People's Democratic Republic of Algeria
Ministry of Higher Education and Scientific Research
Abdelhamid ibn Badis University of Mostaganem
Faculty of Exact Sciences and Computer Science
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Thesis presented by

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Submitted in partial fulfilment of the requirements for the degree of

Doctorate in Mathematics

Speciality: Mathematical Analysis

Application of Nevanlinna Theory to Complex Difference Equations

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Defended on 30 October 2025, before the following examination committee

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Academic Year: 2025–2026

THESIS

Abstract

This thesis focuses on the applications of Nevanlinna Theory in studying the uniqueness of meromorphic functions and solutions to complex difference equations. The significant focus in this thesis is on equations derived from meromorphic functions that share specific values, particularly those that share a small entire function with their shifts and difference operators, or share a value with two difference operators.

Key words: uniqueness, entire function, meromorphic function, difference operators, difference equations, Nevanlinna theory, exponent of convergence, order of growth.

Résumé

Cette thèse se concentre sur les applications de la théorie de Nevanlinna dans l'étude de l'unicité des fonctions méromorphes et des solutions des équations aux différences complexes. L'attention principale de cette thèse porte sur les équations issues des fonctions méromorphes qui partagent des valeurs spécifiques, en particulier celles qui partagent une petite fonction entière avec leurs shifts et opérateurs de différence, ou qui partagent une valeur avec deux opérateurs de différence.

Mots clés: unicité, fonction entière, fonction méromorphe, opérateurs de différence, équations aux différences, théorie de Nevanlinna, exposant de convergence, ordre de croissance.

ملخص

تركز هذه الأطروحة على تطبيقات نظرية نيفانلينا في دراسة تفرد الدوال الميرومورفية وحلول معادلات الفرق المركبة. يركز الجزء الأكبر من الدراسة على المعادلات الناتجة من الدوال الميرومورفية التي تشترك في قيم محددة، خاصة تلك التي تشترك في دالة تامة صغيرة مع إزاحاتها وعوامل الفرق، أو تشترك في قيمة مع عاملين للفرق.

الكلمات المفتاحية: التفرّد، الدالة الصحيحة ، الدالة الميرومورفية، مشغلات الفرق، معادلات الفرق، نظرية نيفانلينا، أسّ التقارب، رتبة التزايد.

Acknowledgements

All praise is due to Allah, the Lord of the Worlds. Peace and blessing be upon whom Allah sent as a mercy to the Worlds, upon his Family, his Companions and his Brothers till the Day of Resurrection.

After that,

I wish to extend my heartfelt thanks to my supervisors, Associate Professor Abdallah El Farissi from the National Higher School of Mathematics and co-supervisor, Associate Professor Zinelaabidine Latreuch from the National Higher School of Mathematics, for their patience, guidance, advice, and for providing me with the opportunity to work on this fascinating research topic

I would like to thank all the members of my dissertation committee, including the Chairperson, Professor Benharrat Belaïdi, for agreeing to examine my work and lead the committee. I also extend my gratitude to Associate Professors Mohammed Amine Ghezzar and Maamar Andasmas for their valuable feedback on this dissertation.

I also extend my heartfelt thanks to all my teachers, faculty, and staff in the Department of Mathematics at the University of Mostaganem. They have created an excellent environment that fosters success and excellence. This supportive setting has equipped me with invaluable skills and experiences.

I extend my special and heartfelt thanks to all my family members and my sincere appreciation to all my friends.

This work would not have been possible without their support, faith, and confidence. I hope this serves as a testament to my gratitude.

Contents

Introduction

Nevanlinna Theory, developed by Finnish mathematician Rolf Nevanlinna[1] in the 1920s, is considered one of the most significant theories in complex analysis, particularly in the study of entire or meromorphic functions and solutions to difference equations. This theory investigates the value distribution of meromorphic functions within the complex plane, offering profound insights into their properties.

Nevanlinna published his work on the value distribution of meromorphic functions was published in 1925. Since then, the theory has seen substantial advancement, with numerous researchers contributing to its development. A key innovation introduced by Nevanlinna was the characteristic function, $T(r, f)$, which has become an essential tool for analyzing the growth of meromorphic functions.

Nevanlinna Theory has had a profound impact on complex analysis, especially in the study and resolution of complex differential and difference equations. It plays a crucial role in understanding the growth and oscillation of solutions to these equations, underscoring its importance in the field. The first and second fundamental theorems of Nevanlinna, established in 1929, were significant breakthroughs in the development of the theory.

Additionally, Nevanlinna Theory has become an indispensable tool for studying the uniqueness of meromorphic functions and solutions to difference equations. A significant focus in this domain is the relationship between meromorphic functions that share specific values, particularly those that share two values with the difference operator, share a small entire function with their shifts and difference operators, or share a value with two difference operators, see e.g. [**3, 5, 6, 10, 11, 12, 22, 23, 26, 27, 28, 38**].

In 1986, Jank, Mues and Volkmann [**19**] proved that if $f(z)$ is a nonconstant meromorphic functions, and $a \neq 0$ is a finite constant, and If $f(z)$, $f'(z)$ and $f''(z)$ share the value a CM, then $f(z) \equiv f'(z)$. Later, In [**3**], B. Chen and al. proved a difference analogue of this result and obtained the following results: if $f(z)$ is a nonconstant entire function of finite order, and $a(z)$ ($\not\equiv 0$) is a small periodic entire function with period c, and If $f(z)$, $\Delta_c f(z)$ and $\Delta_c^2 f(z)$ share $a(z)$ CM, then $\Delta_c f(z) \equiv \Delta_c^2 f(z)$. And In 2015, A. El Farissi, Z. Latreuch and A. Asiri [**10**] obtained the following results: if $f(z)$ is a transcendental entire function of finite order and $f(z) \not\equiv f_c(z)$, and if $f\left(z\right)$, $f\left(z+c\right)$ and $\Delta_c f\left(z\right)$, $\left(n \geq 1\right)$, share $a\ CM$, then $a = 0$ and $f(z) = h(z)e^{\frac{\beta}{c}z}$, where $\beta \neq 0$ and $h(z)$ is periodic entire function of period c.

In this thesis, we mainly focus on applications of Nevanlinna Theory to difference operators, particularly on entire functions sharing a small function with two difference operators.

[1] Rolf Herman Nevanlinna (October 22, 1895 – May 28, 1980) was one of the most famous Finnish mathematicians.

This work is structured into three chapters.

Chapter One introduces the fundamental concepts of Nevanlinna theory. It covers key definitions, examples, and formulas, including the Poisson[2] and Poisson-Jensen[3] formulas. The chapter explains the Nevanlinna functions and discusses the first and second fundamental theorems, which are crucial for analyzing meromorphic functions. It also addresses the classification of meromorphic and entire functions by their order, type, and the exponent of convergence of their zeros. The chapter ends with a discussion on meromorphic functions with exceptional values.

Chapter Two shows the relationship between Nevanlinna Theory and the difference operator, including the difference analogue of the Lemma on the Logarithmic Derivative and the Second Main Theorem. Additionally, the chapter discusses the characteristics of polynomial and rational functions in meromorphic functions and presents theorems related to combinations of meromorphic functions.

Chapter Three contains the most important results obtained in our research. We focused on applying Nevanlinna theory to entire functions that share a small periodic entire function with their shifts and with difference operators, or with two difference operators. We achieved improved results for one of the theories discussed in [**10**].

[2]Siméon Denis Poisson (June 21, 1781 – April 25, 1840) was a French mathematician, engineer, and physicist.

(June 21, 1781 – April 25, 1840)

[3]Johan Ludwig William Valdemar Jensen (May 8, 1859 – March 5, 1925) was a Danish mathematician and engineer.

Chapter 1

Basic Nevanlinna theory

In this chapter, we introduce the essential results and foundational concepts of Nevanlinna theory for entire and meromorphic functions. For more details, see [8, 15, 18, 20, 21, 37].

Throughout this chapter, we denote the open disc of radius R $(0 < R < \infty)$ centred at the origin as $D(R) = \{z \in \mathbb{C} : |z| < R\}$, and the closed disc of radius R centred at the origin as $\overline{D(R)} = \{z \in \mathbb{C} : |z| \leq R\}$.

1.1 Definitions and examples

Definition 1.1 *Suppose that $f \not\equiv 0$ is an analytic function in a neighborhood of a point $z_0 \in \mathbb{C}$ such that $f(z_0) = 0$, we define the order or multiplicity of the zero at that point as the least $n \in \mathbb{N}$ such that the coefficient of $(z - z_0)^n$ in the Taylor expansion of f about z_0 is non-zero.*

Example 1.1 *Consider*

$$g(z) = \sin z - z,$$

then g has a zero of order 3 at $z_0 = 0$, since $g(0) = 0$,

and

$$g(z) = -\frac{1}{3!}z^3 + \frac{1}{5!}z^5 + \cdots, |z| < +\infty.$$

Definition 1.2 *Suppose that z_0 is a point from $\mathbb{C}$. We say that f has an isolated singularity at z_0 if f is defined and analytic on a neighborhood of z_0 except at z_0 itself. There are three kinds of singularities:*

1) We say that f has a removable singularity at z_0 if $\lim_{z \to z_0} f(z) = L \in \mathbb{C}$ (the limit exists and is finite), and setting $f(z_0) = L$ makes f analytic at z_0.

2) f has a pole at z_0 if $\lim_{z \to z_0} f(z) = \infty$, in this case $\frac{1}{f}$ has a zero at z_0. We say that f has a pole of order n at z_0 if $\frac{1}{f}$ has a zero of order n at z_0. A pole of order 1 is called a simple pole.

3) *We say that* z_0 *is an essential singularity of* f *if it is neither a removable singularity nor a pole of* f, *or if* $\lim\limits_{z\to z_0} f(z)$ *does not exist.*

Example 1.2 *In this example, we will consider three cases to illustrate the previous definition*

1) *The function* $g(z) = \dfrac{\sin z}{z}$ *has a removable singularity at* $z_0 = 0$, *since* $\lim\limits_{z\to 0} \dfrac{\sin z}{z} = 1$.

2) *Consider* $g(z) = \dfrac{e^z}{(z-1)^n}$, *where* $n \in \mathbb{N}^*$, *we have*

$$\lim_{z\to 1} g(z) = \infty, \quad (z-1)^n g(z) = e^z \text{ and } \lim_{z\to 1} e^z = e$$

then $z_0 = 1$ *is a pole of order* n *of* g.

3) *Consider* $g(z) = e^{\frac{1}{z}}$, *then* $\lim\limits_{\substack{z \xrightarrow{>} 0 \\ z\in\mathbb{R}}} g(z) = +\infty$ *and* $\lim\limits_{\substack{z \xrightarrow{<} 0 \\ z\in\mathbb{R}}} g(z) = 0$, *hence* $z_0 = 0$ *is an essential singularity of* g.

Definition 1.3 *Suppose that* B *is an open set from* $\mathbb{C}$. *A meromorphic function in* B *is an analytic function in the open set* $B \setminus A$, *where* A *is closed and discrete set of* B *such that every point of* A *is a pole of* f. *In the case where* $B = \mathbb{C}$, *the function is said to be meromorphic.*

Example 1.3 1) *Every rational function is meromorphic.*

2) *The function* $\tan z = \dfrac{\sin z}{\cos z}$ *is meromorphic.*

3) *The function* $e^{\frac{1}{z}}$ *is not meromorphic.*

Definition 1.4 *In complex analysis, an entire function, is a complex-valued function that is analytic on the whole complex plane.*

Example 1.4 1) *Every polynomial function is entire.*

2) The function $e^{p(z)}$ is entire, where $p(z)$ is a polynomial function.

1.2 Formula of Poisson And Poisson-Jensen

Our first theorem, called the Poisson formula, allows us to express a meromorphic function in a disc as an integral around the boundary circle.

Theorem 1.1 (Poisson formula, [8]) *Assume that u is meromorphic in the open disc $D(R)$, and continuous on the closed disc $\overline{D(R)}$. Let z be a point inside the disc. Then*

$$u(z) = \int_0^{2\pi} P(z, Re^{i\theta}) u(Re^{i\theta}) \frac{d\theta}{2\pi},$$

where $P(z,\zeta)$ denotes the Poisson kernel.

Before presenting the precise statement of the Poisson-Jensen formula, we will introduce some additional notation. Considering a non-constant meromorphic function f and a point a in the complex plane, we can always write $f(z) = (z-a)^n g(z)$, where n is an integer, and g is analytic and non-vanishing in a neighborhood of a. The integer n is called the order of f at the point a and is denoted $ord_a f$. We will refer to the non-zero complex number $g(a)$ as the initial Laurent coefficient of f at a, denoted $ilc(f,a)$.

Theorem 1.2 (Poisson-Jensen formula, [15]) *Suppose that $f \not\equiv 0$ is meromorphic on $\overline{D(R)}$. Let a_j $(j = 1, 2, \ldots, p)$ (respectively b_k $(k = 1, 2, \ldots, q)$) denote the zeros (respectively the poles) of f in the open disc $D(R)$, each zero (respectively pole) repeated according to its multiplicity. For any z in $D(R)$ which is not a zero or a pole of f, we have*

$$\begin{aligned}\log|f(z)| &= \int_0^{2\pi} \frac{R^2-|z|^2}{|Re^{i\theta}-z|^2} \log\left|f\left(Re^{i\theta}\right)\right| \frac{d\theta}{2\pi} - \sum_{j=1}^{p} \log\left|\frac{R^2-\overline{a_j}z}{R(z-a_j)}\right| + \sum_{k=1}^{q} \log\left|\frac{R^2-\overline{b_k}z}{R(z-b_k)}\right| \\ &= \int_0^{2\pi} \frac{R^2-|z|^2}{|Re^{i\theta}-z|^2} \log\left|f\left(Re^{i\theta}\right)\right| \frac{d\theta}{2\pi} - \sum_{\zeta\in D(R)} (ord_\zeta f) \log\left|\frac{R^2-\overline{\zeta}z}{R(z-\zeta)}\right|.\end{aligned}$$

This formula was introduced and named by Rolf Nevanlinna.

Remark 1.1 *In the case where f has no zeros or poles on $\overline{D(R)}$ we get the Poisson formula for the harmonic function $\log|f(z)|$.*
If $z=0$ and $f(0) \neq 0, \infty$, we have

$$\begin{aligned}\log|f(0)| &= \int_0^{2\pi} \log\left|f\left(Re^{i\theta}\right)\right| \frac{d\theta}{2\pi} - \sum_{j=1}^{p} \log\left|\frac{R}{a_j}\right| + \sum_{k=1}^{q} \log\left|\frac{R}{b_k}\right| \\ &= \int_0^{2\pi} \log\left|f\left(Re^{i\theta}\right)\right| \frac{d\theta}{2\pi} - \sum_{\zeta\in D(R)} (ord_\zeta f) \log\left|\frac{R}{\zeta}\right|,\end{aligned}$$

which is said to be the Jensen's formula.

Corollary 1.1 ([8, 15]) *Assume that $f \not\equiv 0$ is a meromorphic function on $\overline{D(R)}$. Suppose that a_j $(j = 1, 2, \ldots, p)$ denote the non-zero zeros of f in $D(R)$, repeated according to multiplicity, and let b_k $(k = 1, 2, \ldots, q)$ denote the non-zero poles of f in $D(R)$, repeated according to multiplicity. Then*

$$\begin{aligned}\log|ilc(f,0)| &= \int_0^{2\pi} \log\left|f\left(Re^{i\theta}\right)\right| \frac{d\theta}{2\pi} - \sum_{j=1}^{p} \log\left|\frac{R}{a_j}\right| + \sum_{k=1}^{q} \log\left|\frac{R}{b_k}\right| - (ord_0 f)\log R \\ &= \int_0^{2\pi} \log\left|f\left(Re^{i\theta}\right)\right| \frac{d\theta}{2\pi} - \sum_{\substack{\zeta\in D(R)\\ \zeta\neq 0}} (ord_\zeta f)\log\left|\frac{R}{\zeta}\right| - (ord_0 f)\log R. \qquad (1.1)\end{aligned}$$

Remark 1.2 *Equation* (1.1) *in Corollary **1.1** as well as the equation in Remark **1.1** are referred to simply as the Jensen formula.*

1.3 The Nevanlinna functions

In this section, we begin by defining Nevanlinna functions and then state the first fundamental relationship between them, known as the "First Fundamental Theorem" of Nevanlinna theory. To proceed with these definitions and theorems, we will always assume that f is an entire or meromorphic function. Note that most of our definitions will not be applicable to certain constant functions f, so we exclude such constant functions from consideration.

Definition 1.5 ([15]) *For every real number $x \geq 0$, we introduce $\log^+ x$ by*

$$\log^+ x = \max(\log x,\ 0) = \begin{cases} \log x & \text{if } x \geq 1, \\ 0 & \text{if } 0 \leq x < 1. \end{cases}$$

Note that

$$\log x = \log^+ x - \log^+ \frac{1}{x}, \quad x > 0.$$

Lemma 1.1 ([20]) *We have the following properties*

a) $$\log x \le \log^+ x.$$

b) $$\log^+ x \le \log^+ y \quad \text{for} \;\; 0 \le x \le y.$$

c) $$\log x = \log^+ x - \log^+ \frac{1}{x}.$$

d) $$|\log x| = \log^+ x + \log^+ \frac{1}{x}.$$

e) $$\log^+ \left(\prod_{j=1}^{n} x_j \right) \le \sum_{j=1}^{n} \log^+ x_j.$$

f) $$\log^+ \left(\sum_{j=1}^{n} x_j \right) \le \log n + \sum_{j=1}^{n} \log^+ x_j.$$

Definition 1.6 (Counting function, [20]) *Consider $a \in \mathbb{C}$, and suppose that f is a meromorphic function such that $f \not\equiv a$, we denote by $n(t, a, f)$ the number of zeros of $f(z) - a$ (i.e. the number of a-points of f) on the closed disc $\overline{D(t)}$, each a-point counted according to its multiplicity and by $n(t, \infty, f)$ the number of poles of f on the closed disc $\overline{D(t)}$, each pole counted according to its multiplicity. We define*

$$N(r, a, f) = \int_0^r \frac{n(t, a, f) - n(0, a, f)}{t} dt + n(0, a, f) \log r, \qquad a \neq \infty,$$

and

$$N(r, f) = N(r, \infty, f) = \int_0^r \frac{n(t, \infty, f) - n(0, \infty, f)}{t} dt + n(0, \infty, f) \log r.$$

$N(r, a, f)$ is called the counting function of f at value a, and $N(r, f)$ is called the counting function of poles of f.

Example 1.5 *For every an entire function g, we have*

$$N(r, g) = 0.$$

Example 1.6 *Consider*

$$g(z) = \sin z.$$

g has simple zeros at $z_k = k\pi, \quad k \in \mathbb{Z}$, then

$$n(0,0,g) = 1,$$

and

$$n(t,0,g) - n(0,0,g) = 2\left[\frac{t}{\pi}\right].$$

Hence

$$N(r,0,g) = \int_0^r \frac{2\left[\frac{t}{\pi}\right]}{t} dt + \log r = \int_0^\pi \frac{2\left[\frac{t}{\pi}\right]}{t} dt + \sum_{j=1}^{\left[\frac{r}{\pi}\right]-1} \int_{\pi j}^{\pi(j+1)} \frac{2j}{t} dt + \int_{\pi\left[\frac{r}{\pi}\right]}^{r} \frac{2\left[\frac{t}{\pi}\right]}{t} dt + \log r$$

we have

$$\left[\frac{t}{\pi}\right] = 0 \ , \textit{for } 0 \le t \le \pi,$$

then

$$\begin{aligned}
N(r,0,g) &= \sum_{j=1}^{\left[\frac{r}{\pi}\right]-1} \int_{\pi j}^{\pi(j+1)} \frac{2j}{t} dt + \int_{\pi\left[\frac{r}{\pi}\right]}^{r} \frac{2\left[\frac{t}{\pi}\right]}{t} dt + \log r \\
&= 2\sum_{j=1}^{\left[\frac{r}{\pi}\right]-1} \int_{\pi j}^{\pi(j+1)} \frac{j}{t} dt + \log r + O(1) = 2\sum_{j=1}^{\left[\frac{r}{\pi}\right]-1} (j\log(j+1) - j\log j) + \log r + O(1) \\
&= 2\sum_{j=1}^{\left[\frac{r}{\pi}\right]-1} ((j+1)\log(j+1) - j\log j) - 2\sum_{j=1}^{\left[\frac{r}{\pi}\right]-1} \log(j+1) + \log r + O(1) \\
&= 2\left[\frac{r}{\pi}\right]\log\left[\frac{r}{\pi}\right] - 2\log\left(\left[\frac{r}{\pi}\right]!\right) + \log r + O(1), \ r \to +\infty
\end{aligned}$$

using Sterling's formula

$$n! \sim n^n e^{-n}\sqrt{2\pi n},$$

we get

$$\begin{aligned}
N(r,0,g) &= 2\left[\frac{r}{\pi}\right]\log\left[\frac{r}{\pi}\right] - 2\left[\frac{r}{\pi}\right]\log\left[\frac{r}{\pi}\right] + 2\left[\frac{r}{\pi}\right] - \log\left[\frac{r}{\pi}\right] + \log r + O(1) \\
&= 2\left[\frac{r}{\pi}\right] - \log\left[\frac{r}{\pi}\right] + \log r + O(1) \\
&= \frac{2r}{\pi} + O(1), \ r \to +\infty.
\end{aligned}$$

Definition 1.7 (Proximity function, [20]) *Consider $a \in \mathbb{C}$, and Assume that f is a meromorphic function such that $f \not\equiv a$. We define*

$$m(r,a,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \frac{1}{|f(re^{i\theta}) - a|} d\theta, \qquad a \neq \infty,$$

and

$$m(r,f) = m(r,\infty,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|f\left(re^{i\theta}\right)\right| d\theta.$$

$m(r,a,f)$ is called the proximity function of f at value a, and $m(r,f)$ is called the proximity function of f.

