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Sur La géométrie des surfaces dans les espaces homogènes de dimension 3

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Abstract

In this thesis, we first investigate the minimal translation surfaces i.e (surfaces with null mean curvature $H = 0$) in the 3-dimensional Lorentzian Heisenberg space $\mathcal{H}_3$ endowed with a left invariant metric g_i, $i = 1,\ 2,\ 3$. We classify them in the three cases $(\mathcal{H}_3\,, g_1)$, $(\mathcal{H}_3\,, g_2)$ and $(\mathcal{H}_3\,, g_3)$. Moreover, we give their explicit expressions in each case given previously.

The last chapter of this thesis studies unit speed spacelike biharmonic curves in the 3-dimensional Lorentzian Heisenberg space $\mathcal{H}_3$ according to the metric g_1, g_2 and g_3 respectively.

The concepts: curvature and tension field introduced in this thesis are essential and effective to study the characters and properties of curves and surfaces in $\mathcal{H}_3$ which present a clear way to understand the geometry of this space.

Keywords: Lorentzian Heisenberg $3-$ Space, Lorentzian metric, Translation surfaces, Minimal surfaces, Mean curvature.

Résumé

Dans cette thèse, nous allons tout d'abord étudier les surfaces minimales de translation i.e (surfaces de courbure moyenne nulle $H = 0$) dans l'espace 3-dimensionnel de Lorentz Heisenberg $\mathcal{H}_3$ muni d'une métrique Lorentzienne invariante à gauche g_i, $i = 1,\ 2,\ 3$, les classer dans les trois cas de cet espace $(\mathcal{H}_3, g_1), (\mathcal{H}_3, g_2)$ et $(\mathcal{H}_3, g_3)$. De plus, nous donnons leurs expressions explicites dans chaque cas donné précédemment.
Le dernier chapitre de cette thèse étudie la vitesse unitaire des courbes biharmoniques spatiales dans l'espace de Lorentz Heisenberg $\mathcal{H}_3$ muni de la métric g_1, g_2 et g_3 respectivement. Les concepts: champ de courbure et de tension introduits dans cette thèse sont essentiels et efficaces pour étudier les caractères et les propriétés des courbes et surfaces dans $\mathcal{H}_3$ qui présentent une manière claire de comprendre la géométrie de cet espace.

Mots clés: L'espace de Lorentz Heisenberg de dimension 3, métrique de Lorentz, surfaces de translation, surfaces minimales, courbure moyenne.

ملخــــــــص الأطـــــــــروحة

هذا العمل مخصص لدراسة الأسطح الانسحابية ذات الحد الأدنى (الدنيا: ذات الانحناءات المتوسطة المنعدمة) في الفضاء ثلاثي الأبعاد لرتز هايزنبرغ H_3 المزود بالمتريات اليسارية الثابتة g_i حيث i=1، 2،3. كما سنصنف هذه المساحات في الحالات الثلاث $(H_3، g_1)$، $(H_3، g_2)$ و $(H_3، g_3)$ على التوالي.

بالإضافة إلى إيجاد عباراتها التحليلية في كل حالة من الحالات المذكورة أعلاه.

في أخر فصل من هذا العمل، سنقوم بدراسة المنحنيات ثنائية التوافق ذات وحدة سرعة الفضاء H_3 المزود بالمتريات g_1، g_2 و g_3 على الترتيب.

الانحناء و مجال التوتر هما مفهومان أساسيان و فعالان لدراسة مميزات و خصائص المنحنيات و الأسطح في الفضاء H_3 باعتبارها الطرقة الواضحة و المباشرة لفهم هندسة هذا الفضاء.

الكلمات المفتاحية: الفضاء ثلاثي الأبعاد لرتز هايزنبرغ H_3، مترية لرتز، الأسطح الانسحابية ذات الحد الأدنى، الانحناءات المتوسطة.

Introduction

0.1 Historic view

Geometry of surfaces is an ideal starting point for fix learning geometry. It had been initiated earlier by Johann Carl Friedrich Gauss (1777, 1855) then generalized by Friedrich Bernhard Riemann $(1826, 1866)$, (See [10], and [36]).

Since the $\overline{\underline{XIX}}^{th}$ century, the geometry of surfaces became a topic of differential geometry which deals with various additional structures in three dimensional spaces most often endowed with a Riemaniann metric.

Surfaces have been extensively studied from various perspectives:

1/ Extrinsically, relating to their embedding in Euclidean space

2/ Intrinsically, reflecting their properties determined solely by the distance within the surface as measured along curves on the surface.

One of the fundamental concepts investigated is curvature. Various analogues of curvatures of surfaces have been studied.

Leonhard Euler $(1707 - 1783)$ ([19]), founded the curvature theory of surfaces, defined the normal curvature $k_n(m, T)$ on an oriented surface S in a tangent direction T at a point m $(m \in S)$ as the curvature, at m, of the planar curve of intersection of the surface with the plane generated by the line T and the positive unit normal vector n to the surface S at m. The principal curvatures at m are the extremal values of $k_n(m, T)$ when T ranges over the tangent directions through the point m. Thus, $k_1(m) = k_n(m, T_1)$ is the minimal and $k_2(m) = k_n(m, T_2)$ is the maximal normal curvatures, attained along the principal directions: $T_1(m)$, the minimal, and $T_2(m)$, the maximal (see figure 1).

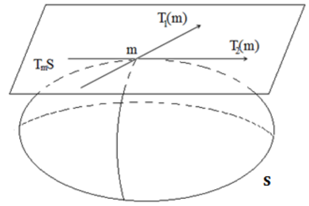

Figure 1: Principal curvature of a surface

We have also the Gaussian curvature, it was studied in depth by Carl Friedrich Gauss (articles of 1825 and 1827), who showed that curvature was an intrinsic property of a surface, independent of its isometric embedding in Euclidian space.

The evolution of the theory of surfaces passed by several stages during the last centuries, the $\overline{\underline{XIX}}^{th}$ is its golden age, in particular of the differential point. Several works of certain mathematicians interested in this theory were gathered in the entitled book "Theory of surfaces" of Darboux $(1887-1896)$. The $\overline{\underline{XX}}\ ^{th}$ century, more exactly from $1920s$, the introduction of the notion connection by Tullio Levi-Civita progressed the theory of surfaces in a more powerful conceptual frame. This theory disposes effective ways to classify the geometrical structures (curves and surfaces). Among these ways, we quote:

1/ The mean curvature $H=\frac{k_1+k_2}{2}$ given by the two fundamentals forms I and II is null for all minimal surface and it is invariant in the case of a surface with constant curvature.

2/ The Gauss curvature $K=k_1k_2$ allows to classify points on a surface (elliptic, hyperbolic, parabolic and planar points).

We recall that $k_1=k_1(m)$ and $k_2=k_2(m)$ are called the principal curvatures of the surface S at the point m defined above. They determine the local shape of a point on a surface. One characterizes the rate of maximum bending of the surface and the tangent direction in which it occurs, while the other characterizes the rate and tangent direction of minimum bending.

0.1.1 Minimal surfaces

Minimal surfaces theory originates with Lagrange. In 1762 he tried to find the surface $Z=Z(x,y)$ of least area stretched across a given closed contour, he derived the Euler-Lagrange equation

$$\frac{d}{dx}\left(\frac{Z_x}{\sqrt{1+Z_x^2+Z_y^2}}\right)+\frac{d}{dy}\left(\frac{Z_y}{\sqrt{1+Z_x^2+Z_y^2}}\right)=0,$$

unfortunately he did not succeed to find any solution beyond the plane. By expanding Lagrange equation to

$$\left(1+Z_x^2\right)Z_{yy}-2Z_xZ_yZ_{xy}+\left(1+Z_y^2\right)Z_{xx}=0, \tag{1}$$

Jean Baptiste Meusnier discovered in 1776 that the helicoid and catenoid satisfy the equation (1), and he concluded that surfaces with zero mean curvature are area-minimizing.

The study of minimal surfaces and translation minimal surfaces in 3−dimensional geometric spaces is the main objects of many researches since the $\overline{\underline{XIX}}^{th}$century. In 1835 H.F. Scherk studied translation surfaces in $\mathbb{E}^3$ defined as graph of the function $z = f(x) + g(y)$, and he proved that, besides the planes, the only minimal translation surfaces are the surfaces given by

$$z(x,y) = \frac{1}{A}\ln\left|\frac{\cos(Ax)}{\cos(Ay)}\right|, \tag{2}$$

where $f(x)$ and $g(y)$ are smooth functions on some domain D of $\mathbb{R}$ and $A \in \mathbb{R}^*$.

Between 1925 and 1950 minimal surfaces theory revived, especially after the complete solution of the Plateau problem by Jesse Douglas and Tibor Rado, without forgetting also Robert Osserman's work on complete minimal surfaces of finite total curvature which was a major milestone in the progress of this theory.

Recently, In 1990 I.V. de Woestijine studied minimal surfaces in the 3−dimensional Minkowski space ([47]), and in 1992 A. Ferrandez and P. Lucas classified in 3−dimensional Lorentz-Minkowski space surfaces which verify the condition $\Delta H = \lambda H$ where Δ is the Laplacian operator, λ is a real constant and H is the mean curvature vector field by proving that these surfaces have zero mean curvature everywhere or they are isoparametric, which means their shape operator have constant characteristic polynomial.

Since $2000's$, there has been an intensive effort to develop the theory of surfaces in homogeneous Riemannian 3−spaces see [([24]), ([31]), ([30]), ([25]), ([48])], partly in the direction of classification of surfaces with constant curvature in Lorentz Heisenberg group $\mathcal{H}_3(\mathbb{R})$ which we note $\mathcal{H}_3$.

0.1.2 Biharmonic curves and surfaces

The theory of biharmonic functions is an old and rich subject. Biharmonic functions have been studied since 1862 by Maxwell and Airy to describe a mathematical model of elasticity. Recently, biharmonic functions on Riemannian manifolds were studied by R. Caddeo and L. Vanhecke ([15]) and ([16]), L. Sario, M. Nakai and C. Wang ([40]).

In the last decade there has been a growing interest in the theory of biharmonic maps which can be divided in two main research directions:

1/ The differential geometric aspect which is based on construction and classification of biharmonic curves and surfaces.

2/ The analytic aspect from the point of view of partial differential equations, because biharmonic maps are solutions of a fourth order strongly elliptic semilinear PDE.

We also denote that the biharmonic character of curves was introduced by J. Eells and J.H. Sampson in 1964, and it was studied by Chen and Ishikawa in three dimentional semi Euclidean spaces.

In 2001 R. Caddeo, S. Montaldo and P. Piu ([13]) found conditions on the Gaussian curvature of the surface along a nongeodesic biharmonic curve, and they studied biharmonic curves in a surface of revolution, giving explicit solutions in the case of surfaces of revolution with constant Gaussian curvature. The biharmonic curves in the Heisenberg group $\mathcal{H}_3$ are investigated in ([14]) by Caddeo in 2004.

In ([21]) Fetcu studied biharmonic curves in the generalized Heisenberg group and obtained two families of proper biharmonic curves.

In the paper "*A short survey on biharmonic maps between Riemaniann manifolds" ([32]),* the authors S. Montaldo and C. Onicius studied biharmonic Riemaniann immersions in particular biharmonic curves in the Heisenberg group $\mathcal{H}_3$ endowed with the left-invariant Riemaniann metric

$$g = dx^2 + dy^2 + \left(dz + \frac{y}{2}dx - \frac{x}{2}dy\right)^2.$$

T. Körpinar and E. Turhan determinated the parametric representation of the spacelike biharmonic curves with timelike binormal according to flat metric in Heisenberg group $\mathcal{H}_3$ ([27]).

In 2013, B. Senoussi and M. Bekkar proved in their paper "*Characterization of general helix in the 3- dimensional Lorentz-Heisenberg space"* ([43]) that all the non-geodesic non-null timelike biharmonic curves in $\mathcal{H}_3$ are helixes.

J.E. Lee in his work entitled *Biharmonic spacelike curves in Lorantzian Heisenberg space* ([28] characterized the proper biharmonic spacelike curve γ in Lorentzian Heisenberg space $(\mathcal{H}_3, g)$, where g is a Lorentzian metric given by

$$g = dx^2 + dy^2 - (dz + (ydx - xdy))^2.$$

0.2 Motivation

Several researchers are interested by classification of geometric structures in Lorentzian Heisenberg space $\mathcal{H}_3$.

M. Belkhelfa, in his paper ([6]) gave a classification of k-parallel surfaces in the three dimensional Heisenberg group.

E. Turhau and G. Atlay studied the minimal and maximal surfaces in the space $\mathcal{H}_3$ endowed with left invariant Lorentz metrics g_1, g_2 and g_3. They obtained characterization of the one parameter subgroups in the three cases $(\mathcal{H}_3, g_1)$, $(\mathcal{H}_3, g_2)$ and $(\mathcal{H}_3, g_3)$, for more details see ([45]).

M. Bekkar and Z. Hanifi showed in their paper ([4]) that the plane, helicoid, hyperbolique, paraboloid and some translation surfaces are defined by elliptic integrals verifying the equation of minimal surfaces in Lorentz Heisenberg 3−space $\mathcal{H}_3$ endowed with a left invariant lorentzian metric g_ξ given by

$$g_\xi = dx^2 + dy^2 - (dz + \xi(ydx - xdy))^2.$$

J. Inoguchi, R. López and M.I. Munteanu defined six types of translation surfaces in the 3-dimensional Heisenberg group Nil_3 endowed with the left invariant metric

$$\tilde{g} = dx^2 + dy^2 + (dz + \frac{1}{2}(ydx - xdy))^2.$$

These translation surfaces are obtained as a product of two planar curves lying in planes which are not orthogonal. In the paper ([25]) the authors studied the condition of minimality

for each type.
D.W. Yoon, C.W. Lee and M.K. Karacan studied some minimal translation surfaces in the 3-dimensional Heisenberg group $\mathcal{H}_3$, see ([48]).
All these works, still others on the minimal surfaces come from the initial problem related to the geometry of soap films and soap bubbles studied first by Joseph Plateau $(1801 - 1883)$.

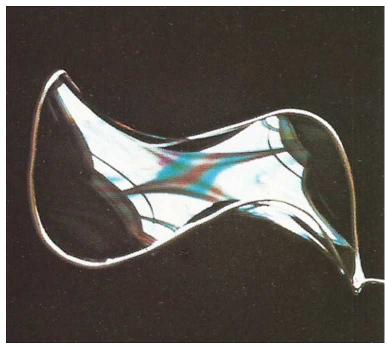

Figure 2: Soap bubbles

Problem: Let C be a closed curve. Find the minimal area of the surface S having C as a boundary, see the figure below

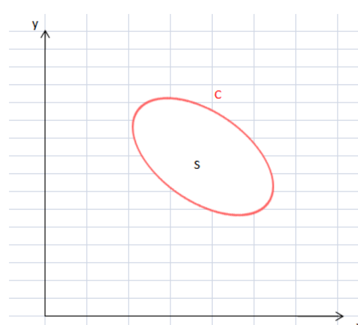

Figure 3: Minimal area of a surface S

Definition 0.2.1. *Let Ω be a domain of $\mathbb{R}^2$ and f a function from Ω to $\mathbb{R}^3$. $S = f\left(\bar{\Omega}\right)$ is* *called the parametric surface. Its area is given by*

$$A(S) = \int_{\Omega} f_X \wedge f_Y.$$

0.3 Presentation of main results

This work is in the context of the theory of surfaces, it has two principal aims:

1/ Determination of some types of minimal translation surfaces in the Lorentzian Heisenberg group $\mathcal{H}_3$ endowed with the three left invariant metrics (1.4) given later.

2/ Characterization of the biharmonic surfaces in the same space.

The thesis is divided to 5 chapters, to complete this introduction we give an overview of the contents of these chapters.

The first chapter consists of reminders, its aim is to set ratings and to serve as references for results that will be used most frequently. These reminders relate to Riemanian geometry, the geometry of the metric spaces and the flot of Ricci. We also give in this chapter some geometric properties of Lorentzian Heisenberg space $\mathcal{H}_3$.

In the second chapter, we determinate the minimality condition in order to characterize minimal surfaces in Lorentzian Heisenberg group $(\mathcal{H}_3, g_1)$. Then we study 4 types of translation minimal surfaces in the same space.

Type	parameterization
1	$(x, y, g(x) + h(y) - xy)$
2	$(x, y, g(x) + h(y))$
3	$(x + h(y), y, g(x) - xy)$
4	$(x + h(y), y, g(x))$

The following theorems will be proved.

Theorem 0.3.1. *The minimal translation surfaces Σ in the $3-$dimensional Lorentzian Heisenberg space $(\mathcal{H}_3, g_1)$ of type 1 are parameterized by $X(x, y) = (x, y, g(x) + h(y) - xy)$ where*

$$g(x) = ax + x_0, \ \textit{with } a, x_0 \in \mathbb{R}$$

and

$$h(y) = \frac{c}{2}\left[(y-a)\sqrt{|(y-a)^2 - 1|} - \ln\left|(y-a) + \sqrt{(y-a)^2 - 1}\right|\right] + y_0, \ \textit{with } (c, y_0 \in \mathbb{R}).$$

Theorem 0.3.2. *The minimal translation surfaces Σ in the $3-$dimensional Lorentzian Heisenberg space $(\mathcal{H}_3, g_1)$ of type 2 are parameterized by $X(x, y) = (x, y, g(x) + h(y))$ where*

$$h(y) = ay + b$$

and

$$g(x) = \frac{1}{2}\left[(x+a)\sqrt{(x+a)^2 + 1} + \sinh^{-1}(x+a)\right]$$

with a, $b \in \mathbb{R}$.

Theorem 0.3.3. *The minimal translation surfaces Σ of type 3 in the 3−dimensional Lorentzian Heisenberg space $(\mathcal{H}_3, g_1)$ are parameterized by*

$$X(x,y) = (x, 0, g(x)) * (h(y), y, 0) = (x + h(y), y, g(x) - xy)$$

where $g(x)$, and $h(y)$ are given by

1/ $h(y) = ay + b$ and $g(x) = \frac{a}{2(1-a^2)}x^2 + cx + x_0$ where $a \in \mathbb{R} - \{-1, 1\}$ and $b,\ c,\ x_0 \in \mathbb{R}$.

2/ Or $g(x) = ax + b$ and $h(y) = -K\left[\pm\sqrt{|(a-y)^2 - 1|} - \arctan(\pm\sqrt{|(a-y)^2 - 1|} + c\right]$, with $K \in \mathbb{R}^{*,+}$ and $a,\ b,\ c \in \mathbb{R}$.

Theorem 0.3.4. *The minimal translation surfaces Σ of type 4 in the 3−dimensional Lorentzian Heisenberg space $(\mathcal{H}_3, g_1)$ are parameterized by*

$$X(x,y) = (h(y), y, 0) * (x, 0, g(x)) = (x + h(y), y, g(x))$$

where $h(y)$ is an affine function $h(y) = ay + b$, a and b are real constants such as $a \neq \pm 1$ and $g(x)$ is given by

- *if $x^2 + 1 - a^2 \geq 0$, then*

$$g(x) = \frac{1}{2}\left[x\sqrt{x^2 + 1 - a^2} + \ln(x + \sqrt{x^2 + 1 - a^2}) - ln(x + \sqrt{x^2 + 1 - a^2})a^2\right] - \frac{a}{1 - a^2}x^2,$$

- *if $x^2 + 1 - a^2 < 0$, then*

$$g(x) = \frac{1}{2}\left[x\sqrt{-x^2 - 1 + a^2} - \tan^{-1}\frac{x}{\sqrt{-x^2 - 1 + a^2}} + \tan^{-1}(\frac{x}{\sqrt{-x^2 - 1 + a^2}})a^2\right] - \frac{a}{1 - a^2}x^2.$$

The chapter 3 is devoted to studying the same 4 types given above in the Lorentzian Heisenberg group $(\mathcal{H}_3, g_2)$, the obtained results are the following

Theorem 0.3.5. *The minimal translation surfaces Σ of type 1 in the 3−dimensional Lorentzian Heisenberg space $(\mathcal{H}_3, g_2)$ are parameterized by $X(x,y) = (x, y, g(x) + h(y) - xy)$, where $g(x)$ and $h(y)$ are given by*

1/ $g(x) = ax + x_0$ and $h(y) = -K[\arcsin(a - y)] + c$, with $c,\ a \in \mathbb{R}$ and $K \in \mathbb{R}^{*,+}$.

2/ or $h(y) = b$ and $g(x) = c_1 x + c_0$, where $c_0,\ c_1$ and b are real constants.

Theorem 0.3.6. *The minimal translation surfaces Σ of type 2 in the 3−dimensional Lorentz Heisenberg space $(\mathcal{H}_3, g_2)$ are parameterized by $X(x,y) = (x, y, g(x) + h(y))$, where $g(x)$ and $h(y)$ are given by*
$h(y) = ay + b$, with $a, b \in \mathbb{R}$ and
$g(x) = \frac{1}{4}(a + x)(a^2 + 2ax + x^2 + 1)^{\frac{3}{2}} + \frac{3}{8}(a + x)\sqrt{a^2 + 2ax + x^2 + 1} + \frac{3}{8}\sinh^{-1}(a + x) + c$, *with* $c \in \mathbb{R}$.

Theorem 0.3.7. *The minimal translation surfaces* Σ *of type* 3 *in the* $3-$*dimensional Lorentzian Heisenberg space* $(\mathcal{H}_3, g_2)$ *are parameterized by*

$$X(x,y) = (x,0,g(x)) * (h(y),y,0) = (x+h(y), y, g(x) - xy),$$

where $g(x) = ax + b$ *with* $a,\ b \in \mathbb{R}$, *for all* $x \in \mathbb{R}$,
and

$$h(y) = \pm K \arctan\left(\frac{1}{\sqrt{|(a-y)^2 - 1|}}\right) + c,$$

with $K \in \mathbb{R}^{*,+}$ *and* $c \in \mathbb{R}$, *for all* $y \in \mathbb{R} - \{a-1, a+1\}$.

Theorem 0.3.8. *The minimal translation surfaces* Σ *of type* 4 *in the* $3-$*dimensional Lorentz Heisenberg space* $(\mathcal{H}_3, g_2)$ *are parameterized by*

$$X(x,y) = (x,0,g(x))(h(y),y,0) = (x+h(y), y, g(x))$$

where $h(y) = ay + b$ *with* a *and* b *are real constants and* $g(x)$ *checks the nonlinear differential equation given by*

$$(x^2 + 1 - a^2)g''(x) + 2a(g'(x))^2 - 3xg'(x) + a = 0.$$

In the chapter 4, we characterize 6 types of translation surfaces in the $3-$dimensional Lorentzian Heisenberg space $(\mathcal{H}_3, g_3)$

Type	parameterization
1	$(x, y, g(x) + h(y) - xy)$
2	$(x, y, g(x) + h(y))$
3	$(x + h(y),\ y,\ g(x) - xy)$
4	$(x + h(y),\ y\ , g(x))$
5	$(x,\ y + g(x),\ h(y) - xy)$
6	$(x,\ y + g(x)\ , h(y))$

where we will prove the following theorems

Theorem 0.3.9. *The minimal translation surfaces* Σ *of type* 1 *in the* $3-$*dimensional Lorentzian Heisenberg space* $(\mathcal{H}_3, g_3)$ *are parameterized by* $X(x,y) = (x, y, g(x) + h(y) - xy)$, *where* $g(x)$ *and* $h(y)$ *are given by*

$$g(x) = ax + x_0,$$

and

$$h(y) = \frac{1}{6}y^3 - \frac{a}{2}y^2 + y_1 y + y_0,$$

where $a,\ x_0,\ y_0$ *and* y_1 *are real constants.*

Theorem 0.3.10. *The minimal translation surfaces* Σ *of type* 2 *in the* $3-$*dimensional Lorentzian Heisenberg space* $(\mathcal{H}_3, g_3)$ *are parameterized by*

$$X(x,y) = (0,y,h(y)) * (x,0,g(x)) = (x, y, g(x) + h(y)),$$

where $g(x)$ *and* $h(y)$ *are given by*

- *$g(x) = ax + b$ and $h(y) = \frac{1}{2}ay^2 + by + c$, where a, b and c are real constants,*
- *or $h(y) = ay + b$ and*

$$g(x) = \begin{cases} K\left[\frac{1}{3}(2x+2a-1)^{\frac{3}{2}} + c_0\right] & \text{if} \quad x \geq \frac{1}{2} - a \\ K\left[\frac{-1}{3}(-2x-2a+1)^{\frac{3}{2}} + c_0\right] & \text{if} \quad x \leq \frac{1}{2} - a \end{cases},$$

where $K \in \mathbb{R}^{*,+}$ and $c_0 \in \mathbb{R}$.

