obtain traveling wave solutions of (4.45) as follows:

Case 1

$$a_1 = 0, a_0 = \tfrac{1}{a_{-1}}\sqrt{-\tfrac{1}{a_{-1}}}, a_{-1} = a_{-1},$$

$$b_1 = 0, b_0 = \tfrac{1}{a_{-1}}\sqrt{-\tfrac{1}{a_{-1}}}, b_{-1} = 0,$$

$$U(\xi) = 1 + \frac{a_{-1}^2}{\sqrt{-\frac{1}{a_{-1}}}} \exp(-\xi).$$

Case 2

$$a_1 = a_1, a_0 = 0, a_{-1} = 0,$$

$$b_1 = b_1, b_0 = 0, b_{-1} = 0,$$

$$U(\xi) = \frac{a_1}{b_1}.$$

Case 3

$$a_1 = \frac{b_1 a_{-1}}{b_{-1}}, a_0 = 0, a_{-1} = a_{-1},$$

$$b_1 = b_1, b_0 = 0, b_{-1} = b_{-1},$$

$$U(\xi) = \frac{\frac{b_1 a_{-1}}{b_{-1}}\exp(\xi) + a_{-1}\exp(-\xi)}{b_1\exp(\xi) + b_{-1}\exp(-\xi)}.$$

Fig.1 presents the graph of solution (Case 1) for (4.45) with $a_{-1} = -3$, $k = 2, \omega = 1, \alpha = \frac{3}{4}$ and $\beta = \frac{3}{5}$.

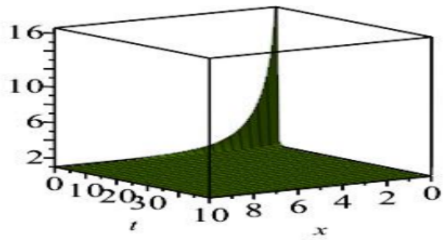


Figure 4.6: *3D plot of traveling wave solution (Case 1) of (4.45) sketched within the intervals* $0 \leq x \leq 10$ *and* $0 \leq t \leq 50$.

Example 2

We consider the following problem:

$$\left(T_x^{\beta} u\right)\left(T_x^{\beta}\left(T_t^{\alpha} u\right)\right) - (u)\left(T_x^{2\beta}\left(T_t^{\alpha} u\right)\right) = 0. \tag{4.49}$$

Substituting (4.2) into (4.49) , this can reduce (4.49) to the equation

$$(U_\xi)(U_{\xi\xi}) - UU_{\xi\xi\xi} = 0, \tag{4.50}$$

and by integrating of (4.50) , we obtain

$$(U_\xi)^2 - UU_{\xi\xi} = 0. \tag{4.51}$$

Hence, we get

$$U(\xi) = \frac{a_1 \exp(\xi) + a_0 + a_{-1}\exp(-\xi)}{b_1 \exp(\xi) + b_0 + b_{-1}\exp(-\xi)}, \tag{4.52}$$

By substitution as before, we have:

$$\begin{cases} -4a_1a_{-1}b_0^2 + 4b_1b_{-1}a_0^2 = 0, \\ -a_{-1}a_0b_1^2 + b_0b_{-1}a_1^2 + b_1b_0a_0^2 + 6a_1a_0b_1b_{-1} - 6a_1a_{-1}b_1b_0 - a_1a_0b_0^2 = 0, \\ b_1b_0a_{-1}^2 + b_1b_{-1}a_0^2 + 6a_{-1}a_0b_1b_{-1} - 6a_{-1}a_1b_0b_{-1} - 6a_{-1}b_0^2 - a_1a_0b_{-1}^2 = 0, \\ -4a_1a_{-1}b_1^2 + 4b_1b_{-1}a_1^2 = 0, \\ 4b_1b_{-1}a_{-1}^2 - 4a_1a_{-1}b_{-1}^2 = 0, \\ -a_0a_1b_1^2 + b_1b_0a_1^2 = 0, \\ b_0b_{-1}a_{-1}^2 - a_{-1}a_0b_{-1}^2 = 0, \end{cases} \tag{4.53}$$

where $A = b_1 \exp(\xi) + b_0 + b_{-1}\exp(-\xi)$.

Solving the system, we find the sets:

Case 1

$$a_1 = a_1, a_0 = 0, a_{-1} = 0,$$

$$b_1 = 0, b_0 = b_0, b_{-1} = 0,$$

$$u(x,t) = \frac{a_1}{b_0} \exp\left(\frac{k}{\alpha}t^\alpha + \frac{\omega}{\beta}x^\beta\right).$$

Case 2

$$a_1 = a_1, a_0 = 0, a_{-1} = 0,$$

$$b_1 = 0, b_0 = 0, b_{-1} = b_{-1},$$

$$u(x,t) = \frac{a_1}{b_{-1}} \exp\left(\frac{2k}{\alpha}t^\alpha + \frac{2\omega}{\beta}x^\beta\right).$$

Case 3

$$a_1 = \frac{a_0b_0}{b_{-1}}, a_0 = a_0, a_{-1} = 0,$$

$$b_1 = 0, b_0 = b_0, b_{-1} = b_{-1},$$

$$u(x,t) = \frac{a_0 + \frac{a_0b_0}{b_{-1}} \exp\left(\frac{k}{\alpha}t^\alpha + \frac{\omega}{\beta}x^\beta\right)}{b_0 + b_{-1}\exp\left(-\frac{k}{\alpha}t^\alpha - \frac{\omega}{\beta}x^\beta\right)}.$$