Definition 1.8 (Nevanlinna characteristic function, [20]) *Consider $a \in \mathbb{C}$, and assume that f is a meromorphic function such that $f \not\equiv a$. We define the Nevanlinna characteristic function of f by*

$$T(r,\infty,f) = T(r,f) = m(r,f) + N(r,f),$$

and

$$T(r,a,f) = T\left(r, \frac{1}{f-a}\right) = m(r,a,f) + N(r,a,f). \qquad a \neq \infty,$$

$T(r,a,f)$ is called the Nevanlinna characteristic function of f with respect to a.

Example 1.7 *Assume that the function g is defined by $g(z) = e^{az^n}$, $a \in \mathbb{C}^*$ and $n \in \mathbb{N}^*$. Then*

$$n(t,\infty,g) = 0 \text{ and } N(r,\infty,g) = 0.$$

In addition

$$\begin{aligned} m(r,g) &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|g\left(re^{i\theta}\right)\right| d\theta = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|e^{|a|e^{i\operatorname{Arg} a} r^n e^{in\theta}}\right| d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|e^{|a|r^n e^{i(n\theta + \operatorname{Arg} a)}}\right| d\theta, \end{aligned}$$

we put

$$\beta = n\theta + \operatorname{Arg} a,$$

then we have

$$d\beta = n d\theta,$$

and

$$\begin{aligned}
m(r,g) &= \frac{1}{2n\pi}\int_{\operatorname{Arg} a}^{\operatorname{Arg} a+2n\pi} \log^{+}\left|e^{|a|r^{n}e^{i\beta}}\right| d\beta \\
&= \frac{1}{2n\pi}\int_{0}^{2n\pi} \log^{+}\left|e^{|a|r^{n}e^{i\beta}}\right| d\beta = \frac{1}{2n\pi}\sum_{j=0}^{n-1}\int_{2j\pi}^{2(j+1)\pi} \log^{+}\left|e^{|a|r^{n}e^{i\beta}}\right| d\beta \\
&= \frac{1}{2n\pi}\sum_{j=0}^{n-1}\int_{2j\pi}^{2j\pi+2\pi} \log^{+}\left|e^{|a|r^{n}e^{i\beta}}\right| d\beta = \frac{1}{2n\pi}\sum_{j=0}^{n-1}\int_{0}^{2\pi} \log^{+}\left|e^{|a|r^{n}e^{i\beta}}\right| d\beta \\
&= \frac{n}{2n\pi}\int_{0}^{2\pi} \log^{+}\left(e^{|a|r^{n}\cos\beta}\right) d\beta = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log^{+}\left(e^{|a|r^{n}\cos\beta}\right) d\beta \\
&= \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |a|\, r^{n}\cos\beta d\beta = \frac{|a|\, r^{n}}{2\pi}2\left[\sin\beta\right]_{0}^{\frac{\pi}{2}} \\
&= \frac{|a|\, r^{n}}{\pi}.
\end{aligned}$$

Hence

$$\begin{aligned}
T\left(r,e^{az^{n}}\right) &= m\left(r,e^{az^{n}}\right)+N\left(r,e^{az^{n}}\right) \\
&= \frac{|a|\, r^{n}}{\pi}.
\end{aligned}$$

Proposition 1.1 ([8, 20]) *Suppose that $f,\ f_1,\ldots,f_n$ are meromorphic functions. We have the following inequalities*

$$\begin{aligned}
&a) & m\left(r,\sum_{j=1}^{n} f_j\right) &\le \sum_{j=1}^{n} m(r,f_j)+\log n. \\
&b) & m\left(r,\prod_{j=1}^{n} f_j\right) &\le \sum_{j=1}^{n} m(r,f_j). \\
&c) & N\left(r,\sum_{j=1}^{n} f_j\right) &\le \sum_{j=1}^{n} N(r,f_j). \\
&d) & N\left(r,\prod_{j=1}^{n} f_j\right) &\le \sum_{j=1}^{n} N(r,f_j). \\
&e) & T\left(r,\sum_{j=1}^{n} f_j\right) &\le \sum_{j=1}^{n} T(r,f_j)+\log n. \\
&f) & T\left(r,\prod_{j=1}^{n} f_j\right) &\le \sum_{j=1}^{n} T(r,f_j). \\
&g) & T(r,f^{n}) &= nT(r,f), \qquad n\in\mathbb{N}.
\end{aligned}$$

Proof. According to Lemma 1.1 we easily get $a)$ and $b)$.

$c)$ If z_0 is a pole of order λ_j of f_j, then z_0 is a pole of order at most $\max_{1\leq j\leq n} \lambda_j \leq \sum_{j=1}^{n} \lambda_j$ of $\sum_{j=1}^{n} f_j$. Hence

$$N\left(r, \sum_{j=1}^{n} f_j\right) \leq \sum_{j=1}^{n} N\left(r, f_j\right).$$

$d)$ If z_0 is a pole of order λ_j of f_j, then z_0 is a pole of order at most $\sum_{j=1}^{n} \lambda_j$ of $\prod_{j=1}^{n} f_j$. Hence

$$N\left(r, \prod_{j=1}^{n} f_j\right) \leq \sum_{j=1}^{n} N\left(r, f_j\right).$$

$e)$ From $a)$ and $c)$ we get

$$\begin{aligned} T\left(r, \sum_{j=1}^{n} f_j\right) &= m\left(r, \sum_{j=1}^{n} f_j\right) + N\left(r, \sum_{j=1}^{n} f_j\right) \\ &\leq \sum_{j=1}^{n} m\left(r, f_j\right) + \sum_{j=1}^{n} N\left(r, f_j\right) + \log n \\ &= \sum_{j=1}^{n} T\left(r, f_j\right) + \log n. \end{aligned}$$

$f)$ From $b)$ and $d)$ we get

$$\begin{aligned} T\left(r, \prod_{j=1}^{n} f_j\right) &= m\left(r, \prod_{j=1}^{n} f_j\right) + N\left(r, \prod_{j=1}^{n} f_j\right) \\ &\leq \sum_{j=1}^{n} m\left(r, f_j\right) + \sum_{j=1}^{n} N\left(r, f_j\right) \\ &= \sum_{j=1}^{n} T\left(r, f_j\right). \end{aligned}$$

$g)$ Noting that $\log^+ |z^n| = n \log^+ |z|$, for all $z \in \mathbb{C}$, we get

$$m\left(r, f^n\right) = nm\left(r, f\right).$$

If z_0 is a pole of order λ of f, then z_0 is a pole of order $n\lambda$ of f^n. Hence

$$N\left(r, f^n\right) = nN\left(r, f\right).$$

Then

$$\begin{aligned} T(r, f^n) &= m(r, f^n) + N(r, f^n) \\ &= n(m(r, f) + N(r, f)) \\ &= nT(r, f). \end{aligned}$$

which completes the proof of proposition **1.1.** ∎

1.4 The first fundamental theorem of Nevanlinna

Lemma 1.2 ([20]) *Assume that f is a meromorphic function with a-points $\alpha_1, \alpha_2, \ldots, \alpha_n$ in $|z| \leq r$ such that $0 < |\alpha_1| \leq |\alpha_2| \leq \cdots \leq |\alpha_n| \leq r$, each a-point repeated according to its multiplicity. Then*

$$\int_0^r \frac{n(t, a, f)}{t} dt = \int_0^r \frac{n(t, a, f) - n(0, a, f)}{t} dt = \sum_{0 < |\alpha_j| \leq r} \log \frac{r}{|\alpha_j|}.$$

Proof. Denoting $r_j = |\alpha_j|, \ \ j = 1, \ldots, n$, we obtain

$$\begin{aligned} \sum_{|\alpha_j| \leq r} \log \frac{r}{|\alpha_j|} &= \sum_{j=1}^{n} \log \frac{r}{r_j} = \log \frac{r^n}{r_1 r_2 \cdots r_n} = \log \left(\left(\frac{r_2}{r_1}\right) \left(\frac{r_3}{r_2}\right)^2 \cdots \left(\frac{r_n}{r_{n-1}}\right)^{n-1} \left(\frac{r}{r_n}\right)^n \right) \\ &= \sum_{j=1}^{n-1} j (\log r_{j+1} - \log r_j) + n (\log r - \log r_n) \\ &= \sum_{j=1}^{n-1} j \int_{r_j}^{r_{j+1}} \frac{dt}{t} + n \int_{r_n}^{r} \frac{dt}{t} = \sum_{j=1}^{n-1} \int_{r_j}^{r_{j+1}} \frac{n(t, a, f)}{t} dt + \int_{r_n}^{r} \frac{n(t, a, f)}{t} dt \\ &= \int_0^r \frac{n(t, a, f)}{t} dt. \end{aligned}$$

This completes the demonstration of lemma **1.2.** ∎

Proposition 1.2 ([20]) *Assume that f is a meromorphic function with the Laurent expansion*

$$f(z) = \sum_{j=m}^{+\infty} c_j z^j, \qquad c_m \neq 0, \qquad m \in \mathbb{Z},$$

at the origin. Then

$$\log |c_m| = \frac{1}{2\pi} \int_0^{2\pi} \log \left| f\left(re^{i\theta}\right) \right| d\theta + N(r, f) - N\left(r, \frac{1}{f}\right).$$

Proof. Assume the function $h(z) = f(z) z^{-m}$. Clearly $h(0) \neq 0, \infty$.
We claim that $m = n(0,0,f) - n(0,\infty,f)$.
If $m > 0$, then

$$n(0,0,f) = m \quad \text{and} \quad n(0,\infty,f) = 0.$$

If $m < 0$, then

$$n(0,0,f) = 0 \quad \text{and} \quad n(0,\infty,f) = -m.$$

If $m = 0$, then

$$n(0,0,f) = 0 \quad \text{and} \quad n(0,\infty,f) = 0.$$

The functions h and f have the same poles and zeros in $0 < |z| \leq r$. The Jensen formula, together with Lemma **1.2**, implies

$$\begin{aligned}
\log|c_m| &= \log|h(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(re^{i\theta}\right) r^{-m}\right| d\theta + \sum_{0<|b_j|\leq r} \log\frac{r}{|b_j|} - \sum_{0<|a_j|\leq r} \log\frac{r}{|a_j|} \\
&= \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(re^{i\theta}\right)\right| d\theta - m\log r + \int_0^r \frac{n(t,\infty,f) - n(0,\infty,f)}{t} dt \\
&\quad - \int_0^r \frac{n(t,0,f) - n(0,0,f)}{t} dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(re^{i\theta}\right)\right| d\theta - \left[n(0,0,f) - n(0,\infty,f)\right]\log r \\
&\quad + \int_0^r \frac{n(t,\infty,f) - n(0,\infty,f)}{t} dt - \int_0^r \frac{n(t,0,f) - n(0,0,f)}{t} dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(re^{i\theta}\right)\right| d\theta + \int_0^r \frac{n(t,\infty,f) - n(0,\infty,f)}{t} dt + n(0,\infty,f)\log r \\
&\quad - \left[\int_0^r \frac{n(t,0,f) - n(0,0,f)}{t} dt + n(0,0,f)\log r\right] \\
&= \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(re^{i\theta}\right)\right| d\theta + N(r,f) - N\left(r,\frac{1}{f}\right).
\end{aligned}$$

This completes the demonstration of proposition **1.2.** ∎

Theorem 1.3 (The first fundamental theorem, [15]) *Assume that f is a meromorphic function, let $a \in \mathbb{C}$ and let*

$$f(z) - a = \sum_{j=m}^{+\infty} c_j z^j, \qquad c_m \neq 0, \qquad m \in \mathbb{Z},$$

be the Laurent expansion of $f-a$ at the origin. Then

$$T\left(r,\frac{1}{f-a}\right)=T\left(r,f\right)-\log\left|c_m\right|+\varphi\left(r,a\right),$$

where

$$\left|\varphi\left(r,a\right)\right|\leq\log 2+\log^+\left|a\right|.$$

Proof. Assume first $a=0$. By Proposition **1.2** and $c)$ from Lemma **1.1** we obtain

$$\begin{aligned}\log\left|c_m\right| &= \frac{1}{2\pi}\int_0^{2\pi}\log^+\left|f\left(re^{i\theta}\right)\right|d\theta-\frac{1}{2\pi}\int_0^{2\pi}\log^+\left|\frac{1}{f\left(re^{i\theta}\right)}\right|d\theta+N\left(r,f\right)-N\left(r,\frac{1}{f}\right)\\ &= m\left(r,f\right)-m\left(r,\frac{1}{f}\right)+N\left(r,f\right)-N\left(r,\frac{1}{f}\right),\end{aligned}$$

hence

$$\begin{aligned}m\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f}\right) &= m\left(r,f\right)+N\left(r,f\right)-\log\left|c_m\right|,\\ T\left(r,\frac{1}{f}\right) &= T\left(r,f\right)-\log\left|c_m\right|,\end{aligned}\tag{1.2}$$

which is the assertion with $\varphi\left(r,0\right)\equiv 0$.
Proceeding now to the general case $a\neq 0$, we define $h=f-a$. Clearly

$$\begin{aligned}N\left(r,\frac{1}{h}\right) &= N\left(r,\frac{1}{f-a}\right)\\ N\left(r,h\right) &= N\left(r,f\right)\\ m\left(r,\frac{1}{h}\right) &= m\left(r,\frac{1}{f-a}\right).\end{aligned}$$

Moreover

$$\begin{aligned}\log^+\left|h\right| &= \log^+\left|f-a\right|\leq\log^+\left|f\right|+\log^+\left|a\right|+\log 2,\\ \log^+\left|f\right| &= \log^+\left|h+a\right|\leq\log^+\left|h\right|+\log^+\left|a\right|+\log 2.\end{aligned}$$

Integrating these inequalities we see that

$$\begin{aligned}m\left(r,h\right) &\leq m\left(r,f\right)+\log^+\left|a\right|+\log 2,\\ m\left(r,f\right) &\leq m\left(r,h\right)+\log^+\left|a\right|+\log 2,\end{aligned}$$

hence

$$\varphi\left(r,a\right)=m\left(r,h\right)-m\left(r,f\right)$$

which satisfies

$$\left|\varphi\left(r,a\right)\right|\leq\log 2+\log^+\left|a\right|.$$

Applying equation (1.2) to h we obtain

$$\begin{aligned} T\left(r, \frac{1}{f-a}\right) &= T\left(r, \frac{1}{h}\right) = T(r,h) - \log|c_m| \\ &= m(r,h) + N(r,h) - \log|c_m| \\ &= m(r,f) + \varphi(r,a) + N(r,f) - \log|c_m| \\ &= T(r,f) + \varphi(r,a) - \log|c_m|. \end{aligned}$$

Where $|\varphi(r,a)| \leq \log 2 + \log^+ |a|$. ∎

Remark 1.3 *The first fundamental theorem may be expressed as*

$$T\left(r, \frac{1}{f-a}\right) = T(r,f) + O(1), \quad (r \to +\infty),$$

for all non-constant meromorphic function f and $a \in \mathbb{C}$.

Proposition 1.3 ([20]) *If f is a non-constant meromorphic function, and if*

$$g = \frac{af+b}{cf+d},$$

for constants a, b, c and d with $ad - bc \neq 0$, then

$$T(r,g) = T(r,f) + O(1).$$

Proof. Since $f = \dfrac{-dg+b}{cg-a}$, then it suffices to prove

$$T(r,g) \leq T(r,f) + O(1).$$

If $c = 0$, by Proposition **1.1**, we get

$$\begin{aligned} T\left(r, \frac{af+b}{d}\right) &= T\left(r, \frac{a}{d}f + \frac{b}{d}\right) \\ &\leq T\left(r, \frac{a}{d}\right) + T(r,f) + T\left(r, \frac{b}{d}\right) + \log 2 \\ &= T(r,f) + \log^+\left|\frac{a}{d}\right| + \log^+\left|\frac{b}{d}\right| + \log 2 \\ &= T(r,f) + O(1). \end{aligned}$$

If $c \neq 0$, then we write

$$\begin{aligned} \frac{af+b}{cf+d} &= \frac{a\left(f + \frac{d}{c}\right) + b - a\frac{d}{c}}{c\left(f+\frac{d}{c}\right)} = \frac{a}{c} + \frac{b - a\frac{d}{c}}{c\left(f+\frac{d}{c}\right)} \\ &= \frac{a}{c} + \frac{bc-ad}{c^2}\frac{1}{f+\frac{d}{c}}, \end{aligned}$$

by Proposition **1.1** and the first fundamental theorem we get

$$\begin{aligned} T(r,g) &= T\left(r, \frac{a}{c} + \frac{bc-ad}{c^2}\frac{1}{f+\frac{d}{c}}\right) \\ &\leq T(r,f) + \log^+\left|\frac{a}{c}\right| + \log^+\left|\frac{bc-ad}{c^2}\right| + \log 2 + O(1) \\ &= T(r,f) + O(1). \end{aligned}$$

This concludes the proof of Proposition **1.3.** ■

Example 1.8 *We have*

$$\begin{aligned} \tan z &= \frac{\sin z}{\cos z} = \frac{e^{iz} - e^{-iz}}{2i}\frac{2}{e^{iz} + e^{-iz}} \\ &= -i\frac{e^{2iz} - 1}{e^{2iz} + 1} = \frac{-ie^{2iz} + i}{e^{2iz} + 1}. \end{aligned}$$

Since $-2i \neq 0$, *then*

$$\begin{aligned} T(r, \tan z) &= T\left(r, e^{2iz}\right) + O(1) \\ &= \frac{2r}{\pi} + O(1), \ r \to +\infty. \end{aligned}$$

1.5 The order, the type and the exponent of convergence of zeros of a meromorphic and entire function in the complex plane

Definition 1.9 (Order of growth, [37]) *Suppose that f is an entire function. The order of f is defined by*

$$\rho(f) = \overline{\lim_{r\to+\infty}} \frac{\log^+ \log^+ M(r,f)}{\log r},$$

where $M(r,f) = \max\{|f(z)|, |z| = r\}$.
If f is a meromorphic function, then the order of f is defined by

$$\rho(f) = \overline{\lim_{r\to+\infty}} \frac{\log^+ T(r,f)}{\log r},$$

Example 1.9 1) *Consider* $g(z) = \dfrac{e^z}{z}$, *then*

$$\begin{aligned} T(r,g) &\leq T(r,e^z) + T(r,z) + O(1) = T(r,e^z) + \log r + O(1) \\ &= T(r,e^z) + O(\log r), \end{aligned}$$

and

$$\begin{aligned} T(r,e^{z}) &= T(r,zg) \leq T(r,z)+T(r,g) \\ &= T(r,g)+O(\log r), \end{aligned}$$

hence

$$\begin{aligned} T(r,g) &= T(r,e^{z})+O(\log r) \\ &= \frac{r}{\pi}+O(\log r). \end{aligned}$$

We calculate $\rho(g)$

$$\begin{aligned} \rho(g) &= \overline{\lim_{r\to+\infty}}\frac{\log^{+}T(r,g)}{\log r} = \overline{\lim_{r\to+\infty}}\frac{\log\left[\frac{r}{\pi}+O(\log r)\right]}{\log r} \\ &= \overline{\lim_{r\to+\infty}}\frac{\log\left[\frac{r}{\pi}\left(1+O\left(\frac{\log r}{r}\right)\right)\right]}{\log r} \\ &= \overline{\lim_{r\to+\infty}}\frac{\log r+\log\left(1+O\left(\frac{\log r}{r}\right)\right)-\log\pi}{\log r} \\ &= 1. \end{aligned}$$

2) *Consider* $g(z)=e^{z^{n}}$ *for* $n\in\mathbb{N}^{*}$, *then*

$$\begin{aligned} \rho(g) &= \overline{\lim_{r\to+\infty}}\frac{\log^{+}T(r,g)}{\log r} = \overline{\lim_{r\to+\infty}}\frac{\log\left(\frac{r^{n}}{\pi}\right)}{\log r} \\ &= \overline{\lim_{r\to+\infty}}\frac{n\log r-\log\pi}{\log r} = n. \end{aligned}$$

Example 1.10 *Consider the entire function* $g(z)=e^{e^{z}}$, g *has infinite order because*

$$\begin{aligned} \rho(g) &= \overline{\lim_{r\to+\infty}}\frac{\log^{+}\log^{+}M(r,g)}{\log r} \\ &= \overline{\lim_{r\to+\infty}}\frac{\log^{+}\log^{+}e^{e^{r}}}{\log r} \\ &= +\infty. \end{aligned}$$

Definition 1.10 ([21], [18]) *Assume that* f *is a meromorphic function in the complex plane of order* $\rho(f)=\rho$ $(0<\rho<\infty)$. *We define the type of* f *by*

$$\tau(f)=\overline{\lim_{r\to+\infty}}\frac{T(r,f)}{r^{\rho}}.$$

If f *is an entire function of order* $\rho(f)=\rho$ $(0<\rho<\infty)$, *we define the type of* f *by*

$$\tau_{M}(f)=\overline{\lim_{r\to+\infty}}\frac{\log M(r,f)}{r^{\rho}}.$$

Remark 1.4 *In general* $\tau(f)$ *does not equal* $\tau_M(f)$.