Theorem 0.3.11. *The minimal translation surfaces Σ of type 3 in the $3-$dimensional Lorentzian Heisenberg space $(\mathcal{H}_3, g_3)$ are parameterized by $X(x,y) = (x+h(y), y, g(x) - xy)$, where g and h are given by*

$$g(x) = bx + c$$

and

$$h(y) = \frac{1}{4}(b-y)^2 - c_1 \ln|b-y| + c_0$$

where b, c, c_1 and c_0 are real constants.

Theorem 0.3.12. *The minimal translation surfaces Σ of type 4 in the $3-$dimensional Lorentzian Heisenberg space $(\mathcal{H}_3, g_3)$ are parameterized by $X(x,y) = (x+h(y), y, g(x))$, where*

- *$g(x) = ax + b$ and $h(y) = K_1 + K_2 \exp(-\frac{1}{a}y) + (-ay + c_0) + a^2$, where $a \in \mathbb{R}^*$ and K_1, K_2, b, $c_0 \in \mathbb{R}$.*
- *$h(y) = ay + b$ and*

$$g(x) = \begin{cases} K\sqrt{a^2 + 2x - 1} + c & \text{if} \quad x \geq \frac{1-a^2}{2} \\ -K\sqrt{-a^2 - 2x + 1} + c & \text{if} \quad x \leq \frac{1-a^2}{2} \end{cases},$$

where $K \in \mathbb{R}^*$ and a, b, $c \in \mathbb{R}$.

Theorem 0.3.13. *The minimal translation surfaces Σ of type 5 in the $3-$dimensional Lorentzian Heisenberg space $(\mathcal{H}_3, g_3)$ are parameterized by $X(x,y) = (x, y+g(x), h(y) - xy)$, where $g(x)$ is a constant function and $h(y) = \frac{1}{6}y^3 + y_1 y + y_0$, with $y_0, y_1 \in \mathbb{R}$.*

Theorem 0.3.14. *The minimal translation surfaces Σ of type 6 in the $3-$dimensional Lorentzian Heisenberg space $(\mathcal{H}_3, g_3)$ are parameterized by $X(x,y) = (x, y+g(x), h(y))$, where $h(y) = ay + b$ is an affine function and $g(x)$ is given by*

$$g(x) = \begin{cases} \frac{K}{3}(2a+2x-1)^{\frac{3}{2}} + c_0 & \text{if} \quad x \geq \frac{1}{2} - a \\ \frac{-K}{3}(-2a-2x+1)^{\frac{3}{2}} + c_0 & \text{if} \quad x \leq \frac{1}{2} - a \end{cases},$$

where $K \in \mathbb{R}^{*,+}$ and a, b, $c_0 \in \mathbb{R}$.

The purpose of the last chapter (chapter 5) is confined only to study biharmonic curves in 3-dimensional Lorentzian Heisenberg spaces $(\mathcal{H}_3, g_1)$, $(\mathcal{H}_3, g_2)$ and $(\mathcal{H}_3, g_3)$.

Chapter 1

Pseudo-Riemannian manifolds

We recall in this chapter the definitions and notions which are constantly useful in this thesis. We start by presenting basic notions of Riemannian geometry. Then, we expose some geometric properties of the Lorentzian Heisenberg space $\mathcal{H}_3$. Finally, we give some number of analytical tools that allow to work locally on this space. For more details, we invite the reader to consult the references ([3], [11], [18], [21], [23], [26], [29], [32], [33], [36], [37], [38], [40], [44], [17] and [46]). This chapter doesn't contain any proof, we refer to the cited references above.

1.1 Riemannian geometry

First of all, we present a short overview of differential manifolds for non-specialist readers.

1.1.1 Differential manifolds

Definition 1.1.1. ***A topological manifold*** *M is a topological space which is locally homeomorphic to a Euclidean space. This means that every point $m \in M$ has a neighborhood V_m for which there exists a homeomorphism (a bijective continuous function whose inverse is also continuous) mapping that neighborhood to an open set in $\mathbb{R}$ for some $n \geq 0$.*
More precisely, ***an n-dimensional topological manifold M*** *is a topological space such that for each point p $p \in M$, there exists an open neighborhood V_m and a homeomorphism $\varphi_m : V_m \to \varphi_m(V_m) \subset \mathbb{R}$. (V_m, φ_m) is called the local chart of M.*
A family of local charts (V_m, φ_m) which covers entirely M constitutes an atlas of the variety M.
An atlas is of class $\mathcal{C}^p$, $1 \leq p \leq +\infty$, if for all m, l such that $V_m \bigcap V_l \neq \emptyset$, the application $\varphi_m o \varphi_l^{-1} : \varphi_l(V_m \bigcap V_l) \to \varphi_m(V_m \bigcap V_l)$ is a diffeomorphism of class $\mathcal{C}^p$.

Definition 1.1.2. ***A differential manifold*** *M of class $\mathcal{C}^p$ is a topological manifold provided with a family of atlases of class $\mathcal{C}^p$ all compatible with a given atlas. A smooth manifold is a differential manifold of class $\mathcal{C}^\infty$.*

In all that follows n, k $(k \leq n)$ and p are non-zero positive integers, M is a smooth manifold of dimension k and class $\mathcal{C}^p$ and m is a point of M. We note $T_m M$ the tangent space to M in m which is the set of all tangent vectors to M at m.

Definition 1.1.3. ***A tangent vector*** *to a manifold M at a point $m \in M$ is a is linear function over $\mathbb{R}$, $X_m : \mathcal{C}^\infty(m) \to \mathbb{R}$ and is a derivation on $\mathcal{C}^\infty(m)$ which satisfies for $\lambda \in \mathbb{R}$ and $f,\ g \in \mathcal{C}^\infty(m)$ on their common domain,*

1/ $X_m(\lambda f + g) = \lambda X_m(f) + X_m(g)$.

2/ $X_m(f.g) = X_m(f).g(m) + f(m).X_m(g)$.

Recall, $f \in \mathcal{C}^\infty(m)$ means that there exists an open neighborhood U of m (depending on f) such that $f \in \mathcal{C}^\infty(U,\ \mathbb{R})$.

Remark 1.1.1. *T_mM is a vector space, i.e T_mM satisfies the following properties:*

1/ If $X_m, Y_m \in T_mM$ and $\lambda \in \mathbb{R}$ then $(X_m + \lambda Y_m) \in T_mM$.

2/ For all $f \in \mathcal{C}^\infty(m)$, $(X_m + \lambda Y_m)f = X_m(f) + \lambda Y_m(f)$.

Definition 1.1.4. *We call a vector field of class $\mathcal{C}^p$ on M, all application $X : M \to TM$ of class $\mathcal{C}^p$ such that $X(m) \in T_mM$ for all $m \in M$. $\Gamma^p(M)$ is the set of $\mathcal{C}^p$ vectors fields on M.*

Remark 1.1.2. *The vector field X on M is identified with the application $v : M \to \mathbb{R}^n$ of class $\mathcal{C}^p$ such that $X(m) = (m, v(m))$ and $v(m) \in T_mM$ for all $m \in M$.*

Proposition 1.1.1. *If we define the addition $(+)$ of two fields as the field $(X + Y)(m) = X(m) + Y(m)$ and the multiplication $(.)$ of a field by a function $f : M \to \mathbb{R}$ as the field $(f.\ X)(m) = f(m)X(m)$, then for all $(X, Y) \in \Gamma^p(M)$ and for all $(f, g) \in \mathcal{C}^p(M, \mathbb{R})$ we have*

1/ $f.(X + Y) = f.X + f.Y$,

2/ $(f + g).X = f.X + g.X$,

3/ $(f.g).X = f.(g.X)$.

It is clear that the set of all tangent spaces of a manifold have a natural manifold structure.

Definition 1.1.5. *The tangent bundle $\mathcal{T}M$ of a manifold M is the union of all tangent spaces to M at all point m of M*

$$\mathcal{T}M = \{(m, X_m) \in M \times T_mM : X_m \in T_mM\}.$$

We note here that it is desirable to distinguish between tangent vectors at different points of M.

Definition 1.1.6. *Let T_mM be the tangent space at m, then* ***the cotangent space*** *at m is defined as the dual space of T_mM. So, we write $T_m^*M = (T_mM)^*$.*

Definition 1.1.7. *The cotangent bundle of a smooth manifold is the vector bundle of all the cotangent spaces at every point in the manifold. It may be described also as the dual bundle to the tangent bundle.*

Definition 1.1.8. *Let M_1 and M_2 be two manifolds of class $\mathcal{C}^{p_1}$ and $\mathcal{C}^{p_2}$ respectively. Let $p \leq min(p_1, p_2)$. An application $f : M_1 \to M_2$ is said to be of class $\mathcal{C}^{p_1}$ at the neighborhood of $m \in M_1$ if it is continuous at m and there exist local charts (V_1, φ_1) of M_1 at m and (V_2, φ_2) of M_2 at $f(m)$ such that $\varphi_2^{-1} o f o \varphi_1 : V_1 \to V_2$ is of class $\mathcal{C}^p$ in the neighborhood of $\varphi_1^{-1}(m)$ in the usual sense (for applications between open vector spaces).*

Remark 1.1.3. *The continuity of f at m ensures that $\varphi_2^{-1} o f o \varphi_1$ is well defined at the neighborhood of $\varphi_1^{-1}(m)$.*

1.1.2 Riemannian metric

Definition 1.1.9. *We call* ***a Riemannian metric*** *of class $\mathcal{C}^p$ on M, the application g_m of class $\mathcal{C}^p$ from M in the set of bilinear symmetric and positive definite forms. In other words, g_m associates to $m \in M$ a scalar product T_mM depending of m*

$$\begin{array}{rccc} g_m: & M & \rightarrow & T_mM \\ & m & \mapsto & g_m\left(\frac{\partial}{\partial x_i}(m), \frac{\partial}{\partial x_j}(m)\right) =: g_{ij}(m) \end{array}$$

The coefficients of the matrix $g_{ij}(m) := (g_{ij})_{\substack{1\le i\le n\\ 1\le j\le n}}$ called the coefficients of the metric in each local representation are functions of class $\mathcal{C}^p$.

Example 1.1.1. *The standard inner product on Euclidean space is a special example of a Riemannian metric.*

Definition 1.1.10. ***A Riemannian manifold*** *(M, g_m) is a smooth manifold M endowed with a (smooth) Riemannian metric g_m .*

Remark 1.1.4. *Let $X = \sum_i X_i \frac{\partial}{\partial x_i} \in \Gamma^p(M)$ and $Y = \sum_i Y_i \frac{\partial}{\partial y_j} \in \Gamma^p(M)$ two vectors fields on M, then we have*

$$g_m(X(m), Y(m)) = \sum_i g_{ij} X_i(m) Y_i(m).$$

In addition, if the metric g_m is of class $\mathcal{C}^p$, then $g_m(X(m), Y(m))$ is also of class $\mathcal{C}^p$.

1.1.3 The Lie bracket

Let M be a smooth manifold, X and Y are two smooth vector fields on an open set U in M.

Definition 1.1.11. *We denote by $[X, Y]$ the operator on $C^\infty(U)$ defined by*

$$[X, Y](f) = X(Y(f)) - Y(X(f))$$

for all smooth real-valued functions f on U.

Remark 1.1.5. *For all functions f and $g \in \mathcal{C}^\infty(U)$, the Lie bracket satisfies the following properties:*

1/ $[X, Y](f + g) = [X, Y](f) + [X, Y](g)$,

2/ for all $\lambda \in \mathbb{R}$,$[X, Y](\lambda f) = \lambda[X, Y](f)$,

3/ $[X, Y](f.g) = [X, Y](f).g + f.[X, Y](g)$,

4/ $[fX, gY] = (f.g)[X, Y] + (f.X(g))Y - (g.Y(f))X$.

Therefore, the operator $[X, Y]$ on $\mathcal{C}^\infty(U)$ is a vector field on U. It is called the Lie bracket of the vector fields X and Y on U.

Lemma 1.1.1. *Let M be a smooth manifold of dimension n and let X and Y be smooth vector fields on some open set U in M. Let $(x_1, x_2, ..., x_n)$ be a smooth coordinate system on U. Suppose that*

$$X = \sum_{i=1}^{n} u_i \frac{\partial}{\partial x_i}, \; Y = \sum_{i=1}^{n} v_i \frac{\partial}{\partial x_i}.$$

Then

$$[X, Y] = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(u_j \frac{\partial v_i}{\partial x_j} - v_j \frac{\partial u_i}{\partial x_j} \right) \frac{\partial}{\partial x_i}.$$

Remark 1.1.6. *Note the particular case, for any smooth coordinate system $(x_1, x_2, ..., x_n)$ on M and for $(i, j = 1, 2, ..., n)$, we have*

$$\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right] = 0.$$

1.1.4 Lie derivative

Definition 1.1.12. *If $\varphi : M \to N$ is a diffeomorphism between differentiable manifolds, and if X is a vector field on M, we define a vector field $Y = \varphi * X$ on N by*

$$Y(p) = d\varphi \left(X \left(\varphi^{-1} (p) \right) \right),$$

and we put

$$\varphi^* Y = \varphi^{-1} * Y.$$

In particular, for a vector field X on M and a local group $(\varphi_t)_{t \in I}, \subseteq \mathbb{R}$, we have

$$\varphi_t^* (X) = \varphi_{-t} * X.$$

If $X = x_i dx^i$ is a 1-form, we can write

$$\varphi_t^* (X) (x) = x_i (\varphi (x)) \frac{\partial \varphi_t^i}{\partial x^k} dx^k,$$

which is a curve in $T_x^* M$.

The particular case of smooth maps of the manifold to itself, $\varphi : M \to M$, leads in a natural way to the notion of Lie derivative. Given a vector X, the Lie derivative L_X measures the change of a tensor field along the integral curves of X .

Definition 1.1.13. *"**Lie derivative**" Let X be a vector field with a local 1-parameter group $(\varphi_t)_{t \in I}$ of local diffeomorphisms, S a tensor field on M. The Lie derivative of S in the direction of X is defined as*

$$L_X S := \left(\frac{d}{dt} \varphi_t^* S \right)_{|t=0}$$

Theorem 1.1.1. *Some properties of Lie derivative*

1/ Let $f : M \to \mathbb{R}$ be a differential function. Then

$$L_X (f) = df(X) = X(f).$$

2/ Let Y be a vector field on M. Then

$$L_X (Y) = [X, Y].$$

3/ Let $w = w_j dx^j$ be a 1-form on M. Then for $X = X^i \frac{\partial}{\partial x^i}$

$$L_X (w) = \left(\frac{\partial w_j}{\partial x^i} X^i + \frac{\partial X^i}{\partial x^j} w_i \right) dx^j .$$

1.1.5 Affine connections on smooth manifolds, tensors and curvature

The connection is a tool that allow to derive vector fields on a manifold of any dimension. Therefore, we give the general definition of connection in Riemannian geometry.

Definition 1.1.14. ***An affine connection*** *∇ on a manifold M (more precisely, on the bundle tangent to M) is a differential operator, sending smooth vector fields X and Y to a smooth vector field $\nabla_X Y$ which satisfies the following bilinear conditions for all smooth vector fields X, Y, Z and real- valued functions f on M*

- **1/** $\nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z$,
- **2/** $\nabla_{fX} Y = f \nabla_X Y$,
- **3/** $\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$,
- **4/** $\nabla_X (fY) = X(f)Y + f \nabla_X Y$, (**Leibniz rule**).

The vector field $\nabla_X Y$ is known as the covariant derivative of the vector field Y along X with respect to the affine connection ∇.

Definition 1.1.15. ***The torsion tensor*** *T of an affine connection ∇ is the operator sending smooth vector fields X and Y on a manifold M to the smooth vector field $T(X, Y)$ given by*

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

Remark 1.1.7. *The affine connection ∇ on a smooth manifold M defined from the derivation in curvilinear coordinates is said to be connection torsion free if its torsion tensor is every where zero. So that $[X, Y] = \nabla_X Y - \nabla_Y X$ for all smooth vector fields X and Y on M.*

Lemma 1.1.2. *Let ∇ be an affine connection on smooth a manifold M. The value of the covariant derivative $\nabla_X Y$ at a point m of M depends only on the vector field Y and on the value X_m of the vector field X at the point m.*

Definition 1.1.16. ***The curvature tensor*** *R of an affine connection ∇ is the operator sending smooth vector fields X, Y and Z on a manifold M to the smooth vector field $R(X,\ Y)Z$ given by*

$$R(X,\ Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{1.1}$$

The curvature tensor R satisfies the following algebraic properties

Proposition 1.1.2. *Let (M,g) be a Riemannian manifold endowed of an affine connection torsion free ∇ . For any $m \in M$ and any $X,\ Y,\ Z,\ W \in T_m M$, we have*

1/ $R(X,Y)Z = R(Y,X)Z$ *(skew-symmetry),*

2/ $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$ *(Bianchi's identity),*

3/ $\langle R(X,Y)Z, W\rangle = -\langle R(X,Y)W, Z\rangle$,

4/ $\langle R(X,Y)Z, W\rangle = \langle R(Z,W)X, Y\rangle$.

Definition 1.1.17. *For all X, Y, Z and W smooth vector fields on M,* ***the Riemann curvature tensor*** *R of the Riemannian manifold (M,g) is given by the formula*

$$R(X,Y,Z,W) = g(R(X,Y)Z, W),$$

where $R(X,Y)Z$ is defined by the expression (1.1).

Definition 1.1.18. *Let (M,g) be a Riemannian manifold and R its curvature tensor. The Ricci curvature tensor is the trace of R. Namely, if $(e_1,...,e_n)$ is an orthonormal basis of $T_m M$, $(m \in M)$, then for any $u,\ v \in T_m M$ we have*

$$\begin{aligned} Ricc(u,v) &= tr\,(x \mapsto R(x,u)\,v) \\ &= \sum_{i=1}^{n} \langle R(e_i,u)\,v, e_i\rangle \\ &= \sum_{i=1}^{n} \langle R(v,e_i)\,e_i, u\rangle \\ &= \sum_{i=1}^{n} \langle R(e_i,v)\,u, e_i\rangle \\ &= Ricc(v,u). \end{aligned} \tag{1.2}$$

Remark 1.1.8. *The Ricci curvature tensor Ricc is symmetric.*

Definition 1.1.19. *M is said to be Ricci flat, if its Ricci tensor is identically zero. A flat manifold is certainly Ricci flat.*

1.1.6 The Levi-Civita connection

Definition 1.1.20. *Let (M, g) be a Riemannian manifold and ∇ be an affine connection on M. The connection ∇ is said to be compatible with the Riemannian metric g if*

$$X.(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z),$$

for all smooth vector fields X, Y and Z on M.

It is showed that on every Riemannian manifold there exists a unique torsion-free connection that is compatible with the Riemannian metric.

Theorem 1.1.2. *For any Riemannian manifold (M, g) there exists a unique linear connection torsion free and compatible with the metric. This connection is called Levi-Civita connection. It is characterized by the identity*

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X\left[g\left(Y, Z\right)\right] + Y\left[g\left(X, Z\right)\right] - Z\left[g\left(X, Y\right)\right] \\ &\quad + g\left(\left[X, Y\right], Z\right) - g\left(\left[X, Z\right], Y\right) - g\left(\left[Y, Z\right], X\right), \end{aligned}$$

for all smooth vector fields X, Y and Z on M.

Lemma 1.1.3. *Let (M, g) be a Riemannian manifold and ∇ be the Levi-Civita connection on M. Let $(x_1, ..., x_n)$ be a smooth coordinates system defined around $m \in M$, then*

$$\nabla_{\partial_i} \partial_j = \sum_{k=1}^{n} \Gamma_{ij}^k \partial_l$$

where Γ_{ij}^k are the Christoffel symbols of Levi-Civita connection given by

$$\underset{\substack{1 \le i \le n \\ 1 \le j \le n}}{\Gamma_{ij}^k} = \frac{1}{2} \sum_{l=1}^{n} g^{kl} \left(\frac{\partial g_{il}}{\partial x_j} + \frac{\partial g_{jl}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_l} \right), \text{ for any } 1 \le k \le n$$

and g^{kl} are the coefficients of the inverse matrix of $(g_{ij})_{\substack{1 \le i \le n \\ 1 \le j \le n}}$.

1.2 Semi-Riemannian geometry

Contrary to basic linear algebra where one typically focusses on positive definite scalar products, semi-Riemannian geometry uses the more general concept of non degenerate bilinear forms. In the following subsection we recall the necessary algebraic foundations.

1.2.1 Non degenerate bilinear forms

Definition 1.2.1. *Let V be a finite dimensional vector space. A bilinear form on V is an $\mathbb{R}-$bilinear mapping $f : V \times V \to \mathbb{R}$. It is called symmetric if*

$$f(x, y) = f(y, x), \quad for\ all\ x, y \in V.$$

A symmetric bilinear form is called

1/ Positive (negative) definite, if $f(x,x) > 0$ (< 0) for all $x \in V$, $x \neq 0$.

2/ Positive (negative) semidefinite, if $f(x,x) \geq 0$ (≤ 0) for all $x \in V$.

3/ Non degenerate, $f(x,y) = 0$, for all $y \in V$ implies $x = 0$.

Definition 1.2.2. ***(Index)*** *We define the index r of a symmetric bilinear form f on V by*

$$r := max\left\{dimW, \text{ where } W \text{ is a subspace of } V \text{ with } f/_{W} \text{ is negative definite}\right\}.$$

Remark 1.2.1. *By definition we have $0 \leq r \leq dimV$ and $r = 0$ if and only if f is positive definite.*

1.2.2 Lorentzian vector spaces

Definition 1.2.3. *A Lorentzian vector space $(V, \langle .,. \rangle)$ is a an $n-$dimensional vector space V endowed with a Lorentzian scalar product $\langle .,. \rangle$ that is, a non degenerate symmetric bilinear form of index 1. This means that we have a basis $\{e_1, ..., e_n\}$ of the space V such that*

$$\begin{cases} \langle e_i, e_i \rangle &= & 1 \\ \langle e_j, e_j \rangle &= & -1 \\ \langle e_i, e_j \rangle &= & 0 \end{cases}$$

for all $1 \leq i, j \leq n$ and $i \neq j$.

Remark 1.2.2. *The metric g associated to a Lorentzian vector space is called pseudo-Riemaniann metric. So, when the metric is definite positive or of signature $(-,\ +, ..., +)$ the group is called semi-Riemannian also called pseudo- Riemannian or Lorentzian.*

Example 1.2.1. *Minkowski space. Let $M = \mathbb{R}^{n+1}$ with variables $x^1, ..., x^n, t$, with the constant semi-Riemannian metric $g_{ij} = diag(1, ..., 1, -1)$.*

Definition 1.2.4. *A metric tensor g on a smooth manifold M is a symmetric non degenerate $(0,2)$ tensor field on M of constant index.*

Definition 1.2.5. *A pseudo-Riemannian manifold (Lorentzian manifold) M is a smooth manifold furnished with a metric tensor g pseudo-Riemannian metric.*

Definition 1.2.6. *If (M, g) is pseudo-Riemannian manifold (for metric g on manifold) then the tangent vectors of each point in the manifold can be classified into three different types. A tangent vector X is*

1/ *timelike if $g\,(X,X) < 0$,*

2/ *null or lightlike if $g\,(X,X) = 0$,*

3/ *spacelike if $g\,(X,X) > 0$.*

1.3 Some geometric properties of Lorentzian Heisenberg space$\mathcal{H}_3$.

Lorentzian spaces, more precisely three dimensional Lie groups equipped with a left-invariant Lorentzian metric constitute the goal of several modern researches in pseudo-Riemannian geometry. Before giving definition and properties of these spaces, let us recall some basic facts about Lie groups and Lie algebras. the details can be obtained from ([35]) and ([41]).

1.3.1 Lie groups and Lie Algebras

Definition 1.3.1. *A Lie group G, is a group endowed with the structure of a $\mathcal{C}^\infty$ manifold such that the inversion map*

$$s: \begin{array}{ccc} G & \rightarrow & G \\ g & \mapsto & g^{-1} \end{array}$$

and the multiplication map

$$m: \begin{array}{ccc} G\times G & \rightarrow & G \\ (g,h) & \mapsto & gh \end{array}$$

are smooth. In other words, a Lie group is a set G with two structures: G is a group and G is a (smooth, real) manifold. These structures agree in the following sense: multiplication and inversion are smooth maps. A morphism of Lie groups is a smooth map which also preserves the group operation: $f(gh) = f(g)f(h)$, $f(1) = 1$.

