CHEBYSHEV TYPE INEQUALITIES FOR GENERALIZED STOCHASTIC FRACTIONAL INTEGRALS

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Abstract. The concept of comonotonic stochastic processes was introduced by Agahi and Yadollahzadeh. In this paper, we establish some Chebyshev type inequalities for comonotonic stochastic processes via generalized mean-square fractional integrals $\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}$ and $\mathcal{J}^{\sigma}_{\rho,\lambda,b-;\omega}$ which were introduced by Budak and Sarikaya.

1. Introduction

In 1980, Nikodem [13] introduced convex stochastic processes and investigated their regularity properties. In 1992, Skwronski [18] obtained some further results on convex stochastic processes.

Let $(\Omega,\mathcal{A},P)$ be an arbitrary probability space. A function $X:\Omega\to\mathbb{R}$ is called a random variable if it is $\mathcal{A}-$measurable. A function $X:I\times\Omega\to\mathbb{R}$, where $I\subset\mathbb{R}$ is an interval, is called a stochastic process if for every $t\in I$ the function $X(t,.)$ is a random variable.

Recall that the stochastic process $X:I\times\Omega\to\mathbb{R}$ is called

(i) continuous in probability in interval I, if for all $t_0\in I$ we have

$$P-\lim_{t\to t_0} X(t,.)=X(t_0,.),$$

where $P-\lim$ denotes the limit in probability.

(ii) *mean-square continuous* in the interval I, if for all $t_0\in I$

$$\lim_{t\to t_0} E\left[(X(t)-X(t_0))^2\right]=0,$$

where $E[X(t)]$ denotes the expectation value of the random variable $X(t,.)$.

Obviously, *mean-square* continuity implies continuity in probability, but the converse implication is not true.

Definition 1. *Suppose we are given a sequence $\{\Delta^m\}$ of partitions, $\Delta^m=\{a_{m,0},...,a_{m,n_m}\}$. We say that the sequence $\{\Delta^m\}$ is a normal sequence of partitions if the length of the greatest interval in the $n-$th partition tends to zero, i.e.,*

$$\lim_{m\to\infty}\sup_{1\le i\le n_m}|a_{m,i}-a_{m,i-1}|=0.$$

Now we would like to recall the concept of the mean-square integral. For the definition and basic properties, see [19].

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Let $X : I \times \Omega \to \mathbb{R}$ be a stochastic process with $E\left[X(t)^2\right] < \infty$ for all $t \in I$. Let $[a,b] \subset I$, $a = t_0 < t_1 < t_2 < ... < t_n = b$ be a partition of $[a,b]$ and $\Theta_k \in [t_{k-1}, t_k]$ for all $k = 1, ..., n$. A random variable $Y : \Omega \to \mathbb{R}$ is called the mean-square integral of the process X on $[a,b]$, if we have

$$\lim_{n\to\infty} E\left[\left(\sum_{k=1}^{n} X(\Theta_k)(t_k - t_{k-1}) - Y\right)^2\right] = 0$$

for all normal sequence of partitions of the interval $[a,b]$ and for all $\Theta_k \in [t_{k-1}, t_k]$, $k = 1, ..., n$. Then, we write

$$Y(\cdot) = \int_a^b X(s,\cdot)\, ds \text{ (a.e.)}.$$

For the existence of the mean-square integral it is enough to assume the mean-square continuity of the stochastic process X.

Throughout this paper, we will frequently use the monotonicity of the mean-square integral. If $X(t,\cdot) \leq Y(t,\cdot)$ (a.e.) in some interval $[a,b]$, then

$$\int_a^b X(t,\cdot)\, dt \leq \int_a^b Y(t,\cdot)\, dt \text{ (a.e.)}.$$

Of course, this inequality is the immediate consequence of the definition of the mean-square integral.

Definition 2. *We say that a stochastic processes $X : I \times \Omega \to \mathbb{R}$ is convex, if for all $\lambda \in [0,1]$ and $u, v \in I$ the inequality*

$$X(\lambda u + (1-\lambda) v, \cdot) \leq \lambda X(u,\cdot) + (1-\lambda) X(v,\cdot) \quad \text{(a.e.)} \tag{1.1}$$

is satisfied. If the above inequality is assumed only for $\lambda = \frac{1}{2}$, then the process X is Jensen-convex or $\frac{1}{2}-$convex. A stochastic process X is concave if $(-X)$ is convex. Some interesting properties of convex and Jensen-convex processes are presented in [13, 18, 19].