Example 1.11 *Consider* $g(z) = e^{z^n}$, $n \in \mathbb{N}^*$ *we have* $T(r,g) = \frac{r^n}{\pi}$ *and* $M(r,g) = e^{r^n}$. *Therefore* $\tau(g) = \frac{1}{\pi}$ *and* $\tau_M(g) = 1$.

Theorem 1.4 ([37]) *Assume that* $f(z)$ *and* $g(z)$ *are two non-constant meromorphic functions in the complex plane with,* $\rho(f)$ *and* $\rho(g)$ *as their orders respectively. Then*

$$\rho(f.g) \leq \max\{\rho(f), \rho(g)\},$$

and

$$\rho(f+g) \leq \max\{\rho(f), \rho(g)\}.$$

Proof. Without loss of generality, we suppose that

$$\rho(f) \leq \rho(g) < \infty.$$

Using the definition of the order, for any $\varepsilon > 0$, There is a positive number R such that

$$T(r,f) \leq r^{\rho(f)+\varepsilon},$$

and

$$T(r,g) \leq r^{\rho(g)+\varepsilon},$$

hold for $r \geq R$. Note that

$$T(r,gf) \leq T(r,f) + T(r,g).$$

We have

$$T(r,gf) \leq 2r^{\rho(g)+\varepsilon}, \ r \geq R,$$

This means that

$$\overline{\lim_{r\to+\infty}} \frac{\log^+ T(r,fg)}{\log r} \leq \rho(g) + \varepsilon,$$

holds for any $\varepsilon > 0$.Hence

$$\rho(f.g) \leq \rho(g).$$

Similarly, from

$$T(r,g+f) \leq T(r,f) + T(r,g) + \ln 2,$$

Hence we get

$$\rho(f+g) \leq \rho(g).$$

This concludes the proof of Theorem **1.4.** ■

Example 1.12 *Consider* $g(z) = e^{z^n}$ *and* $h(z) = \frac{1}{g(z)}$, *we obtain*

$$\rho(h.g) = 0 \leq \max\{\rho(h), \rho(g)\} = n.$$

Example 1.13 *Consider* $g(z) = e^z$ *and* $h(z) = -g(z)$,

$$\rho(h+g) = 0 \leq \max\{\rho(h), \rho(g)\} = 1.$$

Theorem 1.5 ([37], See Theorem 1.44) *Assume that* $h(z)$ *is a non-constant entire function and* $f(z) = e^{h(z)}$. *Let,* ρ *be the order of* $f(z)$. *We have*

1) *If* $h(z)$ *is a polynomial of degree* n, *then,* $\rho = n$.
2) *If* $h(z)$ *is a transcendental entire function, then,* $\rho = \infty$.

Example 1.14 *Consider* $g(z) = e^{z^2-3z+1}$, *then we have*

$$\rho(g) = \rho\left(e^{z^2-3z+1}\right) = 2.$$

Example 1.15 *Consider* $g(z) = Cst \neq 0$, *then we have*

$$\rho(g) = 0.$$

Example 1.16 *Consider* $g(z) = e^{\cos(z)}$, *then we have*

$$\rho(g) = \infty.$$

Theorem 1.6 ([37]) *Assume that* $f(z)$ *and* $g(z)$ *are two non-constant meromorphic functions on the complex plane with,* $\rho(f)$ *and* $\rho(g)$ *as their orders respectively. If* $\rho(f) < \rho(g)$, *then*

$$\rho(f.g) = \rho(f+g) = \rho(g).$$

Proof. We have

$$g = (f.g) \cdot \frac{1}{f},$$

and

$$\rho(f) = \rho\left(\frac{1}{f}\right),$$

by using Theorem **1.4** we get

$$\rho(g) \leq \max\{\rho(f.g), \rho(f)\} \leq \max\{\rho(g), \rho(f)\} = \rho(g).$$

Then

$$\rho(f.g) = \rho(g).$$

Similarly, we can prove

$$\rho(f+g) = \rho(g).$$

This concludes the proof of Theorem **1.6.** ■

Example 1.17 *Consider $g(z) = e^{z^n}$ and $h(z) = e^{z^m}$, $(n < m)$, where $n, m \in \mathbb{N}$, we have*

$$\rho(h) = n < \rho(g) = m,$$

*by using Theorem **1.5** we get*

$$\rho(h.g) = \rho\left(e^{z^n + z^m}\right) = m$$

and

$$\begin{aligned} \rho(h+g) &= \rho\left(e^{z^n} + e^{z^m}\right) \\ &= \rho\left[e^{z^n}\left(e^{z^m - z^n} + 1\right)\right] = m \end{aligned}$$

then

$$\rho(h.g) = \rho(h+g) = \rho(g) = m.$$

Definition 1.11 (Exponent of convergence of zeros, [37]) *Assume that f is a meromorphic function. The exponent of convergence of zeros of f is defined by*

$$\lambda(f) = \overline{\lim_{r \to +\infty}} \frac{\log^+ N\left(r, \frac{1}{f}\right)}{\log r},$$

Definition 1.12 *With this definition, $\lambda(f-a)$ denotes the exponent of convergence of the sequence of a-points of f, and $\lambda(\frac{1}{f})$ is the corresponding exponent of convergence for the poles of f.*

Example 1.18 *For every an entire function g, we have*

$$\lambda\left(\frac{1}{g}\right) = 0.$$

Example 1.19 *Consider $g(z) = e^z - 1$ .We have*

$$\begin{aligned} e^z &= 1 \Leftrightarrow z = 2k\pi i, \ k \in Z \\ |z| &\le t \Rightarrow |(2k\pi)| \le t \\ &\Rightarrow \frac{-t}{2\pi} \le k \le \frac{t}{2\pi} \\ &\Rightarrow n\left(t, \frac{1}{g}\right) = 2\left[\frac{t}{2\pi}\right] + 1. \end{aligned}$$

Thus, using the same method as in Example ***1.6****, we find*

$$N\left(r, \frac{1}{g}\right) = \frac{r}{\pi} + O(1), \ r \longrightarrow \infty.$$

Then we conclude that

$$\lambda(g) = 1.$$

Theorem 1.7 ([37]) *Assume that $f(z)$ is a transcendental meromorphic function of order $\rho(f)$ and let $\lambda(f)$ be the exponent of convergence of zeros for f.Then $\lambda(f) \le \rho(f)$.*

Proof. We have

$$\begin{aligned} N\left(r, \frac{1}{f}\right) &\le T\left(r, \frac{1}{f}\right) \\ &= T(r, f) + O(1), \quad (r \to +\infty) \end{aligned}$$

then

$$\lambda(f) = \overline{\lim_{r\to+\infty}} \frac{\log^+ N\left(r, \frac{1}{f}\right)}{\log r} \le \overline{\lim_{r\to+\infty}} \frac{\log^+ T(r, f)}{\log r} = \rho(f).$$

This concludes the proof of Theorem **1.7**. ■

Example 1.20 *Consider $g(z) = e^z$, then we have*

$$\lambda(g) = 0 \le \rho(g) = 1.$$

1.6 The second fundamental theorem of Nevanlinna

Definition 1.13 (Linear and logarithmic measure, [15]) *The linear measure of a set $E \subset [0, +\infty[$ is defined by*

$$m(E) = \int_0^{+\infty} \chi_E(t)\, dt,$$

where $\chi_E(t)$ *is the characteristic function of the set* E.
The logarithmic measure of a set $F \subset [1, +\infty[$ *is defined by*

$$lm(F) = \int_1^{+\infty} \chi_F(t) \frac{dt}{t}.$$

Example 1.21 1) *The linear measure of* $E = [a, b]$, *where* $a, b \in [0, +\infty[$, *is*

$$m(E) = \int_0^{+\infty} \chi_E(t)\, dt = \int_a^b dt = b - a.$$

2) *The linear measure of any countable subset of* $[0, +\infty[$ *is* 0.
3) *The logarithmic measure of* $F = [a, b]$, *where* $a, b \in [1, +\infty[$, *is*

$$lm(F) = \int_1^{+\infty} \chi_F(t) \frac{dt}{t} = \int_a^b \frac{dt}{t} = \ln(\frac{b}{a}).$$

Definition 1.14 *For a meromorphic function* f, $S(r, f)$ *denotes any quantity that satisfies*

$$S(r, f) = o(T(r, f)), \quad (r \to +\infty, \quad r \notin E),$$

where E *is a set of finite logarithmic measure.*

Lemma 1.3 ([15]) *Assume that* f *is a non-constant meromorphic function. If the order of* f *is finite, then*

$$m\left(r, \frac{f'}{f}\right) = O(\log r), \quad (r \to +\infty).$$

If the order of f *is infinite, then*

$$m\left(r, \frac{f'}{f}\right) = O(\log(rT(r, f))), \quad (r \to +\infty, \quad r \notin E_0),$$

where E_0 *is a set whose linear measure is not greater than* 2.

Remark 1.5 *If* f *is a non-constant rational function, so is* $\frac{f'}{f}$, *and the degree of the numerator is always less than that of the denominator at least by* 1. *Hence*

$$m\left(r, \frac{f'}{f}\right) = o(1), \quad (r \to +\infty).$$

If f is a transcendental meromorphic function, then

$$\log r = o\left(T\left(r, f\right)\right), \quad \left(r \to +\infty\right),$$

and

$$\log T\left(r, f\right) = o\left(T\left(r, f\right)\right), \quad \left(r \to +\infty\right).$$

*Then Lemma **1.3** can be stated as*

$$m\left(r, \frac{f'}{f}\right) = S\left(r, f\right).$$

For any meromorphic function f that is not constant.

Lemma 1.4 ([15, 37]) *Assume that f is a non-constant meromorphic function and a_j $(j = 1, 2, \ldots, q)$ are q complex numbers different from each other. Then*

$$m\left(r, \sum_{j=1}^{q} \frac{1}{f - a_j}\right) = \sum_{j=1}^{q} m\left(r, \frac{1}{f - a_j}\right) + O\left(1\right).$$

Theorem 1.8 (The second fundamental theorem, [15]) *Suppose f is a non-constant meromorphic function and a_j $(j = 1, 2, \ldots, q)$ are q (≥ 2) distinct complex numbers. Then, we have*

$$m\left(r, f\right) + \sum_{j=1}^{q} m\left(r, \frac{1}{f - a_j}\right) \leq 2T\left(r, f\right) - N_1\left(r\right) + S\left(r, f\right),$$

where

$$N_1\left(r\right) = 2N\left(r, f\right) - N\left(r, f'\right) + N\left(r, \frac{1}{f'}\right).$$

Proof. Let

$$F\left(z\right) = \sum_{j=1}^{q} \frac{1}{f\left(z\right) - a_j}.$$

In terms of Lemma **1.4**, we have

$$m\left(r, F\right) = \sum_{j=1}^{q} m\left(r, \frac{1}{f - a_j}\right) + O\left(1\right). \tag{1.3}$$

On the other hand,

$$\begin{aligned} m\left(r, F\right) &\leq m\left(r, f'F\right) + m\left(r, \frac{1}{f'}\right) \\ &\leq m\left(r, f'F\right) + T\left(r, f'\right) - N\left(r, \frac{1}{f'}\right) + O\left(1\right). \end{aligned}$$

Since

$$\begin{aligned} T\left(r, f'\right) &= m\left(r, f'\right)+N\left(r, f'\right) \\ &\leq m(r, f)+m\left(r, \frac{f'}{f}\right)+N\left(r, f'\right) \\ &= T(r, f)+m\left(r, \frac{f'}{f}\right)+N\left(r, f'\right)-N(r, f), \end{aligned} \tag{1.4}$$

it follows from (1.3), (1.4) and Lemma **1.3** that

$$\begin{aligned} m(r, f)+\sum_{j=1}^{q} m\left(r, \frac{1}{f-a_j}\right) &\leq 2T(r, f)-\left\{2N(r, f)-N\left(r, f'\right)+N\left(r, \frac{1}{f'}\right)\right\} \\ &\quad +m\left(r, \frac{f'}{f}\right)+m\left(r, \sum_{j=1}^{q} \frac{f'}{f-a_j}\right)+O(1) \\ &\leq 2T(r, f)-N_1(r)+m\left(r, \frac{f'}{f}\right)+\sum_{j=1}^{q} m\left(r, \frac{f'}{f-a_j}\right)+O(1) \\ &\leq 2T(r, f)-N_1(r)+S(r, f), \end{aligned}$$

This concludes the proof of Theorem **1.8**. ■

Corollary 1.2 ([37]) *Assume that f is a non-constant meromorphic function and $a_1, a_2, \ldots, a_q$ are q (≥ 3) distinct values in $\mathbb{C} \cup \{\infty\}$. Then*

$$(q-2) T(r, f) \leq \sum_{j=1}^{q} N\left(r, \frac{1}{f-a_j}\right)-N_1(r)+S(r, f),$$

where

$$N_1(r)=2N(r, f)-N\left(r, f'\right)+N\left(r, \frac{1}{f'}\right).$$

Proof. Since

$$m\left(r, \frac{1}{f-a_j}\right)=T(r, f)-N\left(r, \frac{1}{f-a_j}\right)+O(1)$$

holds for any finite value a_j, Corollary **1.2** follows immediately from Theorem **1.8**. ■

In the following, we prove a more precise form of the second fundamental theorem.

Definition 1.15 ([20]) *Assume that f is a meromorphic function such that $f \not\equiv a$ and $a \in \mathbb{C}$. We denote by $\overline{n}(t, a, f)$ the number of distinct zeros of $f(z)-a$ on the closed disc $\overline{D(t)}$, any of it be counted only once, and by $\overline{n}(t, f)$ or $\overline{n}(t, \infty, f)$ the number of distinct poles of f on the closed disc $\overline{D(t)}$, any of it be counted only once. We define*

$$\overline{N}(r, a, f)=\int_0^r \frac{\overline{n}(t, a, f)-\overline{n}(0, a, f)}{t} dt+\overline{n}(0, a, f) \log r, \qquad a \neq \infty,$$

and

$$\overline{N}(r,f)=\overline{N}(r,\infty,f)=\int_0^r\frac{\overline{n}(t,\infty,f)-\overline{n}(0,\infty,f)}{t}dt+\overline{n}(0,\infty,f)\log r.$$

Theorem 1.9 ([37]) *Assume that f is a non-constant meromorphic function and $a_1,a_2,\ldots,a_q$ are q (≥ 3) distinct values in $\mathbb{C}\cup\{\infty\}$. Then*

$$(q-2)T(r,f)\leq\sum_{j=1}^{q}\overline{N}\left(r,\frac{1}{f-a_j}\right)+S(r,f).$$

Proof. From Corollary **1.2**, we have

$$(q-2)T(r,f)\leq\sum_{j=1}^{q}N\left(r,\frac{1}{f-a_j}\right)-N_1(r)+S(r,f). \tag{1.5}$$

Let

$$n_1(t)=2n(t,f)-n(t,f')+n\left(t,\frac{1}{f'}\right).$$

We have

$$N_1(r)=\int_0^r\frac{n_1(t)-n_1(0)}{t}dt+n_1(0)\log r.$$

If z_0 is a pole of f of order k in $\overline{D(t)}$, then z_0 be counted $k-1$ times by $n_1(t)$. Similarly, for a finite value a, if z_0 is a zero of $f-a$ of order k in $\overline{D(t)}$, then z_0 be also counted $k-1$ times by $n_1(t)$. Hence

$$\sum_{j=1}^{q}N\left(r,\frac{1}{f-a_j}\right)-N_1(r)\leq\sum_{j=1}^{q}\overline{N}\left(r,\frac{1}{f-a_j}\right). \tag{1.6}$$

From (1.5) and (1.6), we obtain

$$(q-2)T(r,f)\leq\sum_{j=1}^{q}\overline{N}\left(r,\frac{1}{f-a_j}\right)+S(r,f).$$

Thus, this concludes the proof of Theorem **1.9.** ■

1.7 Meromorphic functions with exceptional values

Definition 1.16 ([37]) *Assume that $f(z)$ is a meromorphic function in the complex plane and a any finite value. If $f(z)-a$ has no zero, then a is called a Picard* [4] *exceptional value of $f(z)$.*

[4] Charles E'mile Picard (July 24, 1856 – December 11, 1941) was one of the most famous French mathematicians.

Definition 1.17 ([37]) *Assume that $f(z)$ is a transcendental meromorphic function in the complex plane and a any finite value. If $f(z)-a$ has at most finitely many zeros, then a is called a generalized Picard exceptional value of $f(z)$.*

Example 1.22 *Consider $g(z)=e^z$, then we have 0 is Picard exceptional value of $g(z)$.*

Example 1.23 *Consider $g(z)=(z-1)e^z+a$, for any finite value a, a is a generalized Picard exceptional value of $g(z)$.*

Theorem 1.10 ([37], See Theorem 1.41) *Any transcendental meromorphic function in the complex plane has at most two generalized Picard exceptional values.*

Theorem 1.11 ([37]) *Assume that $f(z)$ is a non-constant entire function in the complex plane. If 0, ∞ are Picard exceptional values of $f(z)$, then $f(z)=e^{h(z)}$, where $h(z)$ is a non-constant entire function.*

Proof. Since $f(z)$ is an entire function without any zero, $\frac{f'(z)}{f(z)}$ is also an entire function. Let z_0 be any fixed point in the complex plane and $\ln f(z_0)$ a fixed branch of $\ln f(z_0)$. Let

$$h(z)=\int_{z_0}^{z}\frac{f'(z)}{f(z)}dz+\ln f(z_0).$$

It is obvious that $h(z)$ is a non-constant entire function and

$$e^{h(z)}=\exp\left\{\int_{z_0}^{z}\frac{f'(z)}{f(z)}dz+\ln f(z_0)\right\}=f(z).$$

With this, This concludes the proof of Theorem **1.11.** ■

Example 1.24 *Consider $g(z)=e^{\cos z}$, then we have, 0 and ∞ are Picard exceptional values of $g(z)$.*

Based on Theorem **1.11**, the following corollary can be obtained

Corollary 1.3 *Assume that $f(z)$ is a non-constant meromorphic function in the complex plane and a any finite value. If a and ∞ are Picard exceptional value of $f(z)$, then $f(z) = e^{h(z)} + a$, where $h(z)$ is a non-constant entire function.*

Example 1.25 *Consider $g(z) = e^{p(z)} + a$, where a is finite value and $p(z)$ is a polynomial, then we have, a and ∞ are Picard exceptional values of $g(z)$.*

Definition 1.18 (Borel's exceptional value, [37]) *Assume that f is a transcendental meromorphic function in the complex plane with the order $\rho(f)$ and let $\lambda(f)$ be the exponent of convergence of zeros for f. A complex number a is said to be a Borel [5] exceptional value if*

$$\lambda(f-a) < \rho(f).$$

Theorem 1.12 ([37]) *Assume that f is a transcendental meromorphic function in the complex plane with finite positive order ρ. Then f has two Borel's exceptional values at most.*

Proof. Assume, to the contrary, that is there are three distinct Borel's exceptional values, say, a_1, a_2 and a_3 at least. Set

$$\overline{\lim_{r\to+\infty}} \frac{\log^+ N\left(r, \frac{1}{f-a_j}\right)}{\log r} = \lambda_j \quad (j=1,2,3)$$

and

$$\lambda = \max\{\lambda_1,\ \lambda_2,\ \lambda_3\}.$$

Then $\lambda < \rho$. Since f is of finite order, then Corollary **1.2** imply

$$(1-o(1))\,T(r,f) \le \sum_{j=1}^{3} N\left(r, \frac{1}{f-a_j}\right) \quad (j=1,2,3),$$

which gives

$$(1-o(1))\,T(r,f) \le 3\max N\left(r, \frac{1}{f-a_j}\right) \quad (j=1,2,3).$$

Using the definition of the exponent of convergence of zeros , For every $\varepsilon > 0$, there is a positive number R such that

$$(1-o(1))\,T(r,f) \le 3\max N\left(r, \frac{1}{f-a_j}\right) \le 3r^{\lambda+\varepsilon}, \quad (j=1,2,3),\ r \ge R$$

[5] E'mile Borel (January 7, 1871 – February 3, 1956) was one of the most famous French mathematicians.

This means that

$$\overline{\lim_{r\to+\infty}} \frac{\log^+ T(r,f)}{\log r} \leq \lambda + \varepsilon,$$

holds for any $\varepsilon > 0$. We have

$$\rho \leq \lambda < \rho.$$

This is a contradiction. ■

From the previous theorem, we conclude the following remark, which we often use to prove some of our results in Chapter Three.