Definition 1.3.2. *Suppose G is a Lie group, a subgroup H of G in the algebraic sense is a Lie subgroup if $H \subset G$ is a locally (with respect to H) closed submanifold of G.*

Remark 1.3.1. *It is not obvious from this definition that a Lie subgroup is even a Lie group.*

Proposition 1.3.1. *A Lie subgroup is a Lie group when equipped with the induced $\mathcal{C}^\infty$ structure and the inclusion $H \rightarrow G$ is a Lie group homomorphism.*

For a Lie group G there are distinguished vector fields. Given an element $a \in G$, we denote

$$L_a: \begin{array}{ccc} G & \rightarrow & G \\ x & \mapsto & ax \end{array}, \text{ respectively } R_a: \begin{array}{ccc} G & \rightarrow & G \\ x & \mapsto & xa \end{array},$$

the left respectively the right multiplication (translation). They define diffeomorphism (resp. biholomorphic maps) $G \rightarrow G$ and induce Lie algebra isomorphisms

$$(L_a)_*, (R_a)_* : \theta(G) \rightarrow \theta(G).$$

Definition 1.3.3. *A vector field $X \in \theta(G)$ on a Lie group G is called left, respectively, right invariant if $(L_a)_*(X) = X$, respectively, $(R_a)_*(X) = X$ for all $g \in G$. We denote $Lie(G) \subset \theta(G)$ or simply $\mathfrak{g} := Lie(G)$ the subspace of all left invariant vector fields.*

Since a Lie group G is a smooth manifold as well as a group, it is customary to use Riemannian metrics that link the geometry of G with the group structure.

Definition 1.3.4. *A Riemannian metric on a Lie group G is called left-invariant if*

$$\langle u, v\rangle_x = \langle (L_a\ )_* u, (L_a)_* v\rangle\,,\ \forall a, x \in G \quad and\ u, v \in T_x G. \tag{1.3}$$

Similarly, a Riemannian metric is right-invariant if each $R_a : G \to G$ is an isometry.

Remark 1.3.2. *Since the tangent space at any point can be translated to the tangent space at the identity element of the group, the above relation (1.3) for left-invariance can be simply written as*

$$\langle u, v\rangle = \langle (L_a\ )_* u, (L_a)_* v\rangle\,,\ \forall a \in G \quad and\ u, v \in T_x G.$$

In the following we consider uniquely left invariant metrics.

1.3.2 Lorentzian Heisenberg space $\mathcal{H}_3$

Definition 1.3.5. *Lorentzian Heisenberg space $\mathcal{H}_3$ is a subgroup of the real Lie group $Gl(3, \mathbb{R})$. It is formed of square matrix in $M_3(\mathbb{R})$*

$$\mathcal{H}_3 = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \in M_3(\mathbb{R})\ /\ x,\ y,\ z \in \mathbb{R} \right\}.$$

Hence, this space is a three dimensional Cartesian space with respect to the following product

$$(x, y, z) * (x', y', z') = (x + x',\ y + y',\ z + z' - xy'),$$

for any (x, y, z) and (x', y', z') of $\mathcal{H}_3$. The identity of this group is $(0, 0, 0)$ and the inverse of each element $(x, y, z) \in \mathcal{H}_3$ is $(-x, -y, -xy - z)$.

Remark 1.3.3. *It 's clear that the product $(*)$ is not commutative.*

Since a Lie group is a smooth manifold, we can endow it with a Riemannian metric. N. Rahmani and S. Rahmani proved in their articles [([37]), ([38])] that modulo an automorphism of the Lie algebra of the Heisenberg group $\mathcal{H}_3$ there exist three classes of left invariant Lorentzian metrics

$$\begin{aligned} g_1 &= -dx^2 + dy^2 + (xdy + dz)^2\,, \\ g_2 &= dx^2 + dy^2 - (xdy + dz)^2\,, \\ g_3 &= dx^2 + (xdy + dz)^2 - [(1 - x)\,dy - dz]^2\,. \end{aligned} \tag{1.4}$$

They also claimed that each left invariant Lorentzian metric on the Heisenberg group $\mathcal{H}_3$ is isometric to one of the three metrics mentioned above and the metric g_3 is flat.
It is important to note that each metric g_i, $i = 1, 2, 3$ is an induced non degenerate metric, i.e, $det(g_i) \neq 0$.

Remark 1.3.4. *If G is a Lie group equipped with a left-invariant metric, then it is possible to express the Levi-Civita connection in terms of quantities defined over the Lie algebra of G, at least for left-invariant vector fields.*

These results show that we can distinguish three different cases of Lorentzian Heisenberg space $\mathcal{H}_3$ and it's important to calculate the Levi-Civita connection of each one.

Case 1: The space $(\mathcal{H}_3, g_1)$

Let $(\mathcal{H}_3, g_1)$ be the Lorentzian Heisenberg space endowed with a left invariant Lorentzian metric g_1 which is given by

$$\begin{aligned} g_1 &= -dx^2 + dy^2 + (xdy + dz)^2 \\ &= (dx\ dy\ dz)(g_{1ij}) \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}, \end{aligned}$$

where $(dx\ dy\ dz)$ is a vector field and $(g_{ij})_{1\le i,j\le 3}$ is the matrix given by

$$(g_{1ij}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1+x^2 & x \\ 0 & x & 1 \end{pmatrix}.$$

The Lie-algebra of $\mathcal{H}_3$ has a pseudo-orthonormal basis $B = (e_1, e_2, e_3)$ such as

$$e_1 = \tfrac{\partial}{\partial z}, \quad e_2 = \tfrac{\partial}{\partial y} - x\tfrac{\partial}{\partial z}, \quad e_3 = \tfrac{\partial}{\partial x}, \tag{1.5}$$

which verify

$$g_1(e_1, e_1) = g_1(e_2, e_2) = 1, \quad g_1(e_3, e_3) = -1.$$

We can easily calculate the Lie product

$$[e_2, e_3] = e_1, \quad [e_3, e_1] = [e_2, e_1] = 0.$$

The Levi-Civita connection ∇ of g_1 satisfies

$$\nabla_{\partial/\partial x_i}\partial/\partial x_j = \sum_{k=1}^{3} \Gamma_{ij}^k \frac{\partial}{\partial x_k}$$

where $\left(\Gamma_{ij}^k\right)_{1\le k\le 3}$ are the Christoffel symbols of g_1. We have

$$\begin{cases} \nabla_{e_1} e_1 = 0, & \nabla_{e_1} e_2 = \frac{1}{2} e_3, & \nabla_{e_1} e_3 = \frac{1}{2} e_2, \\ \nabla_{e_2} e_1 = \frac{1}{2} e_3, & \nabla_{e_2} e_2 = 0, & \nabla_{e_2} e_3 = \frac{1}{2} e_1, \\ \nabla_{e_3} e_1 = \frac{1}{2} e_2, & \nabla_{e_3} e_2 = -\frac{1}{2} e_1, & \nabla_{e_3} e_3 = 0. \end{cases} \tag{1.6}$$

Since g_1 is a Lorentzian metric then the Ricci tensor defined by the formula (1.2) becomes

$$Ricc\,(X, Y) = \sum_{i=1}^{3} \epsilon_i g_1\left(R\left(e_i, X\right) Y, e_i\right),$$

where X, Y are two vectors fields on $\mathcal{H}_3$, $\epsilon_1 = \epsilon_2 = 1$ and $\epsilon_3 = -1$. Thus its components are

$$R_{11} = R_{33} = \frac{1}{2}, \; R_{22} = -\frac{1}{2}, \; R_{ij} = 0 \text{ for all } i \ne j.$$

Case 2: The space $(\mathcal{H}_3, g_2)$

We consider the Lorentzian Heisenberg space $(\mathcal{H}_3, g_2)$ endowed with the following left invariant Lorentzian metric

$$\begin{aligned} g_2 &= dx^2 + dy^2 - (xdy + dz)^2 \\ &= (dx\ dy\ dz)(g_{2ij}) \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}, \end{aligned}$$

where (dx, dy, dz) is a vector field and $(g_{2ij})_{1 \leq i,j \leq 3}$ is the matrix given by

$$(g_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 - x^2 & -x \\ 0 & -x & -1 \end{pmatrix}.$$

The Lie-algebra of $\mathcal{H}_3$ has a pseudo-orthonormal basis $B = (e_1, e_2, e_3)$ such as

$$e_1 = \tfrac{\partial}{\partial z}, \quad e_2 = \tfrac{\partial}{\partial y} - x\tfrac{\partial}{\partial z}, \quad e_3 = \tfrac{\partial}{\partial x}, \tag{1.7}$$

which verify

$$g_2(e_1, e_1) = g_2(e_2, e_2) = 1, \quad g_2(e_3, e_3) = -1.$$

The Lie product brackets of this basis are

$$[e_1, e_2] = [e_1, e_3] = 0 \text{ and } [e_2, e_3] = e_1.$$

∇ the Levi-Civita connection of g_2 is given by

$$\begin{cases} \nabla_{e_1} e_1 = 0, & \nabla_{e_1} e_2 = -\frac{1}{2} e_3, & \nabla_{e_1} e_3 = \frac{1}{2} e_2, \\ \nabla_{e_2} e_1 = -\frac{1}{2} e_3, & \nabla_{e_2} e_2 = 0, & \nabla_{e_2} e_3 = \frac{1}{2} e_1, \\ \nabla_{e_3} e_1 = \frac{1}{2} e_2, & \nabla_{e_3} e_2 = -\frac{1}{2} e_1, & \nabla_{e_3} e_3 = 0. \end{cases} \tag{1.8}$$

The components of the Ricci tensor are

$$R_{11} = R_{22} = R_{33} = \frac{1}{2}, \ R_{ij} = 0 \text{ for all } i \neq j.$$

Case 3: The space $(\mathcal{H}_3, g_3)$

Let $(\mathcal{H}_3, g_2)$ be the Lorentzian Heisenberg space $(\mathcal{H}_3, g_3)$ endowed with the following left invariant Lorentzian metric

$$\begin{aligned} g_3 &= dx^2 + (xdy + dz)^2 - [(1 - x)dy - dz]^2 \\ &= (dx\ dy\ dz)(g_{ij}) \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}, \end{aligned}$$

where (dx, dy, dz) is a vector field and $(g_{ij})_{1 \leq i,j \leq 3}$ is the matrix given by

$$(g_{3ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2x - 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

The Lie-algebra of $\mathcal{H}_3$ has a pseudo-orthonormal basis $B = (e_1, e_2, e_3)$ such as

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = \frac{\partial}{\partial y} + (1 - x)\frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}, \tag{1.9}$$

which verify

$$g_3(e_1, e_1) = g_3(e_2, e_2) = 1, \quad g_3(e_3, e_3) = -1.$$

We can easily calculate the Lie product

$$[e_2, e_3] = 0, \quad [e_3, e_1] = e_2 - e_3, \quad \text{and } [e_2, e_1] = e_2 - e_3.$$

∇ the Levi-Civita connection of g_3 is given by

$$\begin{cases} \nabla_{e_1} e_1 = 0, & \nabla_{e_1} e_2 = 0, & \nabla_{e_1} e_3 = 0, \\ \nabla_{e_2} e_1 = e_2 - e_3, & \nabla_{e_2} e_2 = -e_1, & \nabla_{e_2} e_3 = -e_1, \\ \nabla_{e_3} e_1 = e_2 - e_3, & \nabla_{e_3} e_2 = -e_1, & \nabla_{e_3} e_3 = -e_1. \end{cases} \tag{1.10}$$

In this case all components of the Ricci tensor are zero.

1.4 Curves and surfaces

Curves and surfaces in two and three-dimensional space occur naturally in various fields. For example, car modeling in computer-aided design software (CAO), the trajectory of an object, the route of a road and many other examples are modeled by curves or by surfaces. In this section,we will present some basic notions of parametric curves and surfaces. These notions are defined through differentiation of the parametrization, and they are related to first and second derivatives, respectively.

1.4.1 Parameterized curves and surfaces

Definition 1.4.1. *A parameterized continuous curve in* $\mathbb{R}^n$ $(n = 2, 3, ...)$ *is a continuous map* $\gamma : I \to \mathbb{R}^n$*, where* $I \subset R$ *is an open interval (of end points* $-\infty \leq a < b \leq \infty$*). A parameterized continuous curve, for which the map* $\gamma : I \to \mathbb{R}^n$ *is differentiable up to all orders, is called a parameterized smooth curve. The image set* $C = \gamma(I) \subset \mathbb{R}^n$ *is called the trace of the curve. It is important to notice that we distinguish the curve and its trace. Physically, a curve describes the motion of a particle in n-space, and the trace is its trajectory.*

In this work we will mainly be concerned with plane curves $(n = 2)$ and space curves $(n = 3)$.

Definition 1.4.2. *Let* $\gamma : I \to \mathbb{R}^3$*, where* $I \subset R$ *be a parametrization of a curve. The vector function* $t(s) = \gamma'(s)$ *is called the unit tangent vector function of* γ *and we can define the normal vector* $n(s)$ *as the unique unit vector such that* $\{t(s), n(s)\}$ *is an orthonormal direct basis of* $\mathbb{R}^2$*.*

Definition 1.4.3. *A parameterized continuous surface in* $\mathbb{R}^3$ *is a continuous map* $\Sigma : U \to \mathbb{R}^3$*, where* $U \subset \mathbb{R}^2$ *is an open, non-empty set. We call a parameterized continuous surface smooth if the map* $\Sigma : U \to \mathbb{R}^3$ *is smooth, that is, if the components* Σ_i*,* $i = 1, 2, 3$*, of*

$$\Sigma(u, v) = (\Sigma_1(u, v),\ \Sigma_2(u, v),\ \Sigma_3(u, v))$$

have continuous partial derivatives with respect to u and v of U up to all orders. We adopt the convention that a parameterized surface is smooth, unless otherwise mentioned.

Definition 1.4.4. *A surface Σ in the Euclidian space is called **a translation surface** if it is given by the graph $Z(x, y) = f(x) + g(y)$, where f and g are smooth functions on some domain $D \subseteq \mathbb{R}$.*

1.4.2 The first fundamental form

Definition 1.4.5. *Let $\Sigma : U \to \mathbb{R}^3$ be a surface, where $U \subset \mathbb{R}^2$ which represents the graph of the function $z = f(x, y)$, Σ is parameterized by*

$$\begin{array}{rccc} X: & U \subseteq \mathbb{R}^2 & \to & \mathbb{R}^3 \\ & (x,y) & \mapsto & (x, y, f(x,y)) \end{array}$$

$X(x, y) = (x, y, f(x, y))$ is the position vector and

$$X_x = \frac{\partial X}{\partial x} \quad and \quad X_y = \frac{\partial X}{\partial y}$$

are the components of the tangent vector are given by

$$\begin{cases} X_x(x,y) = \frac{\partial X}{\partial x}(x,y) \\ X_y(x,y) = \frac{\partial X}{\partial y}(x,y) \end{cases}.$$

For any $m \in U$, we define the following three functions on U, associated with X:

$$E = \|X_x(m)\|^2, \quad F = \langle X_x(m), X_y(m)\rangle, \quad G = \|X_y(m)\|^2. \tag{1.11}$$

Definition 1.4.6. *The map $I_m : T_mX \to \mathbb{R}$ that associates to w a tangent vector at m the square of its length, $w \mapsto I_m(w) = I = w^2$ given by*

$$\begin{aligned} I &= w^2 \\ &= Edx^2 + 2Fdxdy + Gdy^2 \end{aligned} \tag{1.12}$$

is called the first fundamental form of γ in m. The coefficients E, F and G are called the component functions which are conveniently arranged as the entries of a symmetric matrix

$$\mathcal{F}_I = \begin{pmatrix} E & F \\ F & G \end{pmatrix}.$$

1.4.3 The second fundamental form

The second fundamental form is another fundamental object through which we describe the curvature of a surface in a given point. Let

$$\begin{array}{rccc} X: & U \subseteq \mathbb{R}^2 & \to & \mathbb{R}^3 \\ & (x,y) & \mapsto & (x, y, f(x,y)) \end{array}$$

be a parametrization of the regular surface $\Sigma : U \to \mathbb{R}^3$, $U \subset \mathbb{R}^2$, and let $m \in U$.

Definition 1.4.7. *The map $w \in T_m X \longmapsto II_m(w) = II \in \mathbb{R}$ such as*

$$II = Ldx^2 + 2Mdxdy + Ndy^2 \tag{1.13}$$

is called the second fundamental form of Σ in m. The coefficients L, M, N are conveniently arranged as the entries of a symmetric matrix

$$\mathcal{F}_{II} = \begin{pmatrix} L & M \\ M & N \end{pmatrix}.$$

They can be computed using the equations

$$L = \langle X_{xx}, n\rangle, \quad M = \langle X_{xy}, n\rangle, \quad N = \langle X_{yy}, n\rangle \tag{1.14}$$

where

$$X_{xx} = \tfrac{\partial^2 X}{\partial x^2}, \quad X_{xy} = \tfrac{\partial^2 X}{\partial x \partial y}, \quad X_{yy} = \tfrac{\partial^2 X}{\partial y^2}$$

and n is the unit normal vector

$$N = \frac{X_x \wedge X_y}{\|X_x \wedge X_y\|}$$

which satisfies

$$\begin{cases} \langle X_x, n\rangle & = & 0 \\ \langle X_y, n\rangle & = & 0 \\ \langle n, n\rangle & = & 1 \end{cases}. \tag{1.15}$$

In the Lorentzian Heisenberg space $(\mathcal{H}_3, g_i)$ with $i = 1, 2, 3$, the coefficients of

1/ the first fundamental form I, given by the formula (1.11) becomes

$$E = g_i(X_x, X_x), \quad F = g_i(X_x, X_y), \quad G = g_i(X_y, X_y), \tag{1.16}$$

2/ the second fundamental form II, given by the formula (1.14) becomes

$$L = g_i(\nabla_{X_x} X_x, n), \quad M = g_i(\nabla_{X_y} X_x, n), \quad N = g_i(\nabla_{X_y} X_y, n). \tag{1.17}$$

Where ∇ is the Levi-Civita connection and the unit normal vector n satisfies the system

$$\begin{cases} g_i(X_x, N) = 0 \\ g_i(X_y, N) = 0 \\ g_i(N, N) = 1 \end{cases} \tag{1.18}$$

1.4.4 Curvature of curves on a surfaces

Curvature is a central notion of classical differential geometry, and various curvatures of surfaces have been studied. Since the first aim of the present work, is the study of minimal translation surfaces in the Lorentzian Heisenberg group $\mathcal{H}_3$, Our main focus will be only on the mean curvature of surfaces.

Definition 1.4.8. *Let $\Sigma : U \to \mathbb{R}^3$ be a surface. The mean curvature H of Σ given by*

$$H = \frac{1}{2} trace \left(\mathcal{F}_I^{-1} \mathcal{F}_{II} \right) \tag{1.19}$$

where

$$\mathcal{F}_I = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \text{ and } \mathcal{F}_{II} \begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

are matrix of the coefficients of the the first and second fundamental form respectively. The equation (1.19) can be reduced to

$$H = \frac{1}{2} \left[\frac{LG - 2MF + NE}{EG - F^2} \right]. \tag{1.20}$$

Remark 1.4.1. *The mean curvature H corresponds to the mean of the principal curvatures.*

Definition 1.4.9. *In $3-$ dimensional Lorentzian Heisenberg space $\mathcal{H}_3$ the surfaces that locally minimize the areas are called minimal surfaces, they satisfy the condition $H = 0$, where H is the mean curvature vector field given by the formula (1.20).*

1.4.5 Biharmonic curves on a surface

Definition 1.4.10. *Biharmonic curves on a surface are critical points of the bienergy functional given by*

$$E_2(\varphi) = \frac{1}{2} \int_M |\tau_\varphi|^2 \, dv$$

where $\varphi : (M, g) \to (N, h)$ is a map between two Riemannian manifolds and

$$\tau(\varphi) = Trace_g \nabla d\varphi$$

is the vanishing of the tension field.

In a different setting, B. Y. Chen ([17]) defined biharmonic submanifolds M of the Euclidean space as those with harmonic mean curvature vector field, that is $\Delta H = 0$, where Δ is the rough Laplacian and H is the mean curvature vector field. This stated the following

Conjecture: Any biharmonic submanifold of the Euclidean space is harmonic, that is minimal.

Theorem 1.4.1. *([12]) Let $N^3(c)$ be a manifold with constant sectional curvature c and let S be a surface of $N^3(c)$. Then S is biharmonic if and only if it is minimal.*

In order to describe proper biharmonic curves on Riemaniann manifold, we need to recall the definition of the Frenet frame.

Definition 1.4.11. *([32]) Let $\gamma : I \subset \mathbb{R} \longrightarrow (N^n, h)$ be a curve parameterized by arc length from an open interval $I \subset \mathbb{R}$ to a Riemannian manifold N^n. The Frenet frame $\{F_i\}_{i=1,\dots,n}$*

associated to γ is the orthonormalisation of the $(n+1)$-uple $\left\{\nabla^{(k)}_{\frac{\partial}{\partial t}} d\gamma\left(\frac{\partial}{\partial t}\right)\right\}_{k=0,\dots,n}$, described by

$$\begin{cases} F_1 & = & d\gamma\left(\frac{\partial}{\partial t}\right), \\ \nabla^{\gamma}_{\frac{\partial}{\partial t}} F_1 & = & k_1 F_2, \\ \nabla^{\gamma}_{\frac{\partial}{\partial t}} F_i & = & -k_{i-1}F_{i-1} + k_i F_{i+1}, \quad \forall i = 2, ..., n-1, \\ \nabla^{\gamma}_{\frac{\partial}{\partial t}} F_n & = & -k_{n-1}F_{n-1}, \end{cases}$$

where the functions $\{k_1 = k > 0,\ k_2 = -\tau,\ k_3, ..., k_{n-1}\}$ are called the curvatures of γ and ∇^{γ} is the connection on the pull-back bundle $\gamma^{-1}(TN)$. Note that $F_1 = T = \gamma'$ is the unit tangent vector field along the curve γ.

Using the Frenet frame, we get that a curve is proper $(k_1 \neq 0)$ biharmonic if and only if

$$\begin{cases} k_1 & = & consatnt \neq 0, & \\ k_1^2 + k_2^2 & = & R(F_1, F_2, F_1, F_2), & \\ k_2' & = & -R(F_1, F_2, F_1, F_3), & \\ k_2 k_3 & = & -R(F_1, F_2, F_1, F_4), & \\ R(F_1, F_2, F_1, F_j) & = & 0, & j = 5, ..., n. \end{cases}$$

Chapter 2

Minimal translation surfaces in $(\mathcal{H}_3, g_1)$

In this Chapter, we present some results on the characterization of the curvature of translation surfaces in the 3− dimensional Lorentzian Heisenberg space $\mathcal{H}_3$ endowed with the following left invariant Lorentzian metric

$$g_1 = -dx^2 + dy^2 + (xdy + dz)^2.$$

2.1 Minimal surface equations in $(\mathcal{H}_3, g_1)$

Let Σ be a surface in the Lorentzian Heisenberg 3−space $\mathcal{H}_1$ which represents the graph of the function $z = f(x, y)$, it is parameterized by

$$\begin{array}{cccc} X: & U \subseteq \mathbb{R}^2 & \to & \mathbb{R}^3 \\ & (x, y) & \mapsto & (x, y, f(x, y)) \end{array}$$

$X(x, y) = (x, y, f(x, y))$ is the position vector. We calculate the components of the tangent vector

$$\begin{cases} X_x(x, y) = \frac{\partial}{\partial x} + f_x \frac{\partial}{\partial z} = e_3 + Pe_1 \\ X_y(x, y) = \frac{\partial}{\partial y} + f_y \frac{\partial}{\partial z} = e_2 + Qe_1 \end{cases}$$

With $B = (e_1, e_2, e_3)$ given by (1.5) is a pseudo-orthonormal basis of the Lie-algebra of $(\mathcal{H}_3, g_1)$.