Now, we present some results proved by Kotrys [8] about Hermite-Hadamard inequality for convex stochastic processes.

Lemma 1. *If $X : I \times \Omega \to \mathbb{R}$ is a stochastic process of the form $X(t,\cdot) = A(\cdot)t + B(\cdot)$, where $A, B : \Omega \to \mathbb{R}$ are random variables, such that $E\left[A^2\right] < \infty, E\left[B^2\right] < \infty$ and $[a,b] \subset I$, then*

$$\int_a^b X(t,\cdot)\, dt = A(\cdot)\frac{b^2 - a^2}{2} + B(\cdot)(b-a) \text{ (a.e.)}.$$

Proposition 1. *Let $X : I \times \Omega \to \mathbb{R}$ be a convex stochastic process and $t_0 \in intI$. Then there exist a random variable $A : \Omega \to \mathbb{R}$ such that X is supported at t_0 by the process $A(\cdot)(t - t_0) + X(t_0,\cdot)$. That is*

$$X(t,\cdot) \geq A(\cdot)(t - t_0) + X(t_0,\cdot) \text{ (a.e.)}.$$

for all $t \in I$.

Definition 3. [6] *We say that two stochastic processes* $X_1,\ X_2 : I \times \Omega \to \mathbb{R}$ *are comonotonic, if for any* $s, t \in I$, *the following inequality holds:*

$$[X_1(t,\cdot) - X_1(s,\cdot)][X_2(t,\cdot) - X_2(s,\cdot)] \geq 0 \qquad (a.e.).$$

In [10], Hafiz gave the following definition of stochastic mean-square fractional integrals:

For the stochastic process $X : I \times \Omega \to \mathbb{R}$, the concept of stochastic mean-square fractional integrals I_{a+}^{α} and I_{b+}^{α} of X of order $\alpha > 0$ is defined by

$$I_{a+}^{\alpha}[X](t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} X(x,s)ds \quad (a.e.), \quad t > a$$

and

$$I_{b-}^{\alpha}[X](t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} X(x,s)ds \quad (a.e.), \quad t < b.$$

In [6], Agahi and Yadollahzadeh gave the following Chebyshev type inequality for comonotonic stoctastic processes:

Theorem 1. *Let* $X_1,\ X_2 : I \times \Omega \to \mathbb{R}$ *be comonotonic stoctastic processesin the interval* I. *Then for all* $t \in I$ *and* $\alpha > 0$, *the following Chebyshev inequality holds*

$$I_{a+}^{\alpha}[1](t) I_{a+}^{\alpha}[X_1 X_2](t) \geq I_{a+}^{\alpha}[X_1](t) I_{a+}^{\alpha}[X_2](t) \quad (a.e.). \tag{1.2}$$

In [14], Raina studied a class of functions defined formally by

$$\mathcal{F}_{\rho,\lambda}^{\sigma}(x) = \mathcal{F}_{\rho,\lambda}^{\sigma(0),\sigma(1),\ldots}(x) = \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\rho k + \lambda)} x^k \qquad (\rho, \lambda > 0; |x| < \mathcal{R}), \tag{1.3}$$

where the coefficients $\sigma(k)$ $(k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$ is a bounded sequence of positive real numbers and $\mathcal{R}$ is the set of real numbers. With the help of (1.3), Budak and Sarikaya give the following definition.

Definition 4. [3] *Let* $X : I \times \Omega \to \mathbb{R}$ *be a stochastic process. The generalized mean-square fractional integrals* $\mathcal{J}_{\rho,\lambda,a+;\omega}^{\alpha}$ *and* $\mathcal{J}_{\rho,\lambda,b-;\omega}^{\alpha}$ *of* X *are defined by*

$$\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma}[X](x) = \int_a^x (x-t)^{\lambda-1} \mathcal{F}_{\rho,\lambda}^{\sigma}[\omega(x-t)^{\rho}] X(t,\cdot)dt, \quad (a.e.)\ \ x > a, \tag{1.4}$$

and

$$\mathcal{J}_{\rho,\lambda,b-;\omega}^{\sigma}[X](x) = \int_x^b (t-x)^{\lambda-1} \mathcal{F}_{\rho,\lambda}^{\sigma}[\omega(t-x)^{\rho}] X(t,\cdot)dt, \quad (a.e.)\ \ x < b, \tag{1.5}$$

where $\lambda, \rho > 0, \omega \in \mathbb{R}$.