Remark 1.6 *Assume that f is a transcendental meromorphic function in the complex plane with the order $\rho(f)$ and let $\lambda(f)$ be the exponent of convergence of zeros for f. A complex number a is not a Borel exceptional value, then*

$$\lambda(f-a) = \rho(f).$$

Example 1.26 *Consider $g(z) = e^{p(z)} + a$, where a is a non-zero finite value and $p(z)$ is a polynomial of degree n, then*

$$\lambda(g) = \lambda(e^p + a) = \rho(e^p) = n.$$

Chapter 2

Nevanlinna theory for the difference operator and Theorems related to combinations of meromorphic functions

In this chapter, we present some important theory that connect Nevanlinna theory with difference equations. We clarify the difference analogue of the Lemma on the Logarithmic Derivative, as well as the difference analogue of the Second Main Theorem, and we show some theorems related to combinations of meromorphic functions.

2.1 Introduction

The difference analogue of the Lemma on the Logarithmic Derivative and the difference analogue of the Second Main Theorem in Nevanlinna theory play a key role in studying meromorphic solutions of complex difference equations. In 2006, Halburd–Korhonen [13] and Chiang–Feng [9] independently obtained the difference version of the logarithmic derivative lemma (commonly referred to as the logarithmic difference lemma), and Halburd–Korhonen [13] obtained the difference analogue of the Second Main Theorem for meromorphic functions with finite order on the complex plane. We say that a meromorphic function $a(z)$ is a small function of $f(z)$ if $T(r,a) = S(r,f)$, we use $S\left(f\right)$ to denote the family of all small functions with respect to $f\left(z\right)$. For a meromorphic function $f\left(z\right)$ we define its shift by $f_c\left(z\right) = f\left(z+c\right)$, and its difference operators by

$$\Delta_c f\left(z\right) = f\left(z+c\right) - f\left(z\right),\ \Delta_c^n f\left(z\right) = \Delta_c^{n-1}\left(\Delta_c f\left(z\right)\right),\ n \in \mathbb{N},\ n \geq 2.$$

Theorems related to combinations of meromorphic functions play a crucial role in studying their uniqueness. Results derived from these theories are used to investigate the behavior of meromorphic functions in difference equations. Over the years, these theories have evolved to include advanced results that deepen our understanding of meromorphic functions and provide new tools for addressing complex mathematical problems. As a result, they have become a fundamental component of research in complex analysis and have been widely cited in numerous papers. The theory and results discussed here are sourced from [1, 2, 7, 8, 13, 14, 15, 34, 37].

2.2 Difference analogue of the Lemma on the Logarithmic Derivative

Lemma 2.1 ([14]) *Suppose that f is a meromorphic function such that $f(0) \neq 0, \infty$ and let $c \in \mathbb{C}$. Then for all $\alpha > 1$, $\delta < 1$ and $r \geq 1$,*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) \leq \frac{K(\alpha, \delta, c)}{r^{\delta}} \left(T\left(\alpha\left(r+|c|\right), f\right) + \log^{+} \frac{1}{|f(0)|}\right),$$

where

$$K(\alpha, \delta, c) = \frac{8|c|(3\alpha+1) + 8\alpha(\alpha-1)|c|^{\delta}}{\delta(1-\delta)(\alpha-1)^{2}}.$$

Proof. Consider $\{a_n\}$ denote the sequence of all zeros of f , and similarly let $\{b_m\}$ be the pole sequence of f , where $\{a_n\}$ and $\{b_m\}$ are listed according to their multiplicities and ordered by increasing modulus. By applying Poisson–Jensen formula with $s = \frac{\alpha+1}{2}(r+|c|)$, see, for instance, [15, Theorem 1.1], we obtain

$$\begin{aligned}
\log\left|\frac{f(z+c)}{f(z)}\right| &= \int_0^{2\pi} \log\left|f\left(se^{i\theta}\right)\right| \operatorname{Re}\left(\frac{se^{i\theta}+z+c}{se^{i\theta}-z-c} - \frac{se^{i\theta}+z}{se^{i\theta}-z}\right)\frac{d\theta}{2\pi} \\
&\quad + \sum_{|a_n|<s} \log\left|\frac{s(z+c-a_n)}{s^2-\overline{a}_n(z+c)} \frac{s^2-\overline{a}_n z}{s(z-a_n)}\right| \\
&\quad - \sum_{|b_m|<s} \log\left|\frac{s(z+c-b_m)}{s^2-\overline{b}_m(z+c)} \frac{s^2-\overline{b}_m z}{s(z-b_m)}\right| \\
&= S_1 + S_2 - S_3.
\end{aligned}$$

where

$$\begin{aligned}
S_1 &= \int_0^{2\pi} \log\left|f\left(se^{i\theta}\right)\right| \operatorname{Re}\left(\frac{se^{i\theta}+z+c}{se^{i\theta}-z-c} - \frac{se^{i\theta}+z}{se^{i\theta}-z}\right)\frac{d\theta}{2\pi}, \\
S_2 &= \sum_{|a_n|<s} \log\left|\frac{s(z+c-a_n)}{s^2-\overline{a}_n(z+c)} \frac{s^2-\overline{a}_n z}{s(z-a_n)}\right|, \\
S_3 &= \sum_{|b_m|<s} \log\left|\frac{s(z+c-b_m)}{s^2-\overline{b}_m(z+c)} \frac{s^2-\overline{b}_m z}{s(z-b_m)}\right|.
\end{aligned}$$

Therefore, by denoting

$$E = \left\{\varphi \in [0, 2\pi] : \left|\frac{f\left(re^{i\varphi}+c\right)}{f\left(re^{i\varphi}\right)}\right| \geq 1\right\},$$

we have

$$\begin{aligned} m\left(r, \frac{f(z+c)}{f(z)}\right) &= \int_E \log\left|\frac{f\left(re^{i\varphi}+c\right)}{f\left(re^{i\varphi}\right)}\right|\frac{d\varphi}{2\pi} \\ &\leq \int_0^{2\pi}\left|S_1\left(re^{i\varphi}\right)\right|+\left|S_2\left(re^{i\varphi}\right)\right|+\left|S_3\left(re^{i\varphi}\right)\right|\frac{d\varphi}{2\pi}. \end{aligned}$$

We will now proceed to estimate each

$$\int_0^{2\pi}\left|S_j\left(re^{i\varphi}\right)\right|\frac{d\varphi}{2\pi},\ j=1,\ 2,\ 3$$

separately. Since

$$\begin{aligned} |S_1| &= \left|\int_0^{2\pi}\log\left|f\left(se^{i\theta}\right)\right|\operatorname{Re}\left(\frac{2cse^{i\theta}}{\left(se^{i\theta}-z-c\right)\left(se^{i\theta}-z\right)}\right)\frac{d\theta}{2\pi}\right| \\ &\leq \frac{2|c|s}{(s-r-|c|)^2}\int_0^{2\pi}\left|\log\left|f\left(se^{i\theta}\right)\right|\right|\frac{d\theta}{2\pi} \\ &= \frac{2|c|s}{(s-r-|c|)^2}\left(m(s,f)+m\left(s,\frac{1}{f}\right)\right), \end{aligned}$$

we have

$$\int_0^{2\pi}\left|S_1\left(re^{i\varphi}\right)\right|\frac{d\varphi}{2\pi}\leq\frac{4|c|s}{(s-r-|c|)^2}\left(T(s,f)+\log^+\frac{1}{|f(0)|}\right). \tag{2.1}$$

Next we consider the cases $j=2,3$ combined together. First, by denoting $\{q_k\}=\{a_n\}\cup\{b_m\}$ and using Lemma **1.1**, we have

$$\begin{aligned} \int_0^{2\pi}\left|S_2\left(re^{i\varphi}\right)\right|+\left|S_3\left(re^{i\varphi}\right)\right|\frac{d\varphi}{2\pi} \leq\ & \sum_{|q_k|<s}\int_0^{2\pi}\log^+\left|1+\frac{c}{re^{i\theta}-q_k}\right|\frac{d\theta}{2\pi} \\ & +\sum_{|q_k|<s}\int_0^{2\pi}\log^+\left|1-\frac{c}{re^{i\theta}+c-q_k}\right|\frac{d\theta}{2\pi} \\ & +\sum_{|q_k|<s}\int_0^{2\pi}\log^+\left|1+\frac{\overline{q}_k c}{s^2-\overline{q}_k(z+c)}\right|\frac{d\theta}{2\pi} \\ & +\sum_{|q_k|<s}\int_0^{2\pi}\log^+\left|1-\frac{\overline{q}_k c}{s^2-\overline{q}_k z}\right|\frac{d\theta}{2\pi}. \end{aligned} \tag{2.2}$$

Second, for any $a\in\mathbb{C}$, and for all $\delta<1$,

$$\int_0^{2\pi}\frac{1}{\left|re^{i\theta}-a\right|^\delta}d\theta\leq 4\int_0^{\frac{\pi}{2}}\frac{1}{\left|re^{i\theta}-|a|\right|^\delta}d\theta\leq\frac{2\pi}{(1-\delta)r^\delta}$$

since

$$\left|re^{i\theta}-|a|\right|\geq r\theta\frac{2}{\pi}\text{ for all }0\leq\theta\leq\frac{\pi}{2}.$$

Therefore

$$\begin{aligned}\int_0^{2\pi} \log^+\left|1+\frac{c}{re^{i\theta}-a}\right|\frac{d\theta}{2\pi} &\leq \frac{1}{\delta}\int_0^{2\pi}\log^+\left(1+\left|\frac{c}{re^{i\theta}-a}\right|^{\delta}\right)\frac{d\theta}{2\pi}\\ &\leq \frac{1}{\delta}\int_0^{2\pi}\left|\frac{c}{re^{i\theta}-a}\right|^{\delta}\frac{d\theta}{2\pi}\\ &\leq \frac{|c|^{\delta}}{\delta\left(1-\delta\right)r^{\delta}},\end{aligned}\tag{2.3}$$

and similarly

$$\int_0^{2\pi}\log^+\left|1-\frac{c}{re^{i\theta}+c-a}\right|\frac{d\theta}{2\pi}\leq\frac{|c|^{\delta}}{\delta\left(1-\delta\right)r^{\delta}}.\tag{2.4}$$

Third, since for all a such that $|a|<s$,

$$\left|\frac{a}{s^2-\overline{a}z}\right|\leq\frac{1}{s-r},$$

we have

$$\int_0^{2\pi}\log^+\left|1+\frac{\overline{a}c}{s^2-\overline{a}\left(z+c\right)}\right|\frac{d\theta}{2\pi}\leq\frac{|c|}{s-r-|c|}\tag{2.5}$$

and

$$\int_0^{2\pi}\log^+\left|1-\frac{\overline{a}c}{s^2-\overline{a}z}\right|\frac{d\theta}{2\pi}\leq\frac{|c|}{s-r}.\tag{2.6}$$

Finally, by combining inequalities (2.1), (2.2), (2.3), (2.4), (2.5) and (2.6), we obtain

$$\begin{aligned}m\left(r,\frac{f(z+c)}{f(z)}\right) &\leq \left(\frac{2\,|c|}{s-r-|c|}+\frac{2\,|c|^{\delta}}{\delta\left(1-\delta\right)r^{\delta}}\right)\left(n\left(s,f\right)+n\left(s,\frac{1}{f}\right)\right)\\ &\quad+\left(\frac{4\,|c|\,s}{\left(s-r-|c|\right)^2}\right)\left(T\left(s,f\right)+\log^+\frac{1}{|f\left(0\right)|}\right).\end{aligned}$$

Therefore, using the fact that

$$n\left(s,f\right)+n\left(s,\frac{1}{f}\right)\leq\frac{4\alpha}{\alpha-1}\left(T\left(\alpha\left(r+|c|\right),f\right)+\log^+\frac{1}{|f\left(0\right)|}\right),$$

see [15, p 37], and $s=\frac{\alpha+1}{2}(r+|c|)$, we conclude

$$\begin{aligned}m\left(r,\frac{f(z+c)}{f(z)}\right) &\leq \left(\frac{8\,|c|\left(3\alpha+1\right)}{\left(\alpha-1\right)^2\left(r+|c|\right)}+\frac{8\alpha\,|c|^{\delta}}{\delta\left(1-\delta\right)\left(\alpha-1\right)r^{\delta}}\right)\\ &\quad\times\left(T\left(\alpha\left(r+|c|\right),f\right)+\log^+\frac{1}{|f\left(0\right)|}\right)\\ &\leq \frac{8\,|c|\left(3\alpha+1\right)+8\alpha\left(\alpha-1\right)|c|^{\delta}}{\delta\left(1-\delta\right)\left(\alpha-1\right)^2r^{\delta}}\left(T\left(\alpha\left(r+|c|\right),f\right)+\log^+\frac{1}{|f\left(0\right)|}\right).\end{aligned}$$

This concludes the proof of lemma **2.1**. ■

Lemma 2.2 ([8, Lemma 3.3.1]) *Let $F(r)$ and $\varphi(r)$ be positive, nondecreasing, continuous functions defined for $r_0 \leq r < \infty$, and assume that $F(r) \geq e$ for $r \geq r_0$ Let $\zeta(x)$ be a positive, nondecreasing, continuous function defined for $e \leq x < \infty$. Let $C > 1$ be a constant, and let E be the closed subset of $[r_o\ , +\infty[$ defined by*

$$E = \left\{ r \in [r_o\ , +\infty[\, : F(r + \frac{\varphi(r)}{\zeta(F(r))}) \geq CF(r) \right\}.$$

Then,for all $R < \infty$,

$$\int\limits_{E\cap[r_0,R]} \frac{dr}{\varphi(r)} \leq \frac{1}{\zeta(e)} + \frac{1}{\log C} \int\limits_{e}^{F(R)} \frac{dx}{x\zeta(x)}.$$

Theorem 2.1 ([14]) *Assume that $f(z)$ is a non-constant meromorphic function, $c \in \mathbb{C}$, $\delta < 1$ and $\varepsilon > 0$.*
then

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = o\left(\frac{T\left(r+|c|\,, f\right)^{1+\varepsilon}}{r^{\delta}}\right) \tag{2.7}$$

for all r outside f a possible exceptional set E with finite logarithmic measure

$$\int\limits_{E} \frac{dr}{r} < \infty.$$

Proof. Consider $\zeta(x)$ and $\varphi(r)$ be positive, nondecreasing, continuous functions defined for $e \leq x \leq \infty$ and $r_0 \leq r \leq \infty$, respectively, where r_0 is such that $T(r+|c|, f) \geq e$ for all $r \geq r_0$. Then by Lemma **2.2**

$$T\left(r+|c|+\frac{\varphi(r)}{\zeta\left(T\left(r+|c|\,, f\right)\right)}, f\right) \leq 2T\left(r+|c|\,, f\right)$$

for all r outside of a set E satisfying

$$\int\limits_{E\cap[r_0,R]} \frac{dr}{\varphi(r)} \leq \frac{1}{\zeta(e)} + \frac{1}{\log 2} \int\limits_{e}^{T(r+|c|,f)} \frac{dx}{x\zeta(x)}$$

where $R < \infty$. Therefore, by choosing $\varphi(r) = r$ and $\zeta(x) = x^{\frac{\varepsilon}{2}}$ with $\varepsilon > 0$, and defining

$$\alpha = 1 + \frac{r}{(r+|c|)\, T\left(r+|c|\,, f\right)^{\frac{\varepsilon}{2}}}, \tag{2.8}$$

we have

$$T\left(\alpha\left(r+|c|\right), f\right) = T\left(r+|c|+\frac{\varphi(r)}{\zeta\left(T\left(r+|c|\,, f\right)\right)}, f\right) \leq 2T\left(r+|c|\,, f\right) \tag{2.9}$$

for all r outside of a set E with finite logarithmic measure. Hence, if $f(0) \neq 0, \infty$, the assertion follows by combining (2.8) and (2.9) with Lemma **2.1**. Otherwise we apply Lemma **2.1** with the function $g(z) = z^p f(z)$, $p \in \mathbb{Z}$ is chosen such that $g(0) \neq 0, \infty$. ■

When f is of finite order, the right-hand side of (2.7) is small compared to $T(r, f)$, and therefore relation (2.7) is a natural analogue of the Lemma on the Logarithmic Derivative

$$m\left(r, \frac{f'}{f}\right) = O\left(\log T\left(r, f\right) + \log r\right).$$

Concerning the sharpness of Theorem **2.1**, the finite-order functions, $exp(z^n)$ and $\tan(z)$ show that δ in (2.7) cannot be replaced by any number strictly greater than one. If f is of infinite order, the quantity $T(r + |c|, f)r^{-\delta}$ may be comparable to $T(r, f)$. For instance, by choosing $f(z) = exp(exp(z))$, we have

$$\begin{aligned} m(r, \frac{f(z+c)}{f(z)}) &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left| e^{e^{re^{i\theta}}(e^c-1)} \right| d\theta, \ c > 0 \\ &= \frac{(e^c-1)}{2\pi}\int_0^{2\pi} \log^+ \left| f\left(re^{i\theta}\right)\right| d\theta, \ c > 0 \\ &= (e^c-1)T(r, f), \ c > 0. \end{aligned}$$

Therefore Theorem **2.1** is mostly useful when applied to functions with finite order, although the assertion remains valid for all meromorphic functions. In the finite-order case we can also remove the ε in Theorem **2.1**.

Corollary 2.1 ([7]) *Assume that f is a non-constant meromorphic function of finite order, $c \in \mathbb{C}$ and $\delta < 1$. Then*

$$m(r, \frac{f(z+c)}{f(z)}) = o\left(\frac{T(r+|c|, f)}{r^\delta}\right) \tag{2.10}$$

In addition, we can conclude the following:

$$m(r, \frac{f(z+c)}{f(z)}) = o\left(\frac{T(r, f)}{r^\delta}\right)$$

for all r outside of a possible exceptional set E with finite logarithmic measure. Then, applying this result to the function $f(z) - a(z)$, where $a(z)$ is a finite-order periodic function with period c, we have

$$\begin{aligned} m(r, \frac{\Delta_c f(z)}{f(z)-a(z)}) &= m(r, \frac{f(z+c)-a(z+c)}{f(z)-a(z)}) + O(1) \\ &= o\left(\frac{T(r, f(z)-a(z))}{r^\delta}\right) + O(1) \end{aligned}$$

Then, we can conclude the following

$$m(r, \frac{\Delta_c f}{f-a}) = S(r, f-a).$$

Proof. Choose any $\delta < 1$ and denote $\delta' = \frac{(1+\delta)}{2}$. Since f is of finite order, we have $T(r + |c|, f) \leq r^{\rho}$ for some $\rho > 0$ and for all r sufficiently large. Therefore, by Theorem **2.1**

$$m(r, \frac{f(z+c)}{f(z)}) = o\left(\frac{T(r+|c|, f)}{r^{\delta' - \varepsilon\rho}}\right),$$

where $\varepsilon > 0$. The assertion follows by choosing $\varepsilon = \frac{(\delta' - \delta)}{\rho}$. ■

Remark 2.1 ([7]) *by replacing z by $z+h$, where $h \in \mathbb{C}$, and c by $c-h$ in* (2.10)*, and using the inequality*

$$T(r, f\,(z+h)) \leq (1+\varepsilon)\, T(r+|h|, f\,(z)),\ \ \varepsilon > 0,\ \ r > r_0,$$

see [1] or [2], we immediately have

$$m(r, \frac{f(z+c)}{f(z+h)}) = o\left(\frac{T(r+|c-h|+|h|, f)}{r^{\delta}}\right)$$

for all $\delta < 1$ outside of a possible exceptional set E with finite logarithmic measure.

2.3 Characteristics of Polynomial and Rational functions in meromorphic functions

For polynomials in meromorphic functions, we have the following characteristic

Theorem 2.2 ([37]) *Assume that $f(z)$ is a meromorphic function in the complex plane and*

$$P(z) = a_0 f^n + a_1 f^{n-1} + ... + a_n,$$

where $a_0\,(\not\equiv 0)$, a_1,...,a_n are small functions of $f(z)$. Then

$$T(r, P(f)) = nT(r, f) + S\,(r, f)\,. \tag{2.11}$$

Proof. Without loss of generality, we assume that $a_0(z) \equiv 1$.

For $n = 1$, the formula (2.11) is obvious. Therefore from

$$\begin{aligned} T(r, P(f)) &\leq T(r, f \cdot \sum_{k=1}^{n} a_{k-1} f^{n-k}) + T(r, a_n) + O\,(1) \\ &\leq T(r, f) + T(r, \sum_{k=1}^{n} a_{k-1} f^{n-k}) + S\,(r, f) \end{aligned}$$

and by using induction in the integer n, we get

$$T(r, P(f)) \leq nT(r, f) + S\,(r, f)\,, \tag{2.12}$$

Now we prove the opposite inequality of above inequality. It is obvious that

$$N(r, P(f)) = nN(r, f) + S(r, f) \tag{2.13}$$

In order to estimate $m(r, P(f))$, let

$$A\left(z\right) = \max_{1 \leq i \leq n} \left|a_i(z)\right|^{\frac{1}{i}},\ i = 1, 2, ..., n$$

where $|z| = r$. For a fixed value of r, we denote E_1 the set of $\theta \in [0,\ 2\pi]$ such that

$$\left|f\left(re^{i\theta}\right)\right| \geq 2A\left(re^{i\theta}\right)$$

and E_2 the complement of E_1. On E_1, we have

$$\begin{aligned} |P(f)| &= |f|^n \cdot \left|1 + \frac{a_1}{f} + ... + \frac{a_n}{f^n}\right| \\ &\geq |f|^n \cdot \left\{1 - \left|\frac{a_1}{f}\right| - ... - \left|\frac{a_n}{f^n}\right|\right\} \\ &\geq |f|^n \cdot \left\{1 - \frac{1}{2} - ... - \frac{1}{2^n}\right\} \\ &= \frac{1}{2^n} \cdot |f|^n . \end{aligned}$$

Therefore

$$\begin{aligned} n \cdot m\left(r, f\right) &= m\left(r, f^n\right) \\ &= \frac{1}{2\pi} \int_{E_1} \log^+ |f^n|\, d\theta + \frac{1}{2\pi} \int_{E_2} \log^+ |f^n|\, d\theta \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} \log^+ |2^n P(f)|\, d\theta + \frac{1}{2\pi} \int_0^{2\pi} \log^+ |2A|^n\, d\theta \\ &\leq m\left(r, P(f)\right) + S(r, f). \end{aligned} \tag{2.14}$$

From (2.13) and (2.14), we get

$$T\left(r, P(f)\right) \geq nT\left(r, f\right) + S(r, f). \tag{2.15}$$

Hence (2.11) follows from (2.12) and (2.15). ■

The following theorem is fundamentally attributed to Valiron. The proof we present here was derived by Mokhon'ko .