By posing

$$\begin{cases} P = f_x \\ Q = x + f_y \end{cases},$$

we get

$$\begin{cases} X_x(x, y) = e_3 + Pe_1 \\ X_y(x, y) = e_2 + Qe_1 \end{cases}$$

The first fundamental form I of X is

$$I = Edx^2 + 2Fdxdy + Gdy^2,$$

with

$$E = g_1(X_x, X_x) = P^2 - 1, \quad F = g_1(X_x, X_y) = PQ, \quad G = g_1(X_y, X_y) = Q^2 + 1. \tag{2.1}$$

The second fundamental form II of X is

$$II = Ldx^2 + 2Mdxdy + Ndy^2,$$

where

$$L = g_1(\nabla_{X_x} X_x, n), \quad M = g_1(\nabla_{X_y} X_x, n), \quad N = g_1(\nabla_{X_y} X_y, n)$$

such as n is a unit vector field normal to Σ, then it satisfies the system (1.18). Therefore, we get

$$n = \frac{e_1 - Qe_2 + Pe_3}{\sqrt{1 + Q^2 - P^2}} = \frac{1}{W}(e_1 - Qe_2 + Pe_3),$$

where

$$W = \sqrt{|1 + Q^2 - P^2|}.$$

Consequently

$$L = \tfrac{1}{W}(f_{xx} - PQ), \quad M = \tfrac{1}{2W}(1 + 2f_{xy} - Q^2 - P^2), \quad N = \tfrac{1}{W}(f_{yy} - PQ). \tag{2.2}$$

The formulas (2.1) et (2.2) allow to compute the mean curvature of Σ, we get

$$H = \frac{1}{2W^3}\left[(P^2 - 1)f_{yy} + (Q^2 + 1)f_{xx} - 2PQf_{xy} - PQ\right].$$

Applying the Definition 1.4.9, we have the following result

Proposition 2.1.1. *The surface Σ defined above is a minimal surface in $3-$dimensional Lorentzian Heisenberg space $\mathcal{H}_3$ if and only if it's mean curvature H satisfies the following condition*

$$H = \frac{1}{2W^3}\left[(P^2 - 1)f_{yy} + (Q^2 + 1)f_{xx} - 2PQf_{xy} - PQ\right] = 0. \tag{2.3}$$

Remark 2.1.1. *If $f(x, y) = -\frac{1}{2}xy + c_1 x + c_2 y$, with c_1 and c_2 are two real constants, then we can check easily that Σ is a minimal surface in $3-$dimensional Lorentzian Heisenberg space $\mathcal{H}_3$.*

Example 2.1.1. *We take $c_1 = c_2 = 0$, then the surface Σ_1 in the Lorentzian Heisenberg $3-$space $\mathcal{H}_3$ which represents the graph of the function $z = f(x, y) = -\frac{1}{2}xy$ is a minimal surface, see the figure below.*

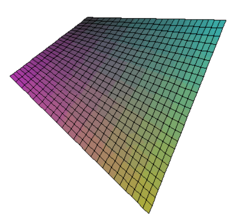

Figure 2.1: Minimal surface in $(\mathcal{H}_3, g_1)$, example (2.1.1)

2.2 Some types of minimal translation surfaces in $(\mathcal{H}_3, g_1)$

A translation surface $\Sigma\ (\gamma_1, \gamma_2)$ in $\mathcal{H}_3$ is a surface parameterized by

$$X: \begin{array}{ccc} \Sigma & \to & \mathcal{H}_3 \\ (x,y) & \mapsto & X(x,y) = \gamma_1(x) * \gamma_2(y) \end{array},$$

and obtained as a product of two generating not orthogonal curves γ_1 and γ_2 situated in the planes of coordinates of $\mathbb{R}^3$. Since the multiplication $*$ in the Lorentzian Heisenberg space is not commutative, then for each choice of curves γ_1 and γ_2 we may construct two translation surfaces, namely $\Sigma\ (\gamma_1, \gamma_2)$ and $\Sigma\ (\gamma_2, \gamma_1)$ which are different. In this section we define and study four types of translation surfaces in the $3-$dimensional Lorentzian Heisenberg space $(\mathcal{H}_3, g_1)$.

2.2.1 Surfaces of type 1 and 2

Let the curves γ_1 and γ_2 be given by $\gamma_1(x) = (x, 0, g(x))$ and $\gamma_2(y) = (0, y, h(y))$, where g and h are two arbitrary surfaces.

Type 1

The translation surface $\Sigma\ (\gamma_1, \gamma_2) = \gamma_1 * \gamma_2$ of type 1 is thus parameterized as

$$X(x,y) = (x, 0, g(x)) * (0, y, h(y)) = (x, y, g(x) + h(y) - xy).$$

Note that $f(x,y) = g(x) + h(y) - xy$, so

$$\begin{cases} P = g'(x) - y \\ Q = h'(y) \end{cases}$$

The minimality condition given by the equation (2.3) becomes

$$\left[(g'(x) - y)^2 - 1\right] h''(y) + ((h'^2(y) + 1)g''(x) + h'(y)(g'(x) - y) = 0. \tag{2.4}$$

We divide (2.4) by $[h'^2(y) + 1]$, we obtain

$$g''(x) + \left[\frac{(g'(x) - y)^2 - 1}{h'^2(y) + 1}\right] h''(y) + \frac{h'(y)(g'(x) - y)}{h'^2(y) + 1} = 0. \tag{2.5}$$

Taking the derivative of (2.5) with respect to x, we get

$$g'''(x) + \left[\frac{2h''(y)}{h'^2(y) + 1}\right] g''(x)(g'(x) - y) + \frac{h'(y)g''(x)}{h'^2(y) + 1} = 0 \tag{2.6}$$

We suppose that $g''(x) \neq 0$, the case $g''(x) = 0$ will be studied separately later. We divide (2.6) by $g''(x)$ then we take the derivative of the obtained equation with respect to y, we find

$$2\left[-\frac{h''(y)}{h'^2(y) + 1} + (g'(x) - y)\frac{(h'''(y))(h'^2(y) + 1) - 2h''^2(y)h'(y)}{(h'^2(y) + 1)^2}\right] + \frac{d}{dy}\left(\frac{h'(y)}{h'^2(y) + 1}\right) = 0 \tag{2.7}$$

which implies

$$\frac{h''(y)}{h'^2(y)+1} - \frac{1}{2}\left[\frac{h''(y)\,(1-h'^2(y))}{(h'^2(y)+1)^2}\right] = (g'(x)-y)\frac{(h'''(y))(h'^2(y)+1)-2h''^2(y)h'(y)}{(h'^2(y)+1)^2} \quad (2.8)$$

Therefore

$$\frac{\frac{h''(y)}{h'^2(y)+1} - \frac{1}{2}\left[\frac{h''(y)\left(1-h'^2(y)\right)}{(h'^2(y)+1)^2}\right]}{\frac{(h'''(y))(h'^2(y)+1)-2h''^2(y)h'(y)}{(h'^2(y)+1)^2}} + y = g'(x) \quad (2.9)$$

This equality (2.9) is satisfied if $g'(x)$ is a constant, so $g''(x) = 0$ which is contradiction.

Hence we have $h''(y) = 0$ which implies $h(y) = cy + d$, $c, d \in \mathbb{R}$ then from the condition of minimality (2.4), we get

$$(c^2+1)g''(x) = -c(g'(x)-y)$$

this last equation is verified only if $c = 0$, therefore we have $g''(x) = 0$.

Finally we conclude that Σ is a minimal translation surface if and only if $g''(x) = 0$ or $h''(y) = 0$.

If $g''(x) = 0$ then $g(x) = ax + x_0$, a et x_0 are two real constants. Replacing $g(x)$ in (2.4), we obtain the differential equation of second order

$$\left((a-y)^2 - 1\right) h''(y) + h'(y)(a-y) = 0$$

which admits the solution

$$h(y) = \frac{c}{2}\left[(y-a)\sqrt{|(y-a)^2-1|} - \ln\left|(y-a) + \sqrt{|(y-a)^2-1|}\right|\right] + y_0, \text{ with } (c, y_0 \in \mathbb{R}).$$

We summarize with the following theorem

Theorem 2.2.1. *The minimal translation surfaces Σ in the $3-$dimensional Lorentzian Heisenberg space $(\mathcal{H}_3, g_1)$ of type 1 are parameterized by $X(x,y) = (x, y, g(x) + h(y) - xy)$ where*

$$g(x) = ax + x_0, \text{ with } a, x_0 \in \mathbb{R}$$

and

$$h(y) = \frac{c}{2}\left[(y-a)\sqrt{|(y-a)^2-1|} - \ln\left|(y-a) + \sqrt{|(y-a)^2-1|}\right|\right] + y_0, \text{ with } (c, y_0 \in \mathbb{R}).$$

Remark 2.2.1. *The translation surfaces Σ in the $3-$dimensional Lorentz Heisenberg space $\mathcal{H}_3$ of type 1 parameterized by $X(x,y) = (x, y, g(x) + h(y) - xy)$ is minimal if and only if g is an affine function.*

Example 2.2.1. *If we take $a = 1$, $x_0 = 1$, $c = 2$ and $y_0 = -1$, then*

$$\begin{aligned} f(x,y) &= g(x) + h(y) - xy \\ &= x + \left[(y-1)\sqrt{(y-1)^2-1} - \ln\left|(y-1) + \sqrt{(y-1)^2-1}\right|\right] - xy \end{aligned}$$

and the translation surfaces Σ in the $3-$dimensional Lorentzian Heisenberg space $\mathcal{H}_3$ parameterized by $X(x,y) = (x, y, f(x,y))$ is minimal, $D_f = D_- \cup D_+$ is the domain of definition of f.

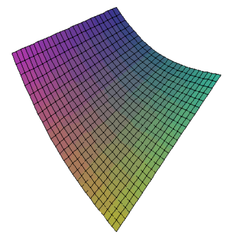

Figure 2.2: Minimal translation surface of type 1 in $(\mathcal{H}_3, g_1)$ on D^-

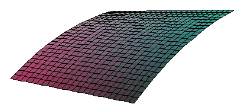

Figure 2.3: Minimal translation surface of type 1 in $(\mathcal{H}_3, g_1)$ on D^+

Type 2

Taking the two curves as in the previous case (type 1), $\gamma_1(x) = (x, 0, g(x))$ and $\gamma_2(y) = (0, y, h(y))$,where g and h are two arbitrary surfaces. We consider the translation surface Σ $(\gamma_2, \gamma_1) = \gamma_2 * \gamma_1$ of type 2 which is parameterized by

$$X(x, y) = (0, y, h(y)) * (x, 0, g(x)) = (x, y, g(x) + h(y)) .$$

In this case, we have $f(x, y) = g(x) + h(y)$. Then

$$\begin{cases} P = g'(x) \\ Q = h'(y) + x \end{cases} \tag{2.10}$$

The minimality condition given by the equation (2.3) becomes

$$(g'^2(x) - 1)h''(y) + ((x + h'(y))^2 + 1)g''(x) - g'(x)(x + h'(y)) = 0. \tag{2.11}$$

We take the derivative of the equation (2.11) with respect to y, we find

$$(g'^2(x) - 1)h'''(y) + 2h''(y)(x + h'(y))g''(x) - g'(x)h''(y) = 0. \tag{2.12}$$

We suppose that $h''(y) \neq 0$ and we divide (2.12) by $h''(y)$ then we take the derivative again with respect to y, we obtain

$$(g'^2(x) - 1)\left[\frac{h'''(y)}{h''(y)}\right]' = -2h''(y)g''(x), \tag{2.13}$$

which implies

$$-\frac{\left[\frac{h'''(y)}{h''(y)}\right]'}{h''(y)} = \frac{2g''(x)}{g'^2(x)-1}.$$

Since the left hand side depends only on y and the right hand side depends only on x, then we get the differential system below

$$-\frac{\left[\frac{h'''(y)}{h''(y)}\right]'}{h''(y)} = \frac{2g''(x)}{g'^2(x)-1} = \lambda, \tag{2.14}$$

where λ is a real constant. In order to solve the system (2.14), we distinguish two cases:

1/ First, if $\lambda = 0$, then $g''(x) = 0$ and $g(x) = ax + b$ with a and b are real constants. Replacing $g(x)$ in the minimality condition (2.11), we obtain

$$(a^2 - 1)h''(y) = a(x + h'(y))$$

which is a contradiction.

2/ If $\lambda \neq 0$ then, by solving the following equation

$$\frac{2g''(x)}{g'^2(x)-1} = \lambda,$$

we obtain

$$g(x) = \frac{1}{\lambda}\left[2\sqrt{K\exp(\lambda x)+1} + \ln\frac{\sqrt{K\exp(\lambda x)+1}-1}{\sqrt{K\exp(\lambda x)+1}+1}\right].$$

This situation also yields a contradiction as h depends only on y.

It remains for us to study the particular case $h''(y) = 0$. Therefore if $h(y) = ay + b$ with a and b are two real constants, then the minimality condition (2.11) becomes

$$((x+a)^2+1)g''(x) = g'(x)(x+a). \tag{2.15}$$

Solving (2.15), we get

$$g(x) = \frac{1}{2}\left[(x+a)\sqrt{(x+a)^2+1} + \sinh^{-1}(x+a)\right].$$

We summarize with the following theorem

Theorem 2.2.2. *The minimal translation surfaces Σ in the $3-$dimensional Lorentzian Heisenberg space $(\mathcal{H}_3, g_1)$ of type 2 are parameterized by $X(x,y) = (x, y, g(x) + h(y))$ where*

$$h(y) = ay + b$$

and

$$g(x) = \frac{1}{2}\left[(x+a)\sqrt{(x+a)^2+1} + \sinh^{-1}(x+a)\right]$$

with a, $b \in \mathbb{R}$.

Example 2.2.2. *If we take $a = 1$, $b = -2$ then*

$$f(x,y) = \frac{1}{2}\left[(x+a)\sqrt{(x+a)^2+1} + \sinh^{-1}(x+a)\right] + y - 2$$

and the translation surfaces Σ in the $3-$dimensional Lorentzian Heisenberg space $\mathcal{H}_3$ parameterized by $X(x,y) = (x, y, f(x,y))$ is minimal.

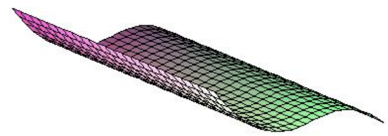

Figure 2.4: Minimal translation surface of type 2 in $(\mathcal{H}_3, g_1)$

2.2.2 Surfaces of type 3 and 4

Let the curves γ_1 and γ_2 be given by $\gamma_1(x) = (x, 0, g(x))$ and $\gamma_2(y) = (h(y), y, 0)$, where g and h are two arbitrary surfaces.

Type 3

The translation surface $\Sigma\,(\gamma_1, \gamma_2)$ of type 3 is given by the product

$$\begin{aligned} \gamma_1(x) * \gamma_2(y) &= (x + h(y),\ y,\ g(x) - xy) \\ &= X(x, y) \end{aligned}$$

is parameterized by

$$\begin{array}{llll} X: & \mathbb{R}^2 & \to & \mathbb{R}^3 \\ & (x,y) & \mapsto & X(x,y) = (x + h(y), y, g(x) - xy) \end{array},$$

where $X(x, y) = (x + h(y), y, g(x) - xy)$ is the position vector and the components of the tangent vector are given by

$$\begin{cases} X_x(x,y) = (g'(x) - y)e_1 + e_3 \\ X_y(x,y) = e_2 + h'(y)e_3 \end{cases}$$

We pose

$$\begin{cases} P = g'(x) - y \\ Q = h'(y) \end{cases},$$

The coefficients of the first fundamental form I of X are

$$E = g_1(X_x, X_x) = P^2 - 1, \quad F = g_1(X_x, X_y) = -Q, \quad G = g_1(X_y, X_y) = 1 - Q^2. \tag{2.16}$$

The coefficients of the second fundamental form II of X are

$$L = \tfrac{1}{W}(g''(x) + P^2 Q), \quad M = \tfrac{1}{2W}(-1 + Q^2P^2 - P^2), \quad N = -\tfrac{1}{W}(Ph''(y)). \tag{2.17}$$

with

$$W = \sqrt{|1 + P^2Q^2 - P^2|}.$$

We follow the same steps as the previous types (1 and 2) to calculate the mean curvature of the translation surface Σ, we yield

$$\begin{aligned} H &= \frac{1}{2W^3}\left[\left(1-P^2\right)Ph''(y) + \left(1-Q^2\right)g''(x) - Q\right] \\ &= \frac{1}{2W^3}\left[(g'(x)-y)\left(1-(g'(x)-y)^2\right)h''(y) + (1-h'^2(y))g''(x) - h'(y)\right]. \end{aligned}$$

The minimality condition $H = 0$ implies the following equation

$$(g'(x)-y)\,(1-(g'(x)-y)^2)h''(y) + (1-h'^2(y))g''(x) = h'(y). \tag{2.18}$$

Taking the derivative with respect to x, we obtain

$$g''(x)h''(y)\left[1-3(g'(x)-y)^2\right] + \left[1-h'^2(y)\right]g'''(x) = 0. \tag{2.19}$$

We suppose now that $g''(x) \neq 0, \forall x \in \mathbb{R}$, the case $g''(x) = 0$,will be treated separately. We divide the equation (2.19) by $g''(x)$ and we take the derivative with respect to x, which we gives

$$-6g''(x)(g'(x)-y)h''(y) + \left[1-h'^2(y)\right]\left(\frac{g'''(x)}{g''(x)}\right)' = 0.$$

We repeat the same operation again, we get

$$\frac{h''(y)}{1-h'^2(y)} = \frac{1}{6g''(x)}\left[\left(\frac{\left(\frac{g'''(x)}{g''(x)}\right)'}{g''(x)}\right)'\right] \tag{2.20}$$

Since the left hand side of the equality (2.20) depends only on y and the right hand side depends only on x, thus

$$\frac{h''(y)}{1-h'^2(y)} = \frac{1}{6g''(x)}\left[\left(\frac{\left(\frac{g'''(x)}{g''(x)}\right)'}{g''(x)}\right)'\right] = \lambda, (\ \lambda \in \mathbb{R}),$$

which implies the following differential system

$$\begin{cases} \frac{h''(y)}{1-h'^2(y)} = \lambda \\ \frac{1}{g''(x)}\left[\left(\frac{\left(\frac{g'''(x)}{g''(x)}\right)'}{g''(x)}\right)'\right] = 6\lambda \end{cases}, \ (\ \lambda \in \mathbb{R}). \tag{2.21}$$

To solve the system (2.21), we distinguish two cases

1/ If $\lambda = 0$ then we have $h''(y) = 0$, which implies $h(y) = ay + b$ with $a \in \mathbb{R} - \{-1, 1\}$ and $b \in \mathbb{R}$. Replacing this result in (2.18) we find $g(x) = \frac{a}{2(1-a^2)}x^2 + cx + x_0$ where $a \in \mathbb{R} - \{-1, 1\}$ and $c,\ x_0 \in \mathbb{R}$.

2/ For all $\lambda \neq 0$, we take the integration with respect to y of the first differential equation of the system (2.21), we obtain

$$\begin{aligned} h'(y) &= \tanh(\lambda y + y_0) \\ &= \frac{1 - \exp(-2\lambda y + 2y_0)}{1 + \exp(-2\lambda y + 2y_0)}, \end{aligned}$$

where y_0 is a real constant. Consequently

$$h(y) = -\frac{1}{2\lambda} \ln \left| \frac{\tanh(\lambda y + y_0) - 1}{\tanh(\lambda y + y_0) + 1} \right|.$$

Replacing $h'(y)$ and $h''(y)$ in the minimality condition (2.18), we obtain

$$(g'(x) - y)(1 - (g'(x) - y)^2)\lambda \left[1 - \tanh^2(\lambda y + y_0)\right] = \tanh(\lambda y + y_0),$$

which is a contradiction as g depends only on x.

We study now the particular case $g''(x) = 0$, then we have $g(x) = ax + b$ where a and b are real constants. Replacing this result in (2.18), we get

$$(a - y)(1 - (a - y)^2)h''(y) = h'(y). \tag{2.22}$$

Solving the differential equation (2.22), we find

$$h(y) = -K\left[\pm\sqrt{|(a-y)^2 - 1|} - \arctan(\pm\sqrt{|(a-y)^2 - 1|} + c\right],$$

where $K \in \mathbb{R}^{*,+}$ and $c \in \mathbb{R}$.

Finally, we conclude that the minimality condition given by the equation (2.18) is verified if and only if $h''(y) = 0$ or $g''(x) = 0$ and we summarize with the following theorem

Theorem 2.2.3. *The minimal translation surface* Σ *of type* 3 *in the* 3*−dimensional Lorentzian Heisenberg space* $(\mathcal{H}_3, g_1)$ *are parameterized by*

$$X(x, y) = (x, 0, g(x)) * (h(y), y, 0) = (x + h(y), y, g(x) - xy)$$

where $g(x)$, *and* $h(y)$ *are given by*

1/ $h(y) = ay + b$ and $g(x) = \frac{a}{2(1-a^2)}x^2 + cx + x_0$ where $a \in \mathbb{R} - \{-1, 1\}$ and $b,\ c,\ x_0 \in \mathbb{R}$.

2/ Or $g(x) = ax + b$ and $h(y) = -K\left[\pm\sqrt{|(a-y)^2 - 1|} - \arctan(\pm\sqrt{|(a-y)^2 - 1|} + c\right]$, with $K \in \mathbb{R}^{*,+}$ and $a,\ b,\ c \in \mathbb{R}$.

Remark 2.2.2. *The translation surface Σ of type 3 in the $3-$dimensional Lorentz Heisenberg space $(\mathcal{H}_3, g_1)$ parameterized by*

$$X(x,y) = (x, 0, g(x)) * (h(y), y, 0) = (x + h(y), y, g(x) - xy)$$

is minimal if and only if h (or g) is an affine function.

Example 2.2.3.

1/ First, we take $a = 2,\ b = 0$, and $c = x_0 = 1$, so

$$h(y) = 2y \text{ and } g(x) = -3x^2 + x + 1.$$

The translation surface Σ of type 3 in this case is parameterized by

$$X(x,y) = (x + 2y,\ y,\ -3x^2 + x + 1 - xy),$$

which is represented in the figure below

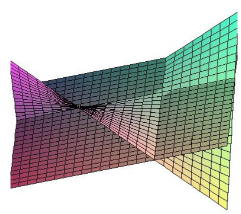

Figure 2.5: Minimal translation surface of type 3 in $(\mathcal{H}_3, g_1)$, example (2.2.3), case 1

2/ The case where $g''(x) = 0$, we take $a = b = K = 1$ and $c = 0$. Then we get

$$g(x) = x + 1 \text{ and } h(y) = -\sqrt{y^2 - 2y} + \arctan\left(\sqrt{y^2 - 2y}\right).$$

The parametrization of the translation surface Σ in this case is given by

$$X(x,y) = (x - \sqrt{y^2 - 2y} + \arctan\sqrt{y^2 - 2y},\ y,\ x + 1 - xy), \tag{2.23}$$

see figure 2.6

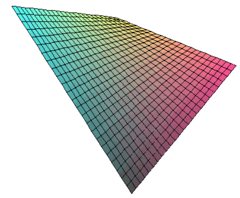

Figure 2.6: Minimal translation surface of type 3 in $(\mathcal{H}_3, g_1)$, example (2.2.3), case 2

Type 4

The translation surface $\Sigma\,(\gamma_2, \gamma_1)$ of type 4 given by the product

$$\begin{aligned} \gamma_2(x) * \gamma_1(y) &= (x + h(y), y, g(x)) \\ &= X(x, y) \end{aligned}$$

is parameterized by

$$\begin{array}{cccc} X: & U \subseteq \mathbb{R}^2 & \to & \mathbb{R}^3 \\ & (x, y) & \mapsto & X(x, y) = (x + h(y), y, g(x)) \end{array},$$

where $X(x, y) = (x + h(y), y, g(x))$ is the position vector and the components of the tangent vector are given by

$$\begin{cases} X_x(x, y) = (g'(x))e_1 + e_3 \\ X_y(x, y) = xe_1 + e_2 + h'(y)e_3 \end{cases}$$

From (1.16) the coefficients of the first fundamental form I of X are

$$\begin{cases} E &= (g'(x) + y)^2 - 1 \\ F &= x(g'(x) + y) - h'(y) \\ G &= x^2 + 1 - h'(y) \end{cases} \tag{2.24}$$

From (1.17) the coefficients of the second fundamental form II of $X$are

$$\begin{cases} L &= \frac{1}{W}\left[g''(x) + g'(x)(g'(x)h'(y) - x)\right], \\ M &= \frac{1}{2W}\left[1 + ((g'(x)h'(y))^2 - x^2) - g'^2(x)\right], \\ N &= \frac{1}{W}\left[xh'(y)(g'(x)h'(y) - x) - g'(x)(h''(y) + x)\right]. \end{cases} \tag{2.25}$$

with

$$W = \sqrt{\left|1 + (g'(x)h'(y) - x)^2 - (g'^2(x) + 1)\right|}. \tag{2.26}$$

Replacing (2.24) and (2.25) in (1.20), we obtain the mean curvature H of the translation surface surface $\Sigma\,(\gamma_2, \gamma_1)$ of type 4 given below
$H = \frac{1}{W^3}\left[g'(x)(g'^2(x) - 1)h''(y) + (x^2 + 1 - h'^2(y))g''(x) + h'(y) - xg'(x)\right]$.

