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Analytical and Numerical Study of Certain Fractional Boundary Problems

On 2024 in front of the committee composed of

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THESIS

Dedications

I dedicate this modest work:

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"A. Amira 2024"

♡

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- **A. Abdelnebi, Z. Dahmani**. *New Van der Pol–Duffing Jerk Fractional Differential Oscillator of Sequential Type.* Mathematics, **10**, 3546, (2022).
- **A. Abdelnebi, Z. Dahmani**. *Existence and Stability Results for a Pantograph Problem With Sequential Caputo-Hadamard Derivatives.* Fractional Differential Calculus, **14(1)**, 21-38, (2024).
- **A. Abdelnebi, I. M. Batiha, I. H. Jebril, Z. Dahmani, S. Alkhazaleh, S. Momani**. *An Approach For Dealing With Fractional Linear Boundary Value Problems.* Submitted.

International and National Conferences

- **A. Abdelnebi**. *New Duffing Fractional Differential Equation of Sequential Type.* The international conference on "Recent Developments in Ordinary and Partial Differential Equations RDOPDE22" held on May 22-26, 2022, at Abderrahmane Mira University, Bejaia, Algeria.
- **A. Abdelnebi**. *Solution and Stability for a 3D-non Homogeneous Van der Pol Duffing Jerk Fractional Differential Oscillator of Sequential Type.* The national conference on "New Trends in Theoretical and Computational Mathematics", held on November 08-09, 2022, at Tamanghasset University, Algeria.
- **A. Abdelnebi**. *Nonlinear Fractional Differential Problem of Van Der Pol-Duffing Jerk Type: Solvability and Stability Analysis.* The National Conference on "Mathematics and Applications", held on November 29-30, 2022, at Abdelhafid Boussouf University Center of Mila, Algeria.
- **A. Abdelnebi, Z. Dahamni**. *Uniqueness and Ulam-Hyers stability results for sequential fractional pantograph differential equation.* The first national Conference on "Differential Geometry and Dynamical Systems DGDS 2023", organized by the Department of Mathematics on December 19-20, 2023 in Relizane, Algeria .

Notations index

For practical purposes, we deemed it necessary to elucidate certain notations utilized in this thesis.

$\mathbb{N}$	:	The set of natural numbers.
$\mathbb{N}^*$	:	The set of natural numbers excluding zero.
$\mathbb{R}$	:	The set of real numbers.
$\mathbb{R}^*$	:	The set of real numbers excluding zero.
$\mathbb{R}^+$	:	The set of positive real numbers.
$\mathbb{C}$	:	The set of complex numbers.
$\Gamma(.)$	:	The Gamma function of Euler.
$\mathscr{B}(.,.)$	:	The Beta function of Euler.
$m!$	:	The factorial of m where $m \in \mathbb{N}$.
$[.]$	:	The integer part of a real number.
$\log(.)$	:	The natural logarithm with base number e ($e \approx 2.718$).
$\Vert . \Vert_\infty$	:	The infinity norm.
D^m	:	The derivative of integer order m.
${}^{RL}\mathrm{I}_a^\alpha$	:	The Riemann-Liouville fractional integral of order α (${}^{RL}\mathrm{I}^\alpha$ for $a=0$).
${}^{H}\mathrm{I}_a^\alpha$	:	The Hadamard fractional integral of order α (${}^{H}\mathrm{I}^\alpha$ for $a=1$).
${}^{RL}\mathrm{D}_a^\alpha$	:	The Riemann-Liouville fractional derivative of order α (${}^{RL}\mathrm{D}^\alpha$ for $a=0$).
${}^{c}\mathrm{D}_a^\alpha$	:	The Caputo fractional derivative of order α (${}^{c}\mathrm{D}^\alpha$ for $a=0$).
${}^{H}\mathrm{D}_a^\alpha$	:	The Hadamard fractional derivative of order α (${}^{H}\mathrm{D}^\alpha$ for $a=1$).
${}^{CH}\mathrm{D}_a^\alpha$	:	The Caputo-Hadamard fractional derivative of order α (${}^{CH}\mathrm{D}^\alpha$ for $a=1$).
$\mathrm{I}d$	:	The identity map.

General Introduction

Mathematics has been described as the "*Queen of Sciences*" because of its key role in enabling scientific and technological progress. It is considered an effective tool for understanding scientific theories and solving practical problems. Whether we are talking about spatial geometry, number theory, or applied mathematics in diverse fields like physics, engineering and economics, mathematics reflects the beauty that can be found in every part of our world. Then, every new development in an aspect of mathematics brings new applications, and the theory of fractional calculus is no exception. So, what is the historical background of this theory?

The field of fractional calculus involves integrating or differentiating with orders that are non-integer, and is recognized as a powerful tool of mathematical analysis. It arose at the end of the seventeenth century, after a conversation between the two mathematicians Gottfried Wihelm Leibniz and L'Hopital in 1695. The topic of this conversation revolved around the notation $\frac{d^n f}{dx^n}$ established by Leibniz in terms of the nth derivative of the function f for $n \in \mathbb{N}$, where L'Hopital asked: "what if n be $\frac{1}{2}$". Leibniz replied: "This would lead to a paradox from which one day we will be able to draw useful conclusions" [54].

Afterward, fractional calculus underwent further development, marked by significant progress in both its theoretical foundations and its practical applications, which resulted from the multiplicity of its fractional oparators. Then, many researchers become interested in this subject for developing the fractional differential equations (FDEs) like Kilbas and al. [40], Podlubny [57], Baleanu and al. [13].

Problems involving fractional order differential equations have gained great interest in various practical and engineering fields during recent years, especially fractional boundary value problems (FBVPs). In this context, certain studies is directed towards the analytical aspects of solving these problems, which includes examining uniqueness, existence, stability, and other related factors. Concurrently, other research efforts have focused on developing numerical methods to solve linear and nonlinear (FBVPs). This is due to the fact that some problems are challenging to solve analytically.

This thesis explores the analytical solvability of certain boundary value problems involving fractional derivatives. Some of these problems draw inspiration from physics, particularly when they align with classical case. Then in the numerical direction, we propose an approach to solve a class of fractional linear boundary value problems. Furthermore, we offer applications to illustrate its effectiveness and validity.

The manuscript consists of a general introduction and four chapters, concluding with a general conclusion and offering some perspectives.

- In the first chapter, we will start by recalling some essential notions of functional analysis. Then, we will present some basic concepts and properties of fractional calculus, including special functions, integral operators, derivative operators and certain auxiliary Lemmas. Finaly, we will introduce some fixed point theorems that are frequently utilized in our thesis.

- In the second chapter, we will suggest a suitable presentation for a third-order fractional problem of the Van Der Pol-Duffing (VDPD) Jerk type according to the caputo-Hadamard approach, which we formulate in the following manner:

$$\begin{cases} {}^{CH}D^{\alpha}\left({}^{CH}D^{2-\beta}+\eta\,{}^{CH}D^{\alpha}\right)y(t)+\kappa_1\phi_1\left(t,y(t),{}^{CH}D^{\alpha}y(t)\right)+\kappa_2\phi_2\left(t,y(t),{}^{H}I^{p}y(t)\right)=h(t). \\ y(1)=0, \qquad \left({}^{CH}D^{1-(\alpha-\beta)}\,{}^{CH}D^{\alpha-\beta}y\right)(1)=\mathscr{B}^{*}\in\mathbb{R}, \qquad y(\mathscr{T})=0, \\ 0\le\beta<\alpha\le 1, \quad 0\le\alpha+\beta<1, \quad 0<p, \quad t\in J, \quad \mathscr{T}>1, \end{cases} \tag{1}$$

where ${}^{CH}D^{\alpha}$ and ${}^{CH}D^{2-\beta}$ are the Caputo-Hadamard fractional derivatives, ${}^{H}I^{p}$ is the Hadamard fractional integral $J=[1,\mathscr{T}]$, $\kappa_1,\kappa_2\in\mathbb{R}$, η is a positive parameter, the functions $\phi_1: J\times\mathbb{R}^2\to\mathbb{R}$, $\phi_2: J\times\mathbb{R}^2\to\mathbb{R}$ and $h: J\to\mathbb{R}$ are continuous. Then, we will prove the uniqueness result via Banach contraction, the existence result via Krasnoselskii theorem and stability results via Ulam type theorems. Two examples are included to show the applicability of our study.

- In the third chapter, we will address the question of existence and uniqueness of the solution using Banach contraction, for the following three-sequential pantograph-type of fractional order:

$$\begin{cases} {}^{CH}D^{\beta_1}[{}^{CH}D^{\beta_2}({}^{CH}D^{\beta_3}\varkappa(t))]=f(t,\varkappa(t),\varkappa(\lambda t),{}^{H}I^{\delta}\varkappa(\lambda t),{}^{CH}D^{\rho}\varkappa(\lambda t)). \\ \varkappa(1)-\mathscr{A}_1=0, \qquad D^{\beta_3}\varkappa(1)-\mathscr{A}_2=0, \qquad D^{\beta_2}(D^{\beta_3}\varkappa(T))=0, \\ 0<\delta<\beta_i<1, \quad i=1,2,3, \quad \beta_3>\rho, \quad 0<\lambda<1, \quad \mathscr{A}_1,\mathscr{A}_2\in\mathbb{R}, \quad t\in J, \quad T>1, \end{cases} \tag{2}$$

where ${}^{CH}D^{\beta_i}$, ${}^{CH}D^{\rho}$ are the Caputo-Hadamard fractional derivatives, ${}^{H}I^{\delta}$ is the Hadamard fractional integral, $J=[1,T]$, the function $f: J\times\mathbb{R}^4\to\mathbb{R}$ is continuous. Also, we will examine the existence of at least one solution for this problem by applying Leray-

Schauder theorem. Then, we discuss some Ulam type stabilities. At the end, an example is presented to illustrate the results obtained.

- The last chapter is devoted to the presentation of a new approach to give an approximate numerical solution for the following fractional linear boundary value problems (FLBVPs):

$$\begin{cases} {}^{c}\mathrm{D}_{a}^{\alpha}\left({}^{c}\mathrm{D}_{a}^{\alpha}y\right)(t)+p(t)\,{}^{c}\mathrm{D}_{a}^{\alpha}y(t)+q(t)y(t)=r(t), \quad t\in \mathrm{J}:=[a,b], \\ \\ y(a)=\mathscr{A}, \quad y(b)=\mathscr{B}, \end{cases} \tag{3}$$

  where $0<\alpha\leq 1$, ${}^{c}\mathrm{D}_{a}^{\alpha}$ is the Caputo fractional derivative, $\mathscr{A},\mathscr{B}\in\mathbb{R}$, the functions p,q and r are continous, such that $q>0$. This approach combines the linear fractional shooting method (FLSM) and the modified fractional Euler method (MFEM). Then, we will propose two physical applications in order to demonstrate the benefits and effectiveness of our approach. Several comparison between the exact solutions and the numerical solutions of these applications will be provided via graphical representations using MATLAB code.

Chapter 1

Fundamental and Basic Concepts in Functional Analysis & Fractional Calculus

1 Introduction

This chapter serves as a preface to our work, recapitulating basic concepts, properties and findings related to the elements of functional analysis and fractional calculus, which are crucial instruments in this research.

We have organized this chapter in three main sections. The first section is dedicated to the fundamental notions and materials of functional analysis theory. The second section is intended for certain special functions of fractional calculus, as well as some definitions and properties of fractional integrals and derivatives according to different approaches. The third section presents some classical fixed point theorems that are significant for a full comprehension of our results. For the good structure of this chapter, we have surrounded its sections by an introduction and conclusion.

2 Essential Notions of Functional Analysis

The purpose of this section is to provide clear information regarding the basic definitions and tools of functional analysis theory, as well as the functional spaces used in the chapters that follow. For additional details, we motivate readers to consult these references [21, 23, 43, 47, 48, 50, 59].

2.1 Concepts and Elementary Results

Definition 2.1 (Norm) *[59] Let $\mathbb{X}$ be a vector space over the field $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We call norm on $\mathbb{X}$ any application $\|.\| : \mathbb{X} \to \mathbb{R}_+$ verifying the following axioms:*

1. $\|\varkappa\|_{\mathbb{X}} = 0$ *if and only if* $\varkappa = 0$.
2. $\|\lambda\varkappa\|_{\mathbb{X}} = |\lambda|\|\varkappa\|_{\mathbb{X}}$ *for all* $\lambda \in \mathbb{K}, \varkappa \in \mathbb{X}$.
3. $\|\varkappa_1 + \varkappa_2\|_{\mathbb{X}} \leq \|\varkappa_1\|_{\mathbb{X}} + \|\varkappa_2\|_{\mathbb{X}}$ *for all* $\varkappa_1, \varkappa_2 \in \mathbb{X}$.

Definition 2.2 (Normed Vector Space) *[21] The pair $(\mathbb{X}, \|.\|_{\mathbb{X}})$ is called a normed vector space, where $\mathbb{X}$ is a vector space and $\|.\|_{\mathbb{X}}$ a norm on $\mathbb{X}$.*

Definition 2.3 (Cauchy Sequence) *[48] Let $(\mathbb{X}, \|.\|_{\mathbb{X}})$ be a normed vector space. We say that the sequence $(\varkappa_m)_m$ of elements of $\mathbb{X}$ is a Cauchy sequence if and only if*

$$\forall \epsilon > 0, \exists \, N \in \mathbb{N}, \forall p, q \geq N \Rightarrow \|\varkappa_p - \varkappa_q\|_{\mathbb{X}} < \epsilon.$$

Definition 2.4 (Complete Space) *[48] We say that $(\mathbb{X}, \|.\|_{\mathbb{X}})$ is complete, if every Cauchy sequence $(\varkappa_m)_m$ in $\mathbb{X}$ is convergent.*

Definition 2.5 (Banach Space) *[48] Every complete normed vector space is called Banach space.*

Definition 2.6 (Contraction Mapping) *[23] Let $(\mathbb{X}, \|.\|_{\mathbb{X}})$ be a normed vector space. A map ϕ of $\mathbb{X}$ in $\mathbb{X}$ is called a contraction, if there exists a positive number $\mathcal{K} \in [0, 1[$, such that for all $\varkappa_1, \varkappa_2 \in \mathbb{X}$, we have:*

$$\|\phi(\varkappa_1) - \phi(\varkappa_2)\|_{\mathbb{X}} \leq \mathcal{K} \|\varkappa_1 - \varkappa_2\|_{\mathbb{X}}.$$

Definition 2.7 (Fixed point) *[23] Let $(\mathbb{X}, \|.\|_{\mathbb{X}})$ be a Banach space and let a map $\phi : \mathbb{X} \to \mathbb{X}$. We say that $\varkappa \in \mathbb{X}$ is a fixed point of ϕ, if $\varkappa$ satisfies the equation*

$$\phi(\varkappa) = \varkappa.$$

Definition 2.8 (Completely Continuous Operator) *[47] Let $\mathbb{X}$ and $\mathbb{Y}$ two Banach spaces. The continuous operator $\phi : \mathbb{X} \to \mathbb{Y}$ is completely continuous if it transforms any bounded set of $\mathbb{X}$ into a relatively compact set of $\mathbb{Y}$. In other words, the operator ϕ is completely continuous, if it is compact and continuous.*

Next, we focus about Ascoli-Arzela theorem, which is among the most widely applied theorems in fixed point theory.

Theorem 2.9 (Arzela-Ascoli Theorem) *[43] Let $\Omega = [a,b] \subseteq \mathbb{R}$ and let $\mathscr{A}$ be a subset of* $\mathrm{C}(\Omega,\mathbb{X})$. *Then $\mathscr{A}$ is relatively compact in* $\mathrm{C}(\Omega,\mathbb{X})$ *if and only if the following conditions are verified*

1. *$\mathscr{A}$ is uniformly bounded, i.e.*

$$\exists \mathscr{C} > 0 : \|\phi(\varkappa)\|_\infty \leq \mathscr{C}, \quad \forall \varkappa \in \Omega \text{ and } \phi \in \mathscr{A}.$$

2. *$\mathscr{A}$ is equicontinuous, i.e*

$$\forall \varepsilon > 0, \exists \eta > 0, \forall \phi \in \mathscr{A} : |\varkappa_1 - \varkappa_2| < \eta \Rightarrow \|\phi(\varkappa_1) - \phi(\varkappa_2)\|_\infty < \varepsilon, \quad \forall \varkappa_1, \varkappa_2 \in \Omega.$$

Finally, we concentrate on Fubini's theorem, a result that gives a possibility to compute a double integral by using an iterated integral. This theorem is important for proving various properties in this chapter.

Theorem 2.10 (Fubini's theorem) *[48] If $f(x,y)$ is a continuous function on $\mathscr{R} = [a,b] \times [c,d]$, then*

$$\int\int_{\mathscr{R}} f(x,y)d(x,y) = \int_c^d \left(\int_a^b f(x,y)dx\right)dy = \int_a^b \left(\int_c^d f(x,y)dy\right)dx. \tag{1.1}$$

2.2 Some Functional Spaces

Let $\Omega = [a,b]$ $(-\infty < a < b < +\infty)$ be a finite interval on the real axis $\mathbb{R}$, and let $(\mathbb{X}, \|.\|_{\mathbb{X}})$ be a Banach space, then we have the following definitions (See [8, 18, 40, 42]).

Definition 2.11 ($\mathrm{C}(\Omega,\mathbb{X})$ **Space)** *We note by* $\mathrm{C}(\Omega,\mathbb{X})$ *the Banach space of continuous functions $\phi : \Omega \to \mathbb{X}$, with the norm*

$$\|\phi\|_\infty = \sup_{t\in\Omega} |\phi(t)|.$$

Definition 2.12 ($\mathrm{L}^p(\Omega)$ **Space)** *Let $1 \leq p < +\infty$, we define space* $\mathrm{L}^p(\Omega)$ *to be the vector space of equivalence classes of measurable functions ϕ such that*

$$\mathrm{L}^p(\Omega) = \left\{\phi : \varkappa \in \Omega \longrightarrow \mathbb{R} : \phi \text{ measurable, and } \|\phi\|_p < +\infty\right\},$$

where

$$\|\phi\|_p = \left(\int_a^b |\phi(\varkappa)|^p\right)^{1/p}.$$

Definition 2.13 ($\mathrm{AC}(\Omega)$ **space)** *We denote by* $\mathrm{AC}(\Omega)$ *the space of absolutely continuous functions on Ω consisting of functions ϕ which are primitives of summable Lebesgue functions i.e,*

$$\phi \in \mathrm{AC}(\Omega) \Leftrightarrow \exists \psi \in \mathrm{L}^1(\Omega) \text{ such that } \phi(\varkappa) = c + \int_a^\varkappa \psi(t)dt.$$

Definition 2.14 **($AC^m_\delta(\Omega)$ space)** *Let* $AC(\Omega)$ *the space of absolutely continuous functions on* Ω *and* $m \in \mathbb{N}^*$, *then*

$$AC^m_\delta([\Omega) := \left\{ \phi : \varkappa \in \Omega \longrightarrow \mathbb{R} : \delta^{m-1}\phi \in AC(\Omega), \delta = \varkappa \frac{d}{d\varkappa} \right\}.$$

3 A Brief Overview About Fractional Calculus

This section will be dedicated to the Euler functions. These are the robust tools that can be used to solve a wide variety of problems in fractional calculus. Then, we will recall the fundamental definitions and characteristics of fractional operators.

3.1 Some Special Functions

The Gamma function of Euler is a cornerstone of fractional calculus. It is defined as an integral, and it can be used to generalize the factorial function to non-integer values.

Definition 3.1 (Euler's Gamma Function) *[57] let* $\varkappa \in \mathbb{R}^*_+$, *the Gamma function is given by the following integral form*

$$\Gamma(\varkappa) := \int_0^{+\infty} e^{-t} t^{\varkappa-1} dt.$$

Now, we give the basic properties of this function and some brief steps for its proof [57].

1 ▶ For all $\varkappa \in \mathbb{R}\backslash\{0, -1, -2, ...\}$, the Gamma function satisfies the following recursion

$$\Gamma(\varkappa + 1) = \varkappa\Gamma(\varkappa). \tag{1.2}$$

Indeed, by integration by parts we get

$$\begin{aligned} \Gamma(\varkappa+1) &= \int_0^{+\infty} e^{-t} t^{\varkappa-1} dt \\ &= [-e^{-t} t^{\varkappa}]_0^{+\infty} + \varkappa \int_0^{\infty} e^{-t} t^{\varkappa-1} dt \\ &= \varkappa \int_0^{+\infty} e^{-t} t^{\varkappa-1} dt \\ &= \varkappa\Gamma(\varkappa). \end{aligned}$$

2 ▶ In particular case, the Gamma function becomes

$$\Gamma(n+1) = n!, \quad \forall n \in \mathbb{N}. \tag{1.3}$$

To prove this property, we use the previous functional equation (1.2) where $\varkappa = n \in$

$\mathbb{N}^*$ and $\Gamma(1) = 1$, then we obtain

$$\begin{aligned}\Gamma(n+1) &= n\Gamma(n)\\ &= n(n-1)\Gamma(n-1)\\ &\vdots\\ &= n(n-1)\cdots\Gamma(1)\\ &= n!.\end{aligned}$$

3 ▶ For $\varkappa = -n$, $n \in \mathbb{N}$, Euler's Gamma function admits simple poles in $\varkappa$, i.e

$$\cdots = \frac{1}{\Gamma(-2)} = \frac{1}{\Gamma(-1)} = \frac{1}{\Gamma(0)} = 0.$$

4 ▶ For $-(n+1) < \varkappa < -n$, $n \in \mathbb{N}$, we can define the extension of $\Gamma(\varkappa)$ by

$$\Gamma(\varkappa) = \frac{\Gamma(\varkappa+n+1)}{\varkappa(\varkappa+1)\cdots(\varkappa+n)}.$$

Among the fundamental functions of fractional calculus, we also have Euler's Beta function. This function plays an important role, especially in certain relationships with the Gamma function.

Definition 3.2 (Euler's Beta function) *[57] The Beta function of Euler is defined as the following integral*

$$\mathscr{B}(\eta,\lambda) := \int_0^1 t^{\eta-1}(1-t)^{\lambda-1}dt, \quad (\eta>0, \quad \lambda>0).$$

The most basic characteristics of the Beta function are shown below [57]:

1 ▶ It is associated with the Gamma function as follows

$$\mathscr{B}(\eta,\lambda) = \frac{\Gamma(\eta)\Gamma(\lambda)}{\Gamma(\eta+\lambda)}, \quad (\eta>0, \quad \lambda>0). \tag{1.4}$$

2 ▶ It is symmetric function

$$\mathscr{B}(\eta,\lambda) = \mathscr{B}(\lambda,\eta), \quad (\eta>0, \quad \lambda>0). \tag{1.5}$$

3.2 The Riemann-Liouville Operators

In this part, we present two fractional operators in the Riemann-Liouville sens with some of their properties.

Definition 3.3 (Riemann-Liouville Fractional Integral) *[61] Let a, b be two real numbers and $f \in C([a,b],\mathbb{R})$. The fractional integral in the Riemann-Liouville sens of f of order α is defined by*

$$\left({}^{RL}I_a^{\alpha} f\right)(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\,d\tau, & \alpha > 0\,,\ a < t \le b, \\ \\ f(t), & \alpha = 0, \end{cases}$$

where Γ is the function defined by the Definition 3.1.

⋆ In the particular case, when $\alpha = n \in \mathbb{N}^*$, the Definition 3.3 coincide with the n-th integrals **(Cauchy formula for repeated integration)** of the form

$$\int_a^t ds_1 \int_a^{s_1} ds_2 \cdots \int_a^{s_{n-1}} f(s_n)\,ds_n = \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} f(\tau)\,d\tau.$$

Example 3.4 *Consider $f(t) = (t-a)^{\beta}$ where $(\beta > -1)$, then*

$$\left({}^{RL}I_a^{\alpha} f\right)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} (\tau - a)^{\beta}\,d\tau. \tag{1.6}$$

To evaluate this integral, we set the following transformation: $s = \frac{\tau-a}{t-a}$. Then, the equation (1.6) becomes

$$\begin{aligned} \left({}^{RL}I_a^{\alpha} f\right)(t) &= \frac{(t-a)^{\beta+\alpha}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} s^{\beta}\,ds \\ &= \frac{(t-a)^{\beta+\alpha}}{\Gamma(\alpha)} \mathscr{B}(\alpha, \beta+1). \end{aligned}$$

Using the expression given by (1.4), we obtain

$$\left({}^{RL}I_a^{\alpha} f\right)(t) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} (t-a)^{\beta+\alpha}. \tag{1.7}$$

Remark 3.5 *The relation (1.7) shows that the fractional integral in the sense of Riemann-Liouville of order α of a* ***constant****, is given by*

$$\left({}^{RL}I_a^{\alpha} \mathscr{C}\right) = \frac{\mathscr{C}}{\Gamma(\alpha+1)} (t-a)^{\alpha}, \quad \mathscr{C} \in \mathbb{R}. \tag{1.8}$$

For $\alpha, \beta \in \mathbb{R}^+$, the **Riemann-Liouville fractional integral** has the following property [61]:

Property 3.6 *If $\alpha > 0$ and $\beta > 0$, so the relation*

$$\left({}^{RL}I_a^{\alpha}\, {}^{RL}I_a^{\beta} f\right)(t) = \left({}^{RL}I_a^{\alpha+\beta} f\right)(t) = \left({}^{RL}I_a^{\beta}\, {}^{RL}I_a^{\alpha} f\right)(t),$$

is satisfied at almost every point $t \in [a,b]$ *and* $f \in \mathrm{L}^p(a,b)$ $(1 \le p \le +\infty)$.

Proof. From the Definition 3.3, we have

$$\begin{aligned}\left({}^{RL}\mathrm{I}_a^{\alpha}\, {}^{RL}\mathrm{I}_a^{\beta} f\right)(t) &= \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\left[\frac{1}{\Gamma(\beta)}\int_a^s (s-x)^{\beta-1} f(x)dx\right]ds \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t (t-s)^{\alpha-1}\left[\int_a^s (s-x)^{\beta-1} f(x)dx\right]ds.\end{aligned}$$

We note that $a \le x \le s \le t$. Then, by Fubini's Theorem 2.10 we get

$$\left({}^{RL}\mathrm{I}_a^{\alpha}\, {}^{RL}\mathrm{I}_a^{\beta} f\right)(t) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t f(x)\left[\int_x^t (t-s)^{\alpha-1}(s-x)^{\beta-1} ds\right]dx.$$

We put $\tau = \frac{s-x}{t-x}$ as a change of variables to calculate the following integral

$$\begin{aligned}\int_x^t (t-s)^{\alpha-1}(s-x)^{\beta-1}ds &= \int_0^1 (t-\tau(t-x)-x)^{\alpha-1}(\tau(t-x)+x-x)^{\beta-1}(t-x)d\tau \\ &= \int_0^1 (t-x)^{\alpha-1}(1-\tau)^{\alpha-1}\tau^{\beta-1}(t-x)^{\beta-1}(t-x)d\tau \\ &= (t-x)^{\alpha+\beta-1}\int_0^1 (1-\tau)^{\alpha-1}\tau^{\beta-1}d\tau \\ &= (t-x)^{\alpha+\beta-1}\mathscr{B}(\alpha,\beta).\end{aligned}$$

Therefore

$$\begin{aligned}\left({}^{RL}\mathrm{I}_a^{\alpha}\, {}^{RL}\mathrm{I}_a^{\beta} f\right)(t) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t f(x)(t-x)^{\alpha+\beta-1}\mathscr{B}(\alpha,\beta)dx \\ &= \frac{\mathscr{B}(\alpha,\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t (t-x)^{\alpha+\beta-1}f(x)dx.\end{aligned}$$

Thanks to the relation between Beta and Gamma (1.4), we can write

$$\begin{aligned}\left({}^{RL}\mathrm{I}_a^{\alpha}\, {}^{RL}\mathrm{I}_a^{\beta} f\right)(t) &= \frac{1}{\Gamma(\alpha+\beta)}\int_a^t (t-x)^{\alpha+\beta-1}f(x)dx \\ &= \left({}^{RL}\mathrm{I}_a^{\alpha+\beta} f\right)(t) \\ &= \left({}^{RL}\mathrm{I}_a^{\beta}\, {}^{RL}\mathrm{I}_a^{\alpha} f\right)(t).\end{aligned}$$

The proof is complete. ■

Now, we introduce Fractional derivative of Riemann-Liouville approch.

Definition 3.7 (Riemann-Liouville fractional derivative) *[61] Let* $\alpha \ge 0$, $m-1 < \alpha \le m$, $m \in \mathbb{N}^*$ *and* $f \in \mathrm{C}([a,b],\mathbb{R})$. *The Riemann-Liouville fractional derivative of order* α *of* f, *denoted* $\left({}^{RL}D_a^{\alpha}f\right)(t)$*, is given by*

$$\begin{aligned}\left({}^{RL}D_a^{\alpha}f\right)(t) &= \begin{cases}\frac{1}{\Gamma(m-\alpha)}D^m\int_a^t(t-\tau)^{m-\alpha-1}f(\tau)d\tau, & m-1<\alpha<m\\ \\ D^m f(t), & \alpha=m\end{cases}\\ &= D^m\left({}^{RL}I_a^{m-\alpha}f\right)(t),\end{aligned}$$

where $m=[\alpha]+1$*, and* $[\alpha]$ *is the integer part of* α.

Let us now state some properties of the operator ${}^{RL}D_a^{\alpha}$ [61].

Proposition 3.8 *The Riemann-Liouville derivation operator has the following properties*

1 ▶ ${}^{RL}D_a^{\alpha}$ *is linear.*

2 ▶ ${}^{RL}D_a^{\alpha}\circ{}^{RL}I_a^{\alpha}=Id$.