Theorem 2.3 ([34]) *Assume that $f(z)$ is a non-constant meromorphic function in the complex plane and*

$$R(f) = \frac{P(f)}{Q(f)},$$

where

$$P(f) = \sum_{k=0}^{p} a_k f^k \text{ and } Q(f) = \sum_{j=0}^{q} b_j f^j$$

are two mutually prime polynomials in $f(z)$. If the coefficients $\{a_k(z)\}$, $\{b_j(z)\}$ are small functions of $f(z)$ and $a_p(z) \not\equiv 0$, $b_q(z) \not\equiv 0$, then

$$T(r, R(f)) = \max\{p, q\}\, T(r, f) + S\left(r, f\right). \tag{2.16}$$

Proof. Without loss of generality, we assume that $p \geq q$. For the opposite case, we can consider

$$\frac{1}{R(f)},$$

noting that

$$T(r, R(f)) = T(r, \frac{1}{R(f)}) + O(1).$$

By applying division algorithm, we have

$$\begin{aligned}
P(f) &= S_1(f) \cdot Q(f) + T_1(f), \;\; \deg S_1 = p - q, \;\; \deg T_1 = t_1 < q, \\
Q(f) &= S_2(f) \cdot T_1(f) + T_2(f), \;\; \deg S_2 = q - t_1, \;\; \deg T_2 = t_2 < t_1, \\
&\dots \\
T_{m-2}(f) &= S_m(f) \cdot T_{m-1}(f) + T_m(f), \;\; \deg S_m = t_{m-2} - t_{m-1}, \;\; \deg T_m = t_m = 0.
\end{aligned}$$

Since $P(f)$ and $Q(f)$ are mutually prime, we have

$$T_m(f) \not\equiv 0,$$

and

$$P(f) \cdot U(f) + Q(f) \cdot V(f) = 1, \tag{2.17}$$

where $U(f)$ and $V(f)$ are polynomials in f whose coefficients are small functions of f and

$$u \leq q - 1, \;\; v \leq p - 1, \;\; \text{here } u = \deg U \text{ and } v = \deg V.$$

Since

$$p + u = q + v \text{ and } p \geq q,$$

we see that

$$v \geq u.$$

By applying Theorem **2.2** and the first fundamental theorem repeat, we deduce that

$$\begin{aligned}
T(r, R(f)) &= T(r, \frac{P(f)}{Q(f)}) \\
&\leq T(r, S_1(f)) + T(r, \frac{T_1(f)}{Q(f)}) + O(1) \\
&= T(r, S_1(f)) + T(r, \frac{Q(f)}{T_1(f)}) + O(1) \\
&\leq T(r, S_1(f)) + T(r, S_2(f)) + T(r, \frac{T_2(f)}{T_1(f)}) + O(1) \\
&\quad \dots \\
&\leq T(r, S_1(f)) + T(r, S_2(f)) + \dots \\
&\quad + T(r, S_m(f)) + T(r, \frac{T_{m-1}(f)}{T_m(f)}) + O(1) \\
&= (p - q)\, T(r, f) + (q - t_1)\, T(r, f) + \dots \\
&\quad + (t_{m-2} - t_{m-1})\, T(r, f) + t_{m-1} T(r, f) + S(r, f) \\
&= pT(r, f) + S(r, f).
\end{aligned} \tag{2.18}$$

Noting that $v \geq u$, by similar procedure, we can prove that

$$T(r, \frac{V(f)}{U(f)}) \leq vT(r, f) + S(r, f). \tag{2.19}$$

In the following, we prove the opposite inequality.
By Theorem **2.2**, it follows from (2.17) that

$$\begin{aligned} T(r, \frac{U(f)}{V(f)} + \frac{Q(f)}{P(f)}) &= T(r, \frac{1}{V(f) \cdot P(f)}) \\ &= T(r, V(f) \cdot P(f)) + O(1) \\ &= (p + v) T(r, f) + S(r, f). \end{aligned} \tag{2.20}$$

On the other hand, by applying (2.19), we get

$$\begin{aligned} T(r, \frac{U(f)}{V(f)} + \frac{Q(f)}{P(f)}) &\leq T(r, \frac{U(f)}{V(f)}) + T\left(r, \frac{Q(f)}{P(f)}\right) + O(1) \\ &= T(r, \frac{V(f)}{U(f)}) + T\left(r, \frac{P(f)}{Q(f)}\right) + O(1) \\ &\leq vT(r, f) + T(r, R(f)) + S(r, f). \end{aligned} \tag{2.21}$$

Combining (2.20) and (2.21), we get

$$T(r, R(f)) \geq pT(r, f) + S(r, f). \tag{2.22}$$

From (2.18) and (2.22), we get (2.16) immediately. ■

2.4 Difference analogue of the Second main theorem

Lemma 2.3 ([13]) *Consider $c \in \mathbb{C}$, $n \in \mathbb{N}^*$, and suppose that $f(z)$ is a meromorphic function of finite order. Then for any small periodic function $a(z)$ with period c, with respect to $f(z)$,*

$$m\left(r, \frac{\Delta_c^n f}{f - a}\right) = S(r, f)$$

where the exceptional set associated with $S(r, f)$ is of at most finite logarithmic measure.

Proof. We prove by induction that for all $n \in \mathbb{N}^*$:

$$m\left(r, \frac{\Delta_c^n f}{f - a}\right) = S(r, f)$$

For $n = 1$: By applying Corollary **2.1**, we deduce that

$$m\left(r, \frac{\Delta_c f}{f - a}\right) = S(r, f).$$

Suppose the result is true for n, then

$$\begin{aligned} m\left(r,\frac{\Delta_c^{n+1}f}{f-a}\right) &= m\left(r,\frac{\Delta_c^{n+1}f}{f-a}\cdot\frac{\Delta_c^n f}{\Delta_c^n f}\right)\\ &= m\left(r,\frac{\Delta_c^{n+1}f}{\Delta_c^n f}\cdot\frac{\Delta_c^n f}{f-a}\right)\\ &\leq m\left(r,\frac{\Delta_c^{n+1}f}{\Delta_c^n f}\right)+m\left(r,\frac{\Delta_c^n f}{f-a}\right)\\ &= m\left(r,\frac{\Delta_c\left(\Delta_c^n f\right)}{\Delta_c^n f}\right)+m\left(r,\frac{\Delta_c^n f}{f-a}\right)\\ &= S\left(r,f\right) \end{aligned}$$

Hence the result is true for $n+1$. ∎

Halburd-Korhonen considered a difference analogue of the second main theorem, and proved the following Theorem

Theorem 2.4 ([13]) *Consider $c\in\mathbb{C}$, and suppose that f is a meromorphic function of finite order such that $\Delta_c f\not\equiv 0$. Let $q\geq 2$, and let $a_1(z)$,..., $a_q(z)$ be distinct meromorphic periodic functions with period c such that $a_k\in S(f)$ for all $k=1,...,q$. Then*

$$m\left(r,f\right)+\sum_{k=1}^{q}m\left(r,\frac{1}{f-a_k}\right)\leq 2T\left(r,f\right)-N^*\left(r,f\right)+S(r,f),$$

where

$$N^*\left(r,f\right)=2N\left(r,f\right)-N\left(r,\Delta_c f\right)+N\left(r,\frac{1}{\Delta_c f}\right)$$

and the exceptional set E associated with $S(r,f)$ is of at most finite logarithmic measure.

Proof. By denoting

$$P\left(f\right)=\prod_{k=1}^{q}\left(f-a_k\right).$$

we have

$$\frac{1}{P\left(f\right)}=\sum_{k=1}^{q}\frac{\alpha_k}{f-a_k}$$

where $\alpha_k\in S(f)$ are certain periodic functions with period c. Hence, by Lemma **2.3**, we obtain

$$m\left(r,\frac{\Delta_c f}{P\left(f\right)}\right)\leq\sum_{k=1}^{q}m\left(r,\frac{\Delta_c f}{f-a_k}\right)+S\left(r,f\right)=S\left(r,f\right)$$

and so

$$m\left(r,\frac{1}{P\left(f\right)}\right)=m\left(r,\frac{\Delta_c f}{P\left(f\right)}\frac{1}{\Delta_c f}\right)\leq m\left(r,\frac{1}{\Delta_c f}\right)+S\left(r,f\right)\tag{2.23}$$

By combining the first main theorem, (2.23) and Theorem **2.3**, we have

$$\begin{aligned}
T(r,\Delta_c f) &= m\left(r,\frac{1}{\Delta_c f}\right)+N\left(r,\frac{1}{\Delta_c f}\right)+O(1)\\
&\geq m\left(r,\frac{1}{P(f)}\right)+N\left(r,\frac{1}{\Delta_c f}\right)+S(r,f)\\
&= qT(r,f)-\sum_{k=1}^{q}N\left(r,\frac{1}{f-a_k}\right)+N\left(r,\frac{1}{\Delta_c f}\right)+S(r,f)\\
&= \sum_{k=1}^{q}m\left(r,\frac{1}{f-a_k}\right)+N\left(r,\frac{1}{\Delta_c f}\right)+S(r,f).
\end{aligned}$$

Thus, by Lemma **2.3**

$$\begin{aligned}
m(r,f)+\sum_{k=1}^{q}m\left(r,\frac{1}{f-a_k}\right) &\leq T(r,f)+N(r,\Delta_c f)+m(r,\Delta_c f)\\
&\quad -N\left(r,\frac{1}{\Delta_c f}\right)-N(r,f)+S(r,f)\\
&\leq T(r,f)+N(r,\Delta_c f)+m(r,f)\\
&\quad -N\left(r,\frac{1}{\Delta_c f}\right)-N(r,f)+S(r,f)\\
&= 2T(r,f)+N(r,\Delta_c f)-N\left(r,\frac{1}{\Delta_c f}\right)\\
&\quad -2N(r,f)+S(r,f).
\end{aligned}$$

With this, the proof of Theorem **2.4** is established. ■

2.5 Theorems related to combinations of meromorphic functions

Definition 2.1 ([20, 37]) *Suppose that $f_1, f_2, \ldots, f_n$ are meromorphic functions and*

$$W(f_1,f_2,\ldots,f_n):=\begin{vmatrix} f_1 & f_2 & \cdots & f_n\\ f_1' & f_2' & \cdots & f_n'\\ \vdots & \vdots & \ddots & \vdots\\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)}\end{vmatrix}.$$

The determinant $W(f_1, f_2, \ldots, f_n)$ is called the Wronskian of $f_1, f_2, \ldots, f_n$.

In studying on uniqueness theorems of meromorphic functions, the following Nevanlinna's theorem plays an important role.

Theorem 2.5 ([35, 37]) *Assume that* $f_1(z)$, $f_2(z)$,..., $f_n(z)$ *are linearly independent meromorphic functions satisfying the following identity*

$$\sum_{j=1}^{n} f_j \equiv 1. \tag{2.24}$$

Then for $1 \le j \le n$, *we have*

$$\begin{aligned} T(r, f_j) \le & \sum_{k=1}^{n} N\left(r, \frac{1}{f_k}\right) + N(r, f_j) + N(r, D) \\ & - \sum_{k=1}^{n} N(r, f_k) - N\left(r, \frac{1}{D}\right) + S(r), \end{aligned} \tag{2.25}$$

where D *is the Wronskian* $W(f_1, f_2, \ldots, f_n)$, *and*

$$S(r) = o(T(r)) \quad (r \to +\infty, \quad r \notin E),$$

here

$$T(r) = \max_{1 \le k \le n} \{T(r, f_k)\},$$

and E *is a set with finite linear measure.*

Proof. By taking derivatives in both sides of identity (2.24), we get

$$\sum_{j=1}^{n} f_j^{(k)} \equiv 0 \quad (k = 1, 2, \ldots, n-1). \tag{2.26}$$

Since $f_1(z), f_2(z), \ldots, f_n(z)$ are linearly independent, we see that $D \not\equiv 0$. From (2.24) and (2.26), we get

$$D = D_j \quad (j = 1, 2, \ldots, n),$$

where D_j is the algebraic cofactor of f_j in D. Hence

$$f_1 = \frac{D_1}{f_2 f_3 \cdots f_n} \Big/ \frac{D}{f_1 f_2 f_3 \cdots f_n} = \frac{A_1}{A}, \tag{2.27}$$

where

$$A = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \frac{f_1'}{f_1} & \frac{f_2'}{f_2} & \cdots & \frac{f_n'}{f_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{f_1^{(n-1)}}{f_1} & \frac{f_2^{(n-1)}}{f_2} & \cdots & \frac{f_n^{(n-1)}}{f_n} \end{vmatrix},$$

and A_1 is the algebraic cofactor of the element at the first column and the first row in A. From (2.27), we have

$$\begin{aligned} m(r, f_1) & \le m(r, A_1) + m\left(r, \frac{1}{A}\right) \\ & \le m(r, A_1) + m(r, A) + N(r, A) - N\left(r, \frac{1}{A}\right) + O(1). \end{aligned} \tag{2.28}$$

Since

$$A = \frac{D}{f_1 f_2 \cdots f_n},$$

we have

$$\begin{aligned} N(r, A) - N\left(r, \frac{1}{A}\right) &= \sum_{k=1}^{n} N\left(r, \frac{1}{f_k}\right) - \sum_{k=1}^{n} N(r, f_k) \\ &\quad + N(r, D) - N\left(r, \frac{1}{D}\right). \end{aligned} \tag{2.29}$$

Note that

$$m\left(r, \frac{f_j^{(k)}}{f_j}\right) = S(r, f_j) = S(r), \quad \left(\begin{array}{c} j = 1, 2, \ldots, n, \\ k = 1, 2, \ldots, n-1 \end{array}\right).$$

We have

$$m(r, A_1) + m(r, A) = S(r). \tag{2.30}$$

From (2.28), (2.29) and (2.30), we get

$$\begin{aligned} T(r, f_1) &= m(r, f_1) + N(r, f_1) \\ &\leq \sum_{k=1}^{n} N\left(r, \frac{1}{f_k}\right) + N(r, f_1) + N(r, D) \\ &\quad - \sum_{k=1}^{n} N(r, f_k) - N\left(r, \frac{1}{D}\right) + S(r). \end{aligned} \tag{2.31}$$

By the same method, we can prove other results similar to (2.31) for $f_j \ (2 \leq j \leq n)$. Hence (2.25) holds. ■

By using Theorem **2.5**, Nevanlinna [35] proved the following theorem.

Theorem 2.6 ([35, 37]) *Assume that* $f_1(z),\ f_2(z), \ldots, f_n(z)$, $(n \geq 2)$ *are meromorphic functions satisfying the following conditions:*

(*i*) $\sum_{j=1}^{n} C_j f_j(z) \equiv 0$, *where* $C_j \quad (j = 1, 2, \ldots, n)$ *are constants.*

(*ii*) $f_j(z) \not\equiv 0 \quad (j = 1, 2, \ldots, n)$, *and* $f_j(z)/f_k(z)$ *are not constants for* $1 \leq j < k \leq n$.

(*iii*) $\sum_{j=1}^{n} \left(N(r, f_j) + N\left(r, \frac{1}{f_j}\right)\right) = o(\tau(r)), \quad (r \to +\infty, \quad r \notin E)$,

where $\tau(r) = \min_{1 \leq j < k \leq n} \left\{T\left(r, \frac{f_j}{f_k}\right)\right\}$, *and* E *is a set with finite linear measure.*

Then $C_j = 0 \quad (j = 1, 2, \ldots, n)$.

Proof. The proof can be done by mathematical induction. First of all, let $n = 2$. We have

$$C_1 f_1(z) + C_2 f_2(z) \equiv 0.$$

If one of C_1 and C_2, say C_1, is not zero, then

$$\frac{f_1(z)}{f_2(z)} \equiv -\frac{C_2}{C_1},$$

which contradicts assumption (ii). Hence Theorem **2.6** is true for $n=2$.
Assuming that Theorem **2.6** is true for $n=l \quad (\geq 2)$, we now prove it is still true for $n=l+1$. In fact, if meromorphic functions $f_j(z) \quad (j=1,2,\ldots,l+1)$ satisfy the assumptions of Theorem **2.6**, then we have

$$\sum_{j=1}^{l+1} C_j f_j(z) \equiv 0. \tag{2.32}$$

Suppose that one of $C_j \quad (j=1,2,\ldots,l+1)$ is not zero. We can prove all of them are not zeros. Otherwise, without loss of generality, let $C_{l+1}=0$. From (2.32), we have

$$\sum_{j=1}^{l} C_j f_j(z) \equiv 0.$$

And $f_j(z) \quad (j=1,2,\ldots,l)$ satisfy the assumptions of Theorem **2.6**, where $n=l$. By the induction hypothesis, Theorem **2.6** is true for $n=l$. Hence $C_j=0$ for $j=1,2,\ldots,l$. Therefore all $C_j \quad (j=1,2,\ldots,l+1)$ are zeros. This is a contradiction. Hence $C_j \neq 0$ $(j=1,2,\ldots,l+1)$. Let

$$g_j(z) = -\frac{C_j f_j(z)}{C_{l+1} f_{l+1}(z)} \quad (j=1,2,\ldots,l). \tag{2.33}$$

From (2.32), we have

$$\sum_{j=1}^{l} g_j(z) \equiv 1.$$

If $g_j(z)$ $(j=1,2,\ldots,l)$ are linearly dependent, then there exist constants $a_j \quad (j=1,2,\ldots,l)$ (one of them is not zero) such that

$$\sum_{j=1}^{l} a_j g_j(z) \equiv 0.$$

That is

$$\sum_{j=1}^{l} a_j C_j f_j(z) \equiv 0.$$

Since, by hypothesis, Theorem **2.6** is true for $n=l$, we have $a_j C_j = 0 \quad (j=1,2,\ldots,l)$. Since one of $a_j \quad (j=1,2,\ldots,l)$ is not zero, say $a_1 \neq 0$, we deduce that $C_1=0$, which contradicts the known result $C_j \neq 0 \quad (j=1,2,\ldots,l+1)$. Hence $g_j(z) \quad (j=1,2,\ldots,l)$ are linearly independent.
Let $T(r) = \max\limits_{1\leq k\leq l}\{T(r,g_k)\}$. From (2.33), we get

$$N\left(r,\frac{1}{g_j}\right) + N(r,g_j) \leq N\left(r,\frac{1}{f_j}\right) + N(r,f_j) + N(r,f_{l+1}) + N\left(r,\frac{1}{f_{l+1}}\right),$$

where $j = 1, 2, \ldots, l$. By condition (iii), we have

$$N\left(r, \frac{1}{g_j}\right) + N(r, g_j) = S(r) \quad (j = 1, 2, \ldots, l),$$

where $S(r) = o(T(r)) \quad (r \to +\infty, \quad r \notin E)$. Hence

$$\sum_{j=1}^{l} N\left(r, \frac{1}{g_j}\right) = S(r),$$

and

$$\sum_{j=1}^{l} N(r, g_j) = S(r).$$

Applying Theorem **2.5** to functions $g_j(z) \quad (j = 1, 2, \ldots, l)$, we get

$$T(r, g_k) \leq S(r) \quad (k = 1, 2, \ldots, l).$$

Hence

$$T(r) \leq S(r),$$

which is impossible. This means that all $C_j \quad (j = 1, 2, \ldots, l+1)$ are zeros. Therefore Theorem **2.6** is true for $n = l + 1$. ■

From Theorem **2.6**, we can prove.

Theorem 2.7 ([37]) *Assume that* $f_1(z), f_2(z), \ldots, f_n(z)$, $(n \geq 2)$ *are meromorphic functions and* $g_1(z), g_2(z), \ldots, g_n(z)$ *are entire functions satisfying the following conditions:*

(i) $\sum_{j=1}^{n} f_j(z) e^{g_j(z)} \equiv 0$.

(ii) $g_j(z) - g_k(z)$ *are not constants for* $1 \leq j < k \leq n$.