The translation surface $\Sigma\,(\gamma_2, \gamma_1)$ in the space $(\mathcal{H}_3, g_1)$ is said to be minimal if and only if $H = 0$, witch implies

$$g'(x)(g'^2(x) - 1)h''(y) + (x^2 + 1 - h'^2(y))g''(x) + h'(y) - xg'(x) = 0. \tag{2.27}$$

Taking the derivative with respect to y, we get

$$(g'^3(x) - g'(x))h'''(y) - 2h''(y)h'(y)g''(x) + h''(y) = 0. \tag{2.28}$$

In order to solve the ordinary differential equation (2.28), we suppose first that $h''(y) \neq 0$,for all $y \in \mathbb{R}$, we divide by $h''(y)$ and we take the derivative with respect to y, we obtain

$$(g'^3(x) - g'(x)) \left[\frac{h'''(y)}{h''(y)}\right]' - 2h''(y)g''(x) = 0. \tag{2.29}$$

which implies

$$\frac{1}{2}\frac{\left[\frac{h'''(y)}{h''(y)}\right]'}{h''(y)} = \frac{g''(x)}{g'^3(x) - g'(x)}. \tag{2.30}$$

The left hand side of the equality (2.30) depends only on y and the right hand side depends only on x, then we get the following differential system

$$\begin{cases} \frac{1}{2}\frac{\left[\frac{h'''(y)}{h''(y)}\right]'}{h''(y)} = \lambda \\ \frac{g''(x)}{g'^3(x)-g'(x)} = \lambda \end{cases} \tag{2.31}$$

where λ is a real constant.

1/ We assume first that $\lambda \neq 0$, solving the second ordinary differential equation of the system (2.30), we get

$$g'(x) = \sqrt{\frac{-1}{K\exp(-2\lambda x) + 1} + 1},$$

with $K \in \mathbb{R}^{*,+}$. Replacing $g'(x)$ and its derivative $g''(x))$ in (2.27), we get a contradiction as the function h depends only on y.

2/ Let now study the case $\lambda = 0$, which implies $g''(x)) = 0$, then we have $g(x) = ax + b$ with a, $b \in \mathbb{R}$. Replacing in (2.27), we get a contradiction.

Finally, we study the particular case $h''(y) = 0$, so $h(y) = ay + b$ with a and b are real constants.
The minimality condition (2.27) becomes

$$(x^2 + 1 - a^2)g''(x) - xg'(x) = -a \tag{2.32}$$

Solving the equation (2.27), we obtain

- If $x^2 + 1 - a^2 > 0$, then

$$g(x) = \frac{1}{2}\left[\sqrt{x^2 + 1 - a^2} + \ln(x + \sqrt{x^2 + 1 - a^2}) - ln(x + \sqrt{x^2 + 1 - a^2})a^2\right] - \frac{a}{1 - a^2}x^2.$$

- If $x^2 + 1 - a^2 < 0$, then

$$g(x) = \frac{1}{2}\left[x\sqrt{-x^2 - 1 + a^2} - \tan^{-1}\frac{x}{\sqrt{-x^2 - 1 + a^2}} + \tan^{-1}(\frac{x}{\sqrt{-x^2 - 1 + a^2}})a^2\right] - \frac{a}{1 - a^2}x^2.$$

Finally, we summarize by the following theorem

Theorem 2.2.4. *The minimal translation surfaces Σ of type 4 in the 3−dimensional Lorentzian Heisenberg space $(\mathcal{H}_3, g_1)$ are parameterized by*

$$X(x,y) = (h(y), y, 0) * (x, 0, g(x)) = (x + h(y), y, g(x))$$

where $h(y)$ is an affine function $h(y) = ay + b$, a and b are real constants such as $a \neq \pm 1$ and $g(x)$ is given by

- *if $x^2 + 1 - a^2 \geq 0$, then*

$$g(x) = \frac{1}{2}\left[x\sqrt{x^2+1-a^2} + \ln(x + \sqrt{x^2+1-a^2}) - ln(x + \sqrt{x^2+1-a^2})a^2\right] - \frac{a}{1-a^2}x^2,$$

- *if $x^2 + 1 - a^2 < 0$, then*

$$g(x) = \frac{1}{2}\left[x\sqrt{-x^2-1+a^2} - \tan^{-1}\frac{x}{\sqrt{-x^2-1+a^2}} + \tan^{-1}(\frac{x}{\sqrt{-x^2-1+a^2}})a^2\right] - \frac{a}{1-a^2}x^2.$$

Example 2.2.4. *If we take $a = 2$, $b = 3$, then the translation surfaces Σ in the 3−dimensional Lorentzian Heisenberg space $\mathcal{H}_3$ parameterized by $(x + h(y), y, g(x))$ on the domain $D =]-\infty, -\sqrt{3}[\bigcup]\sqrt{3}, +\infty[\times\mathbb{R}$, where*

$$g(x) = \frac{1}{2}\left[x\sqrt{x^2-3} + \ln(x + \sqrt{x^2-3}) - \ln(x + \sqrt{x^2-3})4\right] + \frac{2}{3}x^2$$

and $h(y) = 2y + 3$ is minimal.

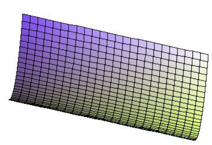

Figure 2.7: Minimal surface of type 4 in $(\mathcal{H}_3, g_1)$.

Chapter 3

Minimal translation surfaces in $(\mathcal{H}_3, g_2)$

As we have already done in the previous chapter, we will study the four types of the translation surfaces in the $3-$dimensional Lorentzian Heisenberg space $\mathcal{H}_3$ endowed with the following left invariant Lorentzian metric

$$g_2 = dx^2 + dy^2 - (xdy + dz)^2.$$

3.1 Minimal surfaces in $(\mathcal{H}_3, g_2)$

Let Σ be a surface in the Lorentzian Heisenberg $3-$space $(\mathcal{H}_3, g_2)$ parameterized by

$$\begin{array}{rccc} X: & U \subseteq \mathbb{R}^2 & \to & \mathbb{R}^3 \\ & (x,y) & \mapsto & (x,y,f(x,y)) \end{array}$$

$X(x,y) = (x,y,f(x,y))$ is the position vector and the components

$$X_x = \tfrac{\partial}{\partial x} + f_x(x,y)\tfrac{\partial}{\partial z} \quad \text{and} \quad X_y = \tfrac{\partial}{\partial y} + f_y(x,y)\tfrac{\partial}{\partial z}$$

of the tangent vector are given by

$$\begin{cases} X_x(x,y) = f_x(x,y)e_1 + e_3, \\ X_y(x,y) = (f_y(x,y)+x)e_1 + e_2. \end{cases}$$

Where $B = (e_1, e_2, e_3)$ given by (1.7) is a pseudo-orthonormal basis of the Lie-algebra of $(\mathcal{H}_3, g_2)$.

The coefficients of the first fundamental form I of the surface Σ are

$$\begin{cases} E &= -1 + (f_x)^2(x,y), \\ F &= f_x(x,y)(f_y(x,y)+x), \\ G &= 1 + (f_y(x,y)+x)^2. \end{cases}$$

n : the unit vector field normal to Σ is

$$n = \frac{1}{W}[e_1 - (f_y(x,y)+x)e_2 + f_x(x,y)e_3],$$

where

$$W = \sqrt{|-f_x^2(x,y) + (f_y(x,y)+x)^2 + 1|}.$$

In order to calculate the coefficients of the second fundamental form II of the surface Σ , we need the following expressions

$$\begin{cases} \nabla_{X_x} X_x &= f_{xx}(x,y)e_1 + f_x(x,y)e_2, \\ \nabla_{X_x} X_y &= (\frac{1}{2} + f_{xy}(x,y))e_1 + \frac{1}{2}(f_y(x,y)+x)e_2 - \frac{1}{2}f_x(x,y)e_3, \\ \nabla_{X_y} X_y &= f_{yy}(x,y)e_1 - (f_y(x,y)+x)e_3. \end{cases}$$

Therefore, the coefficients of the second fundamental form are

$$\begin{cases} L &= \frac{1}{W}[-f_x(x,y)(f_y(x,y)+x) + f_{xx}(x,y)], \\ M &= \frac{1}{W}[\frac{1}{2}f_x^2(x,y) - \frac{1}{2}(f_y(x,y)+x)^2 + (\frac{1}{2} + f_{xy}(x,y))], \\ N &= \frac{1}{W}[f_x(x,y)(f_y(x,y)+x) + f_{yy}(x,y)]. \end{cases}$$

Hence the minimality condition (see definition 1.4.9) in 3−dimensional Lorentzian Heisenberg space $(\mathcal{H}_3, g_2)$ is

$$[f_x^2(x,y)-1]f_{yy}(x,y)+[(f_y(x,y)+x)^2+1]f_{xx}(x,y)-2f_x(x,y)(f_y(x,y)+x)(f_{xy}(x,y)+\frac{3}{2}) = 0. \tag{3.1}$$

Remark 3.1.1. *All surfaces parameterized by $X(x,y) = (x,y,f(x,y))$ with*

$$f(x,y) = -\frac{3}{2}xy + ax + by + c$$

where a, b and c are real constants, are minimal surfaces in $(\mathcal{H}_3, g_2)$.

Example 3.1.1. *We take $a = b = c = 1$, then the surface Σ in the Lorentzian Heisenberg 3−space $(\mathcal{H}_3, g_2)$ parameterized by $\left(x, y, -\frac{3}{2}xy + x + y + 1\right)$ is a minimal surface.*

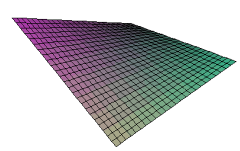

Figure 3.1: Minimal surface in $(\mathcal{H}_3, g_2)$.

3.2 Some types of minimal translation surfaces in $(\mathcal{H}_3, g_2)$

Now we study four types of translation surfaces in the 3− dimensional Lorentzian Heisenberg space $(\mathcal{H}_3, g_2)$, which are parameterized by a product of two generating curves γ_1 and γ_2 situated in the plane of coordinates of $\mathbb{R}^3$ and which are not orthogonal.

3.2.1 Surfaces of type 1 and 2

We consider the curves γ_1 and γ_2 be given by $\gamma_1(x) = (x, 0, g(x))$ and $\gamma_2(y) = (0, y, h(y))$, where g and h are two arbitrary surfaces.

Type 1

Let first consider a translation surface $\Sigma(\gamma_1, \gamma_2)$ parameterized by

$$X(x, y) = (x, 0, g(x)) * (0, y, h(y)) = (x, y, g(x) + h(y) - xy),$$

where g and h are two arbitrary surfaces, the minimality condition given by the equation (3.1) becomes

$$[(g'^2(x) - 1]h''(y) + (h'^2(y) + 1)g''(x) - h'(y)(g'(x) - y) = 0. \tag{3.2}$$

Taking the derivative with respect to x of the equation (3.2), we obtain

$$2g''(x)(g'(x) - y)h''(y) + (h'^2(y) + 1)g'''(x) - h'(y)g''(x) = 0. \tag{3.3}$$

We suppose that $g''(x) \neq 0$, we divide (3.3) by $g''(x)$ then we take the derivative of the obtained equation with respect to x. The case $g''(x) = 0$ will be studied separately later. We get

$$2g''(x)h''(y) + (h'^2(y) + 1)\left[\frac{g'''(x)}{g''(x)}\right]' = 0. \tag{3.4}$$

We divide again by $g''(x)$, we find the following equation

$$\frac{h''(y)}{(h'^2(y) + 1)} = -\frac{1}{2}\left[\frac{\left(\frac{g'''(x)}{g''(x)}\right)'}{g''(x)}\right], \tag{3.5}$$

Since the right hand side of the equality (3.5) depends only on x and the left hand side depends only on y, then we have the following differential system

$$\frac{h''(y)}{(h'^2(y) + 1)} = -\frac{1}{2}\left[\frac{\left(\frac{g'''(x)}{g''(x)}\right)'}{g''(x)}\right] = A, \tag{3.6}$$

where A is a real constant. Consequently, we distinguish two cases:

1/ If $A = 0$, we have $h''(y) = 0$ which implies $h(y) = ay + b$ with a and b are two real constants.
Replacing $h'(y)$ and $h''(y)$ in the minimality condition (3.2), we find

$$(a^2 + 1)g''(x) - ag'(x) = -ay.$$

This last equality is satisfied if and only if $a = 0$. Therefore, we have $g(x) = c_1 x + c_0$ and $h(y) = b$, where c_0, c_1 and b are real constants.
Consequently

$$f(x, y) = c_1 x - xy + c_2$$

with c_1, $c_2 \in \mathbb{R}$.

2/ We suppose now that $A \neq 0$, solving the equation $\frac{h''(y)}{h'^2(y)+1} = A$, we find

$$h(y) = -\frac{1}{A} \ln|\cos(Ay + y_0)| + y_1$$

where $A \in \mathbb{R}^*$, y_0 and $y_1 \in \mathbb{R}$.
Replacing in the equation (3.2), we get

$$g''(x) = -\frac{A}{\cos^2(Ay + y_0)} - \tan^2(Ay + y_0) g''(x) + (g'(x) - y)\tan(Ay + y_0)$$

which is a contradiction as the function g depends only on x.

Now, we have to handle the particular case $g''(x) = 0$ which we gives $g(x) = ax + x_0$ with x_0, $a \in \mathbb{R}$. Replacing in (3.2), we obtain the following equation

$$\frac{h''(y)}{h'(y)} = \frac{(a - y)}{(a - y)^2 - 1}. \tag{3.7}$$

Solving (3.7), we find

$$h(y) = -K[\arcsin(a - y) + y_0],$$

with $K \in \mathbb{R}^{*,+}$ and $y_0 \in \mathbb{R}$.
Consequently, for any minimal translation surface of type 1, we have $g''(x) = 0$ or $h(y) = constant$. We summarize with the following theorem

Theorem 3.2.1. *The minimal translation surfaces Σ of type 1 in the 3− dimensional Lorentz Heisenberg space $(\mathcal{H}_3, g_2)$ are parameterized by $X(x, y) = (x, y, g(x) + h(y) - xy)$ where $g(x)$, and $h(y)$ are given by*

1/ $g(x) = ax + x_0$ and $h(y) = -K[\arcsin(a - y)] + c$, with c, $a \in \mathbb{R}$ and $K \in \mathbb{R}^{*,+}$.

2/ or $h(y) = b$ and $g(x) = c_1 x + c_0$, where c_0, c_1 and b are real constants.

Remark 3.2.1. *The translation surfaces Σ of type 1 in the 3− dimensional Lorentzian Heisenberg space $(\mathcal{H}_3, g_2)$ parameterized by*

$$X(x, y) = (x, y, g(x) + h(y) - xy)$$

is minimal if and only if g is an affine function.

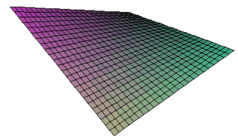

Figure 3.2: Minimal translation surface of type 1 in $(\mathcal{H}_3, g_2)$, example (3.2.1)

Example 3.2.1. *If $a = x_0 = -1$ and $K = b = c = 1$, then the surface Σ in the Lorentzian Heisenberg $3-$space $(\mathcal{H}_3, g_2)$ parameterized by $(x, y, -x - [\arcsin(-1-y)] - xy)$ is a minimal surface.*

Example 3.2.2. *We take $b = -2$, $c_1 = -1$, $c_0 = 3$, then the minimal surface Σ in the Lorentzian Heisenberg $3-$space $(\mathcal{H}_3, g_2)$ in this case is parameterized by $(x, y, -x + 1 - xy)$, it's graph is below*

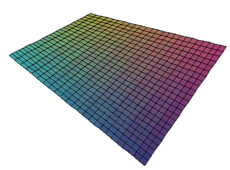

Figure 3.3: Minimal translation surface of type 1 in $(\mathcal{H}_3, g_2)$, example (3.2.2)

Type 2

The translation surface $\Sigma(\gamma_2, \gamma_1)$ of type 2 in $(\mathcal{H}_3, g_2)$ is parameterized by

$$X(x, y) = (0, y, h(y)) * (x, 0, g(x)) = (x, y, g(x) + h(y))$$

where g and h are two arbitrary surfaces. So, the minimality condition given by the equation (3.1) becomes

$$[(g'(x))^2 - 1]h''(y) + [(h'(y) + x)^2 + 1]g''(x) - 3g'(x)(h'(y) + x) = 0. \tag{3.8}$$

We take the derivative of (3.8) with respect to y, we obtain

$$h'''(y)[(g'(x))^2 - 1] + 2g''(x)h''(y)(h'(y) + x) - 3g'(x)h''(y) = 0. \tag{3.9}$$

We suppose that $h''(x) \neq 0$, we divide (3.9) by $h''(y)$ then we take the derivative of the obtained equation with resect to y. The case $h''(y) = 0$ will be studied separately later.

$$[(g'(x))^2 - 1]\left[\frac{h'''(y)}{h''(y)}\right]' = -2g''(x)h''(y)h''(y) = 0. \tag{3.10}$$

So, we get

$$\frac{g''(x)}{g'(x))^2 - 1} = -\frac{\left[\frac{h'''(y)}{h''(y)}\right]'}{2h''(y)} = \lambda \tag{3.11}$$

where λ is a real constant.

- First, we suppose that $\lambda = 0$, then we have $g(x) = ax + b$ with a and b are real constants. Replacing in the minimality condition equation (3.8), we obtain

$$[a^2 - 1]h''(y) = 3a(h'(y) + x). \tag{3.12}$$

  which is a contradiction because the function h depends only y.

- If $\lambda \neq 0$, solving the differential equation

$$\frac{g''(x)}{g'(x))^2 - 1} = \lambda, \lambda \in \mathbb{R},$$

  we find

$$g(x) = x - \frac{1}{\lambda}\ln|1 - K\exp 2\lambda x| + c,$$

  with $K \in \mathbb{R}^{*,+}$ and $c \in \mathbb{R}$. Replacing $g(x)$ in the minimality condition equation (3.8), we get also a contradiction.

We still have to study the particular case $h''(y) = 0$, which implies $h(y) = ay + b$, with a and b are real constants. In this case, the minimality condition given by the equation (3.8) becomes

$$[(a + x)^2 + 1]g''(x) - 3g'(x)(a + x) = 0. \tag{3.13}$$

Solving this last equation, we obtain

$$g(x) = \frac{1}{4}(a + x)(a^2 + 2ax + x^2 + 1)^{\frac{3}{2}} + \frac{3}{8}(a + x)\sqrt{a^2 + 2ax + x^2 + 1} + \frac{3}{8}\sinh^{-1}(a + x) + c$$

where c is a real constant. We deduce that the minimality condition (3.2) is satisfied only if $h''(y) = 0$. Therefore we conclude with the following theorem

Theorem 3.2.2. *The minimal translation surfaces Σ of type 2 in the $3-$ dimensional Lorentz Heisenberg space $(\mathcal{H}_3, g_2)$ are parameterized by $X(x, y) = (x, y, g(x) + h(y))$ where $g(x)$, and $h(y)$ are given by:*
$h(y) = ay + b$, with $a, b \in \mathbb{R}$ and
$g(x) = \frac{1}{4}(a+x)(a^2+2ax+x^2+1)^{\frac{3}{2}} + \frac{3}{8}(a+x)\sqrt{a^2+2ax+x^2+1} + \frac{3}{8}\sinh^{-1}(a+x) + c$, with $c \in \mathbb{R}$.

Example 3.2.3. *We take $a = 3$, $b = -1$, $c = 0$, then the minimal surface Σ in the Lorentzian Heisenberg $3-$space $(\mathcal{H}_3, g_2)$ in this case is parameterized by $(x, y, f(x, y))$, with*

$$f(x,y) = 3y - 1 + \frac{1}{4}(3+x)(6x + x^2 + 10)^{\frac{3}{2}} + \frac{3}{8}(3+x)\sqrt{6x + x^2 + 10} + \frac{3}{8}\sinh^{-1}(3+x)$$

it's graph is below

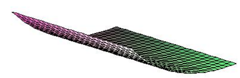

Figure 3.4: Minimal translation surface of type 2 in $(\mathcal{H}_3, g_2)$

3.2.2 Surfaces of type 3 and 4

Let the curves γ_1 and γ_2 be given by $\gamma_1(x) = (x, 0, g(x))$ and $\gamma_2(y) = (h(y), y, 0)$, where g and h are two arbitrary surfaces

Type 3

We consider the translation surface $\Sigma(\gamma_1, \gamma_2)$ parameterized by

$$X(x,y) = (x, 0, g(x)) * (h(y), y, 0) = (x + h(y), y, g(x) - xy)$$

in the Lorentzian Heisenberg $3-$space $(\mathcal{H}_3, g_2)$ obtained as the product of the two curves $\gamma_1(x) = (x, 0, g(x))$ and $\gamma_2(y) = (h(y), y, 0)$. The components of the tangent vector are given by

$$\begin{cases} X_x(x,y) = (g'(x) - y)e_1 + e_3, \\ X_y(x,y) = e_2 + h'(y)e_3. \end{cases}$$

The coefficients of the first fundamental form I of the surface Σ are

$$\begin{cases} E = & (g'(x) - y)^2 - 1 \\ F = & -h'(y) \\ G = & 1 - h'^2(y) \end{cases}$$

We have also

$$\begin{cases} \nabla_{X_x} X_x &= & g''(x)e_1 + (g'(x) - y)e_2, \\ \nabla_{X_x} X_y &= & -\frac{1}{2}e_1 + \frac{1}{2}h'(y)(g'(x) - y)e_2 - \frac{1}{2}(g'(x) - y)e_3, \\ \nabla_{X_y} X_y &= & h''(y)e_3. \end{cases}$$

The unit vector field n normal to Σ is given by

$$n = \frac{1}{W}[e_1 + h'(y)(g'(x) - y)e_2 + (g'(x) - y)e_3]$$

where

$$W = \sqrt{|(g'(x) - y)^2 [-1 + h'^2(y)] + 1|}. \quad (3.14)$$

Therefore, the coefficients of the second fundamental form II of the surface Σ are

$$\begin{cases} L &= \frac{1}{W}[g''(x) + h'(y)(g'(x) - y)^2], \\ M &= \frac{1}{2W}[h'^2(y)(g'(x) - y)^2 + (g'(x) - y)^2 - 1], \\ N &= \frac{1}{W}[-h''(y)(g'(x) - y)]. \end{cases}$$

Following the same steps as above, we get the minimality condition

$$-h''(y)(g'(x)-y)((g'(x)-y)^2-1)+g''(x)(-h'^2(y)+1)+2h'(y)(g'(x)-y)^2-h'(y) = 0. \quad (3.15)$$

Taking the derivative with respect to x, we get

$$-h''(y)[3g''(x)(g'(x)-y)^2 - g''(x)] + g'''(x)(-h'^2(y)+1) + 4g''(x)h'(y)(g'(x)-y) = 0. \quad (3.16)$$

We suppose that $g''(x) \neq 0$, we divide (3.16) by $g''(x)$ then we take the derivative with respect to x. The case $g''(x) = 0$ will be treated separately

$$h''(y)[-6g''(x)(g'(x) - y)] + (-h'^2(y) + 1) + \left[\frac{g'''(x)}{g''(x)}\right]' + 4h'(y)g''(x) = 0. \quad (3.17)$$

We repeat the same last step, we obtain

$$-6h''(y)g''(x) + (-h'^2(y) + 1)\left[\frac{\left(\frac{g'''(x)}{g''(x)}\right)'}{g''(x)}\right]' = 0. \quad (3.18)$$

Hence, we obtain the following differential system

$$\frac{h''(y)}{1 - h'^2(y)} = \frac{\left[\frac{\left(\frac{g'''(x)}{g''(x)}\right)'}{g''(x)}\right]'}{6g''(x)} = A$$

where A is a real constant. Applying the same technic already using to solve the differential system (3.6) in the case of translation surfaces of type 1, we find that the situation $\frac{h''(y)}{1-h'^2} = A$ yields to a contradiction $\forall A \in \mathbb{R}$.
In the particular case $g''(x) = 0$, we have $g(x) = ax + b$ with $a, b \in \mathbb{R}$.
Solving the differential equation obtained by replacing $g'(x)$ and $g''(x)$ in (3.15), we obtain

$$h(y) = \pm K \arctan\left(\frac{1}{\sqrt{|(a-y)^2 - 1|}}\right) + c$$

with $K \in \mathbb{R}^{*,+}$ and $a, c \in \mathbb{R}$.
Finally, we conclude that the minimality condition given by the equation (3.15) is verified if and only if $g''(x) = 0$ and we summarize with the following theorem

Theorem 3.2.3. *The minimal translation surfaces* Σ *of type* 3 *in the* $3-$*dimensional Lorentzian Heisenberg space* $(\mathcal{H}_3, g_2)$ *are parameterized by*

$$X(x,y) = (x,0,g(x)) * (h(y),y,0) = (x+h(y), y, g(x) - xy) \tag{3.19}$$

where $g(x) = ax + b$ *with* a, $b \in \mathbb{R}$, *for all* $x \in \mathbb{R}$.
And

$$h(y) = \pm K \arctan\left(\frac{1}{\sqrt{|(a-y)^2 - 1|}}\right) + c$$

with $K \in \mathbb{R}^{*,+}$ *and* $c \in \mathbb{R}$, *for all* $y \in \mathbb{R} - \{a-1, a+1\}$.