3 ▶ *If* $\alpha>\beta>0$*, we have* ${}^{RL}D_a^{\beta}\circ{}^{RL}I_a^{\alpha}={}^{RL}I_a^{\alpha-\beta}$.

4 ▶ *In a specific case, for* $\alpha,\beta>0$*, we can obtain the following equality*

$$\left({}^{RL}D_a^{\alpha}(t-a)^{\beta}\right) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha},$$

where we applied the definition of Riemann-Liouville, then the following formula

$$\begin{aligned}D^m(t-a)^{\lambda} &= \lambda(\lambda-1)....(\lambda-m+1)(x-a)^{\lambda-m}\\ &= \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-m)}(x-a)^{\lambda-m}, \quad \forall m\in\mathbb{N}^*, \quad \lambda>m.\end{aligned}$$

3.3 The Caputo Operator

In this part, we give the definition of the fractional derivative in Caputo's sense as well as some essential properties.

Definition 3.9 (Caputo fractional derivative) *[40] For a function* $f\in C^m([a,b],\mathbb{R})$*,* $m\in\mathbb{N}^*$ *and* $m-1<\alpha\leq m$*, we define the* ***Caputo fractional differentiator*** *by*

$$\begin{aligned}\left({}^{c}D_a^{\alpha}f\right)(t) &= \begin{cases}\frac{1}{\Gamma(m-\alpha)}\int_a^t(t-\tau)^{m-\alpha-1}f^{(m)}(\tau)d\tau, & m-1<\alpha<m\\ \\ f^{(m)}(t), & \alpha=m\end{cases}\\ &= {}^{RL}I_a^{m-\alpha}f^{(m)}(t),\end{aligned}$$

where $m=[\alpha]+1$*, and* $[\alpha]$ *is the integer part of* α.

Remark 3.10 *We note that [40]:*

★ *The Caputo derivative of a constant is equal to zero, while that of Riemann-Liouville is not. Indeed, $\forall \mathscr{C} \in \mathbb{R}$, we have*

$$^{c}\mathrm{D}_{a}^{\alpha}\mathscr{C} = \frac{1}{\Gamma(m-\alpha)}\int_{a}^{x}(x-t)^{m-\alpha-1}\mathrm{D}^{(m)}\mathscr{C}\,dt = 0.$$

Contrariwise

$$\begin{aligned} ^{\mathrm{RL}}\mathrm{D}_{a}^{\alpha}\mathscr{C} &= \frac{\mathscr{C}}{\Gamma(m-\alpha)}\mathrm{D}^{m}\int_{a}^{x}(x-t)^{m-\alpha-1}dt \\ &= \frac{\mathscr{C}}{\Gamma(1-\alpha)}(x-a)^{-\alpha}. \end{aligned}$$

★ *The link between the Riemann-Liouville and Caputo derivative is defined by*

$$\left({}^{c}\mathrm{D}_{a}^{\alpha}f\right)(t) = {}^{\mathrm{RL}}\mathrm{D}_{a}^{\alpha}\left[f(t) - \sum_{i=0}^{m-1}\frac{(t-a)^{i}}{i!}f^{(i)}(a)\right], \tag{1.9}$$

where $f \in \mathrm{C}^{m}([a,b],\mathbb{R})$, and $m = [\alpha]+1$.

★ *For $\alpha > 0$, if $f^{(i)}(a) = 0$ in the previous relation (1.9) where $i = 0,1,\cdots m-1$, $(m = [\alpha]+1)$, then the Caputo fractional derivative concur with the Riemann-Liouville fractional derivative as follows*

$$\left({}^{\mathrm{RL}}\mathrm{D}_{a}^{\alpha}f\right)(t) = \left({}^{c}\mathrm{D}_{a}^{\alpha}f\right)(t).$$

Next, we mention some important properties of the fractional Caputo derivative [40].

Proposition 3.11 *Let $f \in \mathrm{C}^{m}([a,b],\mathbb{R})$, and $m-1 < \alpha < m$, $m \in \mathbb{N}^{*}$, then*

1 ▶ *$^{c}\mathrm{D}_{a}^{\alpha}$ is a linear operator, thanks to the linearity of fractional integration and classical differentiation.*

2 ▶ *We have:*

$$^{c}\mathrm{D}_{a}^{\alpha}\left[{}^{\mathrm{RL}}\mathrm{I}_{a}^{\alpha}f(t)\right] = f(t).$$

This result is based on the Definition 3.9 of the Caputo fractional derivative and also the relation (1.9) .

3 ▶ *We have also*

$$^{\mathrm{RL}}\mathrm{I}_{a}^{\alpha}\left[{}^{c}\mathrm{D}_{a}^{\alpha}f\right](t) = \mathrm{I}^{m}[\mathrm{D}^{m}f(x)] = f(t) - \sum_{i=0}^{m-1}\frac{(t-a)^{i}}{i!}f^{(i)}(a).$$

4 ▶ *In particular, for* $f(t) = (t-a)^{\beta}$ *where* $(\beta > -1)$, *we get this expression*

$$\left({}^{c}\mathrm{D}_{a}^{\alpha}(t-a)^{\beta}\right) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha},$$

which is similar to that of Riemann Liouville only for β *non integer.*

3.4 The Hadamard Operators

Basic definitions and characteristics of the fractional integral and Hadamard type derivative will be presented in this subsection.

Definition 3.12 (Hadamard fractional integral) *[40] Given a continuous function* $f : [a,b] \longrightarrow \mathbb{R}$, *the Hadamard fractional integral of order* $\alpha \geq 0$ *is defined as*

$$\left({}^{\mathrm{H}}\mathrm{I}_{a}^{\alpha} f\right)(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} f(s) \frac{ds}{s}, & \alpha > 0\,,\ a < t \leq b, \\ f(t), & \alpha = 0 \end{cases}$$

where Γ *is the Gamma function.*

Example 3.13 *Let* $f(t) = \left(\log \frac{t}{a}\right)^{\beta}$ *where* $\beta > -1$, *then for* $\alpha > 0$, *we have*

$$ {}^{\mathrm{H}}\mathrm{I}_{a}^{\alpha}\left(\log \frac{t}{a}\right)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}\left(\log \frac{t}{a}\right)^{\alpha+\beta}. \tag{1.10}$$

In effect,

$$ {}^{\mathrm{H}}\mathrm{I}_{a}^{\alpha}\left(\log \frac{t}{a}\right)^{\beta} = \frac{1}{\Gamma(\alpha)}\int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1}\left(\log \frac{s}{a}\right)^{\beta} \frac{ds}{s}.$$

We put $\tau = \frac{(\log \frac{s}{a})}{(\log \frac{t}{a})}$, *we subsequently obtain*

$$\begin{aligned} {}^{\mathrm{H}}\mathrm{I}_{a}^{\alpha}\left(\log \frac{t}{a}\right)^{\beta} &= \frac{1}{\Gamma(\alpha)}\int_0^1 \left((1-\tau)\log \frac{t}{a}\right)^{\alpha-1}\left(\tau \log \frac{t}{a}\right)^{\beta}\left(\log \frac{t}{a}\right) d\tau \\ &= \frac{1}{\Gamma(\alpha)}\int_0^1 \left(\log \frac{t}{a}\right)^{\alpha+\beta}(1-\tau)^{\alpha-1}\tau^{\beta} d\tau \\ &= \frac{1}{\Gamma(\alpha)}\left(\log \frac{t}{a}\right)^{\alpha+\beta}\mathscr{B}(\alpha,\beta+1). \end{aligned}$$

Then, according to (1.4), we get

$$
\begin{aligned}
{}^{H}I_a^{\alpha}\left(\log\frac{t}{a}\right)^{\beta} &= \frac{\Gamma(\alpha)\Gamma(\beta+1)}{\Gamma(\alpha)\Gamma(\alpha+\beta+1)}\left(\log\frac{t}{a}\right)^{\alpha+\beta} \\
&= \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}\left(\log\frac{t}{a}\right)^{\alpha+\beta}.
\end{aligned}
$$

Particularly, when $\beta=0$*, the previous relation (1.10) becomes*

$$
{}^{H}I_a^{\alpha}(1) = \frac{1}{\Gamma(\alpha+1)}\left(\log\frac{t}{a}\right)^{\alpha}.
$$

Now, let us list some properties of the operator ${}^{H}I_a^{\alpha}$.

Property 3.14 *Let* $f, g \in L^p(a,b)$ *where* $1 \le p \le \infty$*. Then, for* $\alpha, \beta > 0$*, we have [40]*

1. *The Hadamard fractional integral is a linear operator, i.e.,*

$$
{}^{H}I_a^{\alpha}\left(\eta_1 f + \eta_2 g\right)(t) = \eta_1\left({}^{H}I_a^{\alpha} f\right)(t) + \eta_2\left({}^{H}I_a^{\alpha} g\right)(t), \quad \forall \eta_1, \eta_2 \in \mathbb{R}.
$$

2. *The Hadamard fractional integral satisfy the following semigroup property*

$$
\left({}^{H}I_a^{\alpha}\,{}^{H}I_a^{\beta} f\right)(t) = \left({}^{H}I_a^{\alpha+\beta} f\right)(t) = \left({}^{H}I_a^{\beta}\,{}^{H}I_a^{\alpha} f\right)(t). \tag{1.11}
$$

Proof.

1. $\forall \alpha \in \mathbb{R}_+^*$ and $\forall \eta_1, \eta_2 \in \mathbb{R}$, we have

$$
\begin{aligned}
{}^{H}I_a^{\alpha}\left(\eta_1 f + \eta_2 g\right)(t) &= \frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1}\left(\eta_1 f(s) + \eta_2 g(s)\right)\frac{ds}{s} \\
&= \frac{\eta_1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1} f(s)\frac{ds}{s} + \frac{\eta_2}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1} g(s)\frac{ds}{s} \\
&= \eta_1\left({}^{H}I_a^{\alpha} f\right)(t) + \eta_2\left({}^{H}I_a^{\alpha} g\right)(t).
\end{aligned}
$$

2. Based on the Definition 3.15, we can obtain

$$
\begin{aligned}
\left({}^{H}I_a^{\alpha}\,{}^{H}I_a^{\beta} f\right)(t) &= \frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1}\left[\frac{1}{\Gamma(\beta)}\int_a^s \left(\log\frac{s}{\tau}\right)^{\beta-1} f(\tau)\frac{d\tau}{\tau}\right]\frac{ds}{s} \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t\int_a^s \left(\log\frac{t}{s}\right)^{\alpha-1}\left(\log\frac{s}{\tau}\right)^{\beta-1} f(\tau)\frac{d\tau}{\tau}\frac{ds}{s}.
\end{aligned}
$$

At this point, it should be remarked that $a \le \tau \le s \le t$. Then, the preceding expression transforms into

$$
\left({}^{H}I_a^{\alpha}\,{}^{H}I_a^{\beta} f\right)(t) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t \left[\int_\tau^t \left(\log\frac{t}{s}\right)^{\alpha-1}\left(\log\frac{s}{\tau}\right)^{\beta-1}\frac{ds}{s}\right] f(\tau)\frac{d\tau}{\tau}.
$$

By using the substitution $\varkappa = \frac{(\log \frac{s}{\tau})}{(\log \frac{t}{\tau})}$, we calculate the following integral

$$\begin{aligned}
\int_\tau^t \left(\log \frac{t}{s}\right)^{\alpha-1} \left(\log \frac{s}{\tau}\right)^{\beta-1} \frac{ds}{s} &= \int_0^1 \left((1-\varkappa)\log \frac{t}{\tau}\right)^{\alpha-1} \left(\varkappa \log \frac{t}{\tau}\right)^{\beta-1} \left(\log \frac{t}{\tau}\right) d\varkappa \\
&= \left(\log \frac{t}{\tau}\right)^{\alpha+\beta-1} \int_0^1 (1-\varkappa)^{\alpha-1} \varkappa^{\beta-1} d\varkappa \\
&= \left(\log \frac{t}{\tau}\right)^{\alpha+\beta-1} \mathscr{B}(\alpha,\beta).
\end{aligned}$$

Consequently,

$$\begin{aligned}
\left({}^{\mathrm{H}}\mathrm{I}_a^\alpha \, {}^{\mathrm{H}}\mathrm{I}_a^\beta f\right)(t) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \left[\left(\log \frac{t}{\tau}\right)^{\alpha+\beta-1} \mathscr{B}(\alpha,\beta)\right] f(\tau) \frac{d\tau}{\tau} \\
&= \frac{\mathscr{B}(\alpha,\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \left(\log \frac{t}{\tau}\right)^{\alpha+\beta-1} f(\tau) \frac{d\tau}{\tau}.
\end{aligned}$$

This immediately indicates that

$$\begin{aligned}
\left({}^{\mathrm{H}}\mathrm{I}_a^\alpha \, {}^{\mathrm{H}}\mathrm{I}_a^\beta f\right)(t) &= \frac{1}{\Gamma(\alpha+\beta)} \int_a^t \left(\log \frac{t}{\tau}\right)^{\alpha+\beta-1} f(\tau) \frac{d\tau}{\tau} \\
&= \left({}^{\mathrm{H}}\mathrm{I}_a^{\alpha+\beta} f\right)(t) \\
&= \left({}^{\mathrm{H}}\mathrm{I}_a^\beta \, {}^{\mathrm{H}}\mathrm{I}_a^\alpha f\right)(t),
\end{aligned}$$

due to Beta's property (1.4), which yields the desired result.

■

Definition 3.15 (Hadamard fractional derivative) *[40] For a function $f \in \mathrm{AC}_\delta^m([a,b])$, $m \in \mathbb{N}^*$ and $m-1 < \alpha \leq m$, the Hadamard fractional derivative of order $\alpha \geq 0$ is defined as*

$$\begin{aligned}
\left({}^{\mathrm{H}}\mathrm{D}_a^\alpha f\right)(t) &= \begin{cases} \frac{1}{\Gamma(m-\alpha)} \left(t \frac{d}{dt}\right)^m \int_a^t \left(\log \frac{t}{s}\right)^{m-\alpha-1} f(s) \frac{ds}{s}, & m-1 < \alpha < m \\ \delta^m x(t), & \alpha = m \end{cases} \\
&= \delta^m \left({}^{\mathrm{H}}\mathrm{I}_a^{m-\alpha} f\right)(t),
\end{aligned}$$

where Γ is the Gamma function.

Next, we present some basic properties of the Hadamard fractional order derivative operator [40].

Proposition 3.16 *The main properties of the operator ${}^{\mathrm{H}}\mathrm{D}_a^\alpha$ are summarized in the following points:*

1 ▶ ${}^{\mathrm{H}}\mathrm{D}_a^\alpha$ *is linear.*

2 ▶ *For* $\alpha > \beta > 0$, *we have* ${}^{\mathrm{H}}\mathrm{D}_a^{\beta} \circ {}^{\mathrm{H}}\mathrm{I}_a^{\alpha} = {}^{\mathrm{H}}\mathrm{I}_a^{\alpha-\beta}$.

3 ▶ *In particular, if* $\alpha = \beta$, *then* ${}^{\mathrm{H}}\mathrm{D}_a^{\alpha} \circ {}^{\mathrm{H}}\mathrm{I}_a^{\alpha} = \mathrm{I}d$.

4 ▶ *Conversely, we have* ${}^{\mathrm{H}}\mathrm{I}_a^{\alpha} \circ {}^{\mathrm{H}}\mathrm{D}_a^{\alpha} \neq \mathrm{I}d$.

5 ▶ *As an example, for* $\alpha, \beta > 0$, *we have the following result*

$$ {}^{\mathrm{H}}\mathrm{D}_a^{\alpha}\left(\log\frac{t}{a}\right)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}\left(\log\frac{t}{a}\right)^{\beta-\alpha}. \tag{1.12} $$

6 ▶ *If* $\beta = 0$ *in (1.12), then the Hadamard fractional derivative of a real constant* $\mathscr{C}$ *is not equal to* 0.

3.5 The Caputo-Hadamard Operator

It is recognized that the Hadamard fractional derivative operator has several problems. To address these issues, the researchers improved this derivative to a version more appropriate to have physically interpretable initial conditions similar to those of Caputo.

Definition 3.17 (Caputo-Hadamard fractional derivative) *[36] Let* f *be a function in the space* $\mathrm{AC}_{\delta}^{m}([a,b])$, *with* $0 < a < b < \infty, m \in \mathbb{N}^*$ *and* $m-1 < \alpha \leq m \quad (\alpha > 0)$, *the Caputo-Hadamard fractional derivative is given by*

$$ \left({}^{\mathrm{CH}}\mathrm{D}_a^{\alpha} f\right)(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)}\int_a^t \left(\log\frac{t}{s}\right)^{m-\alpha-1}\delta^m f(s)\frac{ds}{s}, & m-1<\alpha<m \\ \delta^m f(t), & \alpha = m \end{cases} $$
$$ = {}^{\mathrm{H}}\mathrm{I}_a^{m-\alpha}\left(\delta^m f\right)(t), $$

where $m = [\alpha] + 1$ *and* Γ *is defined in Definition 3.1.*

Example 3.18 *We invoke again the function* $f(t) = \left(\log\frac{t}{a}\right)^{\beta}$, *which we chosed in the Example 3.13, where* $\beta > -1$. *Then for* $\alpha > 0$, *the following result is obtained:*

$$ {}^{\mathrm{CH}}\mathrm{D}_a^{\alpha}\left(\log\frac{t}{a}\right)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}\left(\log\frac{t}{a}\right)^{\beta-\alpha}. \tag{1.13} $$

According to the Definition 3.17, we find

$$ {}^{\mathrm{CH}}\mathrm{D}_a^{\alpha}\left(\log\frac{t}{a}\right)^{\beta} = {}^{\mathrm{H}}\mathrm{I}^{m-\alpha}\delta^m\left(\log\frac{t}{a}\right)^{\beta}. $$

So, we need to calculate the following quantity

$$\begin{aligned}
\delta^m\left(\log\frac{t}{a}\right)^\beta &= \delta^{m-1}\left[t\frac{t}{dt}\left(\log\frac{t}{a}\right)^\beta\right]\\
&= \delta^{m-1}\left[t\beta\frac{1}{t}\left(\log\frac{t}{a}\right)^{\beta-1}\right]\\
&= \beta\delta^{m-1}\left[\left(\log\frac{t}{a}\right)^{\beta-1}\right]\\
&= \beta(\beta-1)\delta^{m-2}\left[\left(\log\frac{t}{a}\right)^{\beta-2}\right]\\
&= \beta(\beta-1)(\beta-2)\delta^{m-3}\left[\left(\log\frac{t}{a}\right)^{\beta-3}\right].
\end{aligned}$$

Therefore, by recurrence we obtain

$$\begin{aligned}
\delta^m\left(\log\frac{t}{a}\right)^\beta &= \beta(\beta-1)(\beta-2)\cdots(\beta-(m-1))\delta^{m-m}\left[\left(\log\frac{t}{a}\right)^{\beta-m}\right]\\
&= \beta(\beta-1)(\beta-2)\cdots(\beta-m+1)\left(\log\frac{t}{a}\right)^{\beta-m}\\
&= \frac{\Gamma(\beta+1)}{\Gamma(\beta-m+1)}\left(\log\frac{t}{a}\right)^{\beta-m}.
\end{aligned}$$

Thus,

$$\begin{aligned}
{}^{\mathrm{CH}}\mathrm{D}_a^\alpha\left(\log\frac{t}{a}\right)^\beta &= {}^{\mathrm{H}}\mathrm{I}^{m-\alpha}\delta^m\left(\log\frac{t}{a}\right)^\beta\\
&= \frac{\Gamma(\beta+1)}{\Gamma(\beta-m+1)}{}^{\mathscr{H}}\mathrm{I}^{m-\alpha}\left(\log\frac{t}{a}\right)^{\beta-m}.
\end{aligned}$$

Using the formula (1.10), we get

$$\begin{aligned}
{}^{\mathrm{CH}}\mathrm{D}_a^\alpha\left(\log\frac{t}{a}\right)^\beta &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-m+1)}\frac{\Gamma(\beta-m+1)}{\Gamma(\beta-m+m-\alpha+1)}\left(\log\frac{t}{a}\right)^{m-\alpha+\beta-m}\\
&= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}\left(\log\frac{t}{a}\right)^{\beta-\alpha}.
\end{aligned}$$

Remark 3.19 *We note that [36]:*

★ *A constant's fractional-order Caputo-Hadamard derivative operator is equal to zero. In fact,* $\forall\mathscr{C}\in\mathbb{R}$

$${}^{\mathrm{CH}}\mathrm{D}_a^\alpha\mathscr{C} = \frac{1}{\Gamma(m-\alpha)}\int_a^t\left(\log\frac{t}{s}\right)^{m-\alpha-1}\delta^m\mathscr{C}\frac{ds}{s} = 0,$$

contrary to

$$\begin{aligned} {}^{\mathrm{H}}\mathrm{D}_a^{\alpha}\mathscr{C} &= \frac{\mathscr{C}}{\Gamma(m-\alpha)}\delta^m \int_a^t \left(\log\frac{t}{s}\right)^{m-\alpha-1}\frac{ds}{s} \\ &= \frac{\mathscr{C}}{\Gamma(1-\alpha)}\left(\log\frac{t}{a}\right)^{-\alpha}. \end{aligned}$$

★ *The relationship between the Caputo-Hadamard derivative and the Hadamard derivative is expressed by the following formula:*

$$\left({}^{\mathrm{CH}}\mathrm{D}_a^{\alpha}f\right)(t) = {}^{\mathrm{H}}\mathrm{D}_a^{\alpha}\left(f(t) - \sum_{i=0}^{m-1}\frac{\delta^i f(a)}{i!}\left(\log\frac{t}{a}\right)^i\right), \tag{1.14}$$

or

$$\left({}^{\mathrm{CH}}\mathrm{D}_a^{\alpha}f\right)(t) = {}^{\mathrm{H}}\mathrm{D}_a^{\alpha}f(t) - \sum_{i=0}^{m-1}\frac{\delta^i f(a)}{\Gamma(i-\alpha+1)}\left(\log\frac{t}{a}\right)^{i-\alpha}, \tag{1.15}$$

where $f \in \mathrm{AC}_{\delta}^{m}([a,b])$, *and* $m = [\alpha]+1$.

In the following, we demonstrate some properties of the fractional differentiation operator according to the Caputo-Hadamard approach. These properties can show the composition of such operator with the Hadamard integral operator [36].

Proposition 3.20 *Let* $\alpha, \beta > 0$ *and* $f \in \mathrm{C}([a,b])$, *we have*

(i) *If* $\alpha > \beta$, *then*

$$ {}^{\mathrm{CH}}\mathrm{D}_a^{\beta}\left({}^{\mathrm{H}}\mathrm{I}_a^{\alpha}f\right)(t) = \left({}^{\mathrm{H}}\mathrm{I}_a^{\alpha-\beta}f\right)(t).$$

(ii) *If* $\alpha = \beta$, *then*

$$ {}^{\mathrm{CH}}\mathrm{D}_a^{\alpha}\left({}^{\mathrm{H}}\mathrm{I}_a^{\alpha}f\right)(t) = f(t).$$

Proof.

(i) By applying relation (1.14) for the function ${}^{\mathrm{H}}\mathrm{I}_a^{\alpha}f$, we find the following result:

$$ {}^{\mathrm{CH}}\mathrm{D}_a^{\beta}\left({}^{\mathrm{H}}\mathrm{I}_a^{\alpha}f\right)(t) = {}^{\mathrm{H}}\mathrm{D}_a^{\beta}\left[\left({}^{\mathrm{H}}\mathrm{I}_a^{\alpha}f\right)(t) - \sum_{i=0}^{m-1}\frac{\delta^i\left({}^{\mathrm{H}}\mathrm{I}_a^{\alpha}f\right)(a)}{i!}\left(\log\frac{t}{a}\right)^i\right].$$

By the linearity of the operator ${}^{\mathrm{H}}\mathrm{D}_a^{\beta}$, we get

$$\begin{aligned}{}^{\mathrm{CH}}\mathrm{D}_a^{\beta}\left({}^{\mathrm{H}}\mathrm{I}_a^{\alpha}f\right)(t) &= {}^{\mathrm{H}}\mathrm{D}_a^{\beta}\left({}^{\mathrm{H}}\mathrm{I}_a^{\alpha}f\right)(t) - {}^{\mathrm{H}}\mathrm{D}_a^{\beta}\left[\sum_{i=0}^{m-1}\frac{\delta^i\left({}^{\mathrm{H}}\mathrm{I}_a^{\alpha}f\right)(a)}{i!}\left(\log\frac{t}{a}\right)^i\right]\\ &= \left({}^{\mathrm{H}}\mathrm{I}_a^{\alpha-\beta}f\right)(t) - {}^{\mathrm{H}}\mathrm{D}_a^{\beta}\left[\sum_{i=0}^{m-1}\frac{\delta^i\left({}^{\mathrm{H}}\mathrm{I}_a^{\alpha}f\right)(a)}{i!}\left(\log\frac{t}{a}\right)^i\right].\end{aligned}$$

In the other hand, for all $i = 0, 1, \ldots, m-1$ with $m \in \mathbb{N}^*$, we have

$$\delta^i\left({}^{\mathrm{H}}\mathrm{I}_a^{\alpha}f\right)(a) = 0.$$

This implies

$${}^{\mathrm{CH}}\mathrm{D}_a^{\beta}\left({}^{\mathrm{H}}\mathrm{I}_a^{\alpha}f\right)(t) = \left({}^{\mathrm{H}}\mathrm{I}_a^{\alpha-\beta}f\right)(t),$$

which is the desired result.

(ii) The proof of this property is similar to the previous one, only we make the assumption $\alpha = \beta$ in all steps.

■

3.6 Auxiliary Lemmas

In this part, we give some important lemmas that will be needed in the remainder of this thesis [36].

Lemma 3.21 *For $f \in AC_{\delta}^{m}([a,b])$, $m \in \mathbb{N}^*$ and $\alpha > 0$. The following fractional equation:*

$${}^{\mathrm{CH}}\mathrm{D}_a^{\alpha}f(t) = 0,$$

has a general solution expressed by

$$f(t) = \sum_{i=0}^{m-1} c_i\left(\log\frac{t}{a}\right)^i,$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, \ldots, m-1$, $m = [\alpha] + 1$.

Proof. In order to prove this result, we suppose that

$${}^{\mathrm{CH}}\mathrm{D}_a^{\alpha}f(t) = 0,$$

then, by definition, we obtain

$${}^{\mathrm{H}}\mathrm{I}_a^{m-\alpha}\left(\delta^m f\right)(t) = 0. \tag{1.16}$$

Next, we apply the operator ${}^{CH}\mathrm{D}_a^{m-\alpha}$ to (1.16), we get

$$ {}^{CH}\mathrm{D}_a^{m-\alpha}\left[{}^{H}\mathrm{I}_a^{m-\alpha}\left(\delta^m f\right)(t)\right]=0 \Rightarrow \delta^m f(t)=0. $$

This consequently implies

$$ f(t)=\sum_{i=0}^{m-1} c_i\left(\log\frac{t}{a}\right)^i. $$

■

Lemma 3.22 *Let $f\in AC_\delta^m([a,b])$, $m\in\mathbb{N}^*$ and $\alpha>0$. Then, we have*

$$ {}^{H}\mathrm{I}_a^{\alpha}\left({}^{CH}\mathrm{D}_a^{\alpha}f\right)(t)=f(t)+\sum_{i=0}^{m-1} c_i\left(\log\frac{t}{a}\right)^i, $$

where $c_i\in\mathbb{R}$, $i=0,1,\ldots,m-1$, $m=[\alpha]+1$.

Proof. For $\alpha>0$ and $f\in AC_\delta^m([a,b])$, we have

$$ {}^{H}\mathrm{I}_a^{\alpha}\left({}^{CH}\mathrm{D}_a^{\alpha}f\right)(t)={}^{H}\mathrm{I}_a^{\alpha}\left({}^{H}\mathrm{I}^{m-\alpha}\delta^m f\right)(t). $$

Then, by applying the semi-group property (1.11) and the relation (1.15), we get

$$ \begin{aligned} {}^{H}\mathrm{I}_a^{\alpha}\left({}^{CH}\mathrm{D}_a^{\alpha}f\right)(t) &= {}^{H}\mathrm{I}_a^{m}\left(\delta^m f\right)(t)\\ &= f(t)+\sum_{i=0}^{m-1} c_i\left(\log\frac{t}{a}\right)^i, \end{aligned} $$

where $(c_i)_{i=0,1,\ldots,m-1}\in\mathbb{R}$. ■

4 About Fixed Point Theorems

This section includes some fixed point theorems that are frequently used in our work.

Theorem 4.1 (Banach contraction principle) *[23] Let $\mathbb{X}$ be a Banach space. If $\phi:\mathbb{X}\to\mathbb{X}$ is a contraction, then ϕ has a unique fixed point in $\mathbb{X}$.*

Theorem 4.2 (Krasnoselskii fixed point theorem) *[64] Let $\mathscr{A}$ be a closed convex and nonempty subset of a Banach space $\mathbb{X}$, ϕ_1 and ϕ_2 be two operators such that*

(1) $\phi_1 x+\phi_2 y\in\mathscr{A}$ whenever $x,y\in\mathscr{A}$;

(2) ϕ_1 is a completely continuous operator;

(3) ϕ_2 is a contractive operator.

Then there exists $\varkappa^*\in\mathscr{A}$ such that $\phi_1\varkappa^*+\phi_2\varkappa^*=\varkappa^*$.