(iii) *For* $1 \leq j \leq n, \quad 1 \leq h < k \leq n$,

$$T(r, f_j) = o\left\{T\left(r, e^{g_h - g_k}\right)\right\}, \quad (r \to +\infty, \quad r \notin E),$$

where E *is a set with finite linear measure.*

Then

$$f_j(z) \equiv 0 \ \ (j = 1, 2, \ldots, n).$$

Proof. The proof can be done by using mathematical induction. When $n = 2$, the condition (i) becomes

$$f_1(z) e^{g_1(z)} + f_2(z) e^{g_2(z)} \equiv 0.$$

If one of $f_j(z) \quad (j = 1, 2)$ is not identically zero, say $f_1(z) \not\equiv 0$, then

$$e^{g_1(z) - g_2(z)} \equiv -\frac{f_2(z)}{f_1(z)}.$$

By condition (iii), we have

$$\begin{aligned} T\left(r, e^{g_1-g_2}\right) &= T\left(r, \frac{f_2}{f_1}\right) \\ &\leq T(r, f_2) + T(r, f_1) + O(1) \\ &= o\left\{T\left(r, e^{g_1-g_2}\right)\right\}, \end{aligned}$$

a contradiction. Hence Theorem **2.7** holds for $n = 2$.
Now we assume that Theorem **2.7** is true for integer n (≥ 2). In the following, we prove that Theorem **2.7** is true for $n+1$. Suppose that $f_j(z), g_j(z)$ $(j = 1, 2, \ldots, n+1)$ satisfy the conditions of Theorem **2.7** and that one of $f_j(z)$ $(j = 1, 2, \ldots, n+1)$ is not identically zero. If one of $f_j(z)$ $(j = 1, 2, \ldots, n+1)$ is identically zero, say $f_{n+1}(z) \equiv 0$, then from the identity

$$\sum_{j=1}^{n+1} f_j(z)\, e^{g_j(z)} \equiv 0, \tag{2.34}$$

we have

$$\sum_{j=1}^{n} f_j(z)\, e^{g_j(z)} \equiv 0.$$

Therefore $f_j(z)$ $(j = 1, 2, \ldots, n)$ satisfy the conditions of Theorem **2.7**. By the induction hypothesis, Theorem **2.7** is true for n. Hence $f_j(z) \equiv 0$ $(j = 1, 2, \ldots, n)$, and thus $f_j(z) \equiv 0$ $(j = 1, 2, \ldots, n+1)$, which contradicts the assumption. Hence $f_j(z) \not\equiv 0$ $(j = 1, 2, \ldots, n+1)$. Let

$$F_j(z) = f_j(z)\, e^{g_j(z)}, \quad C_j = 1, \quad (j = 1, 2, \ldots, n+1). \tag{2.35}$$

From (2.34), we get

$$\sum_{j=1}^{n+1} C_j F_j(z) \equiv 0.$$

Obviously, $F_j(z) \not\equiv 0$ $(j = 1, 2, \ldots, n+1)$. And it is easy to see that $F_j(z)/F_k(z)$ are not constants for $1 \leq j < k \leq n+1$. Furthermore, we have

$$\begin{aligned} N(r, F_j) + N\left(r, \frac{1}{F_j}\right) &= N(r, f_j) + N\left(r, \frac{1}{f_j}\right) \\ &\leq 2T(r, f_j) + O(1) \\ &= o\left\{T\left(r, e^{g_h-g_k}\right)\right\}, \quad (r \to +\infty, \quad r \notin E), \end{aligned} \tag{2.36}$$

for $j = 1, 2, \ldots, n+1, \;\; 1 \leq h < k \leq n+1$. Since

$$\frac{F_h}{F_k} = \frac{f_h}{f_k} e^{g_h-g_k},$$

we have

$$\begin{aligned} T\left(r, e^{g_h-g_k}\right) &= T\left(r, \frac{f_k}{f_h} . \frac{F_h}{F_k}\right) \\ &\leq T(r, f_k) + T(r, f_h) + T\left(r, \frac{F_h}{F_k}\right) + O(1) \\ &= T\left(r, \frac{F_h}{F_k}\right) + o\left\{T\left(r, e^{g_h-g_k}\right)\right\}, \quad (r \to +\infty, \quad r \notin E). \end{aligned}$$

Hence

$$T\left(r, e^{g_h - g_k}\right) = O\left(T\left(r, \frac{F_h}{F_k}\right)\right), \quad (r \notin E). \tag{2.37}$$

From (2.36) and (2.37), we get

$$N(r, F_j) + N\left(r, \frac{1}{F_j}\right) = o\left\{T\left(r, \frac{F_h}{F_k}\right)\right\}, \quad (r \to +\infty, \quad r \notin E),$$

for $j = 1, 2, \ldots, n+1$, $1 \leq h < k \leq n+1$, which means that $F_j(z)$ $(j = 1, 2, \ldots, n+1)$ satisfy the conditions of Theorem **2.6**. Hence $C_j = 0$ $(j = 1, 2, \ldots, n+1)$, which contradicts (2.35). Therefore $f_j(z)$ $(j = 1, 2, \ldots, n+1)$ are identically zeros. ■

Theorem 2.8 ([37]) *Assume that* $f_1(z), f_2(z), \ldots, f_n(z)$, $(n \geq 2)$ *are meromorphic functions and* $g_1(z), g_2(z), \ldots, g_n(z)$ *are entire functions satisfying the following conditions:*

(i) $\sum_{j=1}^{n} f_j(z) e^{g_j(z)} \equiv f_{n+1}(z)$.

(ii) $g_h(z)$ *and* $g_j(z) - g_k(z)$ *are not constants for* $1 \leq h \leq n$ *and* $1 \leq j < k \leq n$.

(iii) For $1 \leq j \leq n+1$, $1 \leq h < k \leq n$, *and* $1 \leq l \leq n$

$$\begin{aligned} \rho(f_j) &< \rho\left(e^{g_h - g_k}\right), \\ \rho(f_j) &< \rho\left(e^{g_l}\right). \end{aligned}$$

Then

$$f_j(z) \equiv 0 \;\; (j = 1, 2, \ldots, n+1).$$

Proof. If we put $g_{n+1}(z) = 0$, (i) can be expressed as

$$\sum_{j=1}^{n} f_j(z) e^{g_j(z)} - f_{n+1}(z) e^{g_{n+1}(z)} \equiv 0.$$

Obviously $f_j(z), g_j(z)$ $(j = 1, 2, \ldots, n+1)$ satisfy the conditions of Theorem **2.7** then $f_j(z) \equiv 0$ $(j = 1, 2, \ldots, n+1)$. ■

Chapter 3

Applications of Nevanlinna Theory to Entire Functions Sharing a Small Function with Two Difference Operators

In this chapter, by applying Nevanlinna theory, we explore uniqueness problems of entire functions that share a small periodic entire function with their shifts and difference operators, or with two difference operators. We also show that the function can be expressed in a special form. Additionally, we enhance some results previously established in 2015 by A. El Farissi, Z. Latreuch, and A. Asiri [**10**].

3.1 Introduction

Nevanlinna theory is regarded as one of the most significant theories in complex analysis, particularly in the study of entire functions and solutions to difference equations. For an entire function $f(z)$ we define its shift by

$$f_c(z) = f(z+c),$$

and its difference operators by

$$L_c^n f(z) = \alpha_n f(z+nc) + ... + \alpha_1 f(z+c) + \alpha_0 f(z), \ n \in \mathbb{N}, \ n \geq 1$$

where $\alpha_n (\neq 0), ..., \alpha_1, \alpha_0$ are complex numbers. In particular for the case

$$\alpha_i = \binom{n}{i} (-1)^{n-i}, \ i \in \mathbb{N}, \ 0 \leq i \leq n$$

we have

$$L_c^1 f(z) = \Delta_c f(z), \ L_c^n f(z) = \Delta_c^n f(z), \ n \in \mathbb{N}, \ n \geq 2.$$

Recently, many authors have focused on the applications of Nevanlinna Theory to complex difference equations, particularly those involving meromorphic functions that share a small entire function with their shifts and difference operators, share a value with two difference operators, or share two values with the operator see e.g.[3, 4, 10, 11, 16, 17, 22, 24, 25, 29, 30, 31, 36, 38].

Definition 3.1 *Assume that f, g and, a are three meromorphic functions. If $f-a$ and $g-a$ have the same zeros with the same multiplicities, we say that f and g share a CM (counting multiplicities).*
If f, g and a are entire functions, then $f-a=e^{h}(g-a)$, where h is an entire function.

In 2015, A. El Farissi, Z. Latreuch and A. Asiri **[10]** proved:
Theorem A Assume that $f(z)$ is a transcendental entire function of finite order such that $f(z)\not\equiv f_c(z)$. If $f(z)$, $f(z+c)$ and $\Delta_c f(z)$, share the value a CM, then

$$a=0 \text{ and } f(z)=h(z)e^{\frac{\beta}{c}z},$$

where β is a nonzero complex number, and $h(z)$ is a periodic entire function of period c.

3.2 Main results

Remark 3.1 *In all the following results, we consider an entire function $f(z)$ that shares a small periodic entire function $a(z)$ or a value a with its shift $f(z+c)$ and with the difference operators $L_c^n f(z)$ or $\Delta_c^n f(z)$, or shares a small periodic entire function $a(z)$ or a value a with two difference operators $L_c^n f(z)$ and $L_c^n f(z+c)$ or $\Delta_c^n f(z)$ and $\Delta_c^n f(z+c)$. In other words, $f(z)$ shares a small periodic entire function $a(z)$ or a value a with two difference operators. Hence, throughout all the results, we impose the condition $f(z)\not\equiv f_c(z)$, so that we do not have $f(z)\equiv f_c(z)$, $L_c^n f(z)\equiv L_c^n f(z+c)$ and $\Delta_c^n f(z)\equiv\Delta_c^n f(z+c)$, that is, we exclude the case where $f(z)$ shares $a(z)$ or a value a with only one of the operators $L_c^n f(z)$ or $\Delta_c^n f(z)$.*

In this Theorem, we generalized Theorem **A** by replacing $\Delta_c f(z)$, with $\Delta_c^n f(z)$, and we obtained the same result.

Theorem 3.1 ([33]) *Assume that $f(z)$ is a transcendental entire function of finite order such that $f(z)\not\equiv f_c(z)$. If $f(z)$, $f(z+c)$ and $\Delta_c^n f(z)$, $(n\geq 1)$, share the value a CM, then*

$$a=0 \text{ and } f(z)=h(z)e^{\frac{\beta}{c}z},$$

where β is a nonzero complex number, and $h(z)$ is a periodic entire function of period c.

Example 3.1 *The entire function*

$$g(z) = e^z,$$

such that

$$g(z+1) = eg(z), \ \Delta_1^n g(z) = (e-1)^n g(z), \ n \in \mathbb{N}^*,$$

and hence $g(z)$, $g(z+1)$ *and* $\Delta_1^n g(z)$ *share* 0 *CM.*

In this corollary, we have replaced $f(z+c)$ with $\Delta_c^n f(z+c)$ in Theorem **3.1**, and we obtained the same result.

Corollary 3.1 ([33]) *Assume that* $f(z)$ *is a transcendental entire function of finite order such that* $f(z) \not\equiv f_c(z)$. *If* $f(z)$, $\Delta_c^n f(z)$ *and* $\Delta_c^n f(z+c)$, $(n \geq 1)$, *share the value a CM, then*

$$a = 0 \text{ and } f(z) = h(z)e^{\frac{\beta}{c}z},$$

where β *is a nonzero complex number, and* $h(z)$ *is a periodic entire function of period c.*

Example 3.2 *The entire function*

$$g(z) = e^z$$

such that

$$\Delta_1^n g(z) = (e-1)^n g(z), \ \Delta_1^n g(z+1) = e(e-1)^n g(z), \ n \in \mathbb{N}^*$$

and hence $g(z)$, $\Delta_1^n g(z)$ *and* $\Delta_1^n g(z+1)$ *share* 0 *CM.*

It is intriguing to observe the current situation where $f(z)$, $f(z+c)$, and $L_c^n f(z)$ $(n \geq 1)$ share $a(z)$ CM. The primary finding of this theorem demonstrates that the conclusion of Theorem **3.1** holds when we substitute $\Delta_c^n f(z)$ with $L_c^n f(z)$, replace a with $a(z)$, leading to results in the following theorem.

In this theorem, the function f share a small periodic entire function with two difference operators. Therefore, in the proof, we make an exception for the case where $f(z+c) = L_c^n f(z)$.

Theorem 3.2 ([32]) *Assume that* $f(z)$ *is an entire function of finite order such that* $f(z) \not\equiv f_c(z)$, *and let* $a(z) \in S(f)$ *be a periodic entire function with period c. If* $f(z)$, $f(z+c)$ *and* $L_c^n f(z)$ $(n \geq 1)$ *share* $a(z)$ *CM, then*

$$f(z) = h(z)e^{\frac{\beta}{c}z} + a(z) \text{ and } a(z) \equiv 0 \text{ or } \sum_{i=0}^{n} \alpha_i - 1 = 0,$$

where β *is a nonzero complex number, and* $h(z)$ *is a periodic entire function of period c.*

In the following examples, we take several cases to illustrate Theorem **3.2**:

Example 3.3 *In this example we illustrate the case* $\sum_{i=0}^{n}\alpha_i - 1 = 0$ *and* $a(z) \not\equiv 0$.*The entire function* $g(z) = \cos(z)\, e^{\frac{1}{2\pi}z} + e$ *satisfies*

$$g(z + 2\pi n) = e^n \cos(z)\, e^{\frac{1}{2\pi}z} + e,\ \ n \in \mathbb{N}.$$

We put $\alpha_i = -1,\ 1 \leq i \leq n,\ \alpha_0 = n+1$ *and* $a(z) = e$, *then*

$$L_{2\pi}^{n} g(z) = -g(z + 2\pi n) - ... - g(z + 2\pi) + (n+1)\, g(z),\ \ n \in \mathbb{N}^*.$$

We can get

$$\frac{g(z + 2\pi) - a(z)}{g(z) - a(z)} = \frac{e\cos(z)\, e^{\frac{1}{2\pi}z}}{\cos(z)\, e^{\frac{1}{2\pi}z}} = e,$$

and

$$\begin{aligned}\frac{L_{2\pi}^{n} g(z) - a(z)}{g(z) - a(z)} &= \frac{-\cos(z)\, e^{\frac{1}{2\pi}z}(e^n + ... + e) - ne}{\cos(z)\, e^{\frac{1}{2\pi}z}} \\ &\quad + \frac{(n+1)\cos(z)\, e^{\frac{1}{2\pi}z} + (n+1)\, e - e}{\cos(z)\, e^{\frac{1}{2\pi}z}} \\ &= \frac{e^{n+1} - e}{1 - e} + n + 1,\end{aligned}$$

and hence $g(z)$, $g(z + 2\pi)$ *and* $L_{2\pi}^{n} g(z)$ *share* e *CM.*

Example 3.4 *In this example we illustrate the case* $a(z) \equiv 0$ *and* $\sum_{i=0}^{n}\alpha_i - 1 \neq 0$. *The entire function* $g(z) = e^z$ *satisfies*

$$g(z + n) = e^n e^z,\ \ n \in \mathbb{N}.$$

We put $\alpha_i = 1,\ 0 \leq i \leq n$, *then*

$$L_1^n g(z) = g(z + n) + ... + g(z + 1) + g(z),\ \ n \in \mathbb{N}^*.$$

We can get

$$\frac{g(z+1)}{g(z)} = \frac{ee^z}{e^z} = e,$$

and

$$\begin{aligned}\frac{L_1^n g(z)}{g(z)} &= \frac{e^z(e^n + ... + 1)}{e^z} \\ &= \frac{1 - e^{n+1}}{1 - e},\end{aligned}$$

and hence $g(z)$, $g(z+1)$ *and* $L_1^n g(z)$ *share* 0 *CM.*

Example 3.5 *In this example we illustrate the case* $\sum_{i=0}^{n}\alpha_i - 1 = 0$ *and* $a(z) \equiv 0$. *The entire function* $g(z) = \sin(z)\, e^z$ *satisfies*

$$g(z + 2\pi n) = e^{2\pi n} \sin(z)\, e^z, n \in \mathbb{N}.$$

We put $\alpha_i = 1$, $1 \le i \le n$, $\alpha_0 = -(n-1)$ *and* $a(z) = 0$, *then*

$$L_{2\pi}^n g(z) = g(z + 2\pi n) + ... + g(z + 2\pi) - (n-1)\, g(z), \; n \in \mathbb{N}^*$$

We can get

$$\frac{g(z+2\pi)}{g(z)} = \frac{e^{2\pi} \sin(z)\, e^z}{\sin(z)\, e^z} = e^{2\pi},$$

and

$$\begin{aligned}\frac{L_{2\pi}^n g(z)}{g(z)} &= \frac{\sin(z)\, e^z \left(e^{2\pi n} + ... + e^{2\pi} - (n-1)\right)}{\sin(z)\, e^z} \\ &= \frac{e^{2\pi(n+1)} - e^{2\pi}}{e^{2\pi} - 1} - n + 1,\end{aligned}$$

and hence $g(z)$, $g(z + 2\pi)$ *and* $L_{2\pi}^n g(z)$ *share* 0 CM.

In this corollary, we have replaced $f(z+c)$ with $L_c^n f(z+c)$ in Theorem **3.2** and we obtained the same result.

Corollary 3.2 ([32]) *Assume that* $f(z)$ *is an entire function of finite order such that* $f(z) \not\equiv f_c(z)$, *and let* $a(z) \in S(f)$ *be a periodic entire function with period* c. *If* $f(z)$, $L_c^n f(z)$ *and* $L_c^n f(z+c)$ $(n \ge 1)$ *share* $a(z)$ *CM, then*

$$f(z) = h(z) e^{\frac{\beta}{c} z} + a(z) \;\text{ and } a(z) \equiv 0 \text{ or } \sum_{i=0}^{n} \alpha_i - 1 = 0,$$

where β *is a nonzero complex number, and* $h(z)$ *is a periodic entire function of period* c.

Example 3.6 *The entire function* $g(z) = e^{\frac{1}{b} z}$, *where* $b \ne 0$ *satisfies*

$$g(z + nb) = e^n e^{\frac{1}{b} z}, n \in \mathbb{N}.$$

We put $\alpha_i = 1$, $0 \le i \le n$ *and* $a(z) = 0$, *then*

$$L_b^n g(z) = g(z + nb) + ... + g(z + b) + g(z), \; n \in \mathbb{N}^*$$

and

$$L_b^n g(z + b) = g(z + (n+1)\, b) + ... + g(z + 2b) + g(z + b), \; n \in \mathbb{N}^*,$$

we can get

$$\frac{L_b^n g(z) - a(z)}{g(z) - a(z)} = \frac{g(z)\left(e^n + ... + e + 1\right)}{g(z)} = \frac{1 - e^{n+1}}{1 - e},$$

and

$$\begin{aligned}\frac{L_b^n g(z+b)-a(z)}{g(z)-a(z)} &= \frac{g(z)\left(e^{n+1}+...+e\right)}{g(z)} \\ &= \frac{e-e^{n+2}}{1-e},\end{aligned}$$

and hence $g(z)$, $L_b^n g(z)$ *and* $L_b^n g(z+b)$ *share* 0 *CM.*

In the following result, we added a condition to Theorem **3.2** and obtained an improved outcome.

Corollary 3.3 *Assume that* $f(z)$ *is an entire function of finite order such that* $f(z) \not\equiv f_c(z)$, *and let* $a(z) \in S(f)$ *be a periodic entire function with period* c. *If* $f(z)$, $f(z+c)$ *and* $L_c^n f(z)$ $(n \geq 1)$ *share* $a(z)$ *CM and* $\sum\limits_{i=0}^{n} \alpha_i = 0$, *then*

$$f(z) = h(z) e^{\frac{\beta}{c} z}, \text{ and } a(z) \equiv 0,$$

where β *is a nonzero complex number, and* $h(z)$ *is a periodic entire function of period* c.

Example 3.7 *The entire function* $g(z) = \sin(z)\, e^z$ *satisfies*

$$g(z+2\pi n) = e^{2\pi n} \sin(z)\, e^z, \ n \in \mathbb{N}.$$

We put $\alpha_i = 1$, $1 \leq i \leq n$, $\alpha_0 = -n$ *and* $a(z) = 0$, *then*

$$L_{2\pi}^n g(z) = g(z+2\pi n) + ... + g(z+2\pi) - n g(z), \ n \in \mathbb{N}^*$$

We can get

$$\frac{g(z+2\pi)}{g(z)} = \frac{e^{2\pi} \sin(z)\, e^z}{\sin(z)\, e^z} = e^{2\pi},$$

and

$$\begin{aligned}\frac{L_{2\pi}^n g(z)}{g(z)} &= \frac{\sin(z)\, e^z \left(e^{2\pi n}+...+e^{2\pi}-n\right)}{\sin(z)\, e^z} \\ &= \frac{e^{2\pi(n+1)}-e^{2\pi}}{e^{2\pi}-1} - n,\end{aligned}$$

and hence $g(z)$, $g(z+2\pi)$ *and* $L_{2\pi}^n g(z)$ *share* 0 *CM.*

In this corollary, we substituted $f(z+c)$ for $L_c^n f(z+c)$ in corollary **3.3** and obtained the same result.

Corollary 3.4 *Assume that $f(z)$ is an entire function of finite order such that $f(z) \not\equiv f_c(z)$, and let $a(z) \in S(f)$ be a periodic entire function with period c. If $f(z)$, $L_c^n f(z)$ and $L_c^n f(z+c)$ $(n \geq 1)$ share $a(z)$ CM and $\sum\limits_{i=0}^{n} \alpha_i = 0$, then*

$$f(z) = h(z) e^{\frac{\beta}{c} z}, \text{ and } a(z) \equiv 0,$$

where β is a nonzero complex number, and $h(z)$ is a periodic entire function of period c.