Many useful generalized mean-square fractional integrals can be obtained by specializing the coefficient $\sigma(k)$. Here, we just point out that the stochastic mean-square fractional integrals I_{a+}^{α} and I_{b+}^{α} can be established by choosing $\lambda = \alpha$, $\sigma(0) = 1$ and $w = 0$.

For more information and recent developments on stochastic process and Chebyshev type inequalities, we refer the reader to ([1]-[13], [15]-[17], [20], [21]).

2. Main Results

In this section, we present some Chebyshev type inequalities for generalized mean-square fractional integrals $\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}$ and $\mathcal{J}^{\sigma}_{\rho,\lambda,b-;\omega}$ of X.

Theorem 2. *Let $X_1, X_2 : I\times\Omega\to\mathbb{R}$ be comonotonic stoctastic processes in I. Then for all $t\in I$ and $\rho,\lambda>0$, the following Chebyshev inequality holds*

$$\mathcal{J}^{\sigma}_{\rho,\lambda,u+;\omega}[1](t)\mathcal{J}^{\sigma}_{\rho,\lambda,u+;\omega}[X_1X_2](t)\geq\mathcal{J}^{\sigma}_{\rho,\lambda,u+;\omega}[X_1](t)\mathcal{J}^{\sigma}_{\rho,\lambda,u+;\omega}[X_2](t)\quad (a.e.), \tag{2.1}$$

where $\sigma(k)$ $(k\in\mathbb{N}_0=\mathbb{N}\cup\{0\})$ is a bounded sequence of positive real numbers.

Proof. Since X_1 and X_2 are comonotonic for all $\eta,\xi\in I$, we have

$$[X_1(\eta,\cdot)-X_1(\xi,\cdot)][X_2(\eta,\cdot)-X_2(\xi,\cdot)]\geq 0\quad \text{(a.e.)}. \tag{2.2}$$

Then we get

$$X_1(\eta,\cdot)X_2(\eta,\cdot)+X_1(\xi,\cdot)X_2(\xi,\cdot)\geq X_1(\eta,\cdot)X_2(\xi,\cdot)+X_1(\xi,\cdot)X_2(\eta,\cdot)\quad \text{(a.e.)}. \tag{2.3}$$

Multiplying both sides of (2.3) by

$$(t-\eta)^{\lambda-1}(t-\xi)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[\omega(t-\eta)^{\rho}]\mathcal{F}^{\sigma}_{\rho,\lambda}[\omega(t-\xi)^{\rho}],$$

we obtain

$$\begin{aligned}
&(t-\eta)^{\lambda-1}(t-\xi)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[\omega(t-\eta)^{\rho}]\mathcal{F}^{\sigma}_{\rho,\lambda}[\omega(t-\xi)^{\rho}]X_1(\eta,\cdot)X_2(\eta,\cdot)\\
&+(t-\eta)^{\lambda-1}(t-\xi)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[\omega(t-\eta)^{\rho}]\mathcal{F}^{\sigma}_{\rho,\lambda}[\omega(t-\xi)^{\rho}]X_1(\xi,\cdot)X_2(\xi,\cdot)\\
\geq\ &(t-\eta)^{\lambda-1}(t-\xi)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[\omega(t-\eta)^{\rho}]\mathcal{F}^{\sigma}_{\rho,\lambda}[\omega(t-\xi)^{\rho}]X_1(\eta,\cdot)X_2(\xi,\cdot)\\
&+(t-\eta)^{\lambda-1}(t-\xi)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[\omega(t-\eta)^{\rho}]\mathcal{F}^{\sigma}_{\rho,\lambda}[\omega(t-\xi)^{\rho}]X_1(\xi,\cdot)X_2(\eta,\cdot)\quad \text{(a.e.)}.
\end{aligned} \tag{2.4}$$