Theorem 4.3 (Leray-Schauder Alternative) *[23] Let $\phi : \mathbb{X} \to \mathbb{X}$ be a completely continuous operator. If we consider the set $\mathfrak{U}_{\phi} := \{\varkappa \in \mathbb{X} : \varkappa = \sigma\phi(\varkappa) \quad \text{for some} \quad 0 < \sigma < 1\}$, then we have*

(1) *Either ϕ has at least fixed point, or*

(2) *The set $\mathfrak{U}_{\phi}$ is unbounded.*

5 Conclusion

In this chapter, we introduced the basic materials of functional analysis theory. Additionally, we presented some fundamental fractional calculus tools, including the two Euler functions, the fractional integrals and derivatives in differents senses as well as their corresponding properties. The chapter ended with a section reserved for the various fixed point theorems which are very useful for solving differential equations.

Chapter 2

Existence and Stability Results For a Fractional Van Der Pol-Duffing Jerk Oscillator of Sequential Type

1 Introduction

[1] Applied mathematics, physics, electronics, engineering and many other fields are importantly affected by nonlinear phenomena, where the majority of them can be represented by nonlinear differential equations. These Mathematical models are used to explain complex situations that occur in natural phenomena, see [4, 9, 24, 30, 35, 38].

Among these problems, we will focus on an important equation that has many applications and it has been thoroughly studied in regards to multiple particular problems, including chaos, control, vibration description in physics, and many other areas. This equation is called the Van Der Pol-Duffing (VDPD) Jerk oscillator, where VDPD equation is a nonlinear differential equation that describes the motion of a forced oscillator with cubic nonlinearity. Its role is to model the behavior of physical systems which exhibit forced oscillation and non-linearity phenomena, and Jerk is a physical quantity that measures the variation of acceleration over time. It is the third derivative of position with respect to time. In other words, Jerk measures the rapidity how the acceleration changes. This quantity is often used in mechanics to describe the movement of objects, for more information see [41, 44, 46, 51, 56, 58, 65, 68].

The VDPD Jerk oscillator is represented mathematically by the three-dimensional fractional differential equation shown as follows [66]:

$$\frac{d^3y}{dt^3}+\frac{d^2y}{dt^2}-\delta(1-y^2)\frac{dy}{dt}+y-\alpha y^3=0, \quad \delta,\alpha>0, \tag{2.1}$$

[1] **A. Abdelnebi and Z. Dahmani**, *New Van der Pol-Duffing Jerk Fractional Differential Oscillator of Sequential Type*, Mathematics, **10**, 3546, (2022).

where y represents the position of the oscillating system, t represents the time, δ is the damping coefficient, α is nonlinear coefficient.

On other case, the use of the fractional-order differential operator has become a hot topic among researchers due to its variety of applications, which are more realistic and accurate than classical integer-order models. In other words, we can say that the memory and genetic properties of many processes can be described by differential equations of arbitrary order. For this reason, one can find the systematic progress of fractional form of Van Der Pol-Duffing Jerk equation in [17, 22, 26, 62, 66].

Motivated by these previous studies, in this work we attempt to propose an appropriate presentation for a third-order fractional problem of the VdPD Jerk type, which we formulate as follows [1]:

$$\begin{cases} {}^{CH}D^{\alpha}\left({}^{CH}D^{2-\beta}+\eta\,{}^{CH}D^{\alpha}\right)y(t)+\kappa_1\phi_1\left(t,y(t),{}^{CH}D^{\alpha}y(t)\right)+\kappa_2\phi_2\left(t,y(t),{}^{H}I^{p}y(t)\right)=h(t). \\ y(1)=0, \qquad \left({}^{CH}D^{1-(\alpha-\beta)}\,{}^{CH}D^{\alpha-\beta}y\right)(1)=\mathscr{B}^*\in\mathbb{R}, \qquad y(\mathscr{T})=0, \\ 0\le\beta<\alpha\le 1, \quad 0\le\alpha+\beta<1, \quad 0<p, \quad t\in J, \quad \mathscr{T}>1, \end{cases} \tag{2.2}$$

where ${}^{CH}D^{\alpha}$ and ${}^{CH}D^{2-\beta}$ are the Caputo-Hadamard fractional derivatives, ${}^{H}I^{p}$ is the Hadamard fractional integral $J=[1,\mathscr{T}]$, $\kappa_1,\kappa_2\in\mathbb{R}$, η is a positive parameter, the functions $\phi_1: J\times\mathbb{R}^2\to\mathbb{R}$, $\phi_2: J\times\mathbb{R}^2\to\mathbb{R}$ and $h: J\to\mathbb{R}$ are continuous.

In the problem (2.2) we introduce the Caputo-Hadamard operator in a sequential manner. This approach is significant due to the fact that combines the properties of two important operators: Hadamard and Caputo. Additionally, we injected Caputo-Hadamard derivatives into both sides of the equation with boundary conditions. This consideration makes the problem considered more interesting in the application side, knowing that for specific values of α and β we recover the (VdPL) Jerk type model.

The main objective of this study is to develop some existence and stability results for the problem (2.2). First, we will start by presenting the integral solution of the problem under consideration, then we apply Banach's contraction principle and Krasnoselskii's theorem to prove the existence and uniqueness of solution for this problem. Other results around stability in the sense of Ulam-Hyers and generalized Ulam-Hyers will be analyzed. At the end, we will present two examples to validate the theoretical results.

2 The Integral Representation

In this section, we can use the fundamental notations, definitions, lemmas and some properties that were introduced in the first chapter in order to give the integral solution of problem (2.2).

At first, for the purpose of simplicity, we define the function $\mathcal{H} : [1,\mathcal{T}] \to \mathbb{R}$, as follows

$$\mathcal{H}(t) := h(t) - \kappa_1\, \phi_1\left(t, y(t), {}^{CH}D^{\alpha} y(t)\right) - \kappa_2 \phi_2\left(t, y(t), {}^{H}I^{p} y(t)\right).$$

Then, the problem (2.2) becomes

$$\begin{cases} & {}^{CH}D^{\alpha}\left({}^{CH}D^{2-\beta} + \eta\, {}^{CH}D^{\alpha}\right) y(t) = \mathcal{H}(t). \\ y(1) = 0, & \left({}^{CH}D^{1-(\alpha-\beta)}\ {}^{CH}D^{\alpha-\beta} y\right)(1) = \mathcal{B}^* \in \mathbb{R}, \qquad y(\mathcal{T}) = 0. \end{cases} \tag{2.3}$$

Hence, the integral representation of (2.3) is given by the following lemma, which holds great significance for the main results.

Lemma 2.1 *Let $\mathcal{H} \in C([1,\mathcal{T}],\mathbb{R})$, $t \in J$, $0 \le \beta < \alpha \le 1$. Then, the solution of the problem (2.3) is given by the following integral expression*

$$\begin{aligned} y(t) &= \int_1^t \frac{\left(\log\frac{t}{s}\right)^{1-\beta+\alpha}}{\Gamma(2-\beta+\alpha)} \mathcal{H}(s)\frac{ds}{s} - \eta \int_1^t \frac{\left(\log\frac{t}{s}\right)^{1-(\alpha+\beta)}}{\Gamma(2-(\alpha+\beta))} y(s)\frac{ds}{s} \\ &- \left(\mathcal{R}_2(\log t)^{2-\beta} + \mathcal{R}_3(\log t)^{2-(\alpha+\beta)}\right) \int_1^{\mathcal{T}} \frac{\left(\log\frac{\mathcal{T}}{s}\right)^{1-\beta+\alpha}}{\Gamma(2-\beta+\alpha)} \mathcal{H}(s)\frac{ds}{s} \\ &+ \left(\mathcal{R}_2(\log t)^{2-\beta} + \mathcal{R}_3(\log t)^{2-(\alpha+\beta)}\right) \eta \int_1^{\mathcal{T}} \frac{\left(\log\frac{\mathcal{T}}{s}\right)^{1-(\alpha+\beta)}}{\Gamma(2-(\alpha+\beta))} y(s)\frac{ds}{s} \\ &- \left(\mathcal{R}_2(\log t)^{2-\beta} + \mathcal{R}_3(\log t)^{2-(\alpha+\beta)}\right) \frac{\mathcal{B}^*}{\mathcal{R}_1} \log\mathcal{T} + \mathcal{B}^* \log t, \end{aligned} \tag{2.4}$$

with

$$\mathcal{R}_1 = \left(\frac{\log(\mathcal{T})^{2-\beta}}{\Gamma(3-\beta)} + \frac{\lambda \log(\mathcal{T})^{2-(\alpha+\beta)}}{\Gamma(3-(\alpha+\beta))}\right), \qquad \mathcal{R}_2 = \frac{1}{\mathcal{R}_1 \Gamma(3-\beta)}, \qquad \mathcal{R}_3 = \frac{\lambda}{\mathcal{R}_1 \Gamma(3-(\alpha+\beta))}.$$

Proof. We consider the following fractional differential equation associated to our problem

$${}^{CH}D^{\alpha}\left({}^{CH}D^{2-\beta} + \eta\, {}^{CH}D^{\alpha}\right) y(t) = \mathcal{H}(t). \tag{2.5}$$

First, we apply the Hadamard-type fractional integral of order α for both sides of the previous equation, we find

$${}^{H}I^{\alpha}\ {}^{CH}D^{\alpha}\left({}^{CH}D^{2-\beta} + \eta\, {}^{CH}D^{\alpha}\right) y(t) = {}^{H}I^{\alpha}\mathcal{H}(t). \tag{2.6}$$

Then, with the help of the Lemma 3.22, we obtain

$${}^{CH}D^{2-\beta} + \eta\, {}^{CH}D^{\alpha} y(t) + c_0 = {}^{H}I^{\alpha}\mathcal{H}(t). \tag{2.7}$$

Similarly, by applying the operator ${}^{\mathrm{H}}\mathrm{I}^{2-\beta}$ to (2.7), where $1 \leq 2-\beta \leq 2$, we get

$$y(t) + c_1 + c_2 \log t + \eta\, {}^{\mathrm{H}}\mathrm{I}^{2-\beta}\, {}^{\mathrm{CH}}\mathrm{D}^{\alpha} y(t) + {}^{\mathrm{H}}\mathrm{I}^{2-\beta} c_0 = {}^{\mathrm{H}}\mathrm{I}^{2-\beta+\alpha} \mathscr{H}(t). \tag{2.8}$$

We note that

$${}^{\mathrm{H}}\mathrm{I}^{2-\beta}\, {}^{\mathrm{CH}}\mathrm{D}^{\alpha} y(t) = {}^{\mathrm{H}}\mathrm{I}^{2-(\alpha+\beta)} \left({}^{\mathrm{H}}\mathrm{I}^{\alpha}\, {}^{\mathrm{CH}}\mathrm{D}^{\alpha} y(t) \right) = {}^{\mathrm{H}}\mathrm{I}^{2-(\alpha+\beta)} \left(y(t) + c_0 \right).$$

So, the problem's general solution can be expressed as

$$y(t) \;=\; {}^{\mathrm{H}}\mathrm{I}^{2-\beta+\alpha} \mathscr{H}(t) - \eta\, {}^{\mathrm{H}}\mathrm{I}^{2-(\alpha+\beta)} y(t) - \eta c_0\, {}^{\mathrm{H}}\mathrm{I}^{2-(\alpha+\beta)}(1) - c_0\, {}^{\mathrm{H}}\mathrm{I}^{2-\beta}(1) - c_1 - c_2 \log t,$$

equivalent to

$$y(t) = {}^{\mathrm{H}}\mathrm{I}^{2-\beta+\alpha} \mathscr{H}(t) - \eta\, {}^{\mathrm{H}}\mathrm{I}^{2-(\alpha+\beta)} y(t) - c_0 \left(\frac{(\log t)^{2-\beta}}{\Gamma(3-\beta)} + \frac{\eta (\log t)^{2-(\alpha+\beta)}}{\Gamma(3-(\alpha+\beta))} \right) - c_1 - c_2 \log t, \tag{2.9}$$

where $c_0, c_1, c_2 \in \mathbb{R}$, are arbitrary real constants to determine.
Using the first condition $y(1) = 0$, we immediately obtain

$$c_1 = 0.$$

Applying the second condition $\left({}^{\mathrm{CH}}\mathrm{D}^{1-(\alpha-\beta)}\, {}^{\mathrm{CH}}\mathrm{D}^{\alpha-\beta} y \right)(1) = \mathscr{B}^*$, we get

$$c_2 = -\mathscr{B}^*.$$

According to the third condition $y(\mathscr{T}) = 0$, we then have

$$c_0 \;=\; \frac{1}{\mathscr{R}_1} \left({}^{\mathrm{H}}\mathrm{I}^{2-\beta+\alpha} \mathscr{H}(\mathscr{T}) - \eta\, {}^{\mathrm{H}}\mathrm{I}^{2-(\alpha+\beta)} y(\mathscr{T}) + \mathscr{B}^* \log \mathscr{T} \right).$$

Finally, by replacing the values of $c_i, i = 0, 1, 2$ in (2.9), we find the formula (2.4). ■

3 Existence Results in Banach space

In this section, we will present two main results for problem (2.2). The first result depends on the existence of a unique solution to the problem under consideration. The second study, will focus about the existence of at least one solution to the equation.
In order to start, let us first introduce the following Banach space:

$$\mathbb{Y} := \left\{ y \in \mathrm{C}(\mathrm{J}, \mathbb{R}),\ {}^{\mathrm{CH}}\mathrm{D}^{\alpha} y \in \mathrm{C}(\mathrm{J}, \mathbb{R}) \right\},$$

provided with the norm

$$\| y \|_{\mathbb{Y}} = \| y \|_\infty + \| {}^{CH}D^\alpha y \|_\infty,$$

where

$$\| y \|_\infty = \sup_{t\in J} | y(t) |, \qquad \| {}^{CH}D^\alpha y \|_\infty = \sup_{t\in J} | {}^{CH}D^\alpha y(t) | .$$

Then, we consider the operator $\mathscr{O}$ defined by

$$\begin{aligned} \mathscr{O} : \mathbb{Y} &\to \mathbb{Y} \\ y &\to \mathscr{O}(y), \end{aligned}$$

such that, $\forall t \in J$, we have

$$\begin{aligned}
(\mathscr{O} y)(t) &= \int_1^t \frac{(\log\frac{t}{s})^{1-\beta+\alpha}}{\Gamma(2-\beta+\alpha)} \left[\begin{array}{l} h(s) - \kappa_1 \phi_1\left(s, y(s), {}^{CH}D^\alpha y(s)\right) \\ -\kappa_2 \phi_2\left(s, y(s), {}^{H}I^p y(s)\right) \end{array} \right] \frac{ds}{s} - \eta \int_1^t \frac{(\log\frac{t}{s})^{1-(\alpha+\beta)}}{\Gamma(2-(\alpha+\beta))} y(s) \frac{ds}{s} \\
&- \left(\mathscr{R}_2 (\log t)^{2-\beta} + \mathscr{R}_3 (\log t)^{2-(\alpha+\beta)}\right) \int_1^{\mathscr{T}} \frac{(\log\frac{\mathscr{T}}{s})^{1-\beta+\alpha}}{\Gamma(2-\beta+\alpha)} \left[\begin{array}{l} h(s) - \kappa_1 \phi_1\left(s, y(s), {}^{CH}D^\alpha y(s)\right) \\ -\kappa_2 \phi_2\left(s, y(s), {}^{H}I^p y(s)\right) \end{array} \right] \frac{ds}{s} \\
&+ \left(\mathscr{R}_2 (\log t)^{2-\beta} + \mathscr{R}_3 (\log t)^{2-(\alpha+\beta)}\right) \eta \int_1^{\mathscr{T}} \frac{(\log\frac{\mathscr{T}}{s})^{1-(\alpha+\beta)}}{\Gamma(2-(\alpha+\beta))} y(s) \frac{ds}{s} \\
&- \left(\mathscr{R}_2 (\log t)^{2-\beta} + \mathscr{R}_3 (\log t)^{2-(\alpha+\beta)}\right) \frac{\mathscr{B}^*}{\mathscr{R}_1} \log \mathscr{T} + \mathscr{B}^* \log t.
\end{aligned}$$

In other word, we converted the problem (2.2) into a fixed point problem, where the solutions of this problem are the fixed points of operator $\mathscr{O}$.

3.1 Uniqueness Result Via Banach Contraction

In this part, we apply the Banach contraction principle to prove that the problem (2.2) has a unique solution. For this reason, we consider the following theorem.

Theorem 3.1 *Assume that:*

($\mathscr{P}_1$) *There exist nonnegative constants $\mathscr{L}_i, i = 1,2$, such that*

$$\begin{aligned} |\phi_1(t, u_1, v_1) - \phi_1(t, u_2, v_2))| &\le \mathscr{L}_1(|u_1 - u_2| + |v_1 - v_2|). \\ |\phi_2(t, u_1, v_1) - \phi_2(t, u_2, v_2))| &\le \mathscr{L}_2(|u_1 - u_2| + |v_1 - v_2|), \end{aligned}$$

for any $u_1, v_1, u_2, v_2 \in \mathbb{R}$ and $t \in J$.

$(\mathscr{P}_2)$ $0 < \mathrm{S} = \mathrm{S}_1 + \mathrm{S}_2 < 1$, *where*

$$\mathrm{S}_1 = \left(1 + |\mathscr{R}_2| (\log \mathcal{T})^{2-\beta} + |\mathscr{R}_3| (\log \mathcal{T})^{2-(\alpha+\beta)}\right) \left[\begin{array}{c} \frac{(|\kappa_1|\mathscr{L}_1 + |\kappa_2|\mathscr{L}_2)(\log \mathcal{T})^{2-\beta+\alpha}}{\Gamma(3-\beta+\alpha)} \\ +\eta \frac{(\log \mathcal{T})^{2-(\alpha+\beta)}}{\Gamma(3-(\alpha+\beta))} \end{array} \right].$$

$$\mathrm{S}_2 = \left[\begin{array}{c} \frac{(|\kappa_1|\mathscr{L}_1 + |\kappa_2|\mathscr{L}_2)(\log \mathcal{T})^{2-\beta}}{\Gamma(3-\beta)} \\ + \frac{\eta(\log \mathcal{T})^{2-(2\alpha+\beta)}}{\Gamma(3-(2\alpha+\beta))} \end{array} \right] + \left[\begin{array}{c} \frac{|\mathscr{R}_2|\Gamma(3-\beta)(\log \mathcal{T})^{2-(\alpha+\beta)}}{\Gamma(3-(\alpha+\beta))} \\ + \frac{|\mathscr{R}_3|\Gamma(3-(\alpha+\beta))(\log \mathcal{T})^{2-(2\alpha+\beta)}}{\Gamma(3-(2\alpha+\beta))} \end{array} \right] \times \left[\begin{array}{c} \frac{(|\kappa_1|\mathscr{L}_1 + |\kappa_2|\mathscr{L}_2)(\log \mathcal{T})^{2-\beta+\alpha}}{\Gamma(3-\beta+\alpha)} \\ + \frac{\eta(\log \mathcal{T})^{2-(\alpha+\beta)}}{\Gamma(3-(\alpha+\beta))} \end{array} \right],$$

are satisfied. Then the problem (2.2) has a unique solution on J.

Proof. In order to prove this result, it is enough to show that the operator $\mathscr{O}$ is contractive over the space $\mathbb{Y}$. We therefore need to consider the next two steps:

Step 1: In this step, we look for a constant $0 < \mathrm{S}_1 < 1$, such that

$$\| \mathscr{O} y - \mathscr{O} z \|_\infty \leq \mathrm{S}_1 \| y - z \|_{\mathbb{Y}}.$$

Let $y, z \in \mathbb{Y}$, then for $t \in \mathrm{J}$ we have

$$\begin{aligned} |(\mathscr{O} y)(t) - (\mathscr{O} z)(t)| \leq\ & \sup_{t\in \mathrm{J}} \int_1^t \frac{(\log \frac{t}{s})^{1-\beta+\alpha}}{\Gamma(2-\beta+\alpha)} |\kappa_1| \left|\phi_1\left(s, y(s), {}^{\mathrm{CH}}\mathrm{D}^\alpha y(s)\right) - \phi_1\left(s, z(s), {}^{\mathrm{CH}}\mathrm{D}^\alpha z(s)\right)\right| \frac{ds}{s} \\ & + \sup_{t\in \mathrm{J}} \left(\begin{array}{c} |\mathscr{R}_2 (\log t)^{2-\beta}| \\ + |\mathscr{R}_3 (\log t)^{2-(\alpha+\beta)}| \end{array} \right) \int_1^{\mathcal{T}} \frac{(\log \frac{\mathcal{T}}{s})^{1-\beta+\alpha}}{\Gamma(2-\beta+\alpha)} |\kappa_1| \left| \begin{array}{c} \phi_1\left(s, y(s), {}^{\mathrm{CH}}\mathrm{D}^\alpha y(s)\right) \\ -\phi_1\left(s, z(s), {}^{\mathrm{CH}}\mathrm{D}^\alpha z(s)\right) \end{array} \right| \frac{ds}{s} \\ & + \sup_{t\in \mathrm{J}} \int_1^t \frac{(\log \frac{t}{s})^{1-\beta+\alpha}}{\Gamma(2-\beta+\alpha)} |\kappa_2| \left|\phi_2\left(s, y(s), {}^{\mathrm{H}}\mathrm{I}^p y(s)\right) - \phi_2\left(s, z(s), {}^{\mathrm{H}}\mathrm{I}^p z(s)\right)\right| \frac{ds}{s} \\ & + \sup_{t\in \mathrm{J}} \left(\begin{array}{c} |\mathscr{R}_2 (\log t)^{2-\beta}| \\ + |\mathscr{R}_3 (\log t)^{2-(\alpha+\beta)}| \end{array} \right) \int_1^{\mathcal{T}} \frac{(\log \frac{\mathcal{T}}{s})^{1-\beta+\alpha}}{\Gamma(2-\beta+\alpha)} |\kappa_2| \left| \begin{array}{c} \phi_2\left(s, y(s), {}^{\mathrm{H}}\mathrm{I}^p y(s)\right) \\ -\phi_2\left(s, z(s), {}^{\mathrm{H}}\mathrm{I}^p z(s)\right) \end{array} \right| \frac{ds}{s} \\ & + \sup_{t\in \mathrm{J}} \eta \int_1^t \frac{(\log \frac{t}{s})^{1-(\alpha+\beta)}}{\Gamma(2-(\alpha+\beta))} |y(s) - z(s)| \frac{ds}{s} \\ & + \sup_{t\in \mathrm{J}} \eta \left(\begin{array}{c} |\mathscr{R}_2 (\log t)^{2-\beta}| \\ + |\mathscr{R}_3 (\log t)^{2-(\alpha+\beta)}| \end{array} \right) \int_1^{\mathcal{T}} \frac{(\log \frac{\mathcal{T}}{s})^{1-(\alpha+\beta)}}{\Gamma(2-(\alpha+\beta))} |y(s) - z(s)| \frac{ds}{s}. \end{aligned}$$

By using the hypothesis $(\mathscr{P}_1)$, we get

$$
\begin{aligned}
|(\mathscr{O}y)(t)-(\mathscr{O}z)(t)| \leq\;& |\kappa_1|\mathscr{L}_1(\| y-z\|_\infty+\|^{CH}D^\alpha y-{}^{CH}D^\alpha z\|_\infty)\frac{(\log\mathscr{T})^{2-\beta+\alpha}}{\Gamma(3-\beta+\alpha)}\\
&+|\kappa_1|\mathscr{L}_1\begin{pmatrix}|\mathscr{R}_2|(\log\mathscr{T})^{2-\beta}\\ +|\mathscr{R}_3|(\log\mathscr{T})^{2-(\alpha+\beta)}\end{pmatrix}(\| y-z\|_\infty+\|^{CH}D^\alpha y-{}^{CH}D^\alpha z\|_\infty)\frac{(\log\mathscr{T})^{2-\beta+\alpha}}{\Gamma(3-\beta+\alpha)}\\
&+|\kappa_2|\mathscr{L}_2(\| y-z\|_\infty+\|{}^{H}I^p y-{}^{H}I^p z\|_\infty)\frac{(\log\mathscr{T})^{2-\beta+\alpha}}{\Gamma(3-\beta+\alpha)}\\
&+|\kappa_2|\mathscr{L}_2\begin{pmatrix}|\mathscr{R}_2|(\log\mathscr{T})^{2-\beta}\\ +|\mathscr{R}_3|(\log\mathscr{T})^{2-(\alpha+\beta)}\end{pmatrix}(\| y-z\|_\infty+\|{}^{H}I^p y-{}^{H}I^p z\|_\infty)\frac{(\log\mathscr{T})^{2-\beta+\alpha}}{\Gamma(3-\beta+\alpha)}\\
&+\eta\| y-z\|_\infty\frac{(\log\mathscr{T})^{2-(\alpha+\beta)}}{\Gamma(3-(\alpha+\beta))}\\
&+\eta\begin{pmatrix}|\mathscr{R}_2|(\log\mathscr{T})^{2-\beta}\\ +|\mathscr{R}_3|(\log\mathscr{T})^{2-(\alpha+\beta)}\end{pmatrix}\| y-z\|_\infty\frac{(\log\mathscr{T})^{2-(\alpha+\beta)}}{\Gamma(3-(\alpha+\beta))}.
\end{aligned}
$$

This implies

$$
\begin{aligned}
\|\mathscr{O}y-\mathscr{O}z\|_\infty \leq\;& |\kappa_1|\mathscr{L}_1\| y-z\|_{\mathbb{Y}}\frac{(\log\mathscr{T})^{2-\beta+\alpha}}{\Gamma(3-\beta+\alpha)}\\
&+|\kappa_1|\mathscr{L}_1\left(|\mathscr{R}_2|(\log\mathscr{T})^{2-\beta}+|\mathscr{R}_3|(\log\mathscr{T})^{2-(\alpha+\beta)}\right)\| y-z\|_{\mathbb{Y}}\frac{(\log\mathscr{T})^{2-\beta+\alpha}}{\Gamma(3-\beta+\alpha)}\\
&+|\kappa_2|\mathscr{L}_2\| y-z\|_{\mathbb{Y}}\frac{(\log\mathscr{T})^{2-\beta+\alpha}}{\Gamma(3-\beta+\alpha)}\\
&+|\kappa_2|\mathscr{L}_2\left(|\mathscr{R}_2|(\log\mathscr{T})^{2-\beta}+|\mathscr{R}_3|(\log\mathscr{T})^{2-(\alpha+\beta)}\right)\| y-z\|_{\mathbb{Y}}\frac{(\log\mathscr{T})^{2-\beta+\alpha}}{\Gamma(3-\beta+\alpha)}\\
&+\eta\| y-z\|_{\mathbb{Y}}\frac{(\log\mathscr{T})^{2-(\alpha+\beta)}}{\Gamma(3-(\alpha+\beta))}\\
&+\eta\left(|\mathscr{R}_2|(\log\mathscr{T})^{2-\beta}+|\mathscr{R}_3|(\log\mathscr{T})^{2-(\alpha+\beta)}\right)\| y-z\|_{\mathbb{Y}}\frac{(\log\mathscr{T})^{2-(\alpha+\beta)}}{\Gamma(3-(\alpha+\beta))}\\
\leq\;& \begin{bmatrix}\frac{(|\kappa_1|\mathscr{L}_1+|\kappa_2|\mathscr{L}_2)\left(1+|\mathscr{R}_2|(\log\mathscr{T})^{2-\beta}+|\mathscr{R}_3|(\log\mathscr{T})^{2-(\alpha+\beta)}\right)(\log\mathscr{T})^{2-\beta+\alpha}}{\Gamma(3-\beta+\alpha)}\\ \\ +\eta\frac{\left(1+|\mathscr{R}_2|(\log\mathscr{T})^{2-\beta}+|\mathscr{R}_3|(\log\mathscr{T})^{2-(\alpha+\beta)}\right)(\log\mathscr{T})^{2-(\alpha+\beta)}}{\Gamma(3-(\alpha+\beta))}\end{bmatrix}\| y-z\|_{\mathbb{Y}}\\
\leq\;& \left(1+|\mathscr{R}_2|(\log\mathscr{T})^{2-\beta}+|\mathscr{R}_3|(\log\mathscr{T})^{2-(\alpha+\beta)}\right)\begin{bmatrix}\frac{(|\kappa_1|\mathscr{L}_1+|\kappa_2|\mathscr{L}_2)(\log\mathscr{T})^{2-\beta+\alpha}}{\Gamma(3-\beta+\alpha)}\\ \\ +\eta\frac{(\log\mathscr{T})^{2-(\alpha+\beta)}}{\Gamma(3-(\alpha+\beta))}\end{bmatrix}\| y-z\|_{\mathbb{Y}}.
\end{aligned}
$$

Thus,

$$\|\mathscr{O}y-\mathscr{O}z\|_\infty \leq S_1\| y-z\|_{\mathbb{Y}}.$$

Step 2: The goal of the second step is to find a constant $0 < \mathrm{S}_2 < 1$, such that

$$\| {}^{\mathrm{CH}}\mathrm{D}^{\alpha}\mathscr{O}y - {}^{\mathrm{CH}}\mathrm{D}^{\alpha}\mathscr{O}z \|_{\infty} \leq \mathrm{S}_2 \| y - z \|_{\mathbb{Y}}.$$