Case 4

$$a_1 = 0, a_0 = 0, a_{-1} = a_{-1},$$

$$b_1 = b_1, b_0 = 0, b_{-1} = 0,$$

$$u(x,t) = \tfrac{a_{-1}}{b_1} \exp\left(-\tfrac{2k}{\alpha} t^{\alpha} - \tfrac{2\omega}{\beta} x^{\beta}\right).$$

Case 5

$$a_1 = 0, a_0 = a_0, a_{-1} = 0,$$

$$b_1 = b_1, b_0 = 0, b_{-1} = 0,$$

$$u(x,t) = \tfrac{a_0}{b_1} \exp\left(-\tfrac{k}{\alpha} t^{\alpha} - \tfrac{\omega}{\beta} x^{\beta}\right)$$

Case 6

$$a_1 = 0, a_0 = \tfrac{a_{-1} b_1}{b_0}, a_{-1} = a_{-1},$$

$$b_1 = b_1, b_0 = b_0, b_{-1} = 0,$$

$$u(x,t) = \frac{\frac{a_{-1} b_1}{b_0} + a_{-1} \exp\left(-\frac{k}{\alpha} t^{\alpha} - \frac{\omega}{\beta} x^{\beta}\right)}{b_1 \exp\left(\frac{k}{\alpha} t^{\alpha} + \frac{\omega}{\beta} x^{\beta}\right) + b_0}.$$

Case 7

$$a_1 = a_1, a_0 = 0, a_{-1} = 0,$$

$$b_1 = b_1, b_0 = 0, b_{-1} = 0,$$

$$u(x,t) = \tfrac{a_1}{b_1}.$$

Case 8

$$a_1 = \tfrac{a_0 b_1}{b_0}, a_0 = a_0, a_{-1} = 0,$$

$$b_1 = b_1, b_0 = b_0, b_{-1} = 0,$$

$$u(x,t) = \frac{\frac{a_0 b_1}{b_0} \exp\left(\frac{k}{\alpha} t^{\alpha} + \frac{\omega}{\beta} x^{\beta}\right) + a_0}{b_1 \exp\left(\frac{k}{\alpha} t^{\alpha} + \frac{\omega}{\beta} x^{\beta}\right) + b_0}.$$

Case 9

$$a_1 = \tfrac{a_{-1} b_1}{b_{-1}}, a_0 = \tfrac{a_{-1} b_0}{b_{-1}}, a_{-1} = a_{-1},$$

$$b_1 = b_1, b_0 = b_0, b_{-1} = b_{-1},$$

$$u(x,t)=\frac{\frac{a_{-1}b_1}{b_{-1}}\exp\left(\frac{k}{\alpha}t^{\alpha}+\frac{\omega}{\beta}x^{\beta}\right)+\frac{a_{-1}b_0}{b_{-1}}+a_{-1}\exp\left(-\frac{k}{\alpha}t^{\alpha}-\frac{\omega}{\beta}x^{\beta}\right)}{b_1\exp\left(\frac{k}{\alpha}t^{\alpha}+\frac{\omega}{\beta}x^{\beta}\right)+b_0+b_{-1}\exp\left(-\frac{k}{\alpha}t^{\alpha}-\frac{\omega}{\beta}x^{\beta}\right)}.$$

Fig. 2 presents the graph of solution (Case 9) for (4.49) with $a_{-1}=1,\ b_1=-2, b_0=1, b_{-1}=-1, k=1, \omega=2, \alpha=\frac{1}{2}$ and $\beta=\frac{1}{2}$.

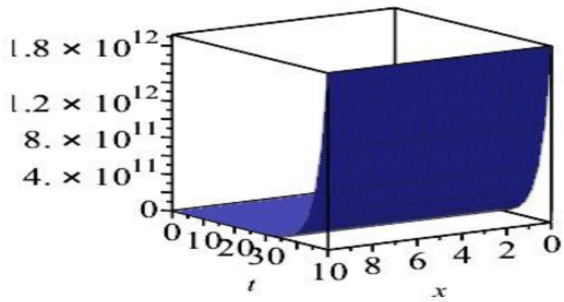


Figure 4.7: *3D plot of traveling wave solution (Case 9) for (4.49) sketched within the intervals* $0 \le x \le 10$ *and* $0 \le t \le 50$.

CHAPTER 5

Conclusion and Perspectives

In our thesis project, we have pursued two distinct approaches. Firstly, we delved into a novel class of fractional differential equations involving sequential Caputo derivatives and intricate right hand side nonlinearities, incorporating positive bounded functions with integral and nonlocal periodic conditions. Initially, we established and validated the unique integral equation governing the problem under study. Subsequently, we demonstrated the existence and uniqueness of solutions using some new sufficient conditions, with an illustrative example provided. In a separate investigation, we focused on a different class of boundary value problems (BVPs) involving sequential Caputo derivatives. Here, we explored the uniqueness of solutions, enabling us to examine the Ulam Hyers (UH) stability of the resulting solutions. Two examples were analyzed to assess UH stability and solution uniqueness. In the second approach of our thesis, we addressed various time and space conformable fractional problems with the aim of discovering traveling wave solutions. We employed both the Tanh method and the EFM method to derive new traveling wave solutions for KdV Burgers, beam deflection, and Ostrowskii-type fractional conformable equations. These solutions were expressed in terms of hyperbolic tangent functions depending on fixed parameters. It is crucial to investigate the analysis of the problem discussed in Chapter 2, particularly due to the presence of nonlinearities represented by fractional functions, especially when K is singular at the origin. Another significant research avenue involves employing Khalil derivatives to study singular sequential problems characterized by time and/or space singularities.

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