Example 3.8 *The entire function $g(z) = \sin(z)\, e^z$ satisfies*

$$g(z + 2\pi n) = e^{2\pi n} \sin(z)\, e^z, \; n \in \mathbb{N}.$$

We put $\alpha_i = 1$, $1 \leq i \leq n$, $\alpha_0 = -n$ and $a(z) = 0$, then

$$L_{2\pi}^n g(z) = g(z + 2\pi n) + \ldots + g(z + 2\pi) - ng(z), \; n \in \mathbb{N}^*$$

and

$$L_{2\pi}^n g(z + 2\pi) = g(z + 2\pi(n+1)) + \ldots + g(z + 4\pi) - ng(z + 2\pi), \; n \in \mathbb{N}^*$$

We can get

$$\frac{g(z + 2\pi)}{g(z)} = \frac{e^{2\pi} \sin(z)\, e^z}{\sin(z)\, e^z} = e^{2\pi},$$

and

$$\begin{aligned} \frac{L_{2\pi}^n g(z + 2\pi)}{g(z)} &= \frac{\sin(z)\, e^z \left(e^{2\pi(n+1)} + \ldots + e^{4\pi} - ne^{2\pi}\right)}{\sin(z)\, e^z} \\ &= \frac{e^{2\pi(n+2)} - e^{4\pi}}{e^{2\pi} - 1} - ne^{2\pi}, \end{aligned}$$

and hence $g(z)$, $g(z + 2\pi)$ and $L_{2\pi}^n g(z + 2\pi)$ share 0 CM.

It is reasonable to wonder what would happen if $L_c^n f(z)$ is replaced by $\Delta_c^n f(z)$ in Theorem **3.2**. In response to this question, we obtain the following result.

Theorem 3.3 ([32]) *Assume that $f(z)$ is an entire function of finite order such that $f(z) \not\equiv f_c(z)$, and let $a(z) \in S(f)$ be a periodic entire function with period c. If $f(z)$, $f(z + c)$ and $\Delta_c^n f(z)$ $(n \geq 1)$ share $a(z)$ CM, then*

$$f(z) = h(z) e^{\frac{\beta}{c} z}, \text{ and } a(z) \equiv 0,$$

where β is a nonzero complex number, and $h(z)$ is a periodic entire function of period c.

Example 3.9 *The entire function* $g(z) = \sin(z)\, e^{\frac{1}{2\pi}z}$ *satisfies*

$$g(z + 2\pi n) = e^n g(z), \ n \in \mathbb{N}.$$

We can get

$$\frac{g(z + 2\pi)}{g(z)} = \frac{eg(z)}{g(z)} = e,$$

and

$$\frac{\Delta_{2\pi}^n g(z)}{g(z)} = \frac{g(z)\sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}e^i}{g(z)} = \sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}e^i,$$

and hence $g(z)$, $g(z + 2\pi)$ *and* $\Delta_{2\pi}^n g(z)$ *share* 0 *CM.*

In this corollary, we substituted $f(z + c)$ for $\Delta_c^n f(z + c)$ in Theorem **3.3** and obtained the same result.

Corollary 3.5 ([32]) *Assume that* $f(z)$ *is an entire function of finite order such that* $f(z) \not\equiv f_c(z)$*, and let* $a(z) \in S(f)$ *be a periodic entire function with period* c*. If* $f(z)$*,* $\Delta_c^n f(z)$ *and* $\Delta_c^n f(z + c)$ $(n \geq 1)$ *share* $a(z)$ *CM, then*

$$f(z) = h(z)e^{\frac{\beta}{c}z}, \text{ and } a(z) \equiv 0,$$

where β *is a nonzero complex number, and* $h(z)$ *is a periodic entire function of period* c*.*

Example 3.10 *The entire function* $g(z) = \sin(z)\, e^{\frac{1}{2\pi}z}$ *satisfies*

$$g(z + 2\pi n) = e^n g(z), \ n \in \mathbb{N}.$$

We can get

$$\frac{\Delta_{2\pi}^n g(z)}{g(z)} = \frac{g(z)\sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}e^i}{g(z)} = \sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}e^i,$$

and

$$\frac{\Delta_{2\pi}^n g(z + 2\pi)}{g(z)} = \frac{g(z)\sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}e^{i+1}}{g(z)} = \sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}e^{i+1},$$

and hence $g(z)$, $g(z + 2\pi)$ *and* $\Delta_{2\pi}^n g(z)$ *share* 0 *CM.*

3.3 Auxiliary lemmas

To prove our results, we require the following lemmas.

Lemma 3.1 ([7]) *Suppose that n is non negative integer, and let $f(z)$ be a meromorphic function*

$$\Delta_c^n f(z) + \Delta_c^{n+1} f(z) = \Delta_c^n f(z+c).$$

Lemma 3.2 ([33]) *Assume that $f(z)$ is an entire function of finite order, and Let $a(z)$ be a periodic entire function of period c. Suppose that f is a solution of the difference equation*

$$\frac{f(z+c) - a(z)}{f(z) - a(z)} = K,$$

where K is a nonzero constant. Then

$$\frac{\Delta_c^n f(z) - a(z)}{f(z) - a(z)} = (K-1)^n - \frac{a(z)}{f(z) - a(z)}.$$

Lemma 3.3 ([33]) *Assume that $f(z)$ and $g(z)$ are entire functions of finite order, and Let $a(z)$ be a periodic entire function of period c. Suppose that f is a solution of the difference equation*

$$f(z+c) - a(z) = [f(z) - a(z)]\, e^{g(z)},$$

then

$$f(z+nc) - a(z) = [f(z) - a(z)]\, e^{\sum\limits_{i=0}^{n-1} g(z+ic)}.$$

3.4 Proof of Lemmas, theorems and corollaries

Proof of Lemma 3.2. We prove by induction that for all $n \in \mathbb{N}^*$:

$$\frac{\Delta_c^n f(z) - a(z)}{f(z) - a(z)} = (K-1)^n - \frac{a(z)}{f(z) - a(z)}.$$

For $n = 1$: We have

$$\frac{f(z+c) - a(z)}{f(z) - a(z)} = K,$$

then

$$\frac{f(z+c) - a(z)}{f(z) - a(z)} - 1 = K - 1,$$

then

$$\frac{f(z+c) - f(z) - a(z)}{f(z) - a(z)} = \frac{\Delta_c f(z) - a(z)}{f(z) - a(z)} = K - 1 - \frac{a(z)}{f(z) - a(z)}.$$

Suppose the result is true for n, then

$$\frac{\Delta_c^n f(z)-a(z)}{f(z)-a(z)}=(K-1)^n-\frac{a(z)}{f(z)-a(z)},$$

then

$$\Delta_c^n f(z)-a(z)=[f(z)-a(z)](K-1)^n-a(z),$$

then

$$\begin{aligned}
\Delta_c^{n+1} f(z) &= [f(z+c)-a(z)](K-1)^n-a(z)-[f(z)-a(z)](K-1)^n+a(z)\\
&= [f(z)-a(z)]K(K-1)^n-[f(z)-a(z)](K-1)^n\\
&= [f(z)-a(z)][K(K-1)^n-(K-1)^n]\\
&= [f(z)-a(z)](K-1)^{n+1}\\
\Delta_c^{n+1} f(z)-a(z) &= [f(z)-a(z)](K-1)^{n+1}-a(z),
\end{aligned}$$

then

$$\frac{\Delta_c^{n+1} f(z)-a(z)}{f(z)-a(z)}=(K-1)^{n+1}-\frac{a(z)}{f(z)-a(z)}.$$

Hence the result is true for $n+1$. ■

Proof of Lemma 3.3. We prove by induction that for all $n\in\mathbb{N}^*$:

$$f(z+nc)-a(z)=[f(z)-a(z)]\,e^{\sum\limits_{i=0}^{n-1} g(z+ic)},$$

For $n=1$:

$$f(z+c)-a(z)=[f(z)-a(z)]\,e^{g(z)}.$$

Suppose the result is true for n

$$\begin{aligned}
f(z+(n+1)c)-a(z) &= [f(z+nc)-a(z)]\,e^{g(z+nc)}\\
&= [f(z)-a(z)]\,e^{\sum\limits_{i=0}^{n-1} g(z+ic)}e^{g(z+nc)}\\
&= [f(z)-a(z)]\,e^{\sum\limits_{i=0}^{n} g(z+ic)}.
\end{aligned}$$

Hence the result is true for $n+1$. ■

Proof of Theorem 3.1. Assume that $f(z)$, $f(z+c)$ and $\Delta_c^n f(z)$ share 0 CM, then

$$\frac{f(z+c)}{f(z)}=e^{p(z)}, \tag{3.1}$$

and

$$\frac{\Delta_c^n f(z)}{f(z)}=\frac{\sum\limits_{i=0}^{n}\binom{n}{i}(-1)^{n-i}f(z+ic)}{f(z)}=e^{q(z)}, \tag{3.2}$$

where p and q are polynomials. By using equation (3.1), and lemma **3.3**, then

$$f(z+nc) = f(z)\, e^{\sum\limits_{i=0}^{n-1} p(z+ic)}. \tag{3.3}$$

By using (3.2) and (3.3), then

$$\begin{aligned} \frac{\Delta_c^n f(z)}{f(z)} &= (-1)^n + n(-1)^{n-1} e^{p(z)} + \ldots + n(-1)\, e^{\sum\limits_{i=0}^{n-2} p(z+ic)} + e^{\sum\limits_{i=0}^{n-1} p(z+ic)} \\ &= e^{q(z)}. \end{aligned} \tag{3.4}$$

In order to prove that $p(z)$ and $q(z)$ are constants, we need to treat the following cases:
a- If $p(z) = Cst$, $\left(e^{p(z)} = K\right)$ and $q(z) \neq Cst$, then we deduce from (3.1), that

$$\frac{f(z+c)}{f(z)} = K. \tag{3.5}$$

If $K = 1$, then we deduce the contradiction

$$f(z+c) = f(z).$$

If $K \neq 1$, by using lemma **3.2**, and equations (3.2) and (3.5), then

$$\frac{\Delta_c^n f(z)}{f(z)} = e^{q(z)} = (K-1)^n,$$

then, we deduce the contradiction

$$\deg e^{q(z)} = \rho\left(e^{q(z)}\right) = \rho\left[(K-1)^n\right] = 0,$$

where $q(z) \neq Cst$.
b- If $q(z) = Cst$, $\left(e^{q(z)} = K\right)$ and $p(z) \neq Cst$, then, from (3.4), we deduce

$$\frac{\Delta_c^n f(z)}{f(z)} = (-1)^n + n(-1)^{n-1} e^{p(z)} + \ldots + n(-1)\, e^{\sum\limits_{i=0}^{n-2} p(z+ic)} + e^{\sum\limits_{i=0}^{n-1} p(z+ic)} = K,$$

then

$$\left[(-1)^n - K\right] e^{-p(z)} + \ldots + n(-1)\, e^{\sum\limits_{i=1}^{n-2} p(z+ic)} + e^{\sum\limits_{i=1}^{n-1} p(z+ic)} = n(-1)^n,$$

by Theorem **2.8**, we deduce the contradiction

$$n(-1)^n = 0.$$

c- If $p(z) \neq Cst$ and $q(z) \neq Cst$, we have two subcases:
i) $\deg p = \deg q$
Case (A): if $p(z) = q(z)$, then, from (3.4), we deduce

$$\left[n(-1)^{n-1} - 1\right] e^{p(z)} + \ldots + n(-1)\, e^{\sum\limits_{i=0}^{n-2} p(z+ic)} + e^{\sum\limits_{i=0}^{n-1} p(z+ic)} = -(-1)^n,$$

by Theorem **2.8**, we deduce the contradiction

$$-(-1)^n = 0.$$

Case (B) if: $p(z) - q(z) = Cst$, then, from equation (3.4), we deduce,

$$(-1)^n e^{-\sum_{i=0}^{n-1} p(z+ic)} + \left[n(-1)^{n-1} - e^{-Cst}\right] e^{-\sum_{i=1}^{n-1} p(z+ic)} + ... + n(-1) e^{-p(z+(n-1)c)} = -1,$$

then, by using Theorem **2.8,** we deduce the contradiction

$$-1 = 0.$$

Case (C): $p(z) - q(z) \neq Cst$, then from (3.4), we deduce

$$n(-1)^{n-1} e^{p(z)} + ... + n(-1) e^{\sum_{i=0}^{n-2} p(z+ic)} + e^{\sum_{i=0}^{n-1} p(z+ic)} - e^{q(z)} = -(-1)^n,$$

by Theorem **2.8**, then we deduce the contradiction

$$-(-1)^n = 0.$$

ii) $\deg p \neq \deg q$:
By using Theorem **2.8** and (3.4), then we deduce the contradiction

$$(-1)^n = 0.$$

Suppose that $e^{p(z)} = e^{\beta}$, where β is complex number, from (3.1), we have

$$f(z+c) = e^{\beta} f(z). \tag{3.6}$$

If $f(z)$ and $g(z)$ are two solution of (3.6), then $h(z) = \frac{f(z)}{g(z)}$ is periodic function of period c. Obviously $g(z) = e^{\frac{\beta}{c}z}$ is solution of (3.6), hence the entire solution of (3.6) must of the form $f(z) = h(z) e^{\frac{\beta}{c}z}$ where where $\beta \neq 0$ and $h(z)$ is periodic entire function of period c.

Suppose now that $f(z)$, $f(z+c)$ and $\Delta_c^n f(z)$ share $a \neq 0$ CM, Then

$$\frac{f(z+c) - a}{f(z) - a} = e^{p(z)}, \tag{3.7}$$

and

$$\frac{\Delta_c^n f(z) - a}{f(z) - a} = e^{q(z)}, \tag{3.8}$$

where p and q are polynomials. from (3.7) and lemma **3.3** then, we deduce that

$$f(z+nc) - a = (f(z) - a) e^{\sum_{i=0}^{n-1} p(z+ic)}, \tag{3.9}$$

by using equations (3.8) and (3.9), then

$$\begin{aligned} & (-1)^n + n(-1)^{n-1} e^{p(z)} + ... \\ & + n(-1) e^{\sum_{i=0}^{n-2} p(z+ic)} + e^{\sum_{i=0}^{n-1} p(z+ic)} - \frac{a}{f(z) - a} \\ = \ & e^{q(z)}, \end{aligned}$$

then

$$\begin{aligned}\frac{a}{f(z)-a} &= (-1)^n + n(-1)^{n-1} e^{p(z)} + \dots \\ &+n(-1)\, e^{\sum_{i=0}^{n-2} p(z+ic)} + e^{\sum_{i=0}^{n-1} p(z+ic)} - e^{q(z)}. \end{aligned} \tag{3.10}$$

In order to prove that $p(z)$ and $q(z)$ are constants, we need to treat the following cases:
a- If $p(z) = Cst$, $\left(e^{p(z)} = K\right)$ and $q(z) \neq Cst$ then, by using lemma **3.2** and equations (3.7) and (3.8), we have

$$\frac{\Delta_c^n f(z) - a(z)}{f(z) - a(z)} = (K-1)^n - \frac{a(z)}{f(z) - a(z)} = e^{q(z)}. \tag{3.11}$$

If $k = e^{p(z)} = 1$, then we deduce from equation (3.7) that

$$f(z+c) - a = f(z) - a,$$

then, we deduce the contradiction

$$f(z+c) = f(z),$$

then

$$K \neq 1.$$

By using equation (3.11), then we deduce the contradiction

$$0 = \lambda\left(\frac{a}{f(z)-a}\right) = \lambda\left((K-1)^n - e^{q(z)}\right) = \rho\left(e^{q(z)}\right) \neq 0.$$

b- If $q(z) = Cst$, $\left(e^{q(z)} = K\right)$ and $p(z) \neq Cst$, Then we deduce from equation (3.7) that

$$\frac{a}{f(z)-a} = \frac{ae^{p(z)}}{f(z+c)-a}, \tag{3.12}$$

by using equation (3.10), we then deduce

$$\begin{aligned}\frac{ae^{p(z)}}{f(z+c)-a} &= [(-1)^n - K]\, e^{p(z)} + n(-1)^{n-1} e^{p(z)+p(z+c)} + \dots \\ &+n(-1)\, e^{\sum_{i=0}^{n-1} p(z+ic)} + e^{\sum_{i=0}^{n} p(z+ic)}, \end{aligned} \tag{3.13}$$

By using equations (3.10), (3.12), and (3.13), we then deduce

$$\begin{aligned} & [(-1)^n - K] + \left[n(-1)^{n-1} - (-1)^n + K\right] e^{p(z)} + \dots \\ & + [1+n]\, e^{\sum_{i=0}^{n-1} p(z+ic)} - e^{\sum_{i=0}^{n} p(z+ic)} \\ = \;& 0, \end{aligned}$$

then

$$\begin{aligned}
& \left[(-1)^n - K\right] e^{-\sum\limits_{i=0}^{n} p(z+ic)} + \left[n(-1)^{n-1} - (-1)^n + K\right] e^{-\sum\limits_{i=1}^{n} p(z+ic)} + ... \\
& + [1+n] e^{-p(z+nc)} \\
= & \ 1,
\end{aligned}$$

by using Theorem **2.8,** then we deduce the contradiction

$$1 = 0.$$

c- if $p(z) \neq Cst$ and $q(z) \neq Cst$, we need to treat the following cases:
By using equation (3.10), then we deduce

$$\begin{aligned}
\frac{ae^{p(z)}}{f(z+c) - a} &= (-1)^n e^{p(z)} + n(-1)^{n-1} e^{p(z)+p(z+c)} + \\
& ... + n(-1) e^{\sum\limits_{i=0}^{n-1} p(z+ic)} + e^{\sum\limits_{i=0}^{n} p(z+ic)} - e^{p(z)+q(z+c)}, \qquad (3.14)
\end{aligned}$$

by using equations (3.10), (3.12), and (3.14), we have

$$\begin{aligned}
& (-1)^n + \left[n(-1)^{n-1} - (-1)^n\right] e^{p(z)} + ... \\
& + (1+n) e^{\sum\limits_{i=0}^{n-1} p(z+ic)} - e^{\sum\limits_{i=0}^{n} p(z+ic)} - e^{q(z)} + e^{p(z)+q(z+c)} \\
= & \ 0, \qquad (3.15)
\end{aligned}$$

i) $\deg p = \deg q$
Case (A): if $q(z) - p(z) = Cst$, then we deduce from equation (3.15) that

$$\begin{aligned}
& (-1)^n + \left[n(-1)^{n-1} - (-1)^n - e^{Cst}\right] e^{p(z)} + ... \\
& + (1+n) e^{\sum\limits_{i=0}^{n-1} p(z+ic)} - e^{\sum\limits_{i=0}^{n} p(z+ic)} + e^{p(z)+q(z+c)} \\
= & \ 0,
\end{aligned}$$

by using Theorem **2.8**, then we deduce the contradiction

$$(-1)^n = 0.$$

Case (B): if $q(z) - p(z) = h(z) \neq Cst$, then we deduce from equation (3.15) that

$$\begin{aligned}
& (-1)^n e^{-p(z)} + ... \\
& + (1+n) e^{\sum\limits_{i=1}^{n-1} p(z+ic)} - e^{\sum\limits_{i=1}^{n} p(z+ic)} - e^{h(z)} + e^{q(z+c)} \\
= & \ -n(-1)^{n-1} + (-1)^n \\
= & \ (-1)^n (n+1),
\end{aligned}$$

by using Theorem **2.8**, then we deduce the contradiction

$$(-1)^n (n+1) = 0.$$

i) $\deg p \neq \deg q$
by using Theorem **2.8** and equation (3.15), we deduce the contradiction

$$(-1)^n = 0.$$

Finally, we deduce that $p(z)$ and $q(z)$ are constants, which is also a contradiction with the fact that f is transcendental entire function. So, there is no entire function that satisfies the hypothesis of Theorem **3.1** when $a \neq 0$. ■

Proof of Corollary 3.1. Assume that $f(z)$, $\Delta_c^n f(z)$ and $\Delta_c^n f(z+c)$ share 0 CM then

$$\frac{\Delta_c^n f(z)}{f(z)} = e^{p(z)}, \tag{3.16}$$

and

$$\frac{\Delta_c^n f(z+c)}{f(z)} = e^{q(z)}, \tag{3.17}$$

where p and q are polynomials. by using lemma **3.1** and from equation (3.17), we have

$$\frac{\Delta_c^n f(z) + \Delta_c^{n+1} f(z)}{f(z)} = e^{q(z)}, \tag{3.18}$$

by using equations (3.16) and (3.18), we have

$$\frac{\Delta_c^{n+1} f(z)}{f(z)} = e^{q(z)} - e^{p(z)},$$

then

$$\Delta_c^{n+1} f(z) = f(z)\, e^{q(z)} - f(z)\, e^{p(z)}, \tag{3.19}$$

by using equation (3.16), we have

$$\Delta_c^n f(z) = f(z)\, e^{p(z)},$$

then

$$\Delta_c^{n+1} f(z) = f(z+c)\, e^{p(z+c)} - f(z)\, e^{p(z)}, \tag{3.20}$$

by using equations (3.19) and (3.20), we have

$$f(z)\, e^{q(z)} - f(z)\, e^{p(z)} = f(z+c)\, e^{p(z+c)} - f(z)\, e^{p(z)},$$

then

$$\frac{f(z+c)}{f(z)} = e^{q(z)-p(z+c)}. \tag{3.21}$$

from (3.16) and (3.21) we deduce $f(z)$, $f(z+c)$ and $\Delta_c^n f(z)$ share 0 CM, by using Theorem **3.1** then $f(z) = h(z) e^{\frac{\beta}{c} z}$, where $\beta \neq 0$ and $h(z)$ is periodic entire function of period c.