For $y, z \in \mathbb{Y}$, we have

$$\begin{aligned}
{}^{\mathrm{CH}}\mathrm{D}^{\alpha}\mathscr{O}y(t) &= \int_1^t \frac{\left(\log\frac{t}{s}\right)^{1-\beta}}{\Gamma(2-\beta)} \left[\begin{array}{l} h(s) - \kappa_1\,\phi_1\left(s, y(s), {}^{\mathrm{CH}}\mathrm{D}^{\alpha}y(s)\right) \\ -\kappa_2\phi_2\left(s, y(s), {}^{\mathrm{H}}\mathrm{I}^{p}y(s)\right) \end{array}\right] \frac{ds}{s} - \eta\int_1^t \frac{\left(\log\frac{t}{s}\right)^{2-(2\alpha+\beta)-1}}{\Gamma(2-(2\alpha+\beta))} y(s)\frac{ds}{s} \\
&- \frac{\mathscr{R}_2\Gamma(3-\beta)(\log t)^{2-(\alpha+\beta)}}{\Gamma(3-(\alpha+\beta))} \int_1^{\mathscr{T}} \frac{\left(\log\frac{\mathscr{T}}{s}\right)^{1-\beta+\alpha}}{\Gamma(2-\beta+\alpha)} \left[\begin{array}{l} h(s) - \kappa_1\,\phi_1\left(s, y(s), {}^{\mathrm{CH}}\mathrm{D}^{\alpha}y(s)\right) \\ -\kappa_2\phi_2\left(s, y(s), {}^{\mathrm{H}}\mathrm{I}^{p}y(s)\right) \end{array}\right] \frac{ds}{s} \\
&- \frac{\mathscr{R}_3\Gamma(3-(\alpha+\beta))(\log t)^{2-(2\alpha+\beta)}}{\Gamma(3-(2\alpha+\beta))} \int_1^{\mathscr{T}} \frac{\left(\log\frac{\mathscr{T}}{s}\right)^{1-\beta+\alpha}}{\Gamma(2-\beta+\alpha)} \left[\begin{array}{l} h(s) - \kappa_1\,\phi_1\left(s, y(s), {}^{\mathrm{CH}}\mathrm{D}^{\alpha}y(s)\right) \\ -\kappa_2\phi_2\left(s, y(s), {}^{\mathrm{H}}\mathrm{I}^{p}y(s)\right) \end{array}\right] \frac{ds}{s} \\
&+ \eta\frac{\mathscr{R}_2\Gamma(3-\beta)(\log t)^{2-(\alpha+\beta)}}{\Gamma(3-(\alpha+\beta))} \int_1^{\mathscr{T}} \frac{\left(\log\frac{\mathscr{T}}{s}\right)^{1-(\alpha+\beta)}}{\Gamma(2-(\alpha+\beta))} y(s)\frac{ds}{s} \\
&+ \eta\frac{\mathscr{R}_3\Gamma(3-(\alpha+\beta))(\log t)^{2-(2\alpha+\beta)}}{\Gamma(3-(2\alpha+\beta))} \int_1^{\mathscr{T}} \frac{\left(\log\frac{\mathscr{T}}{s}\right)^{1-(\alpha+\beta)}}{\Gamma(2-(\alpha+\beta))} y(s)\frac{ds}{s} \\
&- \frac{\mathscr{B}^*\mathscr{R}_2\Gamma(3-\beta)(\log t)^{2-(\alpha+\beta)}}{\mathscr{R}_1\Gamma(3-(\alpha+\beta))}\log\mathscr{T} - \frac{\mathscr{B}^*\mathscr{R}_3\Gamma(3-(\alpha+\beta))(\log t)^{2-(2\alpha+\beta)}}{\mathscr{R}_1\Gamma(3-(2\alpha+\beta))}\log\mathscr{T} \\
&+ \frac{\mathscr{B}^*(\log t)^{1-\alpha}}{\Gamma(2-\alpha)}.
\end{aligned}$$

By applying the same procedure as previously, we find

$$\begin{aligned}
\| {}^{\mathrm{CH}}\mathrm{D}^{\alpha}\mathscr{O}y - {}^{\mathrm{CH}}\mathrm{D}^{\alpha}\mathscr{O}z \|_{\infty} &\leq \frac{(|\kappa_1|\mathscr{L}_1 + |\kappa_2|\mathscr{L}_2)(\log\mathscr{T})^{2-\beta}}{\Gamma(3-\beta)} \| y - z \|_{\mathbb{Y}} + \eta\frac{(\log\mathscr{T})^{2-(2\alpha+\beta)}}{\Gamma(3-(2\alpha+\beta))} \| y - z \|_{\mathbb{Y}} \\
&+ (|\kappa_1|\mathscr{L}_1 + |\kappa_2|\mathscr{L}_2) \left[\begin{array}{l} \frac{|\mathscr{R}_2|\Gamma(3-\beta)(\log\mathscr{T})^{2-(\alpha+\beta)}}{\Gamma(3-(\alpha+\beta))} \\ + \frac{|\mathscr{R}_3|\Gamma(3-(\alpha+\beta))(\log\mathscr{T})^{2-(2\alpha+\beta)}}{\Gamma(3-(2\alpha+\beta))} \end{array}\right] \frac{(\log\mathscr{T})^{2-\beta+\alpha}}{\Gamma(3-\beta+\alpha)} \| y - z \|_{\mathbb{Y}} \\
&+ \eta \left[\begin{array}{l} \frac{|\mathscr{R}_2|\Gamma(3-\beta)(\log\mathscr{T})^{2-(\alpha+\beta)}}{\Gamma(3-(\alpha+\beta))} \\ + \frac{|\mathscr{R}_3|\Gamma(3-(\alpha+\beta))(\log\mathscr{T})^{2-(2\alpha+\beta)}}{\Gamma(3-(2\alpha+\beta))} \end{array}\right] \frac{(\log\mathscr{T})^{2-(\alpha+\beta)}}{\Gamma(3-(\alpha+\beta))} \| y - z \|_{\mathbb{Y}}.
\end{aligned}$$

This is equivalent to

$$\begin{aligned}
\| {}^{\mathrm{CH}}\mathrm{D}^{\alpha}\mathscr{O}y - {}^{\mathrm{CH}}\mathrm{D}^{\alpha}\mathscr{O}z \|_{\infty} &\leq \left[\frac{(|\kappa_1|\mathscr{L}_1 + |\kappa_2|\mathscr{L}_2)(\log\mathscr{T})^{2-\beta}}{\Gamma(3-\beta)} + \frac{\eta(\log\mathscr{T})^{2-(2\alpha+\beta)}}{\Gamma(3-(2\alpha+\beta))}\right] \| y - z \|_{\mathbb{Y}} \\
&+ \left[\begin{array}{l} \frac{|\mathscr{R}_2|\Gamma(3-\beta)(\log\mathscr{T})^{2-(\alpha+\beta)}}{\Gamma(3-(\alpha+\beta))} \\ + \frac{|\mathscr{R}_3|\Gamma(3-(\alpha+\beta))(\log\mathscr{T})^{2-(2\alpha+\beta)}}{\Gamma(3-(2\alpha+\beta))} \end{array}\right] \times \left[\begin{array}{l} \frac{(|\kappa_1|\mathscr{L}_1+|\kappa_2|\mathscr{L}_2)(\log\mathscr{T})^{2-\beta+\alpha}}{\Gamma(3-\beta+\alpha)} \\ + \frac{\eta(\log\mathscr{T})^{2-(\alpha+\beta)}}{\Gamma(3-(\alpha+\beta))} \end{array}\right] \| y - z \|_{\mathbb{Y}}.
\end{aligned}$$

Hence

$$\| {}^{CH}D^{\alpha}\mathscr{O}y - {}^{CH}D^{\alpha}\mathscr{O}z \|_{\infty} \leq S_2 \| y - z \|_{\mathbb{Y}} .$$

Using steps 1 and 2 and the fact that

$$\begin{aligned} \| \mathscr{O}y - \mathscr{O}z \|_{\mathbb{Y}} &= \| \mathscr{O}y - \mathscr{O}z \|_{\infty} + \| {}^{CH}D^{\alpha}\mathscr{O}y - {}^{CH}D^{\alpha}\mathscr{O}z \|_{\infty} \\ &\leq (S_1 + S_2) \| y - z \|_{\mathbb{Y}} . \end{aligned}$$

Then, we conclude that

$$\| \mathscr{O}y - \mathscr{O}z \|_{\mathbb{Y}} \leq S \| y - z \|_{\mathbb{Y}} .$$

According to $(\mathscr{P}_2)$, $\mathscr{O}$ is a contraction. As a result of Banach's contraction principle, problem (2.2) has a unique solution defined on J. The proof is finished. ■

3.2 Existence Result Via Krasnoselskii Theorem

The next main result is centered on Krasnoselskii fixed point Theorem 4.2 to show that there is at least one solution to the problem (2.2). This is what the theorem that follows will demonstrate.

Theorem 3.2 *We suppose that the hypotheses*

$(\mathscr{P}_3)$ *The functions* ϕ_1 *and* ϕ_2 *are continuous in* $J \times \mathbb{R}^2$, *and* h *defined on* J *is also continuous.*

$(\mathscr{P}_4)$ *There exists nonnegative constants* $\mathscr{M}_1, \mathscr{M}_2$ *and* $\mathscr{M}_3$, *such that for all* $t \in J$ *and* $u, v \in \mathbb{R}$, *we have*

$$|\phi_1(t,u,v))| \leq \mathscr{M}_1, \quad |\phi_2(t,u,v)| \leq \mathscr{M}_2, \quad |h(t)| \leq \mathscr{M}_3.$$

are verified. Then, problem (2.2) has at least a solution on J, *provided that* $0 < S^* < 1$, *where*

$$S^* = \left\{ \begin{array}{c} \dfrac{(|\kappa_1|\mathscr{L}_1+|\kappa_2|\mathscr{L}_2)\left(1+|\mathscr{R}_2|(\log\mathscr{T})^{2-\beta}+|\mathscr{R}_3|(\log\mathscr{T})^{2-(\alpha+\beta)}\right)(\log\mathscr{T})^{2-\beta+\alpha}}{\Gamma(3-\beta+\alpha)} \\ +(|\kappa_1|\mathscr{L}_1+|\kappa_2|\mathscr{L}_2)\left[\dfrac{(\log\mathscr{T})^{2-\beta}}{\Gamma(3-\beta)} + \left(\begin{array}{c} \dfrac{|\mathscr{R}_2|\Gamma(3-\beta)(\log\mathscr{T})^{2-(\alpha+\beta)}}{\Gamma(3-(\alpha+\beta))} \\ +\dfrac{|\mathscr{R}_3|\Gamma(3-(\alpha+\beta))(\log\mathscr{T})^{2-(2\alpha+\beta)}}{\Gamma(3-(2\alpha+\beta))} \end{array} \right) \dfrac{(\log\mathscr{T})^{2-\beta+\alpha}}{\Gamma(3-\beta+\alpha)} \right] \end{array} \right. .$$

Proof. To start, let us introduce the convex and non-empty closed subset $\mathscr{B}_r$ of a Banach space $\mathbb{Y}$, which is expressed by

$$\mathscr{B}_r := \left\{ y \in \mathbb{Y} : \|y\|_{\mathbb{Y}} \leq r \right\} \subset \mathbb{Y}.$$

Next, we divide the operator $\mathscr{O}$ into the sum of two operators denoted $\mathscr{O}_1$ and $\mathscr{O}_2$ on $\mathscr{B}_r$, as follows

$$(\mathscr{O}y)(t) = (\mathscr{O}_1 y)(t) + (\mathscr{O}_2 y)(t),$$

where

$$\begin{aligned}(\mathscr{O}_1 y)(t) &= -\eta \int_1^t \frac{\left(\log \frac{t}{s}\right)^{1-(\alpha+\beta)}}{\Gamma(2-(\alpha+\beta))} y(s)\frac{ds}{s} \\ &+ \left(\mathscr{R}_2(\log t)^{2-\beta} + \mathscr{R}_3(\log t)^{2-(\alpha+\beta)}\right)\eta \int_1^{\mathscr{T}} \frac{\left(\log \frac{\mathscr{T}}{s}\right)^{1-(\alpha+\beta)}}{\Gamma(2-(\alpha+\beta))} y(s)\frac{ds}{s},\end{aligned}$$

and

$$\begin{aligned}(\mathscr{O}_2 y)(t) &= \int_1^t \frac{\left(\log \frac{t}{s}\right)^{1-\beta+\alpha}}{\Gamma(2-\beta+\alpha)} \begin{bmatrix} h(s) - \kappa_1 \phi_1\left(s, y(s), {}^{\mathrm{CH}}\mathrm{D}^{\alpha} y(s)\right) \\ -\kappa_2\phi_2\left(s, y(s), {}^{\mathrm{H}}\mathrm{I}^{p} y(s)\right) \end{bmatrix} \frac{ds}{s} \\ &- \left(\mathscr{R}_2(\log t)^{2-\beta} + \mathscr{R}_3(\log t)^{2-(\alpha+\beta)}\right)\int_1^{\mathscr{T}} \frac{\left(\log \frac{\mathscr{T}}{s}\right)^{1-\beta+\alpha}}{\Gamma(2-\beta+\alpha)} \begin{bmatrix} h(s) - \kappa_1 \phi_1\left(s, y(s), {}^{\mathrm{CH}}\mathrm{D}^{\alpha} y(s)\right) \\ -\kappa_2\phi_2\left(s, y(s), {}^{\mathrm{H}}\mathrm{I}^{p} y(s)\right) \end{bmatrix} \frac{ds}{s} \\ &- \left(\mathscr{R}_2(\log t)^{2-\beta} + \mathscr{R}_3(\log t)^{2-(\alpha+\beta)}\right)\frac{\mathscr{B}^*}{\mathscr{R}_1}\log \mathscr{T} + \mathscr{B}^* \log t.\end{aligned}$$

To demonstrate this result, we proceed in three steps.

Step A: We prove that $\mathscr{O}_1 y + \mathscr{O}_2 z \in \mathscr{B}_r$, $\forall y, z \in \mathscr{B}_r$.

For $y, z \in \mathscr{B}_r$ and $t \in \mathrm{J}$, we have

$$\begin{aligned}|(\mathscr{O}_1 y)(t) + (\mathscr{O}_2 z)(t)| \leq \sup_{t\in \mathrm{J}} &\left| \int_1^t \frac{\left(\log \frac{t}{s}\right)^{1-\beta+\alpha}}{\Gamma(2-\beta+\alpha)} \begin{bmatrix} h(s) - \kappa_1 \phi_1\left(s, z(s), {}^{\mathrm{CH}}\mathrm{D}^{\alpha} z(s)\right) \\ -\kappa_2\phi_2\left(s, z(s), {}^{\mathrm{H}}\mathrm{I}^{p} z(s)\right) \end{bmatrix} \frac{ds}{s} \right. \\ &- \begin{pmatrix} \mathscr{R}_2(\log t)^{2-\beta} \\ +\mathscr{R}_3(\log t)^{2-(\alpha+\beta)} \end{pmatrix} \int_1^{\mathscr{T}} \frac{\left(\log \frac{\mathscr{T}}{s}\right)^{1-\beta+\alpha}}{\Gamma(2-\beta+\alpha)} \begin{bmatrix} h(s) - \kappa_1 \phi_1\left(s, z(s), {}^{\mathrm{CH}}\mathrm{D}^{\alpha} z(s)\right) \\ -\kappa_2\phi_2\left(s, z(s), {}^{\mathrm{H}}\mathrm{I}^{p} z(s)\right) \end{bmatrix} \frac{ds}{s} \\ &+ \begin{pmatrix} \mathscr{R}_2(\log t)^{2-\beta} \\ +\mathscr{R}_3(\log t)^{2-(\alpha+\beta)} \end{pmatrix} \eta \int_1^{\mathscr{T}} \frac{\left(\log \frac{\mathscr{T}}{s}\right)^{1-(\alpha+\beta)}}{\Gamma(2-(\alpha+\beta))} y(s)\frac{ds}{s} \\ &\left. - \eta \int_1^t \frac{\left(\log \frac{t}{s}\right)^{1-(\alpha+\beta)}}{\Gamma(2-(\alpha+\beta))} y(s)\frac{ds}{s} - \begin{pmatrix} \mathscr{R}_2(\log t)^{2-\beta} \\ +\mathscr{R}_3(\log t)^{2-(\alpha+\beta)} \end{pmatrix} \frac{\mathscr{B}^*}{\mathscr{R}_1}\log \mathscr{T} + \mathscr{B}^* \log t \right|,\end{aligned}$$

i.e.,

$$
\begin{aligned}
|(\mathcal{O}_1 y)(t)+(\mathcal{O}_2 z)(t)| \leq & \int_1^{\mathcal{T}} \frac{\left(\log\frac{\mathcal{T}}{s}\right)^{1-\beta+\alpha}}{\Gamma(2-\beta+\alpha)} \left| \begin{array}{l} h(s)-\kappa_1\, \phi_1\left(s,z(s),{}^{\mathrm{CH}}\mathrm{D}^{\alpha} z(s)\right) \\ -\kappa_2\phi_2\left(s,z(s),\ {}^{\mathrm{H}}\mathrm{I}^{p} z(s)\right) \end{array} \right| \frac{ds}{s} \\
& - \left(\begin{array}{l} |\mathcal{R}_2|(\log\mathcal{T})^{2-\beta} \\ +|\mathcal{R}_3|(\log\mathcal{T})^{2-(\alpha+\beta)} \end{array} \right) \int_1^{\mathcal{T}} \frac{\left(\log\frac{\mathcal{T}}{s}\right)^{1-\beta+\alpha}}{\Gamma(2-\beta+\alpha)} \left| \begin{array}{l} h(s)-\kappa_1\, \phi_1\left(s,z(s),{}^{\mathrm{CH}}\mathrm{D}^{\alpha} z(s)\right) \\ -\kappa_2\phi_2\left(s,z(s),\ {}^{\mathrm{H}}\mathrm{I}^{p} z(s)\right) \end{array} \right| \frac{ds}{s} \\
& + \left(\begin{array}{l} |\mathcal{R}_2|(\log\mathcal{T})^{2-\beta} \\ +|\mathcal{R}_3|(\log\mathcal{T})^{2-(\alpha+\beta)} \end{array} \right) \eta \int_1^{\mathcal{T}} \frac{\left(\log\frac{\mathcal{T}}{s}\right)^{1-(\alpha+\beta)}}{\Gamma(2-(\alpha+\beta))} |y(s)| \frac{ds}{s} \\
& - \eta \int_1^{\mathcal{T}} \frac{\left(\log\frac{\mathcal{T}}{s}\right)^{1-(\alpha+\beta)}}{\Gamma(2-(\alpha+\beta))} |y(s)| \frac{ds}{s} - \left(\begin{array}{l} |\mathcal{R}_2|(\log\mathcal{T})^{2-\beta} \\ +|\mathcal{R}_3|(\log\mathcal{T})^{2-(\alpha+\beta)} \end{array} \right) \frac{|\mathcal{B}^*|}{|\mathcal{R}_1|} \log\mathcal{T} + \mathcal{B}^* \log\mathcal{T}.
\end{aligned}
$$

Thanks to the assumption $(\mathcal{P}_4)$, we obtain

$$
\begin{aligned}
\|\mathcal{O}_1 y+\mathcal{O}_2 z\|_\infty \leq & (1+\Delta_1)\times \left[\begin{array}{l} \frac{(\mathcal{M}_3+|\kappa_1|\mathcal{M}_1+|\kappa_2|\mathcal{M}_2)(\log\mathcal{T})^{2-\beta+\alpha}}{\Gamma(3-\beta+\alpha)} \\ +\frac{\eta r(\log\mathcal{T})^{2-(\alpha+\beta)}}{\Gamma(3-(\alpha+\beta))} \end{array} \right] + |\mathcal{B}^*|\log\mathcal{T}\left(\frac{\Delta_1}{|\mathcal{R}_1|}+1\right) \\
\leq & (1+\Delta_1)\times \left[\begin{array}{l} \frac{(\mathcal{M}_3+|\kappa_1|\mathcal{M}_1+|\kappa_2|\mathcal{M}_2)(\log\mathcal{T})^{2-\beta+\alpha}}{\Gamma(3-\beta+\alpha)} \\ +\frac{\eta r(\log\mathcal{T})^{2-(\alpha+\beta)}}{\Gamma(3-(\alpha+\beta))} \end{array} \right] + \mathcal{C}_1,
\end{aligned}
$$

where $\Delta_1 = |\mathcal{R}_2|(\log\mathcal{T})^{2-\beta} + |\mathcal{R}_3|(\log\mathcal{T})^{2-(\alpha+\beta)}$.
Similarly, we have

$$
\begin{aligned}
\|{}^{\mathrm{CH}}\mathrm{D}^{\alpha}\mathcal{O}_1 y + {}^{\mathrm{CH}}\mathrm{D}^{\alpha}\mathcal{O}_2 z\|_\infty \leq & (\mathcal{M}_3+|\kappa_1|\mathcal{M}_1+|\kappa_2|\mathcal{M}_2)\left[\frac{(\log\mathcal{T})^{2-\beta}}{\Gamma(3-\beta)} + \Delta_2\frac{(\log\mathcal{T})^{2-\beta+\alpha}}{\Gamma(3-\beta+\alpha)}\right] \\
& + \eta r\left[\frac{(\log\mathcal{T})^{2-(2\alpha+\beta)}}{\Gamma(3-(2\alpha+\beta))} + \Delta_2\frac{(\log\mathcal{T})^{2-(\alpha+\beta)}}{\Gamma(3-(\alpha+\beta))}\right] + |\mathcal{B}^*|\left[\frac{(\log\mathcal{T})^{1-\alpha}}{\Gamma(2-\alpha)} + \Delta_2\log\mathcal{T}\right] \\
\leq & (\mathcal{M}_3+|\kappa_1|\mathcal{M}_1+|\kappa_2|\mathcal{M}_2)\left[\frac{(\log\mathcal{T})^{2-\beta}}{\Gamma(3-\beta)} + \Delta_2\frac{(\log\mathcal{T})^{2-\beta+\alpha}}{\Gamma(3-\beta+\alpha)}\right] \\
& + \eta r\left[\frac{(\log\mathcal{T})^{2-(2\alpha+\beta)}}{\Gamma(3-(2\alpha+\beta))} + \Delta_2\frac{(\log\mathcal{T})^{2-(\alpha+\beta)}}{\Gamma(3-(\alpha+\beta))}\right] + \mathcal{C}_2,
\end{aligned}
$$

where $\Delta_2 = |\mathcal{R}_2|\frac{\Gamma(3-\beta)(\log\mathcal{T})^{2-(\alpha+\beta)}}{\Gamma(3-(\alpha+\beta))} + |\mathcal{R}_3|\frac{\Gamma(3-(\alpha+\beta))(\log\mathcal{T})^{2-(2\alpha+\beta)}}{\Gamma(3-(2\alpha+\beta))}$.
We therefore conclude that

$$\|\mathcal{O}_1 y+\mathcal{O}_2 z\|_{\mathbb{Y}} \leq r,$$

where r satisfies

$$
r \geq \frac{(\mathcal{M}_3+|\kappa_1|\mathcal{M}_1+|\kappa_2|\mathcal{M}_2)\left[(1+\Delta_1)\frac{(\log\mathcal{T})^{2-\beta+\alpha}}{\Gamma(3-\beta+\alpha)} + \frac{(\log\mathcal{T})^{2-\beta}}{\Gamma(3-\beta)} + \Delta_2\frac{(\log\mathcal{T})^{2-\beta+\alpha}}{\Gamma(3-\beta+\alpha)}\right] + \mathcal{C}_1+\mathcal{C}_2}{1-\eta\left[(1+\Delta_1)\frac{(\log\mathcal{T})^{2-(\alpha+\beta)}}{\Gamma(3-(\alpha+\beta))} + \frac{(\log\mathcal{T})^{2-(2\alpha+\beta)}}{\Gamma(3-(2\alpha+\beta))} + \Delta_2\frac{(\log\mathcal{T})^{2-(\alpha+\beta)}}{\Gamma(3-(\alpha+\beta))}\right]}.
$$

Consequently, $\mathscr{O}_1 y + \mathscr{O}_2 z \in \mathscr{B}_r$, for all $y, z \in \mathscr{B}_r$.
Step B: We demonstrate that the operator $\mathscr{O}_2$ is a contraction on $\mathscr{B}_r$.
Let $y, z \in \mathscr{B}_r$, for each $t \in \mathrm{J}$, we have

$$\begin{aligned}
|(\mathscr{O}_2 y)(t) - (\mathscr{O}_2 z)(t)| \leq\ & \sup_{t\in \mathrm{J}} \int_1^t \frac{\left(\log \frac{t}{s}\right)^{1-\beta+\alpha}}{\Gamma(2-\beta+\alpha)} |\kappa_1| \left|\phi_1\left(s, y(s), {}^{\mathrm{CH}}\mathrm{D}^\alpha y(s)\right) - \phi_1\left(s, z(s), {}^{\mathrm{CH}}\mathrm{D}^\alpha z(s)\right)\right| \frac{ds}{s} \\
& + \sup_{t\in \mathrm{J}} \begin{pmatrix} \left|\mathscr{R}_2 (\log t)^{2-\beta}\right| \\ + \left|\mathscr{R}_3 (\log t)^{2-(\alpha+\beta)}\right| \end{pmatrix} \int_1^{\mathscr{T}} \frac{\left(\log \frac{\mathscr{T}}{s}\right)^{1-\beta+\alpha}}{\Gamma(2-\beta+\alpha)} |\kappa_1| \begin{vmatrix} \phi_1\left(s, y(s), {}^{\mathrm{CH}}\mathrm{D}^\alpha y(s)\right) \\ -\phi_1\left(s, z(s), {}^{\mathrm{CH}}\mathrm{D}^\alpha z(s)\right) \end{vmatrix} \frac{ds}{s} \\
& + \sup_{t\in \mathrm{J}} \int_1^t \frac{\left(\log \frac{t}{s}\right)^{1-\beta+\alpha}}{\Gamma(2-\beta+\alpha)} |\kappa_2| \left|\phi_2\left(s, y(s), {}^{\mathrm{H}}\mathrm{I}^p y(s)\right) - \phi_2\left(s, z(s), {}^{\mathrm{H}}\mathrm{I}^p z(s)\right)\right| \frac{ds}{s} \\
& + \sup_{t\in \mathrm{J}} \begin{pmatrix} \left|\mathscr{R}_2 (\log t)^{2-\beta}\right| \\ + \left|\mathscr{R}_3 (\log t)^{2-(\alpha+\beta)}\right| \end{pmatrix} \int_1^{\mathscr{T}} \frac{\left(\log \frac{\mathscr{T}}{s}\right)^{1-\beta+\alpha}}{\Gamma(2-\beta+\alpha)} |\kappa_2| \begin{vmatrix} \phi_2\left(s, y(s), {}^{\mathrm{H}}\mathrm{I}^p y(s)\right) \\ -\phi_2\left(s, z(s), {}^{\mathrm{H}}\mathrm{I}^p z(s)\right) \end{vmatrix} \frac{ds}{s}.
\end{aligned}$$

This implies directly

$$\| \mathscr{O}_2 y - \mathscr{O}_2 z \|_\infty \leq \frac{(|\kappa_1|\mathscr{L}_1 + |\kappa_2|\mathscr{L}_2)\left(1 + |\mathscr{R}_2| (\log \mathscr{T})^{2-\beta} + |\mathscr{R}_3| (\log \mathscr{T})^{2-(\alpha+\beta)}\right) (\log \mathscr{T})^{2-\beta+\alpha}}{\Gamma(3-\beta+\alpha)} \| y - z \|_{\mathbb{Y}} .$$

In addition, we have

$$\begin{aligned}
\| {}^{\mathrm{CH}}\mathrm{D}^\alpha \mathscr{O}_2 y - {}^{\mathrm{CH}}\mathrm{D}^\alpha \mathscr{O}_2 z \|_\infty \leq\ & \frac{(|\kappa_1|\mathscr{L}_1 + |\kappa_2|\mathscr{L}_2)(\log \mathscr{T})^{2-\beta}}{\Gamma(3-\beta)} \| y - z \|_{\mathbb{Y}} \\
& + \frac{(|\kappa_1|\mathscr{L}_1 + |\kappa_2|\mathscr{L}_2)(\log \mathscr{T})^{2-\beta+\alpha}}{\Gamma(3-\beta+\alpha)} \left[\begin{array}{c} \frac{|\mathscr{R}_2|\Gamma(3-\beta)(\log \mathscr{T})^{2-(\alpha+\beta)}}{\Gamma(3-(\alpha+\beta))} + \\ \frac{|\mathscr{R}_3|\Gamma(3-(\alpha+\beta))(\log \mathscr{T})^{2-(2\alpha+\beta)}}{\Gamma(3-(2\alpha+\beta))} \end{array} \right] \| y - z \|_{\mathbb{Y}} .
\end{aligned}$$

Then,

$$\|{}^{\mathrm{CH}}\mathrm{D}^\alpha \mathscr{O}_2 y - {}^{\mathrm{CH}}\mathrm{D}^\alpha \mathscr{O}_2 z \|_\infty \leq (|\kappa_1|\mathscr{L}_1 + |\kappa_2|\mathscr{L}_2) \left[\begin{array}{c} \frac{(\log \mathscr{T})^{2-\beta}}{\Gamma(3-\beta)} + \\ \left(\begin{array}{c} \frac{|\mathscr{R}_2|\Gamma(3-\beta)(\log \mathscr{T})^{2-(\alpha+\beta)}}{\Gamma(3-(\alpha+\beta))} + \\ \frac{|\mathscr{R}_3|\Gamma(3-(\alpha+\beta))(\log \mathscr{T})^{2-(2\alpha+\beta)}}{\Gamma(3-(2\alpha+\beta))} \end{array} \right) \frac{(\log \mathscr{T})^{2-\beta+\alpha}}{\Gamma(3-\beta+\alpha)} \end{array} \right] \| y - z \|_{\mathbb{Y}} .$$

Therefore,

$$\| \mathscr{O}_2 y - \mathscr{O}_2 z \|_{\mathbb{Y}} \leq \mathrm{S}^* \| y - z \|_{\mathbb{Y}} .$$

As a result, the operator $\mathscr{O}_2$ is a contraction given the condition $0 < \mathrm{S}^* < 1$.
Step C: We show that the operator $\mathscr{O}_1$ is a completely continuous. This requires that we focus about the following points:
1: The continuity of the operator $\mathscr{O}_1$ is due to the continuity of the functions ϕ_1, ϕ_2 and h

(refer to the hypothesis $(\mathscr{P}_3)$). This demostration is therefore omitted.