Integrating (2.4) with respect to η and ξ , respectively, over (u,t), we get the desired result. □

Remark 1. *Choosing $\lambda=\alpha$, $\sigma(0)=1$ and $w=0$ in Theorem 2, the inequality (2.1) is reduced to (1.2).*

Theorem 3. *Let $X_1, X_2 : I\times\Omega\to\mathbb{R}$ be comonotonic stoctastic processes in I. Then for all $t\in I$ and $\rho_1, \rho_2, \lambda_1, \lambda_2>0$, the following Chebyshev inequality holds*

$$\begin{aligned}
&\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,u+;\omega_2}[1](t)\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,u+;\omega_1}[X_1X_2](t)+\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,u+;\omega_1}[1](t)\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,u+;\omega_2}[X_1X_2](t)\\
\geq\ &\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,u+;\omega_1}[X_1](t)\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,u+;\omega_2}[X_2](t)+\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,u+;\omega_2}[X_1](t)\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,u+;\omega_1}[X_2](t)\quad (a.e.)
\end{aligned}$$

where $\sigma_1(k)$ and $\sigma_2(k)$ $(k\in\mathbb{N}_0=\mathbb{N}\cup\{0\})$ is a bounded sequence of positive real numbers.

Proof. Similarly, by multiplying both sides of (2.3) by

$$(t-\eta)^{\lambda_1-1}(t-\xi)^{\lambda_2-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}[\omega(t-\eta)^{\rho_1}]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}[\omega(t-\xi)^{\rho_2}],$$

we obtain

$$(t-\eta)^{\lambda_1-1}(t-\xi)^{\lambda_2-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega(t-\eta)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}\left[\omega(t-\xi)^{\rho_2}\right]X_1(\eta,\cdot)X_2(\eta,\cdot)$$
$$+(t-\eta)^{\lambda_1-1}(t-\xi)^{\lambda_2-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega(t-\eta)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}\left[\omega(t-\xi)^{\rho_2}\right]X_1(\xi,\cdot)X_2(\xi,\cdot)$$
$$\geq (t-\eta)^{\lambda_1-1}(t-\xi)^{\lambda_2-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega(t-\eta)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}\left[\omega(t-\xi)^{\rho_2}\right]X_1(\eta,\cdot)X_2(\xi,\cdot)$$
$$+(t-\eta)^{\lambda_1-1}(t-\xi)^{\lambda_2-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega(t-\eta)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}\left[\omega(t-\xi)^{\rho_2}\right]X_1(\xi,\cdot)X_2(\eta,\cdot)\quad \text{(a.e.)}. \tag{2.5}$$

Integrating (2.5) with respect to η and ξ , respectively, over (u,t), we have

$$\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,u+;\omega_2}[1](t)\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,u+;\omega_1}[X_1X_2](t)+\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,u+;\omega_1}[1](t)\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,u+;\omega_2}[X_1X_2](t)$$
$$\geq \mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,u+;\omega_1}[X_1](t)\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,u+;\omega_2}[X_2](t)+\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,u+;\omega_2}[X_1](t)\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,u+;\omega_1}[X_2](t)\quad \text{(a.e.)}.$$

This completes the proof. □

Corollary 1. *Choosing $\lambda_1=\alpha$, $\lambda_2=\beta$, $\sigma_1(0)=\sigma_2(0)=1$ and $w_1=w_2=0$ in Theorem 3, we obtain*

$$I^{\beta}_{u+}[1](t)I^{\alpha}_{u+}[X_1X_2](t)+I^{\alpha}_{u+}[1](t)I^{\beta}_{u+}[X_1X_2](t)$$
$$\geq I^{\alpha}_{u+}[X_1](t)I^{\beta}_{u+}[X_2](t)+I^{\beta}_{u+}[X_1](t)I^{\alpha}_{u+}[X_2](t)\quad \textit{(a.e.)} \tag{2.6}$$

Remark 2. *For $\alpha=\beta$ in (2.6), we get the following Chebyshev inequality*

$$(b-a)\int_u^v X_1(s,\cdot)X_2(s,\cdot)ds \geq \int_u^v X_1(s,\cdot)ds\int_u^v X_2(s,\cdot)ds \quad \textit{(a.e.)},$$

which was given in [6].