Suppose now that $f(z)$, $\Delta_c^n f(z+c)$ and $\Delta_c^n f(z)$ share $a \neq 0$ CM, Then

$$\frac{\Delta_c^n f(z) - a}{f(z) - a} = e^{p(z)}, \tag{3.22}$$

and

$$\frac{\Delta_c^n f(z+c) - a}{f(z) - a} = e^{q(z)}, \tag{3.23}$$

where p and q are polynomials. By using lemma **3.1** and equation (3.23), we have

$$\frac{\Delta_c^{n+1} f(z) + \Delta_c^n f(z) - a}{f(z) - a} = e^{q(z)}, \tag{3.24}$$

by using equations (3.22) and (3.24), we have

$$\frac{\Delta_c^{n+1} f(z)}{f(z) - a} = e^{q(z)} - e^{p(z)},$$

then

$$\Delta_c^{n+1} f(z) = f(z) e^{q(z)} - f(z) e^{p(z)} - a e^{q(z)} + a e^{p(z)}, \tag{3.25}$$

by equation (3.22), we have

$$\Delta_c^{n+1} f(z) = f(z+c) e^{p(z+c)} - f(z) e^{p(z)} - a e^{p(z+c)} + a e^{p(z)}, \tag{3.26}$$

by using equations (3.25) and (3.26), we have

$$\begin{aligned} f(z+c) e^{p(z+c)} - f(z) e^{p(z)} - a e^{p(z+c)} + a e^{p(z)} &= f(z) e^{q(z)} - f(z) e^{p(z)} - a e^{q(z)} + a e^{p(z)} \\ \frac{f(z+c) - a}{f(z) - a} &= e^{q(z) - p(z+c)}, \end{aligned} \tag{3.27}$$

by (3.22), (3.27) and Theorem **3.1,** we deduce a contradiction which $f(z)$, $f(z+c)$ and $\Delta_c^n f(z)$ can not share $a \neq 0$ CM. ■

Proof of Theorem 3.2. Assume that $f(z)$, $f(z+c)$ and $L_c^n f(z)$ share $a(z)$ CM. Then

$$\frac{f(z+c) - a(z)}{f(z) - a(z)} = e^{p(z)}, \tag{3.28}$$

and

$$\frac{L_c^n f(z) - a(z)}{f(z) - a(z)} = e^{q(z)}, \tag{3.29}$$

where p and q are polynomials. From (3.28), and lemma **3.3**, then

$$f(z+nc) - a(z) = [f(z) - a(z)] e^{\sum_{i=0}^{n-1} p(z+ic)}, \tag{3.30}$$

by using equations (3.29), and (3.30), we obtain

$$\begin{aligned} & \frac{\alpha_n [f(z) - a(z)] e^{\sum_{i=0}^{n-1} p(z+ic)} + a(z) \alpha_n}{f(z) - a(z)} + \ldots \\ & + \frac{\alpha_0 [f(z) - a(z)] + a(z) \alpha_0 - a(z)}{f(z) - a(z)} \\ = \; & e^{q(z)}, \end{aligned}$$

then

$$\alpha_n e^{\sum_{i=0}^{n-1} p(z+ic)} + ... + \alpha_1 e^{p(z)} + \alpha_0 + \frac{a(z)\left(\sum_{i=0}^{n}\alpha_i - 1\right)}{f(z)-a(z)} = e^{q(z)}. \tag{3.31}$$

From (3.28) and (3.29), we get

$$\begin{aligned} \frac{L_c^n f(z+c) - L_c^n f(z)}{f(z)-a(z)} &= \frac{\alpha_n \Delta_c f(z+nc) + ... + \alpha_0 \Delta_c f(z)}{f(z)-a(z)} \\ &= e^{p(z)+q_c(z)} - e^{q(z)}, \end{aligned} \tag{3.32}$$

Set

$$\varphi(z) = e^{p(z)+q_c(z)} - e^{q(z)}.$$

We show that $\varphi(z) \not\equiv 0$. If $\varphi(z) \equiv 0$, then

$$e^{p(z)} = e^{q(z)-q_c(z)}, \tag{3.33}$$

thus, by equations (3.28) and (3.33), we have

$$\frac{f(z+c)-a(z)}{f(z)-a(z)} = e^{q(z)-q_c(z)}. \tag{3.34}$$

If $q(z)$ is a constant, then by (3.34) we get the following contradiction

$$f(z+c) = f(z).$$

By equation (3.34), and lemma **3.3**, then

$$\frac{f(z+nc)-a(z)}{f(z)-a(z)} = e^{q(z)-q_{nc}(z)}. \tag{3.35}$$

By using equations (3.30), (3.35) and (3.31), we have

$$\begin{aligned} -\frac{a(z)\left(\sum_{i=0}^{n}\alpha_i - 1\right)}{f(z)-a(z)} &= \alpha_n e^{q(z)-q_{nc}(z)} + ... \\ &\quad + \alpha_1 e^{q(z)-q_c(z)} + \alpha_0 - e^{q(z)}. \end{aligned} \tag{3.36}$$

If $a(z)\left(\sum_{i=0}^{n}\alpha_i - 1\right) = 0$, then

$$e^{q(z)} = \alpha_n e^{q(z)-q_{nc}(z)} + ... + \alpha_1 e^{q(z)-q_c(z)} + \alpha_0,$$

we get the following contradiction

$$T(r, e^q) = S(r, e^q).$$

If $a(z)\left(\sum_{i=0}^{n}\alpha_i - 1\right) \neq 0$, then by using equations (3.34) and (3.36), we have

$$\begin{aligned} -\frac{a(z)\left(\sum_{i=0}^{n}\alpha_i - 1\right)}{f(z)-a(z)} &= \alpha_n e^{q(z)-q_{(n+1)c}(z)} + ... \\ &\quad + \alpha_0 e^{q(z)-q_c(z)} - e^{q(z)}, \end{aligned} \tag{3.37}$$

If deg $(q) = 1$, we get the following contradiction

$$0 = \lambda \left(\frac{1}{f - a} \right) = \rho \left(cst - e^{q} \right) = 1.$$

thus, by equations (3.36) and (3.37), we have

$$\begin{aligned} & \alpha_n e^{q(z) - q_{(n+1)c}(z)} + \left(\alpha_{n-1} - \alpha_n \right) e^{q(z) - q_{nc}(z)} + \ldots \\ & + \left(\alpha_0 - \alpha_1 \right) e^{q(z) - q_c(z)} - \alpha_0 \\ = \; & 0, \end{aligned}$$

as we know from the above that $\deg q(z) \geq 2$, then by Theorem **2.7** we get the following contradiction

$$\alpha_n = \alpha_{j-1} - \alpha_j = \alpha_0 = 0, \; 0 \leq j \leq n$$

thus, we deduce $\varphi(z) \not\equiv 0$.
Since $\varphi(z) \not\equiv 0$, by Lemma **2.3** and equation (3.32), we deduce that

$$\begin{aligned} T(r, \varphi) &= m(r, \varphi) \\ &\leq m\left(r, \frac{\Delta_c f(z + nc)}{f - a(z)} \right) + \ldots + m\left(r, \frac{\Delta_c f(z)}{f - a(z)} \right) + S(r, f) \\ &= S(r, f). \end{aligned} \tag{3.38}$$

Note that $\frac{e^{p(z) + q_c(z)}}{\varphi(z)} - \frac{e^{q(z)}}{\varphi(z)} = 1$. By applying the second main theorem and equation (3.38), we have

$$\begin{aligned} T\left(r, \frac{e^{q}}{\varphi} \right) &\leq \overline{N}\left(r, \frac{e^{q}}{\varphi} \right) + \overline{N}\left(r, \frac{\varphi}{e^{q}} \right) + \overline{N}\left(r, \frac{1}{\frac{e^{q}}{\varphi} + 1} \right) + S\left(r, \frac{e^{q}}{\varphi} \right) \\ &= \overline{N}\left(r, \frac{e^{q}}{\varphi} \right) + \overline{N}\left(r, \frac{\varphi}{e^{q}} \right) + \overline{N}\left(r, \frac{\varphi}{e^{p + q_c}} \right) + S\left(r, \frac{e^{q}}{\varphi} \right) \\ &= S(r, f) + S\left(r, \frac{e^{q}}{\varphi} \right). \end{aligned} \tag{3.39}$$

Thus, by equations (3.38) and (3.39), we have

$$T(r, e^{q}) = S(r, f). \tag{3.40}$$

Similarly, we get

$$T(r, e^{p}) = S(r, f). \tag{3.41}$$

By using the first main theorem, we have

$$T(r, \frac{a(z) \left(\sum_{i=0}^{n} \alpha_i - 1 \right)}{f - a(z)}) = T(r, f) + S(r, f). \tag{3.42}$$

From equations (3.31) and (3.42), we deduce that

$$T(r, f) \leq T(r, e^{\sum_{i=0}^{n-1} p_{ic}}) + \ldots + T(r, e^{p}) + T(r, e^{q}) + S(r, f). \tag{3.43}$$

If $a(z)\left(\sum_{i=0}^{n}\alpha_i - 1\right) \neq 0$, by equations (3.40), (3.41) and (3.43), we deduce the contradiction

$$T(r,f) \leq S(r,f)$$

and from this, we deduce that either $a(z) = 0$ or $\sum_{i=0}^{n}\alpha_i - 1 = 0$, and by equation (3.31), we have

$$\alpha_n e^{\sum_{i=0}^{n-1} p(z+ic)} + ... + \alpha_1 e^{p(z)} + \alpha_0 = e^{q(z)}. \tag{3.44}$$

Next, we prove that $p(z)$ and $q(z)$ are constants. We need to treat the following cases:
First of all, we set

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_0 = a_n z^n + \alpha(z)$$

and

$$q(z) = b_m z^m + b_{m-1} z^{m-1} + ... + b_0 = b_m z^m + \gamma(z),$$

where $a_n \neq 0$, a_{n-1},..., a_0, $b_m \neq 0$, b_{m-1},...,b_0 are constants, α and γ are polynomials where $\deg\alpha \leq n-1$, and $\deg\gamma \leq m-1$.
On the other hand we have

$$\begin{aligned} \sum_{i=0}^{j-1} p(z+ic) &= p(z) + p(z+c) + ... + p(z+(j-1)c) \\ &= j a_n z^n + \lambda_j(z), \end{aligned}$$

where λ_j are polynomials with degree at most $n-1$ for $j = 1, 2, ..., n$. By equation (3.44), we have

$$\alpha_n \left(e^{a_n z^n}\right)^n e^{\lambda_n(z)} + ... + \alpha_1 e^{a_n z^n} e^{\lambda_1(z)} + \alpha_0 = e^{b_m z^m} e^{\gamma(z)}. \tag{3.45}$$

Define functions $H(z) = e^{a_n z^n}$, and $G(z) = e^{b_m z^m}$. Then, equation (3.45) becomes

$$\alpha_n [H(z)]^n e^{\lambda_n(z)} + ... + \alpha_1 H(z) e^{\lambda_1(z)} + \alpha_0 = G(z) e^{\gamma(z)}. \tag{3.46}$$

(i) If $m \neq n$, then we have two subcases:
Case (A): If $m < n$, then by using equation (3.46) and applying Theorem **2.3**, we see that

$$nT(r,H) = S(r,H)$$

which is impossible.
Case (B): If $n < m$, then, by using equation (3.46) and applying Theorem **2.3**, we see that

$$T(r,G) = S(r,G)$$

which is impossible.
(ii) If $n = m \neq 0$, then we have two subcases:
Case (A): If $b_m = j a_n$, $1 \leq j \leq n$, then by using equation (3.45), we have

$$\alpha_n [H(z)]^n e^{\lambda_n(z)} + ... + \alpha_j [H(z)]^j \left(e^{\lambda_j(z)} - e^{\gamma(z)}\right) + ... + H(z) e^{\lambda_1(z)} + \alpha_0 = 0,$$

then by Theorem **2.3,** we deduce the contradiction

$$nT(r,H) = S(r,H).$$

Case (B): If $b_m \neq ja_n$, $1 \leq j \leq n$, then by using equation (3.45), we have

$$\alpha_n e^{na_n z^n + \lambda_n(z)} + ... + \alpha_1 e^{a_n z^n + \lambda_1(z)} + \alpha_0 = e^{b_m z^n + \gamma(z)},$$

then by Theorem **2.7,** we deduce the contradiction

$$1 = \alpha_n = \alpha_j = \alpha_0 = 0, \ 0 \leq j \leq n.$$

Finally, we conclude that $p(z)$ and $q(z)$ are constants, suppose that $e^{p(z)} = e^{\beta}$ where $\beta \neq 0$, from equation (3.28), we have

$$f(z+c) - a(z) = e^{\beta}\left[f(z) - a(z)\right]. \tag{3.47}$$

If $f(z)$ and $g(z)$ are two solutions of the equation (3.47), then $h(z) = \frac{f(z)-a(z)}{g(z)-a(z)}$ is a periodic function of period c. Obviously $g(z) = e^{\frac{\beta}{c}z} + a(z)$ is solution of (3.47). Hence the entire solution of (3.47) must be of the form $f(z) = h(z)e^{\frac{\beta}{c}z} + a(z)$, where $h(z)$ is a periodic entire function of period c. ■

Proof of Corollary 3.2. Assume that $f(z)$, $L_c^n f(z)$ and $L_c^n f(z+c)$ share $a(z)$ CM. Then

$$\frac{L_c^n f(z) - a(z)}{f(z) - a(z)} = e^{p(z)}, \tag{3.48}$$

and

$$\frac{L_c^n f(z+c) - a(z)}{f(z) - a(z)} = e^{q(z)}, \tag{3.49}$$

where p and q are polynomials. By using equation (3.48), we deduce that

$$\frac{L_c^n f(z+c) - a(z)}{f(z+c) - a(z)} = e^{p(z+c)}. \tag{3.50}$$

By equations (3.49) and (3.50), we get the following result

$$\frac{f(z+c) - a(z)}{f(z) - a(z)} = e^{q(z)-p(z+c)}, \tag{3.51}$$

and finally, using equations (3.48) and (3.51), we can deduce $f(z)$, $f(z+c)$ and $L_c^n f(z)$ $(n \geq 1)$ share $a(z)$ CM, then by Theorem **3.2** we conclude that

$$f(z) = h(z)e^{\frac{\beta}{c}z} + a(z) \text{ and } a(z) = 0 \text{ or } \sum_{i=0}^{n} \alpha_i - 1 = 0,$$

Where $\beta \neq 0$ and $h(z)$ is a periodic entire function of period c. ■

Proof of Corollary 3.3. Assume that $f(z)$, $f(z+c)$ and $L_c^n f(z)$ share $a(z)$ CM, and $\sum\limits_{i=0}^{n}\alpha_i = 0$, we follow the same steps as in Proof of Theorem **3.2**. Equation (3.31) becomes

$$\alpha_n e^{\sum\limits_{i=0}^{n-1} p(z+ic)} + ... + \alpha_1 e^{p(z)} + \alpha_0 - \frac{a(z)}{f(z)-a(z)} = e^{q(z)},$$

because we know that

$$\sum_{i=0}^{n}\alpha_i = 0.$$

We continue with the same steps without forgetting that

$$\sum_{i=0}^{n}\alpha_i = 0.$$

In the proof of Theorem **3.2**, following equation (3.43), we concluded that

$$a(z) = 0 \text{ or } \sum_{i=0}^{n}\alpha_i - 1 = 0,$$

but since

$$\sum_{i=0}^{n}\alpha_i = 0,$$

we conclude that

$$a(z) = 0.$$

From this, we continue with the same steps until the end of the previous proof. Finally, we conclude that

$$f(z) = h(z)e^{\frac{\beta}{c}z}, \text{ and } a(z) = 0$$

Where $\beta \neq 0$ and $h(z)$ is a periodic entire function of period c. ■

Proof of Corollary 3.4. We use the same proof as in corollary **3.2**. Finally, we can deduce that $f(z)$, $f(z+c)$ and $L_c^n f(z)$ $(n \geq 1)$ share $a(z)$ CM. We also know that

$$\sum_{i=0}^{n}\alpha_i = 0.$$

Using the theorem **3.2**, we conclude that

$$f(z) = h(z)e^{\frac{\beta}{c}z} \text{ and } a(z) = 0,$$

Where $\beta \neq 0$ and $h(z)$ is a periodic entire function of period c. ■

Proof of Theorem 3.3. It is easy to prove this Theorem because we know that if $f(z)$, $f(z+c)$ and $\Delta_c^n f(z)$ share $a(z)$ CM, then

$$\frac{\Delta_c^n f(z) - a(z)}{f(z)-a(z)} = \frac{L_c^n f(z) - a(z)}{f(z)-a(z)} = e^{p(z)},$$

and

$$\frac{f(z+c)-a(z)}{f(z)-a(z)} = e^{q(z)},$$

where p and q are polynomials and $\sum_{i=0}^{n} \alpha_i = 0$. Following the same steps as in the proof of Theorem **3.2,** we conclude that

$$f(z) = h(z)e^{\frac{\beta}{c}z} \text{ and } a(z) = 0,$$

Where $\beta \neq 0$ and $h(z)$ is a periodic entire function of period c. ■

Proof of Corollary 3.5. Assume that $f(z)$, $\Delta_c^n f(z)$ and $\Delta_c^n f(z+c)$ share $a(z)$ CM. Then

$$\frac{\Delta_c^n f(z)-a(z)}{f(z)-a(z)} = e^{p(z)}, \tag{3.52}$$

and

$$\frac{\Delta_c^n f(z+c)-a(z)}{f(z)-a(z)} = e^{q(z)}, \tag{3.53}$$

where p and q are polynomials. By using equation (3.52), we deduce that

$$\frac{\Delta_c^n f(z+c)-a(z)}{f(z+c)-a(z)} = e^{p(z+c)}, \tag{3.54}$$

By equations (3.53) and (3.54), we get the following result

$$\frac{f(z+c)-a(z)}{f(z)-a(z)} = e^{q(z)-p(z+c)}, \tag{3.55}$$

and finally, using equations (3.52) and (3.55), we can deduce $f(z)$, $f(z+c)$ and $\Delta_c^n f(z)$ $(n \geq 1)$ share $a(z)$ CM, then by Theorem **3.3** we conclude that

$$f(z) = h(z)e^{\frac{\beta}{c}z}, \text{ and } a(z) = 0$$

Where $\beta \neq 0$ and $h(z)$ is a periodic entire function of period c. ■

Conclusion and perspective

The reader of this thesis can gain an overview of the applications of Nevanlinna theory to complex difference equations. In the first chapter, we present the fundamental results and the basic notions of Nevanlinna theory of meromorphic functions, including definitions and fundamental theorems (First and Second). In the second chapter explores the relationship between Nevanlinna theory and difference operators, focusing on the difference analog of the logarithmic derivative lemma and the second main theorem.

The third chapter presents the most significant results obtained in our research. We generalized some results from paper **[10]**, specifically those stating that if $f(z)$ is a transcendental entire function of finite order and $f(z) \not\equiv f_c(z)$, and if $f(z)$, $f_c(z)$, and $\Delta_c f(z)$, $(n \geq 1)$, share a CM, then $a = 0$ and $f(z) = h(z)e^{\frac{\beta}{c}z}$, where $\beta \neq 0$ and $h(z)$ is periodic entire function of period c. We replaced $\Delta_c f(z)$ with $\Delta_c^n f(z)$ and interestingly obtained the same result as before. Additionally, we defined difference operators $L_c^n f(z)$, more general than $\Delta_c^n f(z)$, by

$$L_c^n f(z) = \alpha_n f(z + nc) + ... + \alpha_1 f(z + c) + \alpha_0 f(z), \ n \in \mathbb{N}, \ n \geq 1$$

where $\alpha_n (\neq 0), ..., \alpha_1, \alpha_0$ are complex numbers, and modified the condition from sharing a constant to sharing a periodic entire function. We obtained the following theorem: if $f(z)$ is an entire function of finite order and $f(z) \not\equiv f_c(z)$, and if $f(z)$, $f_c(z)$ and $L_c^n f(z)$, $(n \geq 1)$, share $a(z)$ CM, then $f(z) = h(z)e^{\frac{\beta}{c}z} + a(z)$ and $a(z) = 0$ or $\sum\limits_{i=0}^{n} \alpha_i - 1 = 0$, where $\beta \neq 0$ and $h(z)$ is periodic entire function of period c.

Finally, we believe this work contributes to the field, leaving the extensions and generalizations open to further exploration with the following natural questions:

The first question**:** Can we get the same results if we change the constants α_i, $i \in \mathbb{N}$, in the operator $L_c^n f(z)$ to periodic entire functions ?

The second question: Can we get the same results as before if the function $f(z)$ is sharing two periodic entire functions with the operator $L_c^n f(z)$?

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