2: The operator $\mathscr{O}_1$ is bounded on $\mathscr{B}_r$.

In effect, for $y \in \mathscr{B}_r$ and $\forall\, t \in \mathrm{J}$, we have

$$\begin{aligned}
|(\mathscr{O}_1 y)(t)| &= \eta \int_1^t \frac{\left(\log \frac{t}{s}\right)^{1-(\alpha+\beta)}}{\Gamma(2-(\alpha+\beta))} |y(s)| \frac{ds}{s} \\
&+ \left(\left| \mathscr{R}_2 (\log t)^{2-\beta} \right| + \left| \mathscr{R}_3 (\log t)^{2-(\alpha+\beta)} \right| \right) \eta \int_1^{\mathscr{T}} \frac{\left(\log \frac{\mathscr{T}}{s}\right)^{1-(\alpha+\beta)}}{\Gamma(2-(\alpha+\beta))} |y(s)| \frac{ds}{s}.
\end{aligned}$$

Thus,

$$\|\mathscr{O}_1 y\|_\infty \leq \frac{\eta r (\log \mathscr{T})^{2-(\alpha+\beta)}}{\Gamma(3-(\alpha+\beta))} \left(1 + |\mathscr{R}_2| (\log \mathscr{T})^{2-\beta} + |\mathscr{R}_3| (\log \mathscr{T})^{2-(\alpha+\beta)} \right).$$

In the same way, we have

$$\begin{aligned}
\left| {}^{\mathrm{CH}}\mathrm{D}^{\alpha} \mathscr{O}_1 y(t) \right| &= \eta \int_1^t \frac{\left(\log \frac{t}{s}\right)^{2-(2\alpha+\beta)-1}}{\Gamma(2-(2\alpha+\beta))} |y(s)| \frac{ds}{s} \\
&+ \eta \frac{\Gamma(3-\beta) \left| \mathscr{R}_2 (\log t)^{2-(\alpha+\beta)} \right|}{\Gamma(3-(\alpha+\beta))} \int_1^{\mathscr{T}} \frac{\left(\log \frac{\mathscr{T}}{s}\right)^{1-(\alpha+\beta)}}{\Gamma(2-(\alpha+\beta))} |y(s)| \frac{ds}{s} \\
&+ \eta \frac{\Gamma(3-(\alpha+\beta)) \left| \mathscr{R}_3 (\log t)^{2-(2\alpha+\beta)} \right|}{\Gamma(3-(2\alpha+\beta))} \int_1^{\mathscr{T}} \frac{\left(\log \frac{\mathscr{T}}{s}\right)^{1-(\alpha+\beta)}}{\Gamma(2-(\alpha+\beta))} |y(s)| \frac{ds}{s},
\end{aligned}$$

therefore,

$$\|{}^{\mathrm{CH}}\mathrm{D}^{\alpha} \mathscr{O}_1 y\|_\infty \leq \frac{\eta r (\log \mathscr{T})^{2-(2\alpha+\beta)}}{\Gamma(3-(2\alpha+\beta))} \left(1 + |\mathscr{R}_3| (\log \mathscr{T})^{2-(\alpha+\beta)} \right) + \frac{\eta r |\mathscr{R}_2| \Gamma(3-\beta) (\log \mathscr{T})^{4-2(\alpha+\beta)}}{(\Gamma(3-(\alpha+\beta)))^2}.$$

As a consequence,

$$\|\mathscr{O}_1 y\|_{\mathbb{Y}} < +\infty.$$

Then, $\mathscr{O}_1$ is a bounded operator on $\mathscr{B}_r$.

3: The operator $\mathscr{O}_1$ is equicontinuous.

Let $t_1, t_2 \in \mathrm{J}$ where $t_1 < t_2$. Then, for $y \in \mathscr{B}_r$, we obtain

$$\begin{aligned} |\mathscr{O}_1 y(t_2) - \mathscr{O}_1 y(t_1)| &= \Bigg| -\eta \int_1^{t_2} \frac{\left(\log \frac{t_2}{s}\right)^{1-(\alpha+\beta)}}{\Gamma(2-(\alpha+\beta))} y(s) \frac{ds}{s} \\ &+ \left(\mathscr{R}_2 (\log t_2)^{2-\beta} + \mathscr{R}_3 (\log t_2)^{2-(\alpha+\beta)}\right) \eta \int_1^{\mathscr{T}} \frac{\left(\log \frac{\mathscr{T}}{s}\right)^{1-(\alpha+\beta)}}{\Gamma(2-(\alpha+\beta))} y(s) \frac{ds}{s} \\ &+ \eta \int_1^{t_1} \frac{\left(\log \frac{t_1}{s}\right)^{1-(\alpha+\beta)}}{\Gamma(2-(\alpha+\beta))} y(s) \frac{ds}{s} \\ &- \left(\mathscr{R}_2 (\log t_1)^{2-\beta} + \mathscr{R}_3 (\log t_1)^{2-(\alpha+\beta)}\right) \eta \int_1^{\mathscr{T}} \frac{\left(\log \frac{\mathscr{T}}{s}\right)^{1-(\alpha+\beta)}}{\Gamma(2-(\alpha+\beta))} y(s) \frac{ds}{s} \Bigg|, \end{aligned}$$

thus,

$$\begin{aligned} \| \mathscr{O}_1 y(t_2) - \mathscr{O}_1 y(t_1) \|_\infty &\leq \frac{\eta r}{\Gamma(3-(\alpha+\beta))} \left[(\log t_2)^{2-(\alpha+\beta)} - (\log t_1)^{2-(\alpha+\beta)} \right] \\ &+ \frac{\eta r (\log \mathscr{T})^{2-(\alpha+\beta)}}{\Gamma(3-(\alpha+\beta))} \left[\begin{array}{l} |\mathscr{R}_2| \left((\log t_2)^{2-\beta} - (\log t_1)^{2-\beta} \right) \\ + |\mathscr{R}_3| \left((\log t_2)^{2-(\alpha+\beta)} - (\log t_1)^{2-(\alpha+\beta)} \right) \end{array} \right]. \end{aligned} \tag{2.10}$$

On the other hand,

$$\begin{aligned} |{}^{CH}\mathrm{D}^\alpha \mathscr{O}_1 y(t_2) - {}^{CH}\mathrm{D}^\alpha \mathscr{O}_1 y(t_1)| &= \Bigg| -\eta \int_1^{t_2} \frac{\left(\log \frac{t_2}{s}\right)^{2-(2\alpha+\beta)-1}}{\Gamma(2-(2\alpha+\beta))} y(s) \frac{ds}{s} + \eta \int_1^{t_1} \frac{\left(\log \frac{t_1}{s}\right)^{2-(2\alpha+\beta)-1}}{\Gamma(2-(2\alpha+\beta))} y(s) \frac{ds}{s} \\ &+ \eta \frac{\Gamma(3-\beta) \mathscr{R}_2 (\log t_2)^{2-(\alpha+\beta)}}{\Gamma(3-(\alpha+\beta))} \int_1^{\mathscr{T}} \frac{\left(\log \frac{\mathscr{T}}{s}\right)^{1-(\alpha+\beta)}}{\Gamma(2-(\alpha+\beta))} y(s) \frac{ds}{s} \\ &+ \eta \frac{\Gamma(3-(\alpha+\beta)) \mathscr{R}_3 (\log t_2)^{2-(2\alpha+\beta)}}{\Gamma(3-(2\alpha+\beta))} \int_1^{\mathscr{T}} \frac{\left(\log \frac{\mathscr{T}}{s}\right)^{1-(\alpha+\beta)}}{\Gamma(2-(\alpha+\beta))} y(s) \frac{ds}{s}, \\ &- \eta \frac{\Gamma(3-\beta) \mathscr{R}_2 (\log t_1)^{2-(\alpha+\beta)}}{\Gamma(3-(\alpha+\beta))} \int_1^{\mathscr{T}} \frac{\left(\log \frac{\mathscr{T}}{s}\right)^{1-(\alpha+\beta)}}{\Gamma(2-(\alpha+\beta))} y(s) \frac{ds}{s} \\ &- \eta \frac{\Gamma(3-(\alpha+\beta)) \mathscr{R}_3 (\log t_1)^{2-(2\alpha+\beta)}}{\Gamma(3-(2\alpha+\beta))} \int_1^{\mathscr{T}} \frac{\left(\log \frac{\mathscr{T}}{s}\right)^{1-(\alpha+\beta)}}{\Gamma(2-(\alpha+\beta))} y(s) \frac{ds}{s} \Bigg|. \end{aligned}$$

This implies,

$$\begin{aligned}\|{}^{CH}D^{\alpha}\mathscr{O}_1 y(t_2) - {}^{CH}D^{\alpha}\mathscr{O}_1 y(t_1)\|_{\infty} &\leq \frac{\eta r\left(1+|\mathscr{R}_3|(\log \mathscr{T})^{2-(\alpha+\beta)}\right)}{\Gamma(3-(2\alpha+\beta))}\left[(\log t_2)^{2-(2\alpha+\beta)} - (\log t_1)^{2-(2\alpha+\beta)}\right] \\ &+ \frac{\lambda r|\mathscr{R}_2|\Gamma(3-\beta)(\log \mathscr{T})^{2-(\alpha+\beta)}}{(\Gamma(3-(\alpha+\beta)))^2}\left[(\log t_2)^{2-(\alpha+\beta)} - (\log t_1)^{2-(\alpha+\beta)}\right].\end{aligned} \tag{2.11}$$

In the preceding inequalities (2.10) and (2.11), the right-hand sides tend to zero as $t_1 \to t_2$. Hence, $\mathscr{O}_1$ is equicontinuous. Thus, $\mathscr{O}_1$ is relatively compact based on the Arzela-Ascoli theorem.

In view of the Krasnoselskii fixed point theorem and results of steps A, B, and C, we can deduce that the operator $\mathscr{O}$ has a fixed point, which is a solution to the problem (2.2). The proof is finished. ■

4 Stability Results

The concept of stability is fundamental to the analysis of dynamic system behavior and in the synthesis of control laws for these systems. Therefore, one of the main issues that automation specialists and engineers have and continue to have in their work is the stability of dynamic systems. Several notions of stability are found in the literature, and they are frequently related to the characteristics of the systems under study, as well as their settings, needs, and intended behaviors. One stability theory that has gained a great deal of attention recently is the Ulam theory; for details, see [19, 39, 63, 66].

In (1940), Ulam posed the following question, which relates to the functional equations stability: "under what conditions exists an additive application sufficiently close to an approximate additive application?". Stated differently, the idea of stability for the functional equation emerges when we replace the functional equation by an inequality whose second member is a perturbation of the equation. A year later, in 1941 Hyers provided a satisfactory response in the Banach space case. This explains why this type of stability is called now Ulam-Hyers [37].

In this part, we will define and study some types of Ulam stability for (2.2).

Definition 4.1 *The problem (2.2) is Ulam-Hyers stable if there exists a positive real number λ, such that for each $\epsilon > 0$, $t \in J$ and for all solution $y^* \in \mathbb{Y}$ of the following inequality*

$$\left|{}^{CH}D^{\alpha}\left({}^{CH}D^{2-\beta} + \eta{}^{CH}D^{\alpha}\right) y^*(t) + \kappa_1\phi_1\left(t, y^*(t), {}^{CH}D^{\alpha}y^*(t)\right) + \kappa_2\phi_2\left(t, y^*(t), {}^{H}I^{p}y^*(t)\right) - h(t)\right| \leq \epsilon, \tag{2.12}$$

there exists a solution $y \in \mathbb{Y}$ *of the problem (2.2), i.e,*

$$ {}^{CH}D^{\alpha}\left({}^{CH}D^{2-\beta}+\eta{}^{CH}D^{\alpha}\right)y(t)+\kappa_1\phi_1\left(t,y(t),{}^{CH}D^{\alpha}y(t)\right)+\kappa_2\phi_2\left(t,y(t),{}^{H}I^{p}y(t)\right)=h(t), $$

under the conditions

$$ y(1)=0, \qquad \left({}^{CH}D^{1-(\alpha-\beta)}\,{}^{CH}D^{\alpha-\beta}y\right)(1)=\mathscr{B}^{*}, \qquad y(\mathscr{T})=0, $$

with

$$ \|y^{*}-y\|_{\mathbb{Y}}\leq\lambda\epsilon. $$

Definition 4.2 *The problem (2.2) is generalized Ulam-Hyers stable if there exists* $\chi\in C(\mathbb{R}^{+},\mathbb{R}^{+})$, *with* $\chi(0)=0$, *such that for any* $\epsilon>0$, *and for each solution* $y^{*}\in\mathbb{Y}$ *to the inequality (2.12), there exists a solution* $y\in\mathbb{Y}$ *of (2.2), with*

$$ \|y^{*}-y\|_{\mathbb{Y}}\leq\chi(\epsilon). $$

At this point, we can present the Ulam-Hyers stability results for the problem (2.2).

Theorem 4.3 *Assume that the hypotheses of Theorem 3.1 are satisfied. Then, the problem (2.2) is Ulam-Hyers stable.*

Proof. let $\epsilon>0$ and let $y^{*}\in\mathbb{Y}$ be a solution of (2.12). We put

$$ \begin{aligned} \mathscr{H}_{y^{*}}(s) &:= h(s)-\kappa_1\,\phi_1\left(s,y^{*}(s),{}^{CH}D^{\alpha}y^{*}(s)\right)-\kappa_2\phi_2\left(s,y^{*}(s),{}^{H}I^{p}y^{*}(s)\right). \\ \mathscr{H}_{y}(s) &:= h(s)-\kappa_1\,\phi_1\left(s,y(s),{}^{CH}D^{\alpha}y(s)\right)-\kappa_2\phi_2\left(s,y(s),{}^{H}I^{p}y(s)\right). \end{aligned} $$

Then, we have

$$ \left| \begin{aligned} & y^{*}(t)-\int_1^t\frac{(\log\frac{t}{s})^{1-\beta+\alpha}}{\Gamma(2-\beta+\alpha)}\left[\mathscr{H}_{y^{*}}(s)\right]\frac{ds}{s}+\eta\int_1^t\frac{(\log\frac{t}{s})^{1-(\alpha+\beta)}}{\Gamma(2-(\alpha+\beta))}[y^{*}(s)]\frac{ds}{s} \\ & +\left(\mathscr{R}_2(\log t)^{2-\beta}+\mathscr{R}_3(\log t)^{2-(\alpha+\beta)}\right)\int_1^{\mathscr{T}}\frac{\left(\log\frac{\mathscr{T}}{s}\right)^{1-\beta+\alpha}}{\Gamma(2-\beta+\alpha)}\left[\mathscr{H}_{y^{*}}(s)\right]\frac{ds}{s} \\ & -\left(\mathscr{R}_2(\log t)^{2-\beta}+\mathscr{R}_3(\log t)^{2-(\alpha+\beta)}\right)\eta\int_1^{\mathscr{T}}\frac{\left(\log\frac{\mathscr{T}}{s}\right)^{1-(\alpha+\beta)}}{\Gamma(2-(\alpha+\beta))}[y^{*}(s)]\frac{ds}{s} \\ & +\left(\mathscr{R}_2(\log t)^{2-\beta}+\mathscr{R}_3(\log t)^{2-(\alpha+\beta)}\right)\frac{\mathscr{B}^{*}}{\mathscr{R}_1}\log\mathscr{T}-\mathscr{B}^{*}\log t. \end{aligned} \right| \leq\epsilon\times\frac{(\log t)^{2-\beta+\alpha}}{\Gamma(3-\beta+\alpha)}. $$

According to Theorem 3.1, the problem (2.2) has a unique solution y, which is provided by

$$\begin{aligned} y(t) &= \int_1^t \frac{\left(\log\frac{t}{s}\right)^{1-\beta+\alpha}}{\Gamma(2-\beta+\alpha)}\left[\mathscr{H}_y(s)\right]\frac{ds}{s} - \eta\int_1^t \frac{\left(\log\frac{t}{s}\right)^{1-(\alpha+\beta)}}{\Gamma(2-(\alpha+\beta))}[y(s)]\frac{ds}{s} \\ &- \left(\mathscr{R}_2(\log t)^{2-\beta} + \mathscr{R}_3(\log t)^{2-(\alpha+\beta)}\right)\int_1^{\mathscr{T}} \frac{\left(\log\frac{\mathscr{T}}{s}\right)^{1-\beta+\alpha}}{\Gamma(2-\beta+\alpha)}\left[\mathscr{H}_y(s)\right]\frac{ds}{s} \\ &+ \left(\mathscr{R}_2(\log t)^{2-\beta} + \mathscr{R}_3(\log t)^{2-(\alpha+\beta)}\right)\eta\int_1^{\mathscr{T}} \frac{\left(\log\frac{\mathscr{T}}{s}\right)^{1-(\alpha+\beta)}}{\Gamma(2-(\alpha+\beta))}[y(s)]\frac{ds}{s} \\ &- \left(\mathscr{R}_2(\log t)^{2-\beta} + \mathscr{R}_3(\log t)^{2-(\alpha+\beta)}\right)\frac{\mathscr{B}^*}{\mathscr{R}_1}\log\mathscr{T} + \mathscr{B}^*\log t. \end{aligned}$$

This allows us to write the following expression:

$$\begin{aligned} |y^*(t)-y(t)| \leq \Bigg| & \epsilon\times\frac{(\log t)^{2-\beta+\alpha}}{\Gamma(3-\beta+\alpha)} + \int_1^t \frac{\left(\log\frac{t}{s}\right)^{1-\beta+\alpha}}{\Gamma(2-\beta+\alpha)}\left[\mathscr{H}_{y^*}(s)\right]\frac{ds}{s} - \int_1^t \frac{\left(\log\frac{t}{s}\right)^{1-\beta+\alpha}}{\Gamma(2-\beta+\alpha)}\left[\mathscr{H}_y(s)\right]\frac{ds}{s} \\ &- \eta\int_1^t \frac{\left(\log\frac{t}{s}\right)^{1-(\alpha+\beta)}}{\Gamma(2-(\alpha+\beta))}[y^*(s)]\frac{ds}{s} + \eta\int_1^t \frac{\left(\log\frac{t}{s}\right)^{1-(\alpha+\beta)}}{\Gamma(2-(\alpha+\beta))}[y(s)]\frac{ds}{s}. \\ &- \left(\mathscr{R}_2(\log t)^{2-\beta} + \mathscr{R}_3(\log t)^{2-(\alpha+\beta)}\right)\int_1^{\mathscr{T}} \frac{\left(\log\frac{\mathscr{T}}{s}\right)^{1-\beta+\alpha}}{\Gamma(2-\beta+\alpha)}\left[\mathscr{H}_{y^*}(s)\right]\frac{ds}{s} \\ &+ \left(\mathscr{R}_2(\log t)^{2-\beta} + \mathscr{R}_3(\log t)^{2-(\alpha+\beta)}\right)\int_1^{\mathscr{T}} \frac{\left(\log\frac{\mathscr{T}}{s}\right)^{1-\beta+\alpha}}{\Gamma(2-\beta+\alpha)}\left[\mathscr{H}_y(s)\right]\frac{ds}{s} \\ &+ \eta\left(\mathscr{R}_2(\log t)^{2-\beta} + \mathscr{R}_3(\log t)^{2-(\alpha+\beta)}\right)\int_1^{\mathscr{T}} \frac{\left(\log\frac{\mathscr{T}}{s}\right)^{1-(\alpha+\beta)}}{\Gamma(2-(\alpha+\beta))}[y^*(s)]\frac{ds}{s} \\ &- \eta\left(\mathscr{R}_2(\log t)^{2-\beta} + \mathscr{R}_3(\log t)^{2-(\alpha+\beta)}\right)\int_1^{\mathscr{T}} \frac{\left(\log\frac{\mathscr{T}}{s}\right)^{1-(\alpha+\beta)}}{\Gamma(2-(\alpha+\beta))}[y(s)]\frac{ds}{s} \Bigg|, \end{aligned}$$

which implies

$$
\begin{aligned}
|y^*(t)-y(t)| \;\leq\;& \epsilon \times \frac{(\log t)^{2-\beta+\alpha}}{\Gamma(3-\beta+\alpha)} + \int_1^t \frac{\left(\log \frac{t}{s}\right)^{1-\beta+\alpha}}{\Gamma(2-\beta+\alpha)} \left|\mathscr{H}_{y^*}(s)-\mathscr{H}_y(s)\right| \frac{ds}{s} \\
&+ \eta \int_1^t \frac{\left(\log \frac{t}{s}\right)^{1-(\alpha+\beta)}}{\Gamma(2-(\alpha+\beta))} \left|y^*(s)-y(s)\right| \frac{ds}{s} \\
&+ \left(\mathscr{R}_2 (\log t)^{2-\beta} + \mathscr{R}_3 (\log t)^{2-(\alpha+\beta)}\right) \int_1^{\mathscr{T}} \frac{\left(\log \frac{\mathscr{T}}{s}\right)^{1-\beta+\alpha}}{\Gamma(2-\beta+\alpha)} \left|\mathscr{H}_{y^*}(s)-\mathscr{H}_{y^*}(s)\right| \frac{ds}{s} \\
&+ \eta \left(\mathscr{R}_2 (\log t)^{2-\beta} + \mathscr{R}_3 (\log t)^{2-(\alpha+\beta)}\right) \int_1^{\mathscr{T}} \frac{\left(\log \frac{\mathscr{T}}{s}\right)^{1-(\alpha+\beta)}}{\Gamma(2-(\alpha+\beta))} \left|y^*(s)-y(s)\right| \frac{ds}{s}.
\end{aligned}
$$

Therefore,

$$
\|y^*-y\|_\infty \;\leq\; \frac{\epsilon (\log \mathscr{T})^{2-\beta+\alpha}}{\Gamma(3-\beta+\alpha)} + \mathrm{S}_1 \|y^*-y\|_{\mathbb{Y}}.
$$

Using the same reasoning, we get

$$
\| {}^{\mathrm{CH}}\mathrm{D}^\alpha y^* - {}^{\mathrm{CH}}\mathrm{D}^\alpha y\|_\infty \;\leq\; \frac{\epsilon (\log \mathscr{T})^{2-\beta}}{\Gamma(3-\beta)} + \mathrm{S}_2 \|y^*-y\|_{\mathbb{Y}},
$$

so,

$$
\|y^*-y\|_{\mathbb{Y}} \leq \left[\frac{\epsilon (\log \mathscr{T})^{2-\beta+\alpha}}{\Gamma(3-\beta+\alpha)} + \frac{\epsilon (\log \mathscr{T})^{2-\beta}}{\Gamma(3-\beta)}\right] + \mathrm{S}\|y^*-y\|_{\mathbb{Y}}.
$$

Hence,

$$
\begin{aligned}
\|y^*-y\|_{\mathbb{Y}} \;\leq\;& \left[\frac{\frac{(\log \mathscr{T})^{2-\beta+\alpha}}{\Gamma(3-\beta+\alpha)} + \frac{(\log \mathscr{T})^{2-\beta}}{\Gamma(3-\beta)}}{1-\mathrm{S}}\right] \epsilon \\
:=\;& \lambda \epsilon.
\end{aligned}
$$

As a consequence, the problem (2.2) is Ulam-Hyers stable. Therefore, the proof of Theorem 4.3 is finished. ■

Remark 4.4 *By assuming* $\chi(\epsilon) = \lambda\epsilon$ *where* $\chi(0) = 0$*, we can also conclude that the problem under consideration is generalized Ulam-Hyers stable.*

5 Illustrative Examples

In order to show the application of our main results, two numerical examples are provided in this section.

Example 5.1 *To give an illustration of Theorem 3.1 and Theorem 4.3, we consider the following nonlinear fractional sequential problem of Van Der Pol-Duffing Jerk type, with the Caputo-Hadamard approach:*

$$\begin{cases} {}^{CH}D^{\alpha}\left({}^{CH}D^{2-\beta}+\frac{{}^{CH}D^{\alpha}}{200}\right)y(t)+\kappa_1\left(\frac{\sin(\pi t)}{12t}+\frac{y(t)}{26}+\frac{{}^{CH}D^{\alpha}y(t)}{6}\right)+\kappa_2\left(\frac{9}{t}+\frac{y(t)}{123}+\frac{{}^{H}I^{p}y(t)}{8}\right)=2cos(0.2t). \\ y(1)=0, \qquad \left({}^{CH}D^{1-(\alpha-\beta)}\,{}^{CH}D^{\alpha-\beta}y\right)(1)=-\frac{22}{7}, \qquad y(\mathcal{T})=0, \quad t\in[1,e], \end{cases} \tag{2.13}$$

where

$$\alpha=0.98, \quad \beta=0.01, \quad p=\frac{1}{7}, \quad \eta=\frac{1}{200}, \quad \kappa_1=-\frac{1}{300}, \quad \kappa_2=-\frac{1}{400}, \quad \mathcal{B}^*=-\frac{22}{7}, \quad \mathcal{T}=e,$$

and

$$\begin{aligned} \phi_1(t,u,v) &= \frac{\sin(\pi t)}{12t}+\frac{1}{26}u+\frac{1}{6}v. \\ \phi_2(t,u,v) &= \frac{9}{t}+\frac{1}{123}u+\frac{1}{8}v. \\ h(t) &= 2cos(0.2t). \end{aligned}$$

It is clear that the the functions ϕ_1,ϕ_2 *and h are continuous on* $J=[1,e]$.
For $(u_1,v_1),(u_2,v_2)\in\mathbb{R}^2$ *with* $t\in J$*, we have*

$$\begin{aligned} |\phi_1(t,u_1,v_1)-\phi_1(t,u_2,v_2)| &\leq 0.2051\,[|u_1-u_2|+|v_1-v_2|]. \\ |\phi_2(t,u_1,v_1)-\phi_2(t,u_2,v_2)| &\leq 0.1331\,[|u_1-u_2|+|v_1-v_2|]. \end{aligned}$$

Thus, the condition $(\mathcal{P}_1)$ *of Theorem 3.1 is satisfied, where the Lipschitz constants are determined as follows*

$$\mathcal{L}_1=0.2051, \quad \mathcal{L}_2=0.1331.$$

By using the given data, we obtain also

$$S_1=0.0103, \quad S_2=0.0157,$$

this indicates that the condition $(\mathcal{P}_2)$ *is verified; i.e*

$$0<S=0.0103+0.0157=0.0261<1.$$

As a result, all conditions of Theorem 3.1 are satisfied, then the problem (2.13) has a unique solution on J. *Moreover, the conditions of Theorem 4.3 are also verified, then the problem (2.13) is stable in the sense of Ulam-Hyers.*

Example 5.2 *For Theorem 3.2, we consider the following problem to illustrate the validity of the result dealing with the existence of at least one solution.*

$$\begin{cases} {}^{\mathrm{CH}}\mathrm{D}^{\alpha}\left({}^{\mathrm{CH}}\mathrm{D}^{2-\beta}+\frac{{}^{\mathrm{CH}}\mathrm{D}^{\alpha}}{300}\right)y(t)+\kappa_1\frac{(23t^2-2)}{300}\cos\left(\frac{y(t)}{100}+\frac{{}^{\mathrm{CH}}\mathrm{D}^{\alpha}y(t)}{120}\right)+\kappa_2\frac{(2t-1)}{50}\sin\left(\frac{y(t)}{60}+\frac{{}^{\mathrm{H}}\mathrm{I}^{p}y(t)}{70}\right)=\frac{cos(\frac{1}{30}t)}{200}. \\ y(1)=0, \qquad \left({}^{\mathrm{CH}}\mathrm{D}^{1-(\alpha-\beta)}\ {}^{\mathrm{CH}}\mathrm{D}^{\alpha-\beta}y\right)(1)=\mathscr{B}^*, \qquad y(\mathscr{T})=0, \quad t\in[1,e], \end{cases} \tag{2.14}$$

where

$$\alpha=0.88,\quad \beta=0.1,\quad p=\frac{1}{8},\quad \eta=\frac{1}{300},\quad \kappa_1=-\frac{1}{197},\quad \kappa_2=-\frac{1}{191},\quad \mathscr{B}^*=-\frac{3}{26},\quad \mathscr{T}=e,$$

and

$$\begin{aligned} \phi_1(t,u,v) &= \frac{23t^2-2}{300}\cos\left(\frac{1}{100}u+\frac{1}{120}v\right). \\ \phi_2(t,u,v) &= \frac{2t-1}{50}\sin\left(\frac{1}{60}u+\frac{1}{70}v\right). \\ h(t) &= \frac{1}{200}cos\left(\frac{1}{30}t\right). \end{aligned}$$

Clearly, the functions ϕ_1,ϕ_2 *and h are continuous on* $\mathrm{J}=[1,e]$.
For $(u,v)\in\mathbb{R}^2$ *and* $t\in\mathrm{J}$, *we have*

$$\begin{aligned} |\phi_1(t,u_1,v_1) &\leq \frac{21}{300}. \\ |\phi_2(t,u_1,v_1) &\leq \frac{1}{50}. \\ |h(t) &\leq \frac{1}{200}. \end{aligned}$$

This leads to

$$\mathscr{M}_1=\frac{21}{300},\quad \mathscr{M}_2=\frac{1}{50},\quad \mathscr{M}_3=\frac{1}{200}.$$

Through some calculations, we get

$$0<\mathrm{S}^*=0.372<1,$$

thus, the conditions of Theorem 3.2 are verified, then the problem (2.14) has at least one solution on J.

6 Conclusion

In this chapter, we have proposed a novel sequential (VdPD) Jerk fractional differential oscillator with a Caputo-Hadamard approach. Then, we have established a uniqueness result for the problem (2.2) through the application of the Banach contraction principle, and the existence of at least one solution using the Krasnoselskii theorem. Additionally, we have demonstrated two different types of stability for this problem. An example has been presented to illustrate the practical importance of our results.