Remark 3.2.2. *The minimal translation surfaces* Σ *of type* 3 *in the* $3-$ *dimensional Lorentzian Heisenberg space* $(\mathcal{H}_3, g_2)$ *parameterized by*

$$X(x,y) = (x,0,g(x)) * (h(y),y,0) = (x+h(y), y, g(x) - xy)$$

is minimal if and only if g *is an affine function.*

Type 4

The translation surface Σ of type 4 is the product $\gamma_2(y) * \gamma_1(x)$ of the two curves $\gamma_2(y)$ and $\gamma_1(x)$ defined above so, it is parameterized by

$$X(x,y) = (x+h(y), y, g(x)).$$

Since the components of the tangent vector of this type of surfaces are given by

$$\begin{cases} X_x(x,y) = g'(x)e_1 + e_3 \\ X_y(x,y) = xe_1 + e_2 + h'(y)e_3 \end{cases}$$

Then,the coefficients of the first fundamental form I are

$$\begin{cases} E = & (g'(x))^2 - 1, \\ F = & -h'(y) + xg'(x), \\ G = & -h'^2(y) + x^2 + 1. \end{cases}$$

We have

$$\begin{cases} \nabla_{X_x} X_x &= & g''(x)e_1 + g'(x)e_2, \\ \nabla_{X_x} X_y &= & \frac{1}{2}e_1 + \frac{1}{2}(h'(y)g'(x) + x)e_2 - \frac{1}{2}g'(x)e_3, \\ \nabla_{X_y} X_y &= & xh'(y)e_2 + (h''(y) - x)e_3. \end{cases}$$

The coefficients of the second fundamental form II are

$$\begin{cases} L &= & \frac{1}{W}[g'(x)(h'(y)g'(x) - x) + g''(x)] \\ M &= & \frac{1}{2W}[1 + (g'(x)^2 h'(y)^2 - x^2) + g'(x)^2] \\ N &= & \frac{1}{W}[xh'(y)(g'(x)h'(y) - x) - g'(x)(h''(y) - x)] \end{cases}$$

where

$$W = \sqrt{|-g'(x)^2 + (g'(x)h'(y) - x)^2 + 1|},$$

is the norm of the unit vector field n normal to Σ. Hence, we have

$$n = \frac{1}{W}[e_1 + (g'(x)h'(y) - x)e_2 + g'(x)e_3].$$

Therefore, the minimality condition becomes

$$h''(y)(g'(x) - g'(x)^3) + (x^2 + 1 - h'(y)^2)g''(x) + h'(y)(2g'(x)^2 + 1) - 3xg'(x) = 0. \quad (3.20)$$

Taking the derivative with respect to y, we get

$$h'''(y)(g'(x) - g'(x)^3) - 2h''(y)h'(y)g''(x) + h''(2g'(x)^2 + 1) = 0. \quad (3.21)$$

We suppose that $h''(y) \neq 0$ and we divide (4.18) by $h''(y)$ then we take the derivative with respect to y. The case $h''(y) = 0$ will be treated separately

$$(g'(x) - g'(x)^3)\left[\frac{h'''(y)}{h''(y)}\right]' = 2h''(y)g''(x), \quad (3.22)$$

which implies the following differential system

$$\frac{g''(x)}{g'(x) - g'(x)^3} = \frac{\left[\frac{h'''(y)}{h''(y)}\right]'}{2h''(y)} = \lambda, \lambda \in \mathbb{R}$$

To solve this system, we distinguish two cases

- First, we suppose that $\lambda = 0$, then we have $g(x) = ax+b$ with a and b are real constants. Replacing in the minimality condition equation (3.20), we obtain

$$[a - a^3]h''(y) + h'(y)(a^2 + 1) = 3ax. \quad (3.23)$$

- If $\lambda \neq 0$, solving the differential equation

$$\frac{g''(x)}{g'(x) - g'(x)^3} = \lambda, \lambda \in \mathbb{R},$$

we find

$$g'(x) = \sqrt{\frac{-1}{K\exp(2\lambda x) + 1} + 1} + c,$$

with $K \in \mathbb{R}^{*,+}$ and $c \in \mathbb{R}$.

Since the function h depends only on y, this situation yields a contradiction for all $\lambda \in \mathbb{R}$. So, we only have the particular case $h''y) = 0$, which implies $h(y) = ay + b$ with a and b are real constants.

Replacing h in the minimality condition equation (3.20), we get the differential equation

$$(x^2 + 1 - a^2)g''(x) + 2ag'(x)^2 - 3xg'(x) + a = 0. \quad (3.24)$$

Theorem 3.2.4. *The minimal translation surfaces Σ of type 4 in the $3-$ dimensional Lorentz Heisenberg space $(\mathcal{H}_3, g_2)$ are parameterized by*

$$X(x, y) = (x, 0, g(x))(h(y), y, 0) = (x + h(y), y, g(x))$$

where $h(y) = ay + b$ with a and b are real constants and $g(x)$ checks the nonlinear differential equation given by:

$$(x^2 + 1 - a^2)g''(x) + 2ag'(x)^2 - 3xg'(x) + a = 0.$$

Chapter 4

Minimal translation surfaces in $(\mathcal{H}_3, g_3)$

In this Chapter, we present some results on the characterization of the curvature of translation surfaces in the 3− dimensional Lorentzian Heisenberg space H_3 endowed with the following left invariant Lorentzian metric

$$g_3 \;=\; dx^2 + (xdy + dz)^2 - [(1-x)dy - dz]^2.$$

4.1 Minimal surface equations in $(\mathcal{H}_3, g_3)$

Let Σ be a surface in the Lorentzian Heisenberg 3−space $\mathcal{H}_3$ which represents the graph of the function $z = f(x, y)$, it is parameterized by

$$\begin{array}{rccc} X: & U \subseteq \mathbb{R}^2 & \to & \mathbb{R}^3 \\ & (x,y) & \mapsto & (x, y, f(x,y)) \end{array}$$

$X(x, y) = (x, y, f(x, y))$ is the position vector. The components

$$X_x = \tfrac{\partial}{\partial x} + f_x \tfrac{\partial}{\partial z} \quad \text{and} \quad X_y = \tfrac{\partial}{\partial y} + f_y \tfrac{\partial}{\partial z}$$

of the tangent vector are given by

$$\begin{cases} X_x(x,y) = e_1 + f_x e_2 - f_x e_3, \\ X_y(x,y) = (f_y + x)e_2 + (1 - f_y - x)e_3. \end{cases}$$

Where $B = (e_1, e_2, e_3)$ given by (1.9) is a pseudo-orthonormal basis of the Lie-algebra of $(\mathcal{H}_3, g_3)$.

The coefficients of the first fundamental form I of the surface Σ are

$$E = g_3(X_x, X_x) = 1, \quad F = g_3(X_x, X_y) = f_x(x,y), \quad G = g_3(X_y, X_y) = 2(x + f_y(x,y)) - 1.$$

Using the following

$$\begin{cases} \nabla_{X_x} X_x &=& f_{xx}(x,y)e_2 - f_{xx}(x,y)e_3, \\ \nabla_{X_x} X_y &=& [f_{xy}(x,y) + 1]e_2 - [f_{xy}(x,y) + 1]e_3, \\ \nabla_{X_y} X_y &=& -e_1 + f_{yy}(x,y)e_2 - f_{yy}(x,y)e_3. \end{cases}$$

and n the unit vector field normal to Σ,

$$n = \frac{1}{W}[-f_x(x,y)e_1 + (1 - f_y(x,y) - x)e_2 + (f_y(x,y) + x)e_3],$$

where

$$W = \sqrt{|f_x^2(x,y) + 1 - 2(f_y(x,y) + x)|},$$

we calculate the coefficients of the second fundamental form II of the surface Σ

$$L = \frac{1}{W} f_{xx}(x,y), \quad M = \frac{1}{W}(1 + f_{xy}(x,y)), \quad N = \frac{1}{W}(f_x(x,y) + f_{yy}(x,y)).$$

Consequently the mean curvature H of the surface Σ in the Lorentzian Heisenberg space $(\mathcal{H}_3, g_3)$ is

$$H = \frac{1}{2W^3}\left[f_{yy}(x,y) + [-1 + 2(f_y(x,y) + x)]f_{xx}(x,y) - 2f_x(x,y)f_{xy}(x,y) - f_x(x,y)\right]. \quad (4.1)$$

Proposition 4.1.1. *The surface Σ defined above is a minimal surface in $3-$dimensional Lorentzian Heisenberg space $\mathcal{H}_3$ if and only if it's main curvature H satisfies the following condition*

$$H = \frac{1}{2W^3}\left[f_{yy}(x,y) + [-1 + 2(f_y(x,y) + x)]f_{xx}(x,y) - 2f_x(x,y)f_{xy}(x,y) - f_x(x,y)\right] = 0. \quad (4.2)$$

Remark 4.1.1. *It's clear that if $f(x,y) = -\frac{1}{2}xy + ax + by + c$ where a, b, and c are real constants, then the condition (4.2) is satisfied. Consequently the surfaces parameterized by $X(x,y) = (x, y, -\frac{1}{2}xy + ax + by + c)$ are minimal surfaces in $(\mathcal{H}_3, g_3)$.*

Example 4.1.1. *Let take $f(x,y) = -\frac{1}{2}xy + 3x - 2y + 1$, the surface in the figure below parameterized by $X(x,y) = (x, y, -\frac{1}{2}xy + 3x - 2y + 1)$ is a minimal surface in $(\mathcal{H}_3, g_3)$.*

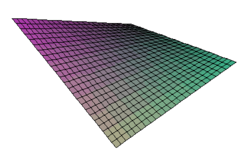

Figure 4.1: Minimal surface in $(\mathcal{H}_3, g_3)$.

4.2 Some types of minimal translation surfaces in $(\mathcal{H}_3, g_3)$

Now we classify six types of translation surfaces in the $3-$dimensional Lorentzian Heisenberg space $(\mathcal{H}_3, g_3)$ obtained as a product of two generating curves γ_1 and γ_2 which are not orthogonal.

4.2.1 Surfaces of type 1 and 2

Type 1

Let first consider a translation surface $\Sigma\,(\gamma_1, \gamma_2)$, γ_1 and γ_2 are curves given by $\gamma_1(x) = (x, 0, g(x))$ and $\gamma_2(y) = (0, y, h(y))$, where g and h are two arbitrary surfaces. $\Sigma\,(\gamma_1, \gamma_2) = \gamma_1(x) * \gamma_2\,(y)$ is parameterized by

$$X(x, y) = (x, 0, g(x)) * (0, y, h(y)) = (x, y, g(x) + h(y) - xy)\,.$$

the minimality condition given by the equation (4.2) becomes

$$h''(y) + (-1 + 2h'(y))g''(x) + (g'(x) - y) = 0 \tag{4.3}$$

Taking the derivative with respect to x of the equation (4.3), we obtain

$$1 - 2h'(y) = \frac{g''(x)}{g'''(x)}. \tag{4.4}$$

Since g depends only on x and h depends only on y, then the equation (4.4) implies the following differential system

$$\begin{cases} 1 - 2h'(y) - \lambda &= 0 \\ g''(x) - \lambda g'''(x) &= 0 \end{cases}, \tag{4.5}$$

with λ is a real constant. To solve the system (4.5), we distinguish two cases:

1/ If $\lambda = 0$, then $g''(x) = 0$, which we gives $g(x) = ax + x_0$ where a and x_0 are two real constants.
Replacing g in the equation (4.3), we obtain $h(y) = \frac{1}{6}y^3 - \frac{a}{2}y^2 + y_1 y + y_0$, where y_0, y_1 $\in \mathbb{R}$.

2/ If $\lambda \neq 0$, we have $h(y) = \frac{1}{2}\,(1 - \lambda)\,y + c_0$ where $c_0 \in \mathbb{R}$. Replacing h in the equation (4.3), we get $-\lambda g''(x) + g'(x) = y$ which is a contradiction.

Consequently, for any minimal translation surface of type 1 in the space c, we have $g''(x) = 0$. We summarize with the following theorem

Theorem 4.2.1. *The minimal translation surfaces Σ of type 1 in the 3−dimensional Lorentzian Heisenberg space $(\mathcal{H}_3, g_3)$ are parameterized by $X(x, y) = (x, y, g(x) + h(y) - xy)$ where $g(x)$, and $h(y)$ are given by:*

$$g(x) = ax + x_0$$

and

$$h(y) = \frac{1}{6}y^3 - \frac{a}{2}y^2 + y_1 y + y_0$$

where a, x_0, y_0 and y_1 are real constants.

Example 4.2.1. *The surface parameterized by*

$$X(x, y) = (x, y, -\frac{1}{6}y^3 + y^2 - 2y - x + 1 - xy)$$

is a minimal surface of type 1 in $(\mathcal{H}_3, g_3)$.

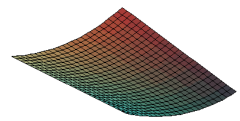

Figure 4.2: Minimal translation surface of type 1 in $(\mathcal{H}_3, g_3)$, example (4.2.1)

Type 2

Now the translation surface $\Sigma\ (\gamma_2, \gamma_1)$ of type 2 in $(\mathcal{H}_3, g_3)$ is parameterized by

$$X(x, y) = (0, y, h(y)) * (x, 0, g(x)) = (x, y, g(x) + h(y))$$

where g and h are two arbitrary surfaces. The minimality condition given by the equation (4.2) becomes

$$h''(y) + (x + 2h'(y))^2 - 1)g''(x) - g'(x) = 0. \tag{4.6}$$

We take the derivative of the equation (4.6) with respect to y, we find

$$h'''(y) + 2h''(y)g''(x) = 0. \tag{4.7}$$

We suppose that $h''(y) \neq 0$ (the case h''(y)=0 will be studied separately later), so the equation (4.7) implies the following differential system

$$\frac{-1}{2}\left[\frac{h'''(y)}{h''(y}\right] = g''(x) = \lambda, \tag{4.8}$$

where $\lambda \in \mathbb{R}$. Therefore

1/ If $\lambda = 0$ then $g(x) = ax + b$ where a and b are two real constants. Replacing in (4.6) we obtain $h(y) = \frac{1}{2}ay^2 + by + c$, with $c \in \mathbb{R}$

2/ If $\lambda \neq 0$ then $g(x) = \frac{1}{2}\lambda x^2 + c_1 x + c_0$ with c_1 and c_0 and are three real constants. Replacing this result in (4.6), we get

$$h''(y) + 2\lambda h'(y) = -\lambda x + \lambda - c_1, \tag{4.9}$$

which is a contradiction since the function $h(y)$ depends only on y and $\lambda \neq 0$.

In the particular case $h''(y) = 0$, we find $h(y) = ay + b$, with a and b are two real constants and

$$g(x) = \begin{cases} K\left[\frac{1}{3}(2x + 2a - 1)^{\frac{3}{2}} + c_0\right] & \text{if} \quad x \geq \frac{1}{2} - a \\ K\left[\frac{-1}{3}(-2x - 2a + 1)^{\frac{3}{2}} + c_0\right] & \text{if} \quad x \leq \frac{1}{2} - a \end{cases},$$

where $K \in \mathbb{R}^{*,+}$ and $c_0 \in \mathbb{R}$. Consequently, we establish the following result

Theorem 4.2.2. *The minimal translation surfaces Σ of type 2 in the 3−dimensional Lorentzian Heisenberg space $(\mathcal{H}_3, g_3)$ are parameterized by*

$$X(x,y) = (0, y, h(y)) * (x, 0, g(x)) = (x, y, g(x) + h(y))$$

where $g(x)$, and $h(y)$ are given by:

- $g(x) = ax + b$ *and* $h(y) = \frac{1}{2}ay^2 + by + c$, *where a, b and c are real constants,*
- *or* $h(y) = ay + b$ *and*

$$g(x) = \begin{cases} K\left[\frac{1}{3}(2x + 2a - 1)^{\frac{3}{2}} + c_0\right] & \text{if} \quad x \geq \frac{1}{2} - a \\ K\left[\frac{-1}{3}(-2x - 2a + 1)^{\frac{3}{2}} + c_0\right] & \text{if} \quad x \leq \frac{1}{2} - a \end{cases},$$

where $K \in \mathbb{R}^{*,+}$ and $c_0 \in \mathbb{R}$.

Example 4.2.2. *Let $f(x,y) = x + \frac{1}{2}y^2 + 3y + 4$. The surface parameterized by $X(x,y) = (x, y, f(x,y))$ is a minimal surface of type 2 in $(\mathcal{H}_3, g_3)$.*

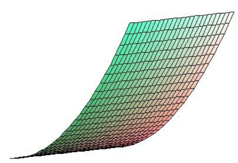

Figure 4.3: Minimal translation surface of type 2 in $(\mathcal{H}_3, g_3)$)

4.2.2 Surfaces of type 3 and 4

We consider the curves γ_1 and γ_2 given by $\gamma_1(x) = (x, 0, g(x))$ and $\gamma_2(y) = (h(y), y, 0)$, where g and h are two arbitrary surfaces

Type 3

We suppose now that the translation surface $\Sigma = \Sigma(\gamma_1, \gamma_2)$ is given by the product

$$X(x,y) = (x, 0, g(x)) * (h(y), y, 0) = (x + h(y), y, g(x) - xy)$$

so it's parameterized by

$$\begin{array}{llll} X: & U \subseteq \mathbb{R}^2 & \to & \mathbb{R}^3 \\ & (x,y) & \mapsto & (x + h(y), y, g(x) - xy) \end{array}$$

where $X(x,y) = (x + h(y), y, g(x) - xy))$ is the position vector and the components of the tangent vector are given by

$$\begin{cases} X_x(x,y) = e_1 + (g'(x) - y)e_2 - (g'(x) - y)e_3, \\ X_y(x,y) = h'(y)e_1 + e_3. \end{cases}$$

The coefficients of the first fundamental form are

$$E = g_3(X_x, X_x) = 1, \quad F = g_3(X_x, X_y) = h'(y) + g'(x) - y, \quad G = g_3(X_y, X_y) = h'^2(y) - 1.$$

Since

$$\begin{cases} \nabla_{X_x} X_x &= g''(x)e_2 - g''(x)e_3 \\ \nabla_{X_x} X_y &= 0 \\ \nabla_{X_y} X_y &= [h''(y) - 1]e_1 + h'(y)e_2 - h'(y)e_3 \end{cases}$$

and n the unit vector field normal to Σ is given by

$$n = \frac{1}{W}[(g'(x) - y)e_1 - (1 + h'(y)(g'(x) - y))e_2 + (h'(y)(g'(x) - y))e_3]$$

with

$$W = \sqrt{|[(g'(x) - y)^2(-1 + h'^2(y)) + 1|}$$

then the coefficients of the second fundamental form of Σ are

$$L = -\tfrac{1}{W} g''(x), \quad M = 0, \quad N = \tfrac{1}{W}[(h''(y) - 1)(g'(x) - y) - h'(y)].$$

We follow the same steps as the previous types to calculate the mean curvature of the translation surface Σ, we yield:

$$H = \frac{1}{2W^3}[(g'(x) - y)[(h''(y) - 1] - h'(y) - g''(x)(h'^2(y) - 1)].$$

The minimality condition $H = 0$ implies the following equation

$$(g'(x) - y)[h''(y) - 1] - h'(y) - g''(x)(h'^2(y) - 1) = 0 \tag{4.10}$$

Taking the derivative with respect to x, we obtain the following differential system

$$\frac{g'''(x)}{g''(x)} = \frac{h''(y) - 1}{h'^2(y) - 1} \tag{4.11}$$

Since the left hand side of the equality (4.11) depends only on y and the right hand side depends only on x, thus for all λ a real constant, we have

$$\frac{g'''(x)}{g''(x)} = \frac{h''(y) - 1}{h'^2(y) - 1} = \lambda, \lambda \in \mathbb{R}$$

which implies the following differential system

$$\begin{cases} g'''(x) = \lambda g''(x) \\ h''(y) - 1 = \lambda[h'^2(y) - 1] \end{cases} \tag{4.12}$$

Solving this system, we obtain first in the particular case $\lambda = 0$, $g'''(x) = 0$, then $g(x) = ax^2 + bx + c$ with a, b and c are real constants. Replacing g in the minimality condition (4.10), we obtain

$$(2ax + b - y)(h''(y) - 1) - h'(y) = 2a[(h'^2 - 1] \tag{4.13}$$

The equation (4.10) is satisfied if $h''(y) - 1 = 0$ or $a = 0$. Moreover,

1/ if $h''(y) - 1 = 0$ then we get $h'(y) = y + y_0$ where $y_0 \in \mathbb{R}$ and $2a[(h'^2 - 1] + h'(y) = 0$, which implies

$$2a[(y + y_0)^2 - 1] + (y + y_0) = 0 \tag{4.14}$$

This last equation is not true for all $y \in \mathbb{R}$, so we conclude that $h''(y) - 1 \neq 0$, consequently we have $a = 0$ and the equation (4.13) becomes

$$(b - y)h''(y) - h'(y) = b - y,$$

which has the solution

$$h(y) = \frac{1}{4}(b - y)^2 - c_1 \ln|b - y| + c_0$$

where c_1, $c_0 \in \mathbb{R}$.

2/ We suppose now that $\lambda \neq 0$. Solving the first differential equation of the system (4.12), we obtain

$$g(x) = C(x + \frac{1}{\lambda}) + C_1 \exp(\lambda x) + C_0$$

where C_1, C_0, $C \in \mathbb{R}$. Replacing g in the equation (4.10), we get

$$(C - y)(h''(y) - 1) - h'(y) = C_1 \lambda \exp(\lambda x)[1 - h''(y) + \lambda h'^2(y) - 1]$$

Since the right hand side side of this equality depends only on y and the left hand side depends on x and y then the minimality condition is satisfied if and only if $C_1 = 0$. So we get

$$(C - y)h''(y) - h'(y) = C - y$$

which is already solved above. Finally we announce the following theorem summarizing this result

Theorem 4.2.3. *The minimal translation surfaces Σ of type 3 in the 3−dimensional Lorentzian Heisenberg space $(\mathcal{H}_3, g_3)$ are parameterized by $X(x, y) = (x + h(y), y, g(x) - xy)$ where g and h are given by*

$$g(x) = bx + c$$

and

$$h(y) = \frac{1}{4}(b - y)^2 - c_1 \ln|b - y| + c_0$$

where b, c, c_1 and c_0 are real constants.

Example 4.2.3. *Let Σ be a minimal translation surface of type 3 in the space $(\mathcal{H}_3, g_3)$ parameterized by*

$$X(x, y) = (x + \frac{1}{4}(2 - y)^2 - \ln(2 - y) - 1, y, 2x - 3 - xy),$$

on the domain $D = \mathbb{R} \times]-\infty, 2[$.