Theorem 4. *Let $(X_i)_{i=\overline{1,n}}$ be n non-negative stochastic processes in the interval I satisfying $[X_i(t,\cdot)-X_i(s,\cdot)][X_j(t,\cdot)-X_j(s,\cdot)]\geq 0$, $i\neq j$ for any $t,s\in I$ (i.e. $(X_i)_{i=\overline{1,n}}$ are pairwise comonotonic stochastic processes). Then for all $t\in I$ and $\rho,\lambda>0$, we have the following inequality*

$$\left(\mathcal{J}^{\sigma}_{\rho,\lambda,u+;\omega}[1](t)\right)^{n-1}\mathcal{J}^{\sigma}_{\rho,\lambda,u+;\omega}\left[\prod_{i=1}^{n}X_i\right](t)\geq \prod_{i=1}^{n}\left(\mathcal{J}^{\sigma}_{\rho,\lambda,u+;\omega}[X_i](t)\right)\quad \textit{(a.e.)} \tag{2.7}$$

where the coefficients $\sigma(k)$ $(k\in\mathbb{N}_0=\mathbb{N}\cup\{0\})$ represent a bounded sequence of positive real numbers.

Proof. We use mathematical induction on n:

Clearly, for $n=1$, we have $\mathcal{J}^{\sigma}_{\rho,\lambda,u+;\omega}[X_1](t)\geq \mathcal{J}^{\sigma}_{\rho,\lambda,u+;\omega}[X_1](t)$.

For $n=2$, from the result of Theorem 2, we obtain:

$$\mathcal{J}^{\sigma}_{\rho,\lambda,u+;\omega}[1](t)\mathcal{J}^{\sigma}_{\rho,\lambda,u+;\omega}[X_1X_2](t)\geq \mathcal{J}^{\sigma}_{\rho,\lambda,u+;\omega}[X_1](t)\mathcal{J}^{\sigma}_{\rho,\lambda,u+;\omega}[X_2](t)\quad \text{(a.e.)}.$$

Inductively, we can show that the stochastic processes $\prod_{i=1}^{n}X_i$ and X_{n+1} are comonotonic stoctastic processes. Suppose that the inequality (2.7) is valid for n.

Then for $n+1$, by Theorem 3, we have

$$\begin{aligned}
& \left(\mathcal{J}^{\sigma}_{\rho,\lambda,u+;\omega}[1](t)\right)^{n}\mathcal{J}^{\sigma}_{\rho,\lambda,u+;\omega}\left[\prod_{i=1}^{n+1}X_i\right](t) \\
= & \left(\mathcal{J}^{\sigma}_{\rho,\lambda,u+;\omega}[1](t)\right)^{n-1}\mathcal{J}^{\sigma}_{\rho,\lambda,u+;\omega}[1](t)\mathcal{J}^{\sigma}_{\rho,\lambda,u+;\omega}\left[\prod_{i=1}^{n}X_i.X_{n+1}\right](t) \\
\geq & \left(\mathcal{J}^{\sigma}_{\rho,\lambda,u+;\omega}[1](t)\right)^{n-1}\mathcal{J}^{\sigma}_{\rho,\lambda,u+;\omega}\left[\prod_{i=1}^{n}X_i\right](t)\mathcal{J}^{\sigma}_{\rho,\lambda,u+;\omega}[X_{n+1}](t) \\
\geq & \prod_{i=1}^{n}\left(\mathcal{J}^{\sigma}_{\rho,\lambda,u+;\omega}[X_i](t)\right)\mathcal{J}^{\sigma}_{\rho,\lambda,u+;\omega}[X_{n+1}](t) \\
= & \prod_{i=1}^{n+1}\left(\mathcal{J}^{\sigma}_{\rho,\lambda,u+;\omega}[X_i](t)\right).
\end{aligned}$$

This ends the proof. □

Remark 3. *If we choose $\lambda=\alpha$, $\sigma(0)=1$ and $w=0$ in Theorem 4, then we get Corollary 12 of* [6]*.*

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