Chapter 3

Solvability and Stability Analysis For a Pantograph Problem With Sequential Caputo-Hadamard Derivatives

1 Introduction

[1] Given the remarkable development in railway networks (trams, for example), we decided to look at the operation of these trains, and how to provide them with electrical energy for transportation. From our research we have determined that the collection of electrical current is ensured by a pantograph, which is an articulated tubular structure.

The objective of the pantograph is to follow the height of the catenary without any shock in order to maintain permanent contact, which prompted the researchers to provide a suitable mathematical model for the pantograph system called pantograph equation [28]. This equation has been modeled by Ockendon and Tayler [52]. The differential equation that follows provides its standard form:

$$\begin{cases} y'(t) = a(t)y(t) + b(t)y(\eta t), \\ y(0) = y_0, \\ 0 \leq t \leq T, \quad 0 < \eta < 1, \end{cases}$$

where the function $a, b : [0, T] \to \mathbb{R}$ are continuous.

In recent years, the topic of fractional integral and differential equations has proved to be valuable tools in modeling the dynamics of various systems and processes in variety of domains. Many researchers have proposed several fractional variants of the above pantograph equation. For more details, see [5, 6, 29, 31, 32, 33, 34, 67, 70] and references therein.

---

[1] **A. Abdelnebi and Z. Dahmani**, *Existence and Stability results for a Pantograph Problem With Sequential Caputo-Hadamard Derivatives*. Fractional Differential Calculus, **14(1)**, 21-38, (2024).

In particular, the authors of the paper [12] have presented the following fractional pantograph equation involving Caputo fractional derivative:

$$\begin{cases} {}^{C}D^{\alpha}[y(\tau)] = \varphi(\tau, y(\tau), y(\eta\tau)), \quad \tau \in [0,T], \\ y(0) = y_0, \\ 0 < \alpha, \eta < 1. \end{cases}$$

Very recently in [16], S. Belarbi and al. have discussed the existence and uniqueness of the following Φ-Caputo sequential pantograph fractional differential problem with integral conditions:

$$\begin{cases} {}^{c}D^{\beta,\Phi}({}^{c}D^{\alpha,\Phi}y(t) + g(t, y(t))) = f(t, y(t), y(\lambda t), {}^{c}D^{\alpha,\Phi}y(t)), \quad t \in [0,1], \\ y(0) = 0, \qquad x(1) = \int_0^1 h(s, y(s))ds, \\ 0 < \alpha, \beta < 1, \quad \lambda > 0, \end{cases}$$

where ${}^{c}D^{\beta,\Phi}$ and ${}^{c}D^{\alpha,\Phi}$ are the Φ-Caputo derivatives, the functions f, g and h are continuous.

Inspired by the previously mentioned results, we focus in this chapter on the following three-sequential pantograph-type problem [2]:

$$\begin{cases} {}^{CH}D^{\beta_1}[{}^{CH}D^{\beta_2}({}^{CH}D^{\beta_3}\varkappa(t))] = f(t, \varkappa(t), \varkappa(\lambda t), {}^{H}I^{\delta}\varkappa(\lambda t), {}^{CH}D^{\rho}\varkappa(\lambda t)). \\ \varkappa(1) - \mathscr{A}_1 = 0, \qquad D^{\beta_3}\varkappa(1) - \mathscr{A}_2 = 0, \qquad D^{\beta_2}(D^{\beta_3}\varkappa(T)) = 0, \\ 0 < \delta < \beta_i < 1, \quad i = 1,2,3, \quad \beta_3 > \rho, \quad 0 < \lambda < 1, \quad \mathscr{A}_1, \mathscr{A}_2 \in \mathbb{R}, \quad t \in J, \quad T > 1, \end{cases} \tag{3.1}$$

where ${}^{CH}D^{\beta_i}$, ${}^{CH}D^{\rho}$ are the Caputo-Hadamard fractional derivatives, ${}^{H}I^{\delta}$ is the Hadamard fractional integral, $J = [1,T]$, the function $f : J \times \mathbb{R}^4 \to \mathbb{R}$ is continuous.

The main motivations and advantages of the current work are:

- The problem (3.1) is very interesting since it can admit as a limiting case of the pantograph equation of order three, which provides a more accurate mathematical model of the motion of the pantograph arms. It takes into account the third derivative of the unknown function x, which represents the curvature of the pantograph arms. This makes the equation more accurate than the equation of order two.

- The use of the Caputo-Hadamard fractional derivative which combines the properties of two significant operators of Caputo and Hadamard.

The organization of this chapter is as follows: In Section 2, we show the integral representation of problem the (3.1). In Section 3, we examine the existence and uniqueness of solutions using the Banach fixed point. We also use the Leray-Schauder theorem to derive a second theorem for the existence of at least one solution. After that, we move on to investigate into certain Ulam stability types. Two of our main results are illustrated with an example in Section 4.

2 The Integral Solution

In this section, we will establish the integral representation of problem (3.1), but prior to that, for the sake of simplicity, we put

$$\mathscr{F}(t) := f(t, \varkappa(t), \varkappa(\lambda t), {}^{\mathrm{H}}\mathrm{I}^{\delta}\varkappa(\lambda t), {}^{\mathrm{CH}}\mathrm{D}^{\rho}\varkappa(\lambda t)),$$

where $\mathscr{F} \in \mathrm{C}([1,\mathrm{T}])$.
After that, the problem (3.1) transforms into

$$\begin{cases} {}^{\mathrm{CH}}\mathrm{D}^{\beta_1}[{}^{\mathrm{CH}}\mathrm{D}^{\beta_2}({}^{\mathrm{CH}}\mathrm{D}^{\beta_3}\varkappa(t))] = \mathscr{F}(t). \\ \varkappa(1) - \mathscr{A}_1 = 0, \qquad \mathrm{D}^{\beta_3}\varkappa(1) - \mathscr{A}_2 = 0, \qquad \mathrm{D}^{\beta_2}(\mathrm{D}^{\beta_3}\varkappa(\mathrm{T})) = 0. \end{cases} \tag{3.2}$$

Now, we present the following lemma in order to give the integral representation for problem (3.2).

Lemma 2.1 *Let $0 < \beta_i \leq 1, i = 1,2,3$. For $\mathscr{F} : [1,\mathrm{T}] \to \mathbb{R}$ and $t \in \mathrm{J}$, the integral equation that follows represents a solution to the problem (3.2):*

$$\begin{aligned} \varkappa(t) &= \frac{1}{\Gamma(\beta_1+\beta_2+\beta_3)} \int_1^t \left(\log\frac{t}{u}\right)^{\beta_1+\beta_2+\beta_3-1} \mathscr{F}(u)\frac{du}{u} \\ &- \frac{(\log t)^{\beta_2+\beta_3}}{\Gamma(\beta_2+\beta_3+1)\Gamma(\beta_1)} \int_1^{\mathrm{T}} \left(\log\frac{\mathrm{T}}{u}\right)^{\beta_1-1} \mathscr{F}(u)\frac{du}{u} \\ &+ \mathscr{A}_1\frac{(\log t)^{\beta_3}}{\Gamma(\beta_3+1)} + \mathscr{A}_2. \end{aligned} \tag{3.3}$$

Proof. For $i = 1,2,3$, we consider

$${}^{\mathrm{CH}}\mathrm{D}^{\beta_1}[{}^{\mathrm{CH}}\mathrm{D}^{\beta_2}({}^{\mathrm{CH}}\mathrm{D}^{\beta_3}\varkappa(t))] = \mathscr{F}(t). \tag{3.4}$$

The previous equation (3.4) can be written as follows by applying the operator ${}^{CH}\mathrm{I}^{\beta_1}$:

$$ {}^{H}\mathrm{I}^{\beta_1}\left[{}^{CH}\mathrm{D}^{\beta_1}({}^{CH}\mathrm{D}^{\beta_2}({}^{CH}\mathrm{D}^{\beta_3}\varkappa(t)))\right] = {}^{H}\mathrm{I}^{\beta_1}\mathscr{F}(t). \tag{3.5} $$

Then, thanks to Lemma 3.22, we obtain

$$ {}^{CH}\mathrm{D}^{\beta_2}({}^{CH}\mathrm{D}^{\beta_3}\varkappa(t)) = {}^{H}\mathrm{I}^{\alpha_1}\mathscr{F}(t) - c_0. \tag{3.6} $$

By the same procedures, we apply ${}^{H}\mathrm{I}^{\beta_2}$ for both sides of (3.6), we get

$$ ({}^{CH}\mathrm{D}^{\beta_3}\varkappa)(t) = {}^{H}\mathrm{I}^{\beta_1+\beta_2}\mathscr{F}(t) - c_0\,{}^{H}\mathrm{I}^{\beta_2}(1) - c_1. \tag{3.7} $$

In the last step, we use the operator ${}^{H}\mathrm{I}^{\beta_3}$ for (3.7) to obtain the following equation

$$ \varkappa(t) = {}^{H}\mathrm{I}^{\beta_1+\beta_2+\beta_3}\mathscr{F}(t) - c_0\,{}^{H}\mathrm{I}^{\beta_2+\beta_3}(1) - c_1\,{}^{H}\mathrm{I}^{\beta_3}(1) - c_2, \tag{3.8} $$

this implies

$$ \varkappa(t) = {}^{H}\mathrm{I}^{\beta_1+\beta_2+\beta_3}\mathscr{F}(t) - c_0\frac{(\log t)^{\beta_2+\beta_3}}{\Gamma(\beta_2+\beta_3+1)} - c_1\frac{(\log t)^{\beta_3}}{\Gamma(\beta_3+1)} - c_2, \tag{3.9} $$

where c_0, c_1 and c_2 are arbitrary constants.
From the condition $\varkappa(1) - \mathscr{A}_1 = 0$, we have

$$ c_2 = -\mathscr{A}_1. $$

Next, according to $\mathrm{D}^{\beta_3}\varkappa(1) - \mathscr{A}_2 = 0$, we get

$$ c_1 = -\mathscr{A}_2. $$

Finally, by the last condition $\mathrm{D}^{\beta_2}(\mathrm{D}^{\beta_3}\varkappa(\mathrm{T})) = 0$, we obtain

$$ c_0 = {}^{H}\mathrm{I}^{\beta_1}\mathscr{F}(\mathrm{T}). $$

The desired solution (3.3) can be obtained by inserting the values of $c_i, i = 0,1,2$ in (3.9). ■

3 Existence and Stability results

In this section, we discuss three main results of the sequential pantograph problem (3.1) which are: existence and uniqueness of solution based on the Banach contraction principle, existence of at least one solution via the Leray-Schauder theorem and some Ulam type stability results.

First, we define the following Banach space:

$$\mathbb{X} := \left\{\varkappa \in C(J,\mathbb{R}),\ {}^{CH}D^{\rho}\varkappa \in C(J,\mathbb{R})\right\},$$

with the norm

$$\| \varkappa \|_{\mathbb{X}} = 2 \| \varkappa \|_{\infty} + \| {}^{CH}D^{\rho}\varkappa \|_{\infty},$$

where by definition,

$$\| x \|_{\infty} = \sup_{t\in J} | \varkappa(t) |, \qquad \| {}^{CH}D^{\rho}\varkappa \|_{\infty} = \sup_{t\in J} | {}^{CH}D^{\rho}\varkappa(t) | .$$

Then, we transform our problem into a fixed point one. For this reason, we define the operator $\mathscr{H}$ as follows:

$$\begin{aligned} \mathscr{H} : \mathbb{X} &\to \mathbb{X} \\ \varkappa &\to \mathscr{H}(\varkappa), \end{aligned}$$

such that, $\forall\, t \in J$, we have

$$\begin{aligned} (\mathscr{H}\varkappa)(t) &= \frac{1}{\Gamma(\beta_1+\beta_2+\beta_3)} \int_1^t \left(\log\frac{t}{u}\right)^{\beta_1+\beta_2+\beta_3-1} f(u,\varkappa(u),\varkappa(\lambda u),{}^{H}I^{\delta}\varkappa(\lambda u),{}^{CH}D^{\rho}\varkappa(\lambda u))\frac{du}{u} \\ &- \frac{(\log t)^{\beta_2+\beta_3}}{\Gamma(\beta_2+\beta_3+1)\Gamma(\beta_1)} \int_1^T \left(\log\frac{T}{u}\right)^{\beta_1-1} f(u,\varkappa(u),\varkappa(\lambda u),{}^{H}I^{\delta}\varkappa(\lambda u),{}^{CH}D^{\rho}\varkappa(\lambda u))\frac{du}{u} \\ &+ \mathscr{A}_1\frac{(\log t)^{\beta_3}}{\Gamma(\beta_3+1)} + \mathscr{A}_2. \end{aligned}$$

To prove the main results of this chapter, we introduce these basic hypothesis:

($\mathscr{P}_1$) $f : J \times \mathbb{R}^4 \to \mathbb{R}$ is continuous function.

($\mathscr{P}_2$) There exists a nonnegative constant Δ_1, such that for all $t \in J$ and $u_i, v_i \in \mathbb{R}$ $(i = \overline{1:4})$, we have:

$$|f(t,u_1,u_2,u_3,u_4) - f(t,v_1,v_2,v_3,v_4)| \leq \Delta_1(|u_1-v_1|+|u_2-v_2|+|u_3-v_3|+|u_4-v_4|).$$

($\mathscr{P}_3$) There are nonnegative constants $\omega_0,\omega_1,\omega_2,\omega_3$ and ω_4, such that for all $u_i \in \mathbb{R}, i = \overline{1:4}$, we have

$$|f(t,u_1,u_2,u_3,u_4))| \leq \omega_0+\omega_1|u_1|+\omega_2|u_2|+\omega_3|u_3|+\omega_4|u_4|.$$

Next, for the sake of brevity, we pose the following notations:

$$
\begin{aligned}
\varDelta &= \varDelta_1\left(1+\frac{(\log \mathrm{T})^{\delta}}{\Gamma(\delta+1)}\right).\\
\Re_1 &= \left(\frac{(\log \mathrm{T})^{\beta_1+\beta_2+\beta_3}}{\Gamma(\beta_1+\beta_2+\beta_3+1)}+\frac{(\log \mathrm{T})^{\beta_2+\beta_3}(\log \mathrm{T})^{\beta_1}}{\Gamma(\beta_2+\beta_3+1)\Gamma(\beta_1+1)}\right).\\
\Re_2 &= \left(\frac{(\log \mathrm{T})^{\beta_1+\beta_2+\beta_3-\rho}}{\Gamma(\beta_1+\beta_2+\beta_3-\rho+1)}+\frac{(\log \mathrm{T})^{\beta_2+\beta_3-\rho}(\log \mathrm{T})^{\beta_1}}{\Gamma(\beta_2+\beta_3-\rho+1)\Gamma(\beta_1+1)}\right).\\
\Re_3 &= \varDelta\Re_1.\\
\Re_4 &= \mathscr{M}\Re_1+|\mathscr{A}_1|\frac{(\log \mathrm{T})^{\beta_3}}{\Gamma(\beta_3+1)}+|\mathscr{A}_2|.\\
\Re_5 &= \varDelta\Re_2.\\
\Re_6 &= \mathscr{M}\Re_2+|\mathscr{A}_1|\frac{(\log \mathrm{T})^{\beta_3-\rho}}{\Gamma(\beta_3-\rho+1)}.
\end{aligned}
$$

3.1 A Unique Solution

In this part, we consider the subsequent theorem to demonstrate that there is only one solution to the problem (3.1).

Theorem 3.1 *Assume that the condition ($\mathscr{P}_2$) hold. If*

$$2\Re_3+\Re_5<1, \tag{3.10}$$

then, the problem (3.1) has a unique solution on J.

Proof. Let us consider the subset $\mathfrak{B}_{\mathfrak{r}}$ of $\mathbb{X}$ given by

$$\mathfrak{B}_{\mathfrak{r}} := \{\varkappa\in\mathbb{X} : \|\varkappa\|_{\mathbb{X}}\le r\},$$

where

$$r\ge\frac{2\Re_4+\Re_6}{1-(2\Re_3+\Re_5)},$$

and, let us put

$$\sup_{t\in \mathrm{J}}|f(t,0,0,0,0)| = \mathscr{M}.$$

So, we shall prove that $\mathscr{H}\mathfrak{B}_{\mathfrak{r}} \subset \mathfrak{B}_{\mathfrak{r}}$.
For $\varkappa \in \mathfrak{B}_{\mathfrak{r}}$, $t \in \mathrm{J}$ and by $(\mathscr{P}_2)$, we have

$$\begin{aligned}
| \quad & f(t,\varkappa(t),\varkappa(\lambda t),{}^{\mathrm{H}}\mathrm{I}^{\delta}\varkappa(\lambda t),{}^{\mathrm{CH}}\mathrm{D}^{\rho}\varkappa(\lambda t))| \\
&\leq |f(t,\varkappa(t),\varkappa(\lambda t),{}^{\mathrm{H}}\mathrm{I}^{\delta}\varkappa(\lambda t),{}^{\mathrm{CH}}\mathrm{D}^{\rho}\varkappa(\lambda t)) - f(t,0,0,0,0) + f(t,0,0,0,0)| \\
&\leq |f(t,\varkappa(t),\varkappa(\lambda t),{}^{\mathrm{H}}\mathrm{I}^{\delta}\varkappa(\lambda t),{}^{\mathrm{CH}}\mathrm{D}^{\rho}\varkappa(\lambda t)) - f(t,0,0,0,0)| + |f(t,0,0,0,0)| \\
&\leq \varDelta_1(|\varkappa(t)| + |\varkappa(\lambda t)| + |{}^{\mathrm{H}}\mathrm{I}^{\delta}\varkappa(\lambda t)| + |{}^{\mathrm{CH}}\mathrm{D}^{\rho}\varkappa(\lambda t)|) + |f(t,0,0,0,0)| \\
&\leq \sup_{t\in\mathrm{J}}\left\{\varDelta_1(|\varkappa(t)| + |\varkappa(\lambda t)| + |{}^{\mathrm{H}}\mathrm{I}^{\delta}\varkappa(\lambda t)| + |{}^{\mathrm{CH}}\mathrm{D}^{\rho}\varkappa(\lambda t)|) + |f(t,0,0,0,0)|\right\},
\end{aligned}$$

which implies

$$\begin{aligned}
|f(t,\varkappa(t),\varkappa(\lambda t),{}^{\mathrm{H}}\mathrm{I}^{\delta}\varkappa(\lambda t),{}^{\mathrm{CH}}\mathrm{D}^{\rho}\varkappa(\lambda t))| &\leq \varDelta_1\left(2\parallel\varkappa\parallel_{\infty} + \parallel{}^{\mathrm{H}}\mathrm{I}^{\delta}\varkappa\parallel_{\infty} + \parallel{}^{\mathrm{CH}}\mathrm{D}^{\rho}\varkappa\parallel_{\infty}\right) + \mathscr{M} \\
&\leq \varDelta_1\left(2\parallel\varkappa\parallel_{\infty} + \parallel{}^{\mathrm{CH}}\mathrm{D}^{\rho}\varkappa\parallel_{\infty}\right) + \varDelta_1\parallel{}^{\mathrm{H}}\mathrm{I}^{\delta}\varkappa\parallel_{\infty} + \mathscr{M} \\
&\leq \varDelta_1\parallel\varkappa\parallel_{\mathbb{X}} + \varDelta_1\frac{(\log\mathrm{T})^{\delta}}{\varGamma(\delta+1)}\parallel\varkappa\parallel_{\mathbb{X}} + \mathscr{M} \\
&\leq \varDelta_1\left(1 + \frac{(\log\mathrm{T})^{\delta}}{\varGamma(\delta+1)}\right)\parallel\varkappa\parallel_{\mathbb{X}} + \mathscr{M} \\
&\leq \varDelta\parallel\varkappa\parallel_{\mathbb{X}} + \mathscr{M},
\end{aligned}$$

thus

$$|f(t,\varkappa(t),\varkappa(\lambda t),{}^{\mathrm{H}}\mathrm{I}^{\delta}\varkappa(\lambda t),{}^{\mathrm{CH}}\mathrm{D}^{\rho}\varkappa(\lambda t))| \leq \varDelta r + \mathscr{M}. \tag{3.11}$$

So, by using the preceding inequality (3.11), we get

$$\begin{aligned}
|(\mathscr{H}\varkappa)(t)| &\leq \sup_{t\in\mathrm{J}}\Bigg\{\Bigg|\frac{1}{\varGamma(\beta_1+\beta_2+\beta_3)}\int_1^t\left(\log\frac{t}{u}\right)^{\beta_1+\beta_2+\beta_3-1} f(u,\varkappa(u),\varkappa(\lambda u),{}^{\mathrm{H}}\mathrm{I}^{\delta}\varkappa(\lambda u),{}^{\mathrm{CH}}\mathrm{D}^{\rho}\varkappa(\lambda u))\frac{du}{u} \\
&\quad - \frac{(\log t)^{\beta_2+\beta_3}}{\varGamma(\beta_2+\beta_3+1)\varGamma(\beta_1)}\int_1^{\mathrm{T}}\left(\log\frac{\mathrm{T}}{u}\right)^{\beta_1-1} f(u,\varkappa(u),\varkappa(\lambda u),{}^{\mathrm{H}}\mathrm{I}^{\delta}\varkappa(\lambda u),{}^{\mathrm{CH}}\mathrm{D}^{\rho}\varkappa(\lambda u))\frac{du}{u} \\
&\quad + \mathscr{A}_1\frac{(\log t)^{\beta_3}}{\varGamma(\beta_3+1)} + \mathscr{A}_2\Bigg|\Bigg\} \\
&\leq \frac{1}{\varGamma(\beta_1+\beta_2+\beta_3)}\int_1^{\mathrm{T}}\left(\log\frac{\mathrm{T}}{u}\right)^{\beta_1+\beta_2+\beta_3-1}\left|f(u,\varkappa(u),\varkappa(\lambda u),{}^{\mathrm{H}}\mathrm{I}^{\delta}\varkappa(\lambda u),{}^{\mathrm{CH}}\mathrm{D}^{\rho}\varkappa(\lambda u))\right|\frac{du}{u} \\
&\quad + \frac{(\log\mathrm{T})^{\beta_2+\beta_3}}{\varGamma(\beta_2+\beta_3+1)\varGamma(\beta_1)}\int_1^{\mathrm{T}}\left(\log\frac{\mathrm{T}}{u}\right)^{\beta_1-1}\left|f(u,\varkappa(u),\varkappa(\lambda u),{}^{\mathrm{H}}\mathrm{I}^{\delta}\varkappa(\lambda u),{}^{\mathrm{CH}}\mathrm{D}^{\rho}\varkappa(\lambda u))\right|\frac{du}{u} \\
&\quad + |\mathscr{A}_1|\frac{(\log\mathrm{T})^{\beta_3}}{\varGamma(\beta_3+1)} + |\mathscr{A}_2|,
\end{aligned}$$

or

$$\begin{aligned}|(\mathscr{H}\varkappa)(t)| &\leq (\Delta r+\mathscr{M})\frac{\left(\log \mathrm{T}\right)^{\beta_1+\beta_2+\beta_3}}{\Gamma(\beta_1+\beta_2+\beta_3+1)}+(\Delta r+\mathscr{M})\frac{\left(\log \mathrm{T}\right)^{\beta_2+\beta_3}\left(\log \mathrm{T}\right)^{\beta_1}}{\Gamma(\beta_2+\beta_3+1)\Gamma(\beta_1+1)}+|\mathscr{A}_1|\frac{(\log \mathrm{T})^{\beta_3}}{\Gamma(\beta_3+1)}+|\mathscr{A}_2|\\ &\leq \Delta\left(\frac{\left(\log \mathrm{T}\right)^{\beta_1+\beta_2+\beta_3}}{\Gamma(\beta_1+\beta_2+\beta_3+1)}+\frac{\left(\log \mathrm{T}\right)^{\beta_2+\beta_3}\left(\log \mathrm{T}\right)^{\beta_1}}{\Gamma(\beta_2+\beta_3+1)\Gamma(\beta_1+1)}\right)r\\ &\quad+\mathscr{M}\left(\frac{\left(\log \mathrm{T}\right)^{\beta_1+\beta_2+\beta_3}}{\Gamma(\beta_1+\beta_2+\beta_3+1)}+\frac{\left(\log \mathrm{T}\right)^{\beta_2+\beta_3}\left(\log \mathrm{T}\right)^{\beta_1}}{\Gamma(\beta_2+\beta_3+1)\Gamma(\beta_1+1)}\right)+|\mathscr{A}_1|\frac{(\log \mathrm{T})^{\beta_3}}{\Gamma(\beta_3+1)}+|\mathscr{A}_2|.\end{aligned}$$

Consequently,

$$\|\mathscr{H}\varkappa\|_\infty \leq \Re_3 r+\Re_4.$$

Similarly, we have

$$\begin{aligned}({}^{\mathrm{CH}}\mathrm{D}^{\rho}\mathscr{H}\varkappa)(t) &= \int_1^t \frac{\left(\log\frac{t}{u}\right)^{\beta_1+\beta_2+\beta_3-\rho-1}}{\Gamma(\beta_1+\beta_2+\beta_3-\rho)} f(u,\varkappa(u),\varkappa(\lambda u),{}^{\mathrm{H}}\mathrm{I}^{\delta}\varkappa(\lambda u),{}^{\mathrm{CH}}\mathrm{D}^{\rho}\varkappa(\lambda u))\frac{du}{u}\\ &- \frac{(\log t)^{\beta_2+\beta_3-\rho}}{\Gamma(\beta_2+\beta_3-\rho+1)}\int_1^{\mathrm{T}}\frac{\left(\log\frac{\mathrm{T}}{u}\right)^{\beta_1-1}}{\Gamma(\beta_1)} f(u,\varkappa(u),\varkappa(\lambda u),{}^{\mathrm{H}}\mathrm{I}^{\delta}\varkappa(\lambda u),{}^{\mathrm{CH}}\mathrm{D}^{\rho}\varkappa(\lambda u))\frac{du}{u}\\ &+ \mathscr{A}_1\frac{(\log t)^{\beta_3-\rho}}{\Gamma(\beta_3-\rho+1)}.\end{aligned}$$

So,

$$\begin{aligned}|{}^{\mathrm{CH}}\mathrm{D}^{\rho}\mathscr{H}\varkappa(t)| &\leq \sup_{t\in \mathrm{J}}\left\{\left|\frac{1}{\Gamma(\beta_1+\beta_2+\beta_3-\rho)}\int_1^t \frac{\left(\log\frac{t}{u}\right)^{\beta_1+\beta_2+\beta_3-\rho-1}}{\Gamma(\beta_1+\beta_2+\beta_3-\rho)} f(u,\varkappa(u),\varkappa(\lambda u),{}^{\mathrm{H}}\mathrm{I}^{\delta}\varkappa(\lambda u),{}^{\mathrm{CH}}\mathrm{D}^{\rho}\varkappa(\lambda u))\frac{du}{u}\right.\right.\\ &\quad-\frac{(\log t)^{\beta_2+\beta_3-\rho}}{\Gamma(\beta_2+\beta_3-\rho+1)}\int_1^{\mathrm{T}}\frac{\left(\log\frac{\mathrm{T}}{u}\right)^{\beta_1-1}}{\Gamma(\beta_1)} f(u,\varkappa(u),\varkappa(\lambda u),{}^{\mathrm{H}}\mathrm{I}^{\delta}\varkappa(\lambda u),{}^{\mathrm{CH}}\mathrm{D}^{\rho}\varkappa(\lambda u))\frac{du}{u}\\ &\quad\left.\left.+\mathscr{A}_1\frac{(\log t)^{\beta_3-\rho}}{\Gamma(\beta_3-\rho+1)}\right|\right\},\end{aligned}$$

then,

$$\begin{aligned}\|{}^{\mathrm{CH}}\mathrm{D}^{\rho}\mathscr{H}\varkappa\|_\infty &\leq \int_1^{\mathrm{T}} \frac{\left(\log\frac{\mathrm{T}}{u}\right)^{\beta_1+\beta_2+\beta_3-\rho-1}}{\Gamma(\beta_1+\beta_2+\beta_3-\rho)} \left|f(u,\varkappa(u),\varkappa(\lambda u),{}^{\mathrm{H}}\mathrm{I}^{\delta}\varkappa(\lambda u),{}^{\mathrm{CH}}\mathrm{D}^{\rho}\varkappa(\lambda u))\right|\frac{du}{u}\\ &\quad+\frac{(\log \mathrm{T})^{\beta_2+\beta_3-\rho}}{\Gamma(\beta_2+\beta_3-\rho+1)}\int_1^{\mathrm{T}}\frac{\left(\log\frac{\mathrm{T}}{u}\right)^{\beta_1-1}}{\Gamma(\beta_1)} \left|f(u,\varkappa(u),\varkappa(\lambda u),{}^{\mathrm{H}}\mathrm{I}^{\delta}\varkappa(\lambda u),{}^{\mathrm{CH}}\mathrm{D}^{\rho}\varkappa(\lambda u))\right|\frac{du}{u}\\ &\quad+|\mathscr{A}_1|\frac{(\log \mathrm{T})^{\beta_3-\rho}}{\Gamma(\beta_3-\rho+1)}.\end{aligned}$$

This indicates

$$\begin{aligned}\|^{CH}D^{\rho}\mathscr{H}\varkappa\|_{\infty}| &\leq \Delta\left(\frac{(\log T)^{\beta_1+\beta_2+\beta_3-\rho}}{\Gamma(\beta_1+\beta_2+\beta_3-\rho+1)}+\frac{(\log T)^{\beta_2+\beta_3-\rho}(\log T)^{\beta_1}}{\Gamma(\beta_2+\beta_3-\rho+1)\Gamma(\beta_1+1)}\right)r \\ &+\mathscr{M}\left(\frac{(\log T)^{\beta_1+\beta_2+\beta_3-\rho}}{\Gamma(\beta_1+\beta_2+\beta_3-\rho+1)}+\frac{(\log T)^{\beta_2+\beta_3-\rho}(\log T)^{\beta_1}}{\Gamma(\beta_2+\beta_3-\rho+1)\Gamma(\beta_1+1)}\right)+|\mathscr{A}_1|\frac{(\log T)^{\beta_3-\rho}}{\Gamma(\beta_3-\rho+1)}.\end{aligned}$$