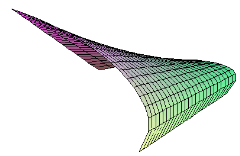

Figure 4.4: Minimal translation surface of type 3 in $(\mathcal{H}_3, g_3)$, example (4.2.3)

Type 4

Now we suppose that the translation surface $\Sigma = \Sigma(\gamma_2, \gamma_1)$ of type 4 in the space $(\mathcal{H}_3, g_3)$ is given by the product

$$X(x,y) = (x, 0, g(x)) * (h(y), y, 0) = (x + h(y), y, g(x)).$$

So it's parameterized by

$$\begin{array}{rccc} X: & U \subseteq \mathbb{R}^2 & \rightarrow & \mathbb{R}^3 \\ & (x,y) & \mapsto & (x+h(y), y, g(x)) \end{array}$$

where $X(x,y) = (x + h(y), y, g(x)))$ is the position vector and the components of the tangent vector are

$$\begin{cases} X_x(x,y) = e_1 + g'(x)e_2 - g'(x)e_3), \\ X_y(x,y) = h'(y)e_1 + xe_2 + (1-x)e_3. \end{cases}$$

The coefficients of the first fundamental form of the surface Σ are

$$\begin{array}{rcl} E & = & 1, \\ F & = & h'(y) + g'(x), \\ G & = & h'^2(y) + 2x - 1. \end{array}$$

Using ∇ the Levi-Civita connection of g_3 given by (1.10), we get

$$\begin{cases} \nabla_{X_x} X_x & = & g''(x)e_2 - g''(x)e_3, \\ \nabla_{X_x} X_y & = & e_2 - e_3, \\ \nabla_{X_y} X_y & = & [h''(y) - 1]e_1 + h'(y)e_2 - h'(y)e_3. \end{cases}$$

n the unit vector field normal to Σ is given by

$$n = \frac{1}{W}[(g'(x)e_1 + (x - h'(y)g'(x) - 1)e_2 + (-x + h'(y)(g'(x)))e_3],$$

with

$$W = \sqrt{|g'(x)^2 + (x - h'(y)g'(x) - 1)^2 - (-x + h'(y)(g'(x)))^2|}.$$

The coefficients of the second fundamental form of Σ are

$$L = \tfrac{-1}{W} g''(x), \quad M = \tfrac{-1}{W}, \quad N = \tfrac{1}{W}[(-1 + h''(y))g'(x) - h'(y)].$$

Therefore, we get the mean curvature of the translation surface Σ of type 4 in the space $(\mathcal{H}_3, g_3)$

$$H = \frac{1}{2W^3}\left[g'(x)h''(y) - (h'^2(y) + 2x - 1)g''(x) + g'(x) + h'(y)\right].$$

The minimality condition $H = 0$ yields the following equation

$$g'(x)h''(y) - (h'^2(y) + 2x - 1)g''(x) + g'(x) + h'(y) = 0 \tag{4.15}$$

Taking the derivative firstly with respect to y, we obtain

$$g'(x) + h'''(y) - 2h''(y)h'(y)g''(x) + h''(y) = 0 \tag{4.16}$$

We suppose that $h''(y) \neq 0$, we divide (4.16) by $h''(y)$, then we take the derivative with respect to y, hence we get

$$\frac{(\frac{h'''(y)}{h''(y)})'}{2h''(y)} = \frac{g''(x)}{g'(x)} \tag{4.17}$$

Since the left hand side of the equality (4.17) depends only on y and the right hand side depends only on x, thus for all λ a real constant, we have

$$\frac{(\frac{h'''(y)}{h''(y)})'}{2h''(y)} = \frac{g''(x)}{g'(x)} = \lambda,$$

which implies the following differential system

$$\begin{cases} g''(x) = \lambda g'(x) \\ \left(\frac{h'''(y)}{h''(y)}\right)' = 2\lambda h''(y) \end{cases}$$

To Solve this system we distinguish two cases

1/ If $\lambda = 0$, we find $g(x) = ax + b$, where a and b are real constants. Replacing g in the minimality condition (4.15), we obtain

$$h(y) = K_1 + K_2 \exp(-\frac{1}{a}y) + (-ay + c_0) + a^2,$$

where $a \in \mathbb{R}^*$ and K_1, K_2, $c_0 \in \mathbb{R}$.

2/ If $\lambda \neq 0$, we have

$$g(x) = \frac{K}{\lambda}\exp(\lambda x) + c,$$

with $K \in \mathbb{R}^{*,+}$ and $c \in \mathbb{R}$. Replacing this result in the minimality condition (4.15), we get a contradiction since $h(y)$ depends only on y.

Now, we study the particular case $h''(y) = 0$, which implies $h(y) = ay + b$ with a and b are two real constants. Replacing $h'(y)$ and $h''(y)$ in (4.15), we find

$$g(x) = \begin{cases} K\sqrt{a^2 + 2x - 1} + c_0 & \text{if} \quad x \geq \frac{1-a^2}{2} \\ -K\sqrt{-a^2 - 2x + 1} + c_0 & \text{if} \quad x \leq \frac{1-a^2}{2} \end{cases},$$

Theorem 4.2.4. *The minimal translation surfaces Σ of type 4 in the 3−dimensional Lorentzian Heisenberg space $(\mathcal{H}_3, g_3)$ are parameterized by $X(x, y) = (x + h(y), y, g(x))$, where*

- $g(x) = ax + b$ *and* $h(y) = K_1 + K_2 \exp(-\frac{1}{a}y) + (-ay + c_0) + a^2$, *where* $a \in \mathbb{R}^*$ *and* K_1, K_2, b, $c_0 \in \mathbb{R}$.
- $h(y) = ay + b$ *and*

$$g(x) = \begin{cases} K\sqrt{a^2 + 2x - 1} + c & \text{if } \ x \geq \frac{1-a^2}{2} \\ -K\sqrt{-a^2 - 2x + 1} + c & \text{if } \ x \leq \frac{1-a^2}{2} \end{cases},$$

where $K \in \mathbb{R}^*$ *and* a, b, $c \in \mathbb{R}$.

Example 4.2.4. *Let Σ in the figure below be a translation minimal surface of type 4 in the space $(\mathcal{H}_3, g_3)$ parameterized by*

$$X(x, y) = (x + \exp(-y) - y + 2, y, x + 2)$$

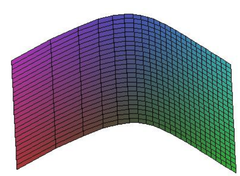

Figure 4.5: Minimal translation surface of type 4 in $(\mathcal{H}_3, g_3)$

4.2.3 Surfaces of type 5 and 6

Let the curves γ_1 and γ_2 be given by $\gamma_1(x) = (x, g(x), 0)$ and $\gamma_2(y) = (0, y, h(y))$, where g and h are two arbitrary surfaces.

Type 5

The translation surface $\Sigma = \Sigma\ (\gamma_1, \gamma_2)$ of type 5 in the 3−dimensional Lorentz Heisenberg space $(\mathcal{H}_3, g_3)$ given by the product $\Sigma = \gamma_1(x) * \gamma_2(y)$ is parameterized by

$$X(x, y) = ((x, g(x), 0) * (0, y, h(y)) = (x, y + g(x), h(y) - xy).$$

So, the position vector is

$$\begin{array}{rccc} X: & U \subseteq \mathbb{R}^2 & \rightarrow & \mathbb{R}^3 \\ & (x, y) & \mapsto & (x, y + g(x), h(y) - xy) \end{array},$$

and using the pseudo-orthonormal basis (1.9) of the Lie-algebra of $(\mathcal{H}_3, g_3)$,we compute the components of the tangent vector are

$$\begin{cases} X_x(x, y) = e_1 + (xg'(x) - y)e_2 - (xg'(x) - y - g'(x))e_3, \\ X_y(x, y) = h'(y)e_2 + [1 - h'(y)]e_3. \end{cases}$$

The coefficients of the first fundamental form of Σ are given by

$$\begin{cases} E &= 1 + g'^2(x)(2x-1) - 2yg'(x), \\ F &= -y + g'(x)[h'(y) + x - 1], \\ G &= 2h'(y) - 1. \end{cases}$$

From ∇ the Levi-Civita connection of g_3 (1.10), we have

$$\begin{cases} \nabla_{X_x} X_x &= -g'^2(x)\, e_1 + (2g'(x) + xg''(x))e_2 - [2g'(x) + xg''(x) - g''(x)]e_3, \\ \nabla_{X_x} X_y &= -g'(x)e_1, \\ \nabla_{X_y} X_y &= -e_1 + h''(y)e_2 - h''(y)e_3. \end{cases}$$

The unit vector field n normal to Σ is given by

$$n = \frac{1}{W}([g'(x)(h'(y) - x) + y]e_1 + [1 - h'(y)]e_2 + h'(y)e_3)$$

with

$$W = \sqrt{|\,[g'(x)\,(h'(y) - x) + y]^2 - 2h'(y) + 1|}.$$

The coefficients of the second fundamental form of the surface Σ are

$$\begin{cases} L &= \frac{1}{W}[-g'^3(x)(h'(y) - x) - y(g'^2(x) + 2g'(x) + g''(x)(x - h'(y)))], \\ M &= \frac{1}{W}[-g'^2(x)(h'(y) - x)], \\ N &= \frac{1}{W}[-g'(x)(h'(y) - x) + h''(y) - y]. \end{cases}$$

Since the mean curvature of the translation surface Σ of type 6 is given by

$$H = \frac{1}{2W^3}[g'^2(x)((2x-1)h''(y) - 2y + (x + h'(y) - 1)) + g'(x)(-2yh''(y) + 3h'^2 + x - 2) + g''(x)(x - h'(y))(2h'(y) - 1) - y + h''(y)]$$

then, the minimality condition yields

$$g'^2(x)\,((2x-1)h''(y) - 2y + (x + h'(y) - 1)) + g'(x)(-2yh''(y) + 3h'^2 + x - 2) + g''(x)\,(x - h'(y))(2h'(y) - 1) = y - h''(y). \tag{4.18}$$

It's clear that the right hand side of (4.18) depends on x and y and the left one depends only on y, therefore, for any minimal translation surface of type 5 in the space $(\mathcal{H}_3, g_3)$, we have $g''(x) = g'(x) = 0$.
Consequently, g is a constant function. and $h = \frac{1}{6}y^3 + y_1 y + y_0$, where y_0 and y_1 are integration constants.

Theorem 4.2.5. *The minimal translation surfaces Σ of type 5 in the 3−dimensional Lorentzian Heisenberg space $(\mathcal{H}_3, g_3)$ are parameterized by $X(x, y) = (x, y + g(x), h(y) - xy)$ where $g(x)$ is a constant function and $h(y) = \frac{1}{6}y^3 + y_1 y + y_0$, with $y_0, y_1 \in \mathbb{R}$.*

Example 4.2.5. *Let Σ be a translation surface of type 5 in the space $(\mathcal{H}_3, g_3)$ parameterized by*

$$X(x, y) = (x, y + 3, \frac{1}{6}y^3 + y - 2 - xy)$$

then, Σ is minimal.

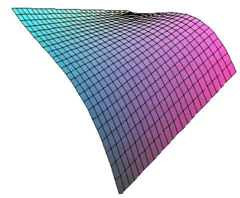

Figure 4.6: Minimal translation surface of type 5 in $(\mathcal{H}_3, g_3)$, example (4.2.5)

Type 6

The translation surface $\Sigma = \Sigma(\gamma_2, \gamma_1)$ is given the product $\Sigma(\gamma_2, \gamma_1) = \gamma_2(x) * \gamma_1(y)$, it is parameterized by

$$X(x, y) = (0, y, h(y)) * (x, g(x), 0) = (x, y + g(x), h(y))$$

so, we have

$$\begin{array}{cccc} X: & U \subseteq \mathbb{R}^2 & \to & \mathbb{R}^3 \\ & (x,y) & \mapsto & (x, y+g(x), h(y)) \end{array}$$

such that

$$\begin{cases} X_x(x,y) = e_1 + (y + xg'(x))e_2 + g'(x)e_3, \\ X_y(x,y) = (h'(y) + x)e_2 - [1 - x - h'(y)]e_3. \end{cases}$$

The coefficients of the first fundamental form of Σ are given by

$$\begin{cases} E &= 1 - g'^2(x)(1-2x), \\ F &= (2x + h'(y) - 1)g'(x), \\ G &= 2h'(y) + 2x - 1. \end{cases}$$

We have also

$$\begin{cases} \nabla_{X_x} X_x &= -g'^2(x)\, e_1 + (2g'(x) + xg''(x))e_2 - [2g'(x) + xg''(x) - g''(x)]e_3, \\ \nabla_{X_x} X_y &= -g'(x)e_1 + 2e_2 - 2e_3, \\ \nabla_{X_y} X_y &= -e_1 + h''(y)e_2 - h''(y)e_3. \end{cases}$$

n the unit vector field normal to Σ is given by

$$n = \frac{1}{W}\left[g'(x)h'(y)e_1 + (1 - x - h'(y))e_2 + (x + h'(y))e_3)\right]$$

with

$$W = \sqrt{|g'^2(x)h'^2(y) + 1 - 2(x + h'(y))|}.$$

The coefficients of the second fundamental form of the surface Σ are

$$\begin{cases} L &= \frac{1}{W}[-g'^3(x)h'(y) + 2g'(x) - h'(y)g''(x)], \\ M &= \frac{1}{W}[-g'^2 h'(y) + 1], \\ N &= \frac{1}{W}[-g'(x)h'(y) + h''(y)]. \end{cases}$$

Following the same steps as in the previous types, we get the minimality condition

$$(1 - g'^2(x) + 2xg'^2(x))h''(y) - h'(y)(2x + 2h'(y) - 1)g''(x) + g'(x) + h'(y) = 0. \quad (4.19)$$

We take the derivative of (4.19) with respect to y, we obtain

$$(1-g'^2(x)+2xg'^2(x))h'''(y)-g''(x)[h''(y)(2x+2h'(y)-1)+2h'(y)h''(y)]+g'(x)+h''(y) = 0. \quad (4.20)$$

We suppose that $h''(y) \neq 0$ (the case $h''(y) = 0$ will be studied separately later).
We divide (4.20) by $h''(y)$, then we take the derivative with respect to y. So, we find

$$(1 - g'^2(x) + 2xg'^2(x)) \left[\frac{h'''(y)}{h''(y)}\right]' = 0, \quad (4.21)$$

which implies

$$\begin{cases} 1 - g'^2(x) + 2xg'^2(x) & = & 0 \\ \text{or} & & \\ \left[\frac{h'''(y)}{h''(y)}\right]' & = & 0 \end{cases} \quad (4.22)$$

Solving the first equation of the system above, we obtain
$g(x) = (2x - 1)\sqrt{\frac{1}{2x-1}} + c$, with $c \in \mathbb{R}$. Replacing this result in the minimality condition (4.19), we get a contradiction.
$h(y)$ the solution of the second equation of the system (4.22) is given by
$h(y) = K \exp(cy) + \frac{K'}{c^2}$, where K, K' and c are real constants. Replacing $h'(y)$ and $h''(y)$ in (4.19) yields also to a contradiction.
In the particular case $h''(y) = 0$, we obtain $h(y) = ay + b$ with a and b are two real constants.
Replacing in the minimality condition (4.19), we obtain

$$g(x) = \begin{cases} \frac{K}{3}(2a + 2x - 1)^{\frac{3}{2}} + c_0 & \text{if} \quad x \geq \frac{1}{2} - a \\ \frac{-K}{3}(-2a - 2x + 1)^{\frac{3}{2}} + c_0 & \text{if} \quad x \leq \frac{1}{2} - a \end{cases},$$

where $K \in \mathbb{R}^{*,+}$ and $c_0 \in \mathbb{R}$.

Theorem 4.2.6. *The minimal translation surfaces Σ of type 6 in the 3−dimensional Lorentzian Heisenberg space $(\mathcal{H}_3, g_3)$ are parameterized by $X(x, y) = (x, y+g(x), h(y))$ where $h(y) = ay+b$ is an affine function and $g(x)$ is given by*

$$g(x) = \begin{cases} \frac{K}{3}(2a + 2x - 1)^{\frac{3}{2}} + c_0 & \text{if} \quad x \geq \frac{1}{2} - a \\ \frac{-K}{3}(-2a - 2x + 1)^{\frac{3}{2}} + c_0 & \text{if} \quad x \leq \frac{1}{2} - a \end{cases},$$

where $K \in \mathbb{R}^{*,+}$ and a, b, $c_0 \in \mathbb{R}$.

Example 4.2.6. *Let Σ be a translation surface of type 6 in the space $(\mathcal{H}_3, g_3)$ parameterized by*

$$X(x, y) = (x, y + (2x + 1)^{\frac{3}{2}} + 1, y - 2)$$

then, Σ is minimal.

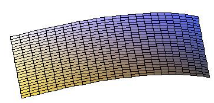

Figure 4.7: Minimal translation surface of type 6 in $(\mathcal{H}_3, g_3)$

Chapter 5

Biharmonic curves in the Lorentzian Heisenberg space $\mathcal{H}_3$

In this chapter, we will characterize the biharmonic spacelike curves in the Lorentzian Heisenberg group $\mathcal{H}_3$.
We note that, this last decade has seen several researches on the study of biharmonic spacelike curves in 3-dimensional Lorentz Heisenberg space endowed with different metrics, among which can be especially cited:

- In 2012: the paper entitled *"Characterization of spacelike biharmonic curves with timelike binormal according to flat metric in Lorentzian Heisenberg group"* ([27]), where the authors T. Körpinar and E. Turhan determined the parametric representation of the spacelike biharmonic curves with timelike binormal in Heisenberg group $\mathcal{H}_3$ endowed with flat metric (5.3).

- In 2018: J.E. Lee in his work entitled *"Biharmonic spacelike curves in Lorantzian Heisenberg space"*([28]) showed that proper biharmonic spacelike curve γ in Lorentzian Heisenberg space $(\mathcal{H}_3, g)$, where g is a Lorentzian metric given by

$$g = dx^2 + dy^2 - (dz + (ydx - xdy))^2$$

  is pseudo helix. He also proved that γ has the spacelike normal vector field and he found its parametric equations.

Since the 3-dimensional Lorentz Heisenberg space $\mathcal{H}_3$ has three classes of left invariant Lorentzian metrics, therefore, we will study the biharmonic character of curves in three cases.

1/ First, in $\mathcal{H}_3$ endowed with the left invariant Lorentzian metric

$$g_1 = -dx^2 + dy^2 + (xdy + dz)^2. \tag{5.1}$$

2/ Secondly, in $\mathcal{H}_3$ endowed with the left invariant Lorentzian metric

$$g_2 = dx^2 + dy^2 - (xdy + dz)^2. \tag{5.2}$$

3/ Finally, in $\mathcal{H}_3$ endowed with the left invariant Lorentzian metric

$$g_3 \quad = \quad dx^2 + (xdy + dz)^2 - [(1-x)dy - dz]^2. \tag{5.3}$$

The last case is the subject of the paper ([27]).
Before starting, let recall some definitions.

Definition 5.0.1. *Let (N, h) between two Riemannian manifolds. A map φ is called harmonic if it is a critical point of the energy functional*

$$E(\varphi, M) = \frac{1}{2}\int_M |d\varphi|^2 dx \tag{5.4}$$

A natural generalization of harmonic maps is given by considering the functional obtained integrating the square of the norm of the tension field

$$\tau(\varphi) = Trace_g \nabla d\varphi \tag{5.5}$$

on a compact subset Ω of M, more precisely φ is biharmonic map if and only if its bitension field vanishes identically. The Euler-Lagrange of this functional gives the biharmonic map equation

$$\tau_2(\varphi) := Trace_g(\nabla^\varphi\nabla^\varphi - \nabla^\varphi_{\nabla^M})\tau(\varphi) - Trace_g R^N(d\varphi, \tau(\varphi))d\varphi = 0, \tag{5.6}$$

where $R^N(X, Y) = [\nabla^N_X, \nabla^N_Y] - \nabla^N_{[X,Y]}$ is the curvature operator on N.
From the equation (5.5) we can see clearly that any harmonic map is biharmonic. The proper biharmonic maps are biharmonic but nonharmonic maps.
Let I be an open interval of $\mathbb{R}$ and $\gamma : I \to (H_3, g_i)$ be a curve parameterized by arc length $s \in I$ such that the geodesic curvature of γ at s given by $H(s) = |\gamma''(s)|$ is no-zero. We shall denote by $t(s) = \gamma'(s)$ the unit tangent vector field of γ at s. Thus $t'(s) = H(s)n(s)$ where $n(s)$ is a unit normal vector in the direction of $\gamma''(s)$. The unit vector $b(s) = t(s) \wedge n(s)$ is normal to the osculating plane and will be called the binormal vector at s. So for each value of the parameter s the three orthogonal unit vectors $t(s), n(s), b(s)$ formed the Frenet trihedron of γ at s which satisfies the following formulas

$$\left\{\begin{array}{ccc} t & = & \gamma' = \frac{d\gamma}{ds}, \\ \nabla^\gamma_s t & = & Hn, \\ \nabla^\gamma_s n & = & -Ht + Kb, \\ \nabla^\gamma_s b & = & -Kn, \end{array}\right. \tag{5.7}$$

where ∇^γ is the connection on the pull-back bundle $\gamma^{-1}(tH_3)$, H is the curvature of γ and K is its torsion function. We have also $t = t_1e_1 + t_2e_2 + t_3e_3$, $n = n_1e_1 + n_2e_2 + n_3e_3$ and $b = b_1e_1 + b_2e_2 + b_3e_3$ with $\{e_1, e_2, e_3\} = B$ is the left invariant pseudo-orthonormal basis with respect to the metric (5.1), (5.2) or (5.3) such that

$$\left\{\begin{array}{llll} g_i(t,t) & = g_i(n,n) & = 1, & g_i(b,b) \;\; = -1 \\ g_i(t,n) & = g_i(t,b) & = g_i(n,b) & = 0 \end{array}\right. , \; for \; i = 1, 2, 3.$$

In these cases the tension field defined in (5.5) becomes $\tau(\gamma) = \nabla_s^\gamma t$ where $t = \gamma'$ and the biharmonic equation (5.6) is reduced to

$$\tau_2(\gamma) := \nabla_s^3 t - R(t, \nabla_s^\gamma t)t = 0. \tag{5.8}$$

In all that follows, let I be an open interval of $\mathbb{R}$ and $\gamma : I \to (\mathcal{H}_3, g_i)$ be a curve parameterized by arc length $s \in I$. The Frenet trihedron of γ at s is formed of the vectors $\{t(s), n(s), b(s)\}$ where $t(s)$ is the unit tangent vector field of γ at s, $n(s)$ is a unit normal vector of γ at s in the direction of $\gamma''(s)$, and $b(s)$ is the binormal vector of γ at s, such that $b(s) = t(s) \wedge n(s)$, with $\wedge$ is the Lorentzian exterior product (vector product) given by

$$X \wedge Y = (x_2 y_3 - x_3 y_2)\, e_1 + (x_3 y_1 - x_1 y_3)\, e_2 + (x_2 y_1 - x_1 y_2)\, e_3,$$

for all tangent vectors

$$X = x_1 e_1 + x_2 e_2 + x_3 e_3 \text{ and } Y = y_1 e_1 + y_2 e_2 + y_3 e_3.$$

Definition 5.0.2. *An arbitrary curve $\gamma : I \to (H_3, g_i)$ is spacelike, timelike or null, if all of its velocity vectors $\gamma'(s)$ (tangent vectors) are respectively, spacelike, timelike or null, for each $s \in I$.*

5.1 Biharmonic curves in the space $(\mathcal{H}_3, g_1)$

We start by the characterization of biharmonic curves in the Lorentzian Heisenberg group $(\mathcal{H}_3, g_1)$.