Therefore,

$$\|^{CH}D^{\rho}\mathscr{H}\varkappa\|_{\infty}\leq\Re_5 r+\Re_6.$$

Using the norm $\|.\|_{\mathbb{X}}$, we obtain

$$\begin{aligned}\|\mathscr{H}\varkappa\|_{\mathbb{X}} &= 2\|\mathscr{H}\varkappa\|+\left\|^{CH}D^{\rho}\mathscr{H}\varkappa\right\| \\ &\leq 2(\Re_3 r+\Re_4)+\Re_5 r+\Re_6 \\ &\leq (2\Re_3+\Re_5)r+2\Re_4+\Re_6 \\ &\leq r.\end{aligned}$$

As a result,

$$\mathscr{H}\mathfrak{B}_{\mathfrak{r}}\subset\mathfrak{B}_{\mathfrak{r}}.$$

Now, let $\varkappa, y\in\mathfrak{B}_{\mathfrak{r}}$. For all $t\in J$, we have

$$\begin{aligned}&|\ \mathscr{H}\varkappa(t)-\mathscr{H}y(t)\,| \\ &\leq \sup_{t\in J}\left\{\frac{1}{\Gamma(\beta_1+\beta_2+\beta_3)}\int_1^t(\log\tfrac{t}{u})^{\beta_1+\beta_2+\beta_3-1}\left|\begin{array}{l}f(u,\varkappa(u),\varkappa(\lambda u),{}^{H}I^{\delta}\varkappa(\lambda u),{}^{CH}D^{\rho}\varkappa(\lambda u))\\-f(u,y(u),y(\lambda u),{}^{H}I^{\delta}y(\lambda u),{}^{CH}D^{\rho}y(\lambda u))\end{array}\right|\frac{du}{u}\right. \\ &\left.-\frac{(\log t)^{\beta_2+\beta_3}}{\Gamma(\beta_2+\beta_3+1)\Gamma(\beta_1)}\int_1^T\left(\log\frac{T}{u}\right)^{\beta_1-1}\left|\begin{array}{l}f(u,\varkappa(u),\varkappa(\lambda u),{}^{H}I^{\delta}\varkappa(\lambda u),{}^{CH}D^{\rho}\varkappa(\lambda u))\\-f(u,y(u),y(\lambda u),{}^{H}I^{\delta}y(\lambda u),{}^{CH}D^{\rho}y(\lambda u))\end{array}\right|\frac{du}{u}\right\} \\ &\leq \frac{1}{\Gamma(\beta_1+\beta_2+\beta_3)}\int_1^T\left(\log\frac{T}{u}\right)^{\beta_1+\beta_2+\beta_3-1}\left|\begin{array}{l}f(u,\varkappa(u),\varkappa(\lambda u),{}^{H}I^{\delta}\varkappa(\lambda u),{}^{CH}D^{\rho}\varkappa(\lambda u))\\-f(u,y(u),y(\lambda u),{}^{H}I^{\delta}y(\lambda u),{}^{CH}D^{\rho}y(\lambda u))\end{array}\right|\frac{du}{u} \\ &+\frac{(\log T)^{\beta_2+\beta_3}}{\Gamma(\beta_2+\beta_3+1)\Gamma(\beta_1)}\int_1^T\left(\log\frac{T}{u}\right)^{\beta_1-1}\left|\begin{array}{l}f(u,\varkappa(u),\varkappa(\lambda u),{}^{H}I^{\delta}\varkappa(\lambda u),{}^{CH}D^{\rho}\varkappa(\lambda u))\\-f(u,y(u),y(\lambda u),{}^{H}I^{\delta}y(\lambda u),{}^{CH}D^{\rho}y(\lambda u))\end{array}\right|\frac{du}{u},\end{aligned}$$

this leads to

$$\|\,\mathscr{H}\varkappa-\mathscr{H}y\,\|_{\infty}\ \leq\ \Delta\left(\frac{(\log T)^{\beta_1+\beta_2+\beta_3}}{\Gamma(\beta_1+\beta_2+\beta_3+1)}+\frac{(\log T)^{\beta_2+\beta_3}(\log T)^{\beta_1}}{\Gamma(\beta_2+\beta_3+1)\Gamma(\beta_1+1)}\right)\left\|\varkappa-y\right\|_{\mathbb{X}}.$$

Therefore,

$$\|\,\mathscr{H}\varkappa-\mathscr{H}y\,\|_{\infty}\leq\Re_3\left\|\varkappa-y\right\|_{\mathbb{X}}. \tag{3.12}$$

Using the same arguments, we have

$$\begin{aligned}
&| \ {}^{CH}D^{\rho}\mathscr{H}\varkappa(t) - {}^{CH}D^{\rho}\mathscr{H}y(t) \,| \\
&\leq \frac{1}{\Gamma(\beta_1+\beta_2+\beta_3-\rho)}\int_1^T \left(\log\frac{T}{u}\right)^{\beta_1+\beta_2+\beta_3-\rho-1} \left| \begin{array}{l} f(u,\varkappa(u),\varkappa(\lambda u),{}^{H}I^{\delta}\varkappa(\lambda u),{}^{CH}D^{\rho}\varkappa(\lambda u)) \\ -f(u,y(u),y(\lambda u),{}^{H}I^{\delta}y(\lambda u),{}^{CH}D^{\rho}y(\lambda u)) \end{array}\right| \frac{du}{u} \\
&+\frac{(\log T)^{\beta_2+\beta_3-\rho}}{\Gamma(\beta_2+\beta_3-\rho+1)\Gamma(\beta_1)}\int_1^T \left(\log\frac{T}{u}\right)^{\beta_1-1} \left| \begin{array}{l} f(u,\varkappa(u),\varkappa(\lambda u),{}^{H}I^{\delta}\varkappa(\lambda u),{}^{CH}D^{\rho}\varkappa(\lambda u)) \\ -f(u,y(u),y(\lambda u),{}^{H}I^{\delta}y(\lambda u),{}^{CH}D^{\rho}y(\lambda u)) \end{array}\right| \frac{du}{u},
\end{aligned}$$

then,

$$\|{}^{CH}D^{\rho}\mathscr{H}\varkappa - {}^{CH}D^{\rho}\mathscr{H}y\,\|_{\infty} \leq \Delta\left(\frac{(\log T)^{\beta_1+\beta_2+\beta_3-\rho}}{\Gamma(\beta_1+\beta_2+\beta_3-\rho+1)} + \frac{(\log T)^{\beta_2+\beta_3-\rho}(\log T)^{\beta_1}}{\Gamma(\beta_2+\beta_3-\rho+1)\Gamma(\beta_1+1)}\right)\|\varkappa-y\|_{\mathbb{X}}.$$

Thus, we get

$$\|{}^{CH}D^{\rho}\mathscr{H}\varkappa - {}^{CH}D^{\rho}\mathscr{H}y\,\|_{\infty} \leq \Re_5\,\|\varkappa-y\|_{\mathbb{X}}. \tag{3.13}$$

Thanks to (3.12) and (3.13), we obtain

$$\begin{aligned}
\|\mathscr{H}\varkappa-\mathscr{H}y\|_{\mathbb{X}} &= 2\,\|\mathscr{H}\varkappa-\mathscr{H}y\|_{\infty} + \|{}^{CH}D^{\rho}\mathscr{H}\varkappa - {}^{CH}D^{\rho}\mathscr{H}y\|_{\infty} \\
&\leq (2\Re_3+\Re_5)\,\|\varkappa-y\|_{\mathbb{X}}.
\end{aligned}$$

So, we conclude that $\mathscr{H}$ is a contraction due to (3.10).
Hence, $\mathscr{H}$ possesses a unique fixed point which is the unique solution of the problem (3.1) based upon the Banach contraction principle Theorem 4.1. ■

3.2 At Least One Solution

In this current subsection, we provide the reader the following second main result, which is based on Theorem 4.3.

Theorem 3.2 *Assume that the hypotheses* $(\mathscr{P}_1)$ *and* $(\mathscr{P}_3)$ *are valid. If*

$$0 < \left[\omega_1+\omega_2+\frac{\omega_3(\log T)^{\delta}}{\Gamma(\delta+1)}+\omega_4\right][2\Re_1+\Re_2] < 1,$$

then, problem (3.1) has at least a solution on J.

Proof. In order to prove this theorem, we proceed as follows:
Step 1: We shall prove that $\mathscr{H}$ is continuous operator on $\mathbb{X}$:

Let $(\varkappa_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{X}$, with $\| \varkappa_n - \varkappa \|_{\mathbb{X}} \to 0$ when $n\to\infty$. For each $t \in \mathrm{J}$, we have

$$\begin{aligned} &| \mathscr{H}\varkappa_n(t) - \mathscr{H}\varkappa(t) | \\ &\leq \sup_{t\in\mathrm{J}} \Bigg\{ \frac{1}{\Gamma(\beta_1+\beta_2+\beta_3)} \int_1^t \left(\log\frac{t}{u}\right)^{\beta_1+\beta_2+\beta_3-1} \left| \begin{array}{c} f(u,\varkappa_n(u),\varkappa_n(\lambda u),{}^{\mathrm{H}}\mathrm{I}^{\delta}\varkappa_n(\lambda u),{}^{\mathrm{CH}}\mathrm{D}^{\rho}\varkappa_n(\lambda u)) \\ -f(u,\varkappa(u),\varkappa(\lambda u),{}^{\mathrm{H}}\mathrm{I}^{\delta}\varkappa(\lambda u),{}^{\mathrm{CH}}\mathrm{D}^{\rho}\varkappa(\lambda u)) \end{array} \right| \frac{du}{u} \\ &- \frac{(\log t)^{\beta_2+\beta_3}}{\Gamma(\beta_2+\beta_3+1)\Gamma(\beta_1)} \int_1^{\mathrm{T}} \left(\log\frac{\mathrm{T}}{u}\right)^{\beta_1-1} \left| \begin{array}{c} f(u,\varkappa_n(u),\varkappa_n(\lambda u),{}^{\mathrm{H}}\mathrm{I}^{\delta}\varkappa_n(\lambda u),{}^{\mathrm{CH}}\mathrm{D}^{\rho}\varkappa_n(\lambda u)) \\ -f(u,\varkappa(u),\varkappa(\lambda u),{}^{\mathrm{H}}\mathrm{I}^{\delta}\varkappa(\lambda u),{}^{\mathrm{CH}}\mathrm{D}^{\rho}\varkappa(\lambda u)) \end{array} \right| \frac{du}{u} \Bigg\}, \end{aligned}$$

so,

$$\begin{aligned} \| \mathscr{H}\varkappa_n - \mathscr{H}\varkappa \|_\infty \leq\ & \frac{(\log\mathrm{T})^{\beta_1+\beta_2+\beta_3}}{\Gamma(\beta_1+\beta_2+\beta_3+1)} \left\| \begin{array}{c} f(u,\varkappa_n(u),\varkappa_n(\lambda u),{}^{\mathrm{H}}\mathrm{I}^{\delta}\varkappa_n(\lambda u),{}^{\mathrm{CH}}\mathrm{D}^{\rho}\varkappa_n(\lambda u)) \\ -f(u,\varkappa(u),\varkappa(\lambda u),{}^{\mathrm{H}}\mathrm{I}^{\delta}\varkappa(\lambda u),{}^{\mathrm{CH}}\mathrm{D}^{\rho}\varkappa(\lambda u)) \end{array} \right\|_\infty \\ &+ \frac{(\log\mathrm{T})^{\beta_2+\beta_3}(\log\mathrm{T})^{\beta_1}}{\Gamma(\beta_2+\beta_3+1)\Gamma(\beta_1+1)} \left\| \begin{array}{c} f(u,\varkappa_n(u),\varkappa_n(\lambda u),{}^{\mathrm{H}}\mathrm{I}^{\delta}\varkappa_n(\lambda u),{}^{\mathrm{CH}}\mathrm{D}^{\rho}\varkappa_n(\lambda u)) \\ -f(u,\varkappa(u),\varkappa(\lambda u),{}^{\mathrm{H}}\mathrm{I}^{\delta}\varkappa(\lambda u),{}^{\mathrm{CH}}\mathrm{D}^{\rho}\varkappa(\lambda u)) \end{array} \right\|_\infty . \end{aligned}$$

Hence, from the continuity of the function f (see $(\mathscr{P}_1)$), we get

$$\| \mathscr{H}\varkappa_n - \mathscr{H}\varkappa \|_\infty \to 0 \quad \text{when} \quad n\to\infty. \tag{3.14}$$

In a similar way, we can obtain

$$\begin{aligned} &| {}^{\mathrm{CH}}\mathrm{D}^{\rho}\mathscr{H}\varkappa_n(t) - {}^{\mathrm{CH}}\mathrm{D}^{\rho}\mathscr{H}\varkappa(t) | \\ \leq\ & \frac{1}{\Gamma(\beta_1+\beta_2+\beta_3-\rho)} \int_1^{\mathrm{T}} \left(\log\frac{\mathrm{T}}{u}\right)^{\beta_1+\beta_2+\beta_3-\rho-1} \left| \begin{array}{c} f(u,\varkappa_n(u),\varkappa_n(\lambda u),{}^{\mathrm{H}}\mathrm{I}^{\delta}\varkappa_n(\lambda u),{}^{\mathrm{CH}}\mathrm{D}^{\rho}\varkappa_n(\lambda u)) \\ -f(u,\varkappa(u),\varkappa(\lambda u),{}^{\mathrm{H}}\mathrm{I}^{\delta}\varkappa(\lambda u),{}^{\mathrm{CH}}\mathrm{D}^{\rho}\varkappa(\lambda u)) \end{array} \right| \frac{du}{u} \\ &+ \frac{(\log\mathrm{T})^{\beta_2+\beta_3-\rho}}{\Gamma(\beta_2+\beta_3-\rho+1)\Gamma(\beta_1)} \int_1^{\mathrm{T}} \left(\log\frac{\mathrm{T}}{u}\right)^{\beta_1-1} \left| \begin{array}{c} f(u,\varkappa_n(u),\varkappa_n(\lambda u),{}^{\mathrm{H}}\mathrm{I}^{\delta}\varkappa_n(\lambda u),{}^{\mathrm{CH}}\mathrm{D}^{\rho}\varkappa_n(\lambda u)) \\ -f(u,\varkappa(u),\varkappa(\lambda u),{}^{\mathrm{H}}\mathrm{I}^{\delta}\varkappa(\lambda u),{}^{\mathrm{CH}}\mathrm{D}^{\rho}\varkappa(\lambda u)) \end{array} \right| \frac{du}{u}, \end{aligned}$$

or

$$\begin{aligned} &\| {}^{\mathrm{CH}}\mathrm{D}^{\rho}\mathscr{H}\varkappa_n - {}^{\mathrm{CH}}\mathrm{D}^{\rho}\mathscr{H}\varkappa \|_\infty \\ \leq\ & \frac{(\log\mathrm{T})^{\beta_1+\beta_2+\beta_3-\rho}}{\Gamma(\beta_1+\beta_2+\beta_3-\rho+1)} \left\| \begin{array}{c} f(u,\varkappa_n(u),\varkappa_n(\lambda u),{}^{\mathrm{H}}\mathrm{I}^{\delta}\varkappa_n(\lambda u),{}^{\mathrm{CH}}\mathrm{D}^{\rho}\varkappa_n(\lambda u)) \\ -f(u,\varkappa(u),\varkappa(\lambda u),{}^{\mathrm{H}}\mathrm{I}^{\delta}\varkappa(\lambda u),{}^{\mathrm{CH}}\mathrm{D}^{\rho}\varkappa(\lambda u)) \end{array} \right\|_\infty \\ &+ \frac{(\log\mathrm{T})^{\beta_2+\beta_3-\rho}(\log\mathrm{T})^{\beta_1}}{\Gamma(\beta_2+\beta_3-\rho+1)\Gamma(\beta_1+1)} \left\| \begin{array}{c} f(u,\varkappa_n(u),\varkappa_n(\lambda u),{}^{\mathrm{H}}\mathrm{I}^{\delta}\varkappa_n(\lambda u),{}^{\mathrm{CH}}\mathrm{D}^{\rho}\varkappa_n(\lambda u)) \\ -f(u,\varkappa(u),\varkappa(\lambda u),{}^{\mathrm{H}}\mathrm{I}^{\delta}\varkappa(\lambda u),{}^{\mathrm{CH}}\mathrm{D}^{\rho}\varkappa(\lambda u)) \end{array} \right\|_\infty . \end{aligned}$$

Therefore,

$$\| {}^{\mathrm{CH}}\mathrm{D}^{\rho}\mathscr{H}\varkappa_n - {}^{\mathrm{CH}}\mathrm{D}^{\rho}\mathscr{H}\varkappa \|_\infty \to 0 \quad \text{when} \quad n\to\infty. \tag{3.15}$$

As a result, according to (3.14) and (3.15), we obtain

$$\| \mathscr{H}\varkappa_n - \mathscr{H}\varkappa \|_{\mathbb{X}} \to 0 \quad \text{when} \quad n \to \infty, \tag{3.16}$$

then, the operator $\mathscr{H}$ is continuous.

Step 2: We show that the operator $\mathscr{H}$ is uniformly bounded.

Let Υ be bounded subset of $\mathbb{X}$. Then, $\forall \varkappa_i \in \Upsilon, i = \overline{1,4}$, $\exists \tau_1 > 0$, such that

$$|f(t, \varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4)| \leq \tau_1.$$

For all $\varkappa \in \Upsilon$, we have

$$\begin{aligned}
|(\mathscr{H}\varkappa)(t)| \leq\ & \frac{1}{\Gamma(\beta_1+\beta_2+\beta_3)} \int_1^T \left(\log\frac{T}{u}\right)^{\beta_1+\beta_2+\beta_3-1} \left| f(u, \varkappa(u), \varkappa(\lambda u), {}^{H}I^{\delta}\varkappa(\lambda u), {}^{CH}D^{\rho}\varkappa(\lambda u)) \right| \frac{du}{u} \\
& + \frac{(\log T)^{\beta_2+\beta_3}}{\Gamma(\beta_2+\beta_3+1)\Gamma(\beta_1)} \int_1^T \left(\log\frac{T}{u}\right)^{\beta_1-1} \left| f(u, \varkappa(u), \varkappa(\lambda u), {}^{H}I^{\delta}\varkappa(\lambda u), {}^{CH}D^{\rho}\varkappa(\lambda u)) \right| \frac{du}{u} \\
& + |\mathscr{A}_1| \frac{(\log T)^{\beta_3}}{\Gamma(\beta_3+1)} + |\mathscr{A}_2|.
\end{aligned}$$

This implies:

$$\| \mathscr{H}\varkappa \|_\infty \leq \tau_1 \left(\frac{(\log T)^{\beta_1+\beta_2+\beta_3}}{\Gamma(\beta_1+\beta_2+\beta_3+1)} + \frac{(\log T)^{\beta_2+\beta_3}(\log T)^{\beta_1}}{\Gamma(\beta_2+\beta_3+1)\Gamma(\beta_1+1)} \right) + |\mathscr{A}_1| \frac{(\log T)^{\beta_3}}{\Gamma(\beta_3+1)} + |\mathscr{A}_2| < +\infty.$$

In analogous way, we find

$$\|{}^{CH}D^{\rho}\mathscr{H}\varkappa\|_\infty \leq \tau_1 \left(\frac{(\log T)^{\beta_1+\beta_2+\beta_3-\rho}}{\Gamma(\beta_1+\beta_2+\beta_3-\rho+1)} + \frac{(\log T)^{\beta_2+\beta_3-\rho}(\log T)^{\beta_1}}{\Gamma(\beta_2+\beta_3-\rho+1)\Gamma(\beta_1+1)} \right) + |\mathscr{A}_1| \frac{(\log T)^{\beta_3-\rho}}{\Gamma(\beta_3-\rho+1)} < +\infty.$$

In view of the above two inequalities, it results

$$\| \mathscr{H}\varkappa \|_{\mathbb{X}} < +\infty,$$

which means that $\mathscr{H}$ is uniformly bounded.

Step 3: We prove the equicontinuity of the operator $\mathscr{H}$.

Let $t_1, t_2 \in \mathrm{J}$, with $t_1 < t_2$, we have

$$\begin{aligned}
| & \mathscr{H}\varkappa(t_2) - \mathscr{H}\varkappa(t_1) | \\
\leq & \left| \frac{1}{\Gamma(\beta_1+\beta_2+\beta_3)} \int_1^{t_2} \left(\log \frac{t_2}{u}\right)^{\beta_1+\beta_2+\beta_3-1} f(u, \varkappa(u), \varkappa(\lambda u), {}^{\mathrm{H}}\mathrm{I}^{\delta}\varkappa(\lambda u), {}^{\mathrm{CH}}\mathrm{D}^{\rho}\varkappa(\lambda u)) \frac{du}{u} \right. \\
& - \frac{(\log t_2)^{\beta_2+\beta_3}}{\Gamma(\beta_2+\beta_3+1)\Gamma(\beta_1)} \int_1^{\mathrm{T}} \left(\log \frac{\mathrm{T}}{u}\right)^{\beta_1-1} f(u, \varkappa(u), \varkappa(\lambda u), {}^{\mathrm{H}}\mathrm{I}^{\delta}\varkappa(\lambda u), {}^{\mathrm{CH}}\mathrm{D}^{\rho}\varkappa(\lambda u)) \frac{du}{u} \\
& + \mathscr{A}_1 \frac{(\log t_2)^{\beta_3}}{\Gamma(\beta_3+1)} \\
& - \frac{1}{\Gamma(\beta_1+\beta_2+\beta_3)} \int_1^{t_1} \left(\log \frac{t_1}{u}\right)^{\beta_1+\beta_2+\beta_3-1} f(u, \varkappa(u), \varkappa(\lambda u), {}^{\mathrm{H}}\mathrm{I}^{\delta}\varkappa(\lambda u), {}^{\mathrm{CH}}\mathrm{D}^{\rho}\varkappa(\lambda u)) \frac{du}{u} \\
& + \frac{(\log t_1)^{\beta_2+\beta_3}}{\Gamma(\beta_2+\beta_3+1)\Gamma(\beta_1)} \int_1^{\mathrm{T}} \left(\log \frac{\mathrm{T}}{u}\right)^{\beta_1-1} f(u, \varkappa(u), \varkappa(\lambda u), {}^{\mathrm{H}}\mathrm{I}^{\delta}\varkappa(\lambda u), {}^{\mathrm{CH}}\mathrm{D}^{\rho}\varkappa(\lambda u)) \frac{du}{u} \\
& \left. - \mathscr{A}_1 \frac{(\log t_1)^{\beta_3}}{\Gamma(\beta_3+1)} \right|,
\end{aligned}$$

which leads us to write

$$\begin{aligned}
| \mathscr{H}\varkappa(t_2) - \mathscr{H}\varkappa(t_1) | \leq & \frac{\tau_1}{\Gamma(\beta_1+\beta_2+\beta_3+1)} \left[(\log t_2)^{\beta_1+\beta_2+\beta_3} - (\log t_1)^{\beta_1+\beta_2+\beta_3} \right] \\
& + \frac{\tau_1 (\log \mathrm{T})^{\beta_1}}{\Gamma(\beta_2+\beta_3+1)\Gamma(\beta_1+1)} \left[(\log t_2)^{\beta_2+\beta_3} - (\log t_1)^{\beta_2+\beta_3} \right] \\
& + \frac{|\mathscr{A}_1|}{\Gamma(\beta_3+1)} \left[(\log t_2)^{\beta_3} - (\log t_1)^{\beta_3} \right].
\end{aligned}$$

Therefore

$$| \mathscr{H}\varkappa(t_2) - \mathscr{H}\varkappa(t_1) | \to 0, \quad \text{as} \quad t_2 \to t_1. \tag{3.17}$$

Similarly, we obtain

$$\begin{aligned}
| {}^{\mathrm{CH}}\mathrm{D}^{\rho}\mathscr{H}\varkappa(t_1) - {}^{\mathrm{CH}}\mathrm{D}^{\rho}\mathscr{H}\varkappa(t_2) | \leq & \frac{\tau_1}{\Gamma(\beta_1+\beta_2+\beta_3-\rho+1)} \left[(\log t_2)^{\beta_1+\beta_2+\beta_3-\rho} - (\log t_1)^{\beta_1+\beta_2+\beta_3-\rho} \right] \\
& + \frac{\tau_1 (\log \mathrm{T})^{\beta_1}}{\Gamma(\beta_2+\beta_3-\rho+1)\Gamma(\beta_1+1)} \left[(\log t_2)^{\beta_2+\beta_3-\rho} - (\log t_1)^{\beta_2+\beta_3-\rho} \right] \\
& + \frac{|\mathscr{A}_1|}{\Gamma(\beta_3-\rho+1)} \left[(\log t_2)^{\beta_3-\rho} - (\log t_1)^{\beta_3-\rho} \right].
\end{aligned}$$

Thus,

$$| {}^{\mathrm{CH}}\mathrm{D}^{\rho}\mathscr{H}\varkappa(t_1) - {}^{\mathrm{CH}}\mathrm{D}^{\rho}\mathscr{H}\varkappa(t_2) | \to 0, \quad \text{as} \quad t_2 \to t_1. \tag{3.18}$$

From (3.17) and (3.18), we deduce that the operator $\mathscr{H}$ is equicontinuous. So, thanks to the Arzela-Ascoli theorem, it follows that $\mathscr{H}$ is completely continuous.

Step 4: Finally, we establish that the set given by

$$\mathfrak{U}_{\mathscr{H}} := \{\varkappa \in \mathbb{X} : \varkappa = \sigma\mathscr{H}(\varkappa), \quad 0 < \sigma < 1\},$$

is bounded.