Theorem 5.1.1. *If $\gamma : I \subseteq \mathbb{R} \to (\mathcal{H}_3, g_1)$ is a unit speed spacelike biharmonic curve according to the metric g_1 (5.1), then*

$$\begin{cases} H = \text{constant} \neq 0 \\ K^2 + H^2 = b_1(-\frac{1}{2}n_1' + \frac{3}{4}b_1) + b_2(\frac{1}{2}n_2' - \frac{1}{4}b_2) + b_3(-\frac{1}{2}n_3' + \frac{1}{4}b_3) \\ K' = -b_1 n_1 + \frac{1}{2}(n_1' n_1 - n_2' n_2 + n_3' n_3) \end{cases} \tag{5.9}$$

Proof. The biharmonic equation (5.8) implies

$$(-3H'H)t + (H''^2 - K^2 H - H^3)n + (2H'K + K'H)b - HR(t,n)t = 0. \tag{5.10}$$

First, we calculate $R(t,n)t$, we have

$$R(t,n)t = [\nabla_t, \nabla_n]t - \nabla_{[t,n]}t.$$

Using the Levi-Civita connection ∇ of g_1(1.6), we get

$$\begin{cases} \nabla_{t_1 e_1 + t_2 e_2 + t_3 e_3} e_1 = \frac{1}{2}(t_2 e_3 + t_3 e_2), \\ \nabla_{t_1 e_1 + t_2 e_2 + t_3 e_3} e_2 = \frac{1}{2}(t_1 e_3 - t_3 e_1), \\ \nabla_{t_1 e_1 + t_2 e_2 + t_3 e_3} e_3 = \frac{1}{2}(t_1 e_2 + t_2 e_1), \end{cases} \tag{5.11}$$

then,

$$\nabla_t^t = \nabla_{t_1 e_1 + t_2 e_2 + t_3 e_3}(t_1 e_1 + t_2 e_2 + t_3 e_3) = t_1' e_1 + t_1 \nabla_t e_1 + t_2' e_2 + t_2 \nabla_t e_2 + t_3' e_3 + t_3 \nabla_t e_3$$

which we gives

$$\nabla_t^t = t_1'e_1 + (t_2' + t_1t_3)e_2 + (t_3' + t_1t_2)e_3. \tag{5.12}$$

We have also

$$\begin{cases} \nabla_{n_1e_1+n_2e_2+n_3e_3}e_1 &= \frac{1}{2}(n_2e_3 + n_3e_2), \\ \nabla_{n_1e_1+n_2e_2+n_3e_3}e_2 &= \frac{1}{2}(n_1e_3 - n_3e_1), \\ \nabla_{n_1e_1+n_2e_2+n_3e_3}e_3 &= \frac{1}{2}(n_1e_2 + n_2e,), \end{cases} \tag{5.13}$$

then

$$\nabla_{n_1e_1+n_2e_2+n_3e_3}(\nabla_t t) = t_1'\nabla_{n_1e_1+n_2e_2+n_3e_3}e_1 + (t_2'+t_1t_3)\nabla_{n_1e_1+n_2e_2+n_3e_3}e_2 + (t_3'+t_1t_2)\nabla_{n_1e_1+n_2e_2+n_3e_3}e_3. \tag{5.14}$$

The expression (5.14) implies

$$\nabla_n(\nabla_t t) = \frac{1}{2}[(n_2t_3'+t_1t_2n_2-n_3t_2'-t_1t_3n_3)e_1+(t_1'n_3+t_3'n_1+t_1t_2n_1)e_2+(t_1'n_2+t_2'n_1+t_1t_3n_1)e_3]. \tag{5.15}$$

Since

$$\nabla_n t = t_1\nabla_n e_1 + t_2\nabla_n e_2 + t_3\nabla_n e_3,$$

then using (5.14), we obtain

$$\nabla_n t = \frac{1}{2}[(n_2t_3 - n_3t_2)e_1 + (t_1n_3 + t_3n_1)e_2 + (t_1n_2 + t_2n_1)e_3].$$

Therefore

$$\begin{array}{rl} \nabla_t(\nabla_n t) = & \frac{1}{2}[(n_2t_3 - n_3t_2)'e_1 + (n_2t_3 - n_3t_2)\nabla_t e_1] \\ + & \frac{1}{2}[(t_1n_3 + t_3n_1)'e_2 + (t_1n_3 + t_3n_1)\nabla_t e_2] \\ + & \frac{1}{2}[(t_1n_2 + t_2n_1)'e_3 + (t_1n_2 + t_2n_1)\nabla_t e_3] \end{array} \tag{5.16}$$

and we get

$$\begin{array}{rl} \nabla_t(\nabla_n t) = & \frac{1}{4}[2(n_2't_3 + n_2t_3' - n_3't_2 - n_3t_2') + t_2(t_1n_2 + t_2n_1) - t_3(t_1n_3 + t_3n_1)]e_1 \\ + & \frac{1}{4}[2(t_1'n_3 + t_1n_3' + t_3'n_1 - t_3n_1') + (t_1(t_1n_2 + t_2n_1) + t_3(n_2t_3 - n_3t_2)]e_2 \\ + & \frac{1}{4}[2(t_1'n_2 + t_1n_2' + t_2'n_1 + t_2n_1') + t_2(n_2t_3 - n_3t_2) + t_1(t_1n_3 + t_3n_1)]e_3. \end{array} \tag{5.17}$$

From (5.15) and (5.17), we have

$$\begin{array}{rl} \nabla_t(\nabla_n t) - \nabla_n(\nabla_t t) = & [\frac{1}{2}(n_2't_3 - n_3't_2) + \frac{1}{4}(t_1t_3n_3 + n_1t_2^2 - n_1t_3^2 - t_1t_2n_2)]e_1 \\ + & [\frac{1}{2}(t_1n_3' + t_3n_1') + \frac{1}{4}(t_1^2n_2 + n_2t_3^2 - n_3t_2t_3 - t_1t_2n_1)]e_2 \\ + & [\frac{1}{2}(t_1n_2' + t_2n_1') + \frac{1}{4}(t_2n_2t_3 + t_1^2n_3 - n_3t_2^2 - t_1t_3n_1)]e_3 \end{array} \tag{5.18}$$

We compute now $\nabla_{[t,n]}t$, we have

$$[t,n] = (t_2n_3 - t_3n_2)e_1$$

and

$$\nabla_{[t,n]}t = \frac{1}{2}[t_3(t_2n_3 - t_3n_2)e_2 + t_2(t_2n_3 - t_3n_2)e_3] \tag{5.19}$$

Using (5.18) and (5.19), we find

$$\begin{array}{rl} R(t,n)t = & [\frac{1}{2}(n_2't_3 - n_3't_2) + \frac{1}{4}(-t_3b_2 + t_2b_3)]e_1 \\ + & [\frac{1}{2}(t_1n_3' + t_3n_1') + \frac{1}{4}(-t_1b_3 - 3t_3b_1)]e_2 \\ + & [\frac{1}{2}(t_1n_2' + t_2n_1') + \frac{1}{4}(-t_1b_2 - 3t_2b_1]e_3. \end{array} \tag{5.20}$$

From (5.20), we have

$$\begin{cases} R(t,n,t,t) &= 0, \\ R(t,n,t,n) &= b_1(-\frac{1}{2}n_1' + \frac{3}{4}b_1) + b_2(\frac{1}{2}n_2' - \frac{1}{4}b_2) + b_3(-\frac{1}{2}n_3' + \frac{1}{4}b_3), \\ R(t,n,t,b) &= -b_1n_1 + \frac{1}{2}(n_1'n_1 - n_2'n_2 + n_3'n_3). \end{cases} \tag{5.21}$$

Replacing (5.21) in (5.10), we get the following system

$$\begin{cases} -3H'H &= 0, \\ H''^2 - K^2H - H^3 &= H[b_1(-\frac{1}{2}n_1' + \frac{3}{4}b_1) + b_2(\frac{1}{2}n_2' - \frac{1}{4}b_2) + b_3(-\frac{1}{2}n_3' + \frac{1}{4}b_3)], \\ 2H'K + K'H &= H[-b_1n_1 + \frac{1}{2}(n_1'n_1 - n_2'n_2 + n_3'n_3)], \end{cases}$$

which implies

$$\begin{cases} H &= constant \neq 0, \\ K^2 + H^2 &= b_1(-\frac{1}{2}n_1' + \frac{3}{4}b_1) + b_2(\frac{1}{2}n_2' - \frac{1}{4}b_2) + b_3(-\frac{1}{2}n_3' + \frac{1}{4}b_3), \\ K' &= -b_1n_1 + \frac{1}{2}(n_1'n_1 - n_2'n_2 + n_3'n_3). \end{cases}$$

Consequently the proof of the theorem is completed.

As a result of this theorem, we have the following corollary

Corollary 5.1.1. *If $\gamma : I \to (H_3, g_1)$ is a unit speed spacelike biharmonic curve according to the metric g_1, then*

$$\begin{cases} H &= constant \neq 0, \\ K &= \frac{[b_1(-\frac{1}{2}n_1' + \frac{3}{4}b_1) + b_2(\frac{1}{2}n_2' - \frac{1}{4}b_2) + b_3(-\frac{1}{2}n_3' + \frac{1}{4}b_3)]'}{-2b_1n_1 + (n_1'n_1 - n_2'n_2 + n_3'n_3)}, \\ K^2 + H^2 &= b_1(-\frac{1}{2}n_1' + \frac{3}{4}b_1) + b_2(\frac{1}{2}n_2' - \frac{1}{4}b_2) + b_3(-\frac{1}{2}n_3' + \frac{1}{4}b_3), \end{cases}$$

with $b_1n_1 \neq \frac{1}{2}(n_1'n_1 - n_2'n_2 + n_3'n_3)$. In the particular case, where

$$b_1n_1 = \frac{1}{2}(n_1'n_1 - n_2'n_2 + n_3'n_3),$$

we get

$$\begin{cases} H &= constant \neq 0, \\ K &= constant, \\ K^2 + H^2 &= b_1(-\frac{1}{2}n_1' + \frac{3}{4}b_1) + b_2(\frac{1}{2}n_2' - \frac{1}{4}b_2) + b_3(-\frac{1}{2}n_3' + \frac{1}{4}b_3). \end{cases}$$

Therefore, γ is a helix.

5.2 Biharmonic curves in the space $(\mathcal{H}_3, g_2)$

Similar to the method applied above and following the same steps, we will study characterization of speed spacelike biharmonic curve with timelike binormal in Lorentzian Heisenberg group $(\mathcal{H}_3, g_2)$.

Theorem 5.2.1. *If $\gamma : I \subseteq \mathbb{R} \to (\mathcal{H}_3, g_2)$ is a unit speed spacelike biharmonic curve according to the metric g_2 (5.2), then*

$$\begin{cases} H &= constant \quad \neq 0, \\ K' &= R(t,n,t,b), \\ K^2+H^2 &= R(t,n,t,n). \end{cases} \tag{5.22}$$

with

$$\begin{aligned} R(t,n,t,n) &= n_1n_2t_3' + \frac{1}{2}n_2'(t_3n_1+t_1n_3) - \frac{1}{2}n_3'b_3 + \frac{1}{2}n_1'(t_3n_2+t_2n_3) \\ &\quad -\frac{1}{2}n_1n_2t_1t_2 + \frac{1}{4}t_3^2\left(3n_2^2-n_1^2\right) - \frac{1}{4}t_2^2\left(3n_3^2+n_1^2\right) + \frac{1}{4}t_1^2\left(n_3^2+n_2^2\right), \end{aligned}$$

and

$$\begin{aligned} R(t,n,t,b) &= b_1n_2t_3' + \frac{1}{2}n_2'(t_3b_1+t_1b_3) + \frac{1}{2}n_1'n_1 + \frac{1}{2}n_3'n_3 + \frac{3}{4}t_2n_2(t_3b_3-t_1b_1) \\ &\quad -\frac{1}{4}t_3t_1t_2 + \frac{1}{4}t_3^2(3b_2n_2-b_1n_1) - \frac{1}{4}t_2^2(-3b_3n_3+b_1n_1) + \frac{1}{4}t_1^2(b_3n_3-b_2n_2) \\ &\quad +\frac{1}{4}t_2b_2(-3t_3n_3+t_1n_1). \end{aligned}$$

Proof. The biharmonic equation (5.8) implies

$$(-3H'H)t + (H''^2 - K^2H - H^3)n + (2H'K + K'H)b = HR(t,n)t. \tag{5.23}$$

First, we calculate $R(t,n)t$, we have

$$R(t,n)t = [\nabla_t, \nabla_n]t - \nabla_{[t,n]}t.$$

Using the Levi-Civita connection ∇ of g_2 (1.8), we get

$$\begin{cases} \nabla_{t_1e_1+t_2e_2+t_3e_3}e_1 &= \frac{1}{2}(t_3e_2 - t_2e_3), \\ \nabla_{t_1e_1+t_2e_2+t_3e_3}e_2 &= -\frac{1}{2}(t_1e_3 + t_3e_1), \\ \nabla_{t_1e_1+t_2e_2+t_3e_3}e_3 &= \frac{1}{2}(t_1e_2 + t_2e_1), \end{cases} \tag{5.24}$$

then, we have

$$\nabla_t^t = t_1'e_1 + (t_2' + t_1t_3)e_2 + (t_3' - t_1t_2)e_3. \tag{5.25}$$

We have also

$$\begin{cases} \nabla_{n_1e_1+n_2e_2+n_3e_3}e_1 &= \frac{1}{2}(n_3e_2 - n_2e_3), \\ \nabla_{n_1e_1+n_2e_2+n_3e_3}e_2 &= -\frac{1}{2}(n_1e_3 + n_3e_1), \\ \nabla_{n_1e_1+n_2e_2+n_3e_3}e_3 &= \frac{1}{2}(n_1e_2 + n_2e_1). \end{cases} \tag{5.26}$$

Therefore, from (5.25) and (5.26) we get

$$\nabla_n(\nabla_t t) = \frac{1}{2}[(-n_2t_3'+t_1t_2n_2-n_3t_2'-t_1t_3n_3)e_1+(t_1'n_3+t_3'n_1-t_1t_2n_1)e_2-(t_1'n_2+t_2'n_1+t_1t_3n_1)e_3]. \tag{5.27}$$

Since

$$\nabla_n t = t_1\nabla_n e_1 + t_2\nabla_n e_2 + t_3\nabla_n e_3,$$

then using (5.26), we obtain

$$\nabla_n t = \frac{1}{2}[(n_2t_3 - n_3t_2)e_1 + (t_1n_3 + t_3n_1)e_2 - (t_1n_2 + t_2n_1)e_3].$$

Therefore

$$\begin{array}{rl} \nabla_t(\nabla_n t) = & \frac{1}{2}[(n_2t_3 - n_3t_2)'e_1 + (n_2t_3 - n_3t_2)\nabla_t e_1] \\ + & \frac{1}{2}[(t_1n_3 + t_3n_1)'e_2 + (t_1n_3 + t_3n_1)\nabla_t e_2] \\ - & \frac{1}{2}[(t_1n_2 + t_2n_1)'e_3 + (t_1n_2 + t_2n_1)\nabla_t e_3] \end{array} \tag{5.28}$$

and we get

$$\begin{array}{rl} \nabla_t(\nabla_n t) = & \frac{1}{4}[2(n_2't_3 + n_2t_3' - n_3't_2 - n_3t_2') - t_2(t_1n_2 + t_2n_1) - t_3(t_1n_3 + t_3n_1)]e_1 \\ + & \frac{1}{4}[2(t_1'n_3 + t_1n_3' + t_3'n_1 + t_3n_1') - (t_1(t_1n_2 + t_2n_1) + t_3(n_2t_3 - n_3t_2)]e_2 \\ + & \frac{1}{4}[-2(t_1'n_2 + t_1n_2' + t_2'n_1 + t_2n_1') - t_2(n_2t_3 - n_3t_2) - t_1(t_1n_3 + t_3n_1)]e_3. \end{array} \tag{5.29}$$

From (5.27) and (5.29), we have

$$\begin{array}{rl} \nabla_t(\nabla_n t) - \nabla_n(\nabla_t t) = & \left[n_2t_3' + \frac{1}{2}\left(n_2't_3 - n_3't_2\right) - \frac{3}{4}t_1t_2n_2 + \frac{1}{4}t_1t_3n_3 - \frac{1}{4}n_1\left(t_2^2 + t_3^2\right)\right] e_1 \\ + & \left[\frac{1}{2}(t_1n_3' + t_3n_1') + \frac{1}{4}(-t_1^2n_2 + n_2t_3^2 - n_3t_2t_3 + t_1t_2n_1)\right] e_2 \\ + & \left[-\frac{1}{2}(t_1n_2' + t_2n_1') + \frac{1}{4}(-t_2n_2t_3 - t_1^2n_3 + n_3t_2^2 + t_1t_3n_1)\right] e_3. \end{array} \tag{5.30}$$

We compute now $\nabla_{[t,n]}t$, where

$$[t, n] = (t_2n_3 - t_3n_2)e_1,$$

we obtain

$$\nabla_{[t,n]}t = \frac{1}{2}(t_2n_3 - t_3n_2)[t_3e_2 - t_2e_3] \tag{5.31}$$

Using (5.30) and (5.31), we find

$$\begin{array}{rl} R(t,n)t = & \left[n_2t_3' + \frac{1}{2}\left(n_2't_3 - n_3't_2\right) - \frac{3}{4}t_1t_2n_2 + \frac{1}{4}t_3\left(t_1n_3 - n_1t_3\right) - \frac{1}{4}n_1t_2^2\right] e_1 \\ + & \left[\frac{1}{2}(t_1n_3' + t_3n_1') + \frac{1}{4}t_1(t_2n_1 - t_1n_2) + \frac{3}{4}t_3\left(n_2t_3 - t_2n_3\right)\right] e_2 \\ + & \left[-\frac{1}{2}(t_1n_2' + t_2n_1') + \frac{1}{4}t_1(t_3n_1 - t_1n_3) + \frac{3}{4}t_2(n_3t_2 - n_2t_3)\right] e_3 \end{array} \tag{5.32}$$

From (5.32), we have

$$R(t,n,t,t) = t_1\left(n_2t_3' + t_3n_2'\right) + t_3t_2n_1' - \frac{3}{2}t_2t_3b_1 - \frac{1}{2}t_1t_3b_2 - t_1^2t_2n_2, \tag{5.33}$$

$$\begin{array}{rcl} R(t,n,t,n) & = & n_1n_2t_3' + \frac{1}{2}n_2'\left(t_3n_1 + t_1n_3\right) - \frac{1}{2}n_3'b_3 + \frac{1}{2}n_1'\left(t_3n_2 + t_2n_3\right) \\ & & -\frac{1}{2}n_1n_2t_1t_2 + \frac{1}{4}t_3^2\left(3n_2^2 - n_1^2\right) - \frac{1}{4}t_2^2\left(3n_3^2 + n_1^2\right) + \frac{1}{4}t_1^2\left(n_3^2 + n_2^2\right), \end{array} \tag{5.34}$$

and

$$R(t,n,t,b) = b_1 n_2 t_3' + \frac{1}{2} n_2' (t_3 b_1 + t_1 b_3) + \frac{1}{2} n_1' n_1 + \frac{1}{2} n_3' n_3 + \frac{3}{4} t_2 n_2 (t_3 b_3 - t_1 b_1) - \frac{1}{4} t_3 t_1 t_2 + \frac{1}{4} t_3^2 (3 b_2 n_2 - b_1 n_1) - \frac{1}{4} t_2^2 (-3 b_3 n_3 + b_1 n_1) + \frac{1}{4} t_1^2 (b_3 n_3 - b_2 n_2) + \frac{1}{4} t_2 b_2 (-3 t_3 n_3 + t_1 n_1). \tag{5.35}$$

From (5.23) and using (5.33), (5.34) and (5.35), we get the following system

$$\begin{cases} H' = -\frac{1}{3}\left[t_1 (n_2 t_3' + t_3 n_2') + t_3 t_2 n_1' - \frac{3}{2} t_2 t_3 b_1 - \frac{1}{2} t_1 t_3 b_2 - t_1^2 t_2 n_2\right], \\ H''^2 - K^2 H - H^3 = \left[\begin{array}{c} n_1 n_2 t_3' + \frac{1}{2} n_2' (t_3 n_1 + t_1 n_3) - \frac{1}{2} n_3' b_3 + \frac{1}{2} n_1' (t_3 n_2 + t_2 n_3) \\ -\frac{1}{2} n_1 n_2 t_1 t_2 + \frac{1}{4} t_3^2 (3 n_2^2 - n_1^2) - \frac{1}{4} t_2^2 (3 n_3^2 + n_1^2) + \frac{1}{4} t_1^2 (n_3^2 + n_2^2) \end{array}\right], \\ 2H'K + K'H = H\left[\begin{array}{c} b_1 n_2 t_3' + \frac{1}{2} n_2' (t_3 b_1 + t_1 b_3) + \frac{1}{2} n_1' n_1 + \frac{1}{2} n_3' n_3 + \frac{3}{4} t_2 n_2 (t_3 b_3 - t_1 b_1) \\ -\frac{1}{4} t_3 t_1 t_2 + \frac{1}{4} t_3^2 (3 b_2 n_2 - b_1 n_1) - \frac{1}{4} t_2^2 (-3 b_3 n_3 + b_1 n_1) + \frac{1}{4} t_1^2 (b_3 n_3 - b_2 n_2) \\ + \frac{1}{4} t_2 b_2 (-3 t_3 n_3 + t_1 n_1). \end{array}\right] \end{cases}$$

Consequently,

$$\begin{cases} H = \mathit{constant} \neq 0, \\ K' = -R(t,n,t,b), \\ K^2 + H^2 = R(t,n,t,n), \end{cases}$$

where $R(t,n,t,n)$ and $R(t,n,t,b)$ are given above by the equations (5.34) and (5.35) respectively.

Corollary 5.2.1. *If $\gamma : I \to (H_3, g_2)$ is a unit speed spacelike biharmonic curve according to the metric g_2, and $R(t,n,t,b) = 0$, then*

$$\begin{cases} H = \mathit{constant} \neq 0, \\ K = \mathit{constant}, \\ K^2 + H^2 = R(t,n,t,n), \end{cases}$$

and γ is a helix.

5.3 Biharmonic curves in the space $(\mathcal{H}_3, g_3)$

Recently (in 2012), T. Körpinar and E. Turhan studied spacelike biharmonic curves with timelike binormal in Heisenberg group $\mathcal{H}_3$ endowed with flat metric (5.3). They got the following results

Theorem 5.3.1. *([27]) Let $\gamma : I \to \mathcal{H}_3$ be a unit speed spacelike curve with timelike binormal and $\{t,n,b\}$ are Frenet vector fields satisfiing Frenet formulas*

$$\begin{cases} \nabla_s^\gamma t = Hn, \\ \nabla_s^\gamma n = -Ht + Kb, \\ \nabla_s^\gamma b = -Kn. \end{cases}$$

If γ is a unit speed spacelike biharmonic curve with timelike binormal according to flat metric, then

$$\begin{cases} H &= \textit{constant} \neq 0, \\ H^2 - K^2 &= 0, \\ K &= \textit{constant}. \end{cases}$$

Corollary 5.3.1. *([27]) If $\gamma : I \to \mathcal{H}_3$ is a unit speed spacelike biharmonic curve with timelike binormal according to flat metric, then $H = \mp K$.*

Corollary 5.3.2. *([27]) If $\gamma : I \to \mathcal{H}_3$ is a unit speed spacelike biharmonic curve with timelike binormal according to flat metric, then γ is a helix.*

The authors of the paper ([27]) also determined the parametric representation of the spacelike biharmonic curves with timelike binormal according to flat metric (5.3).

Theorem 5.3.2. *([27]) Let $\gamma : I \to \mathcal{H}_3$ is a unit speed spacelike biharmonic curve with timelike binormal according to flat metric (5.3). Then the parametric equations of γ are*

$$\begin{aligned} x(s) &= \cosh\varphi s + C_1, \\ y(s) &= \frac{1}{H}\sinh^2\varphi\left[\cosh\left(\frac{Hs}{\sinh\varphi} + C_1\right) + \sinh\left(\frac{Hs}{\sinh\varphi} + C_1\right)\right] + C_2, \\ z(s) &= -\frac{(-1 + C_1 + \cosh\varphi s)\sinh^2\varphi}{H}\cosh\left(\frac{Hs}{\sinh\varphi} + C_1\right) \\ &\quad + \frac{\sinh^2\varphi\cosh\varphi}{H^2}\left[\sinh\left(\frac{Hs}{\sinh\varphi} + C_1\right) + \cosh\left(\frac{Hs}{\sinh\varphi} + C_1\right)\right] \\ &\quad - \frac{\sinh\varphi\,(\cosh\varphi s + C_1)}{H}\sinh\left(\frac{Hs}{\sinh\varphi} + C_1\right) + C_3, \end{aligned}$$

where C_1, C_2, C_3 are constants of integration and φ is constant angle.

Conclusion

This thesis is a short exposition of some geometrical structures in 3-dimensional Lorentzian Heisenberg spaces $\mathcal{H}_3$ endowed with a left invariant metric g_i, $i = 1, 2, 3$. We have given characterization and explicit examples of 4 types of minimal translation surfaces in the space $\mathcal{H}_3$ according to the left invariant metrics g_1 and g_2 respectively and 6 types in the same space endowed with the left invariant metric g_3.

We have also study the biharmonic caractere of unit speed spacelike curves according to the three metrics g_1, g_2 and g_3 respectively.

To enrich our study on the geometry of surfaces in the 3-dimensional Lorentzian Heisenberg spaces $\mathcal{H}_3$, we will focus in our next work on surfaces with mean curvature and Gauss curvature both constant.

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