Let $\varkappa \in \mathfrak{U}_{\mathscr{H}}$ and $\varkappa = \sigma\mathscr{H}(\varkappa)$ for some $0 < \sigma < 1$. Then, for each $t \in \mathrm{J}$, we can write

$$\begin{aligned}
\varkappa(t) &= \frac{\sigma}{\Gamma(\beta_1+\beta_2+\beta_3)} \int_1^t \left(\log\frac{t}{u}\right)^{\beta_1+\beta_2+\beta_3-1} f(u,\varkappa(u),\varkappa(\lambda u),{}^{\mathrm{H}}\mathrm{I}^{\delta}\varkappa(\lambda u),{}^{\mathrm{CH}}\mathrm{D}^{\rho}\varkappa(\lambda u))\frac{du}{u} \\
&- \frac{\sigma(\log t)^{\beta_2+\beta_3}}{\Gamma(\beta_2+\beta_3+1)\Gamma(\beta_1)} \int_1^{\mathrm{T}} \left(\log\frac{\mathrm{T}}{u}\right)^{\beta_1-1} f(u,\varkappa(u),\varkappa(\lambda u),{}^{\mathrm{H}}\mathrm{I}^{\delta}\varkappa(\lambda u),{}^{\mathrm{CH}}\mathrm{D}^{\rho}\varkappa(\lambda u))\frac{du}{u} \\
&+ \mathscr{A}_1\frac{\sigma(\log t)^{\beta_3}}{\Gamma(\beta_3+1)} + \sigma\mathscr{A}_2.
\end{aligned}$$

Then, we have

$$\begin{aligned}
\|\varkappa\|_\infty &\le \left[\omega_1+\omega_2+\frac{(\log \mathrm{T})^{\delta}}{\Gamma(\delta+1)}\omega_3+\omega_4\right]\left[\frac{(\log \mathrm{T})^{\beta_1+\beta_2+\beta_3}}{\Gamma(\beta_1+\beta_2+\beta_3+1)}+\frac{(\log \mathrm{T})^{\beta_2+\beta_3}(\log \mathrm{T})^{\beta_1}}{\Gamma(\beta_2+\beta_3+1)\Gamma(\beta_1+1)}\right]\|\varkappa\|_{\mathbb{X}} \\
&+\omega_0\left[\frac{(\log \mathrm{T})^{\beta_1+\beta_2+\beta_3}}{\Gamma(\beta_1+\beta_2+\beta_3+1)}+\frac{(\log \mathrm{T})^{\beta_2+\beta_3}(\log \mathrm{T})^{\beta_1}}{\Gamma(\beta_2+\beta_3+1)\Gamma(\beta_1+1)}\right]+|\mathscr{A}_1|\frac{(\log \mathrm{T})^{\beta_3}}{\Gamma(\beta_3+1)}+|\mathscr{A}_2|.
\end{aligned}$$

In a similar manner, we get

$$\begin{aligned}
\|{}^{\mathrm{CH}}\mathrm{D}^{\rho}\varkappa\|_\infty &\le \left[\omega_1+\omega_2+\frac{(\log \mathrm{T})^{\delta}}{\Gamma(\delta+1)}\omega_3+\omega_4\right]\left[\frac{(\log \mathrm{T})^{\beta_1+\beta_2+\beta_3-\rho}}{\Gamma(\beta_1+\beta_2+\beta_3-\rho+1)}+\frac{(\log \mathrm{T})^{\beta_2+\beta_3}(\log \mathrm{T})^{\beta_1}}{\Gamma(\beta_2+\beta_3-\rho+1)\Gamma(\beta_1+1)}\right]\|\varkappa\|_{\mathbb{X}} \\
&+\omega_0\left[\frac{(\log \mathrm{T})^{\beta_1+\beta_2+\beta_3-\rho}}{\Gamma(\beta_1+\beta_2+\beta_3-\rho+1)}+\frac{(\log \mathrm{T})^{\beta_2+\beta_3-\rho}(\log \mathrm{T})^{\beta_1}}{\Gamma(\beta_2+\beta_3-\rho+1)\Gamma(\beta_1+1)}\right]+|\mathscr{A}_1|\frac{(\log \mathrm{T})^{\beta_3-\rho}}{\Gamma(\beta_3-\rho+1)}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|\varkappa\|_{\mathbb{X}} &\le 2\left[\omega_1+\omega_2+\frac{(\log \mathrm{T})^{\delta}}{\Gamma(\delta+1)}\omega_3+\omega_4\right]\left[\frac{(\log \mathrm{T})^{\beta_1+\beta_2+\beta_3}}{\Gamma(\beta_1+\beta_2+\beta_3+1)}+\frac{(\log \mathrm{T})^{\beta_2+\beta_3}(\log \mathrm{T})^{\beta_1}}{\Gamma(\beta_2+\beta_3+1)\Gamma(\beta_1+1)}\right]\|\varkappa\|_{\mathbb{X}} \\
&+2\omega_0\left[\frac{(\log \mathrm{T})^{\beta_1+\beta_2+\beta_3}}{\Gamma(\beta_1+\beta_2+\beta_3+1)}+\frac{(\log \mathrm{T})^{\beta_2+\beta_3}(\log \mathrm{T})^{\beta_1}}{\Gamma(\beta_2+\beta_3+1)\Gamma(\beta_1+1)}\right]+2|\mathscr{A}_1|\frac{(\log \mathrm{T})^{\beta_3}}{\Gamma(\beta_3+1)}+2|\mathscr{A}_2|. \\
&+\left[\omega_1+\omega_2+\frac{(\log \mathrm{T})^{\delta}}{\Gamma(\delta+1)}\omega_3+\omega_4\right]\left[\frac{(\log \mathrm{T})^{\beta_1+\beta_2+\beta_3-\rho}}{\Gamma(\beta_1+\beta_2+\beta_3-\rho+1)}+\frac{(\log \mathrm{T})^{\beta_2+\beta_3}(\log \mathrm{T})^{\beta_1}}{\Gamma(\beta_2+\beta_3-\rho+1)\Gamma(\beta_1+1)}\right]\|\varkappa\|_{\mathbb{X}} \\
&+\omega_0\left[\frac{(\log \mathrm{T})^{\beta_1+\beta_2+\beta_3-\rho}}{\Gamma(\beta_1+\beta_2+\beta_3-\rho+1)}+\frac{(\log \mathrm{T})^{\beta_2+\beta_3-\rho}(\log \mathrm{T})^{\beta_1}}{\Gamma(\beta_2+\beta_3-\rho+1)\Gamma(\beta_1+1)}\right]+|\mathscr{A}_1|\frac{(\log \mathrm{T})^{\beta_3-\rho}}{\Gamma(\beta_3-\rho+1)}.
\end{aligned}$$

Hence,

$$\|\varkappa\|_{\mathbb{X}} \leq \frac{\omega_0\left[2\left(\frac{(\log T)^{\beta_1+\beta_2+\beta_3}}{\Gamma(\beta_1+\beta_2+\beta_3+1)} + \frac{(\log T)^{\beta_2+\beta_3}(\log T)^{\beta_1}}{\Gamma(\beta_2+\beta_3+1)\Gamma(\beta_1+1)}\right) + \left(\frac{(\log T)^{\beta_1+\beta_2+\beta_3-\rho}}{\Gamma(\beta_1+\beta_2+\beta_3-\rho+1)} + \frac{(\log T)^{\beta_2+\beta_3-\rho}(\log T)^{\beta_1}}{\Gamma(\beta_2+\beta_3-\rho+1)\Gamma(\beta_1+1)}\right)\right] + |\mathscr{A}_1|\left[2\frac{(\log T)^{\beta_3}}{\Gamma(\beta_3+1)} + \frac{(\log T)^{\beta_3-\rho}}{\Gamma(\beta_3-\rho+1)}\right] + 2|\mathscr{A}_2|}{1-2\left[\omega_1+\omega_2 + \frac{\omega_3(\log T)^{\delta}}{\Gamma(\delta+1)} + \omega_4\right]\left(\frac{(\log T)^{\beta_1+\beta_2+\beta_3}}{\Gamma(\beta_1+\beta_2+\beta_3+1)} + \frac{(\log T)^{\beta_2+\beta_3}(\log T)^{\beta_1}}{\Gamma(\beta_2+\beta_3+1)\Gamma(\beta_1+1)}\right) + \left[\omega_1+\omega_2 + \frac{\omega_3(\log T)^{\delta}}{\Gamma(\delta+1)} + \omega_4\right]\left(\frac{(\log T)^{\beta_1+\beta_2+\beta_3-\rho}}{\Gamma(\beta_1+\beta_2+\beta_3-\rho+1)} + \frac{(\log T)^{\beta_2+\beta_3-\rho}(\log T)^{\beta_1}}{\Gamma(\beta_2+\beta_3-\rho+1)\Gamma(\beta_1+1)}\right)}.$$

As a result,

$$\|\varkappa\|_{\mathbb{X}} \leq \frac{\omega_0\left[2\Re_1+\Re_2\right] + |\mathscr{A}_1|\left[2\frac{(\log T)^{\beta_3}}{\Gamma(\beta_3+1)} + \frac{(\log T)^{\beta_3-\rho}}{\Gamma(\beta_3-\rho+1)}\right] + 2|\mathscr{A}_2|}{1-\left[\omega_1+\omega_2+\frac{\omega_3(\log T)^{\delta}}{\Gamma(\delta+1)}+\omega_4\right]\left[2\Re_1+\Re_2\right]}.$$

So, $\mathfrak{U}_{\mathscr{H}}$ is bounded.

Consequently, we can deduce that $\mathscr{H}$ admits at least one fixed point thanks to Leray-Schauder's alternative theorem. Thus, the problem (3.1) has a solution, which completes the demonstration of Theorem 3.2. ■

3.3 Ulam Type Stabilities

In this part of section, we discuss some Ulam type stabilities of the solutions for the three-sequential problem of pantograph (3.1).

Let $\epsilon > 0$ and the function φ in $\mathbb{X}$. Then, we consider the following inequalities:

$$\left|{}^{CH}D^{\beta_1}[{}^{CH}D^{\beta_2}({}^{CH}D^{\beta_3}\varkappa^*(t))] - f(t,\varkappa^*(t),\varkappa^*(\lambda t),{}^{H}I^{\delta}\varkappa^*(\lambda t),{}^{CH}D^{\rho}\varkappa^*(\lambda t))\right| \leq \epsilon, \quad t\in J. \tag{3.19}$$

$$\left|{}^{CH}D^{\beta_1}[{}^{CH}D^{\beta_2}({}^{CH}D^{\beta_3}\varkappa^*(t))] - f(t,\varkappa^*(t),\varkappa^*(\lambda t),{}^{H}I^{\delta}\varkappa^*(\lambda t),{}^{CH}D^{\rho}\varkappa^*(\lambda t))\right| \leq \epsilon\varphi(t), \quad t\in J. \tag{3.20}$$

To obtain the third main result, we need to present the following definitions.

Definition 3.3 *The problem (3.1) is Ulam-Hyers stable if there exists a real number $\mathfrak{C}_{\mathfrak{f}} > 0$, such that for each $\epsilon > 0$ and for each solution $\varkappa^* \in \mathbb{X}$ of the inequality (3.19), there exists $\varkappa \in \mathbb{X}$ a solution of (3.1), such that*

$$\|\varkappa^* - \varkappa\|_{\mathbb{X}} \leq \mathfrak{C}_{\mathfrak{f}}\epsilon.$$

Definition 3.4 *We say that problem (3.1) is generalized Ulam-Hyers stable if there exists $\mathfrak{F} \in C(\mathbb{R}^+,\mathbb{R}^+)$, with $\mathfrak{F}(0) = 0$, such that for any $\epsilon > 0$, and for any solution $\varkappa^* \in \mathbb{X}$ to the inequality (3.19), there exists a solution $\varkappa \in \mathbb{X}$ of (3.1), with*

$$\|\varkappa^* - \varkappa\|_{\mathbb{X}} \leq \mathfrak{F}(\epsilon).$$

Definition 3.5 *We say that problem (3.1) is Ulam-Hyers-Rassias stable with respect to* $\varphi \in C(J,\mathbb{R}^+)$ *if there exists a real number* $\mathfrak{C}_{\mathfrak{f},\varphi} > 0$ *such that for each* $\epsilon > 0$ *and for each solution* $\varkappa^* \in \mathbb{X}$ *of (3.20), there is a solution* $\varkappa \in \mathbb{X}$ *of (3.1), with*

$$|\varkappa^*(t) - \varkappa(t)| \leq \epsilon \mathfrak{C}_{\mathfrak{f},\varphi}\varphi(t), \quad \forall t \in J.$$

Now, we are ready to examine the stability of solutions for our problem (3.1).

Theorem 3.6 *If the conditions of Theorem 3.1 are satisfied, then the problem of (3.1) is Ulam-Hyers stable.*

Proof. Let $\epsilon > 0$ and let $\varkappa^* \in \mathbb{X}$ be a function that satisfies the inequality (3.19). We put

$$\begin{aligned} \mathscr{F}_{\varkappa^*}(u) &:= f(u, \varkappa^*(u), \varkappa^*(\lambda u), {}^{H}I^{\delta}\varkappa^*(\lambda u), {}^{CH}D^{\rho}\varkappa^*(\lambda u)). \\ \mathscr{F}_{\varkappa}(u) &:= f(u, \varkappa(u), \varkappa(\lambda u), {}^{H}I^{\delta}\varkappa(\lambda u), {}^{CH}D^{\rho}\varkappa(\lambda u)). \end{aligned}$$

Using the same conditions as in (3.1), we integrate (3.19) to obtain

$$\left| \begin{array}{c} \varkappa^*(t) - \frac{1}{\Gamma(\beta_1+\beta_2+\beta_3)} \int_1^t (\log\frac{t}{u})^{\beta_1+\beta_2+\beta_3-1} \mathscr{F}_{\varkappa^*}(u)\frac{du}{u} \\ + \frac{(\log t)^{\beta_2+\beta_3}}{\Gamma(\beta_2+\beta_3+1)\Gamma(\beta_1)} \int_1^T (\log\frac{T}{u})^{\beta_1-1} \mathscr{F}_{\varkappa^*}(u)\frac{du}{u} \\ -\mathscr{A}_1\frac{(\log t)^{\beta_3}}{\Gamma(\beta_3+1)} - \mathscr{A}_2. \end{array} \right| \leq \epsilon \times \frac{(\log t)^{\beta_1+\beta_2+\beta_3}}{\Gamma(\beta_1+\beta_2+\beta_3+1)},$$

With regard to Theorem 3.1, we suppose that $\varkappa$ is the unique solution of problem (3.1) presented by

$$\begin{aligned} \varkappa(t) &= \frac{1}{\Gamma(\beta_1+\beta_2+\beta_3)} \int_1^t \left(\log\frac{t}{u}\right)^{\beta_1+\beta_2+\beta_3-1} \mathscr{F}_{\varkappa}(u)\frac{du}{u} \\ &- \frac{(\log t)^{\beta_2+\beta_3}}{\Gamma(\beta_2+\beta_3+1)\Gamma(\beta_1)} \int_1^T \left(\log\frac{T}{u}\right)^{\beta_1-1} \mathscr{F}_{\varkappa}(u)\frac{du}{u} \\ &+ \mathscr{A}_1\frac{(\log t)^{\beta_3}}{\Gamma(\beta_3+1)} + \mathscr{A}_2. \end{aligned}$$

Then, for each $t \in J$, we have

$$\left|\varkappa^*(t) - \varkappa(t)\right| \leq \left| \begin{array}{c} \epsilon \times \frac{(\log t)^{\beta_1+\beta_2+\beta_3}}{\Gamma(\beta_1+\beta_2+\beta_3+1)} + \frac{1}{\Gamma(\beta_1+\beta_2+\beta_3)} \int_1^t (\log\frac{t}{u})^{\beta_1+\beta_2+\beta_3-1} [\mathscr{F}_{\varkappa^*}(u) - \mathscr{F}_{\varkappa}(u)]\frac{du}{u} \\ + \frac{(\log t)^{\beta_2+\beta_3}}{\Gamma(\beta_2+\beta_3+1)\Gamma(\beta_1)} \int_1^T (\log\frac{T}{u})^{\beta_1-1} [\mathscr{F}_{\varkappa^*}(u) - \mathscr{F}_{\varkappa}(u)]\frac{du}{u} \end{array} \right|.$$

Using the assumption $(\mathscr{P}_2)$, we get

$$\|\varkappa^* - \varkappa\|_\infty \leq \frac{\epsilon(\log T)^{\beta_1+\beta_2+\beta_3}}{\Gamma(\beta_1+\beta_2+\beta_3+1)} + \Re_3\|\varkappa^* - \varkappa\|_{\mathbb{X}}. \tag{3.21}$$

With the same arguments, we can obtain the following estimate

$$\left|{}^{CH}D^{\rho}\varkappa^*(t) - {}^{CH}D^{\rho}\varkappa(t)\right| \leq \left| \begin{array}{l} \frac{\epsilon(\log t)^{\beta_1+\beta_2+\beta_3-\rho}}{\Gamma(\beta_1+\beta_2+\beta_3-\rho+1)} + \frac{1}{\Gamma(\beta_1+\beta_2+\beta_3-\rho)}\int_1^t (\log\frac{t}{u})^{\beta_1+\beta_2+\beta_3-\rho-1}[\mathscr{F}_{\varkappa^*}(u) - \mathscr{F}_{\varkappa}(u)]\frac{du}{u} \\ + \frac{(\log t)^{\beta_2+\beta_3-\rho}}{\Gamma(\beta_2+\beta_3-\rho+1)\Gamma(\beta_1)}\int_1^T (\log\frac{T}{u})^{\beta_1-1}[\mathscr{F}_{\varkappa^*}(u) - \mathscr{F}_{\varkappa}(u)]\frac{du}{u} \end{array} \right|,$$

which implies that

$$\|{}^{CH}D^{\rho}\varkappa^* - {}^{CH}D^{\rho}\varkappa\| \leq \frac{\epsilon(\log T)^{\beta_1+\beta_2+\beta_3-\rho}}{\Gamma(\beta_1+\beta_2+\beta_3-\rho+1)} + \Re_5\|\varkappa^* - \varkappa\|_{\mathbb{X}}. \tag{3.22}$$

From (3.21) and (3.22), we obtain

$$\begin{aligned} \|\varkappa^* - \varkappa\|_{\mathbb{X}} &\leq 2\frac{\epsilon(\log T)^{\beta_1+\beta_2+\beta_3}}{\Gamma(\beta_1+\beta_2+\beta_3+1)} + 2\Re_3\|\varkappa^* - \varkappa\|_{\mathbb{X}} \\ &\quad + \frac{\epsilon(\log T)^{\beta_1+\beta_2+\beta_3-\rho}}{\Gamma(\beta_1+\beta_2+\beta_3-\rho+1)} + \Re_5\|\varkappa^* - \varkappa\|_{\mathbb{X}} \\ &\leq \left[\frac{2\epsilon(\log T)^{\beta_1+\beta_2+\beta_3}}{\Gamma(\beta_1+\beta_2+\beta_3+1)} + \frac{\epsilon(\log T)^{\beta_1+\beta_2+\beta_3-\rho}}{\Gamma(\beta_1+\beta_2+\beta_3-\rho+1)}\right] + (2\Re_3 + \Re_5)\|\varkappa^* - \varkappa\|_{\mathbb{X}}. \end{aligned}$$

Hence, we get

$$\|\varkappa^* - \varkappa\|_{\mathbb{X}} \leq \left[\frac{\frac{2(\log T)^{\beta_1+\beta_2+\beta_3}}{\Gamma(\beta_1+\beta_2+\beta_3+1)} + \frac{(\log T)^{\beta_1+\beta_2+\beta_3-\rho}}{\Gamma(\beta_1+\beta_2+\beta_3-\rho+1)}}{1 - (2\Re_3 + \Re_5)}\right]\epsilon = \mathfrak{C}_{\mathfrak{f}}\epsilon.$$

Therefore, the problem (3.1) is Ulam-Hyers stable. This completes the proof of this theorem. ∎

Remark 3.7 *We can state that the problem under consideration (3.1) is generalized Ulam-Hyers stable if we take* $\mathfrak{F}(\epsilon) = \mathfrak{C}_{\mathfrak{f}}\epsilon$.

Theorem 3.8 *Assume that the hypotheses of Theorem 3.1 and*

(P3): The function $\varphi \in C(J, \mathbb{R}^+)$ *is increasing and there exist* $\theta_{\varphi,\beta} > 0$ *such that, we have*

$${}^{H}I^{\beta}\varphi(t) \leq \theta_{\varphi,\beta}\varphi(t), \tag{3.23}$$

for all $t \in J$.

are valid, then the problem (3.1) is Ulam-Hyers-Rassias stable.

Proof. By integration of (3.20) and using (3.23), we obtain

$$\left| \begin{array}{c} \varkappa^*(t) - \frac{1}{\Gamma(\beta_1+\beta_2+\beta_3)} \int_1^t \left(\log\frac{t}{u}\right)^{\beta_1+\beta_2+\beta_3-1} \mathscr{F}_{\varkappa^*}(u)\frac{du}{u} \\ + \frac{(\log t)^{\beta_2+\beta_3}}{\Gamma(\beta_2+\beta_3+1)\Gamma(\beta_1)} \int_1^T \left(\log\frac{T}{s}\right)^{\beta_1-1} \mathscr{F}_{\varkappa^*}(u)\frac{du}{u} \\ -\mathscr{A}_1 \frac{(\log t)^{\beta_3}}{\Gamma(\beta_3+1)} - \mathscr{A}_2. \end{array} \right| \leq \epsilon^{H}I^{\beta_1+\beta_2+\beta_3}\varphi(t) \leq \epsilon\theta_{\varphi,\beta_1+\beta_2+\beta_3}\varphi(t),$$

where $\varkappa^* \in \mathbb{X}$ is suggested as a solution of (3.20).
Assume that the unique solution to problem (3.1) is $\varkappa \in \mathbb{X}$. Then, for each $t \in \mathrm{J}$, we have

$$|\varkappa^*(t) - \varkappa(t)| \leq \epsilon\theta_{\varphi,\beta_1+\beta_2+\beta_3}\varphi(t) + \Re_3|\varkappa^*(t) - \varkappa(t)|.$$

Similarly, we obtain

$$|{}^{CH}D^{\rho}\varkappa^*(t) - {}^{CH}D^{\rho}\varkappa(t)| \leq \epsilon\theta_{\varphi,\beta_1+\beta_2+\beta_3-\rho}\varphi(t) + \Re_5|\varkappa^*(t) - \varkappa(t)|.$$

As a direct result, we find

$$\begin{aligned} |\varkappa^*(t) - \varkappa(t)| &\leq \left(2\theta_{\varphi,\beta_1+\beta_2+\beta_3} + \theta_{\varphi,\beta_1+\beta_2+\beta_3-\rho}\right)\epsilon\varphi(t) + (2\Re_3 + \Re_5)\,|\varkappa^*(t) - \varkappa(t)| \\ &\leq \epsilon \times \frac{\left(2\theta_{\varphi,\alpha_1+\alpha_2+\alpha_3} + \theta_{\varphi,\beta_1+\beta_2+\beta_3-\rho}\right)}{1 - (2\Re_3 + \Re_5)}\varphi(t). \end{aligned}$$

We put

$$\mathfrak{C}_{\mathfrak{f},\varphi} = \frac{\left(2\theta_{\varphi,\alpha_1+\alpha_2+\alpha_3} + \theta_{\varphi,\beta_1+\beta_2+\beta_3-\rho}\right)}{1 - (2\Re_3 + \Re_5)},$$

thus,

$$|\varkappa^*(t) - \varkappa(t)| \leq \epsilon\mathfrak{C}_{\mathfrak{f},\varphi}\varphi(t).$$

Consequently, the problem (3.1) is Ulam-Hyers-Rassias stable. ■

4 Illustrative Example

In this section, we give an example to show the validity of the result concerning the existence of only one solution to the following three-sequential pantograph problem:

$$\begin{cases} D^{0.9}[D^{0.93}(D^{0.86}\varkappa(t))] = f\left(t, \varkappa(t), \varkappa(\frac{1}{6}t), {}^{H}I^{0.45}\varkappa(\frac{1}{6}t), {}^{CH}D^{0.80}x(\frac{1}{6}t)\right), \quad t \in [1, e], \\ \\ x(1) - \frac{9}{10} = 0, \qquad D^{0.86}x(1) + \frac{26}{3} = 0, \qquad D^{0.93}(D^{0.86}x(e)) = 0, \end{cases} \tag{3.24}$$

where,

$$f(t,u_1,u_2,u_3,u_4) = \frac{\sin(\pi t)}{100\ln(t+1)} + \frac{2}{70}u_1 + \frac{11}{1997}u_2 + \frac{1}{20^3}\sin(t)u_3 + \frac{1}{30^3}u_4,$$

for $u_1, u_2, u_3, u_4 \in \mathbb{R}$.
We remark that

$$\beta_1 = 0.9, \quad \beta_2 = 0.93, \quad \beta_3 = 0.86, \quad \delta = 0.45, \quad \rho = 0.80, \quad \lambda = \frac{1}{6},$$

and

$$\mathscr{A}_1 = \frac{9}{10}, \quad \mathscr{A}_2 = -\frac{26}{3}, \quad \mathrm{T} = e.$$

So, for any $u_i, v_i \in \mathbb{R}(i = 1,2,3,4)$ and $t \in [1,e]$, we have

$$|f(t,u_1,u_2,u_3,u_4) - f(t,v_1,v_2,v_3,v_4)| \leq \frac{2}{70}(|u_1 - v_1| + |u_2 - v_2| + |u_3 - v_3| + |u_4 - v_4|).$$

Through some calculations, we easily get

$$\Delta_1 = \frac{2}{70},$$

then, the condition ($\mathscr{P}_2$) is valid.
From the given data, we find that

$$\Delta = \Delta_1\left(1 + \frac{(\log \mathrm{T})^{\beta}}{\Gamma(\beta+1)}\right) = 0.0608,$$

and

$$\Re_3 = 0.0861, \quad \Re_5 = 0.0971.$$

Thus,

$$2\Re_3 + \Re_5 = 0.2694 < 1.$$

Consequently, Theorem 3.1 implies that (3.24) has a unique solution on $[1,e]$.
Also, thanks to Theorem 3.6, the problem (3.24) is Ulam-Hyers stable.

5 Conclusion

In this work, we have suggested a novel pantograph problem with three-sequential derivatives of Caputo-Hadamard type. This problem has been discussed in terms of its existence, uniqueness, and different types of Ulam-stability. The results were obtained using Leray-Schauder and Banach fixed point theorems. Furthermore, it is demonstrated that the proposed problem is stable in the Ulam-Hyers and Ulam-Hyers-Rassias senses. A numerical example illustrating our main results has been provided for validation.

Chapter 4

An Approach For Dealing With Fractional Linear Boundary Value Problems of Sequential Type

1 Introduction

[1] The modeling of physical applications is a very active research topic, which is justified by the various results obtained by modeling some phenomena such as a neutron diffusion in nuclear reactors, chemical reactions, mechanics of materials, gas dynamics and others, through the two-point boundary value problems (BVPs) of both types: linear and nonlinear.

In this regard, they developed several analytical and numerical methods to adress this gap, which can be judged in terms of speed and reliability. Among the most famous methods is shooting method, which is used to solve the two-point (BVPs), usually by reducing the problem into two initial value problems (IVPs) of second order also, then this problems solved using an appropriate numerical method. For more information, see, for example, the research works [7, 25, 49, 55, 60, 69].

On the other hand, some researchers have investigated the two-point (BVPs) in the field of fractional calculus, as this field has many applications such as simulating systems that have memory effects. To identify few, we refer to [10, 11, 14, 45] and references therein.

The type that caught our interest is the linear one, which is reformulated as follows:

$$\begin{cases} {}^c\mathrm{D}^{\alpha}_{a}\left({}^c\mathrm{D}^{\alpha}_{a}y\right)(t) = f(t, y(t), {}^c\mathrm{D}^{\alpha}_{a}y(t)), \quad t \in \mathrm{J} := [a, b], \\ \\ y(a) = \mathscr{A} \in \mathbb{R}, \quad y(b) = \mathscr{B} \in \mathbb{R}, \end{cases} \tag{4.1}$$

where $0 < \alpha \leq 1$, the operator ${}^c\mathrm{D}^{\alpha}_{a}$ is the Caputo fractional derivative and the function f is continous.

In this work, we present a novel technique to solve a class of two-sequential fractional linear boundary value problems (FLBVPs), which is called Fractional Linear Shooting Method, for short (FLSM), such that the function f of (4.1) is replaced by

$$f(t, y(t), {}^c\mathrm{D}^{\alpha}_{a}y(t)) = -p(t)\,{}^c\mathrm{D}^{\alpha}_{a}y(t) - q(t)y(t) + r(t), \quad t \in \mathrm{J} := [a, b], \tag{4.2}$$

[1] **A. Abdelnebi, I. M. Batiha, I. H. Jebril, Z. Dahmani, S. Alkhazaleh, S. Momani**, *"An Approach For Dealing With Fractional Linear Boundary Value Problems"*. Submitted.

where p, q and r are continous functions.

The proposed technique converts this class of FLBVP into two equivalent fractional initial value problems (FIVPs). The FLBVP is then solved by solving the FIVPs. For this purpose, we chose the Modified Fractional Euler Method (MFEM) developed in [15], to obtain better approximations. To verify the efficiency of our suggested combined methods, numerical comparisons and a discussion of some physical applications are included.

2 Preliminary Results

In this section we will give the basic concepts and techniques that we will need throughout this chapter. So, let's start by introducing the fundamental ideas of the linear shooting method (LSM).

2.1 Description of Linear Shooting Technique

Let us consider the following two-point linear boundary value problem (BVP)[20]:

$$\begin{cases} y''(t) + p(t)y'(t) + q(t)y(t) = r(t), \quad t \in \mathrm{J} := [a,b], \\ \\ y(a) = \mathscr{A}, \quad y(b) = \mathscr{B}, \end{cases} \tag{4.3}$$

where $\mathscr{A}, \mathscr{B} \in \mathbb{R}$, the functions p, q and r are continuous such that $q > 0$ on J.

The basic idea of the linear shooting technique is to convert the two-point linear BVP (4.3) into two initial value problems (IVPs): nonhomogeneous and homogeneous given as follows:

$$\begin{cases} y_1''(t) + p(t)y_1'(t) + q(t)y_1(t) = r(t), \quad t \in \mathrm{J} := [a,b], \\ \\ y_1(a) = \mathscr{A}, \quad y_1'(a) = 0, \end{cases} \tag{4.4}$$

and

$$\begin{cases} y_2''(t) + p(t)y_2'(t) + q(t)y_2(t) = 0, \quad t \in \mathrm{J} := [a,b], \\ \\ y_2(a) = 0, \quad y_2'(a) = 1. \end{cases} \tag{4.5}$$

where the solutions of IVP (4.4) and IVP (4.5) are denoted by y_1 and y_2, respectively.

To approximate the solutions y_1 and y_2, an efficient existing numerical method specialized in solving linear IVPs is applied, such as: Runge-Kutta, Euler,...etc.

Aftar that, the solution of BVP (4.3) is expressed by a linear combination of the previous two solutions as follows:

$$y(t) = y_1(t) + \left(\frac{\mathscr{B} - y_1(b)}{y_2(b)}\right) y_2(t), \quad t \in \mathrm{J}, \tag{4.6}$$

where $y_2(b) \neq 0$.

2.2 Generalized Taylor's Formula

Our main results depend also on the following fundamental theorem.

Theorem 2.1 *[53] Suppose that ${}^{c}\mathrm{D}^{j\alpha}y(t)$ is a continous function on $(0,b]$, for $j = 0,1,2,\dots,n+1$, where* $0 <$

$\alpha \le 1$. *Then, The function y for the node t_0 can then be expanded as follows:*

$$y(t) = \sum_{j=0}^{n} \frac{(t-t_0)^{j\alpha}}{\Gamma(j\alpha+1)}\, {}^c\mathrm{D}^{j\alpha} y(t_0) + \frac{(t-t_0)^{(n+1)\alpha}}{\Gamma((n+1)\alpha+1)}\, {}^c\mathrm{D}^{(n+1)\alpha} y(\xi), \tag{4.7}$$

where $0 < \xi \le t$.

For more illustration, we can express the above expression of the function y as follows:

$$\begin{aligned} y(t) =& y(t_0) + \frac{(t-t_0)^{\alpha}}{\Gamma(\alpha+1)}\, {}^c\mathrm{D}^{\alpha} y(t_0) + \frac{(t-t_0)^{2\alpha}}{\Gamma(2\alpha+1)}\, {}^c\mathrm{D}^{2\alpha} y(t_0) + \cdots + \frac{(t-t_0)^{n\alpha}}{\Gamma(n\alpha+1)}\, {}^c\mathrm{D}^{n\alpha} y(t_0) \\ &+ \frac{(t-t_0)^{(n+1)\alpha}}{\Gamma((n+1)\alpha+1)}\, {}^c\mathrm{D}^{(n+1)\alpha} y(\xi). \end{aligned} \tag{4.8}$$

2.3 Background of Modified Fractional Euler Method (MFEM)

We intend in this subsection to recall the existing numerical method called the Modified Fractional Euler Method (MFEM), which is given in [15] and used to solve the following fractional initial value problem :

$$\begin{cases} {}^c\mathrm{D}^{\alpha} y(t) = f(t, y(t)), \quad t \in [a,b], \\ y(a) = y_0, \end{cases} \tag{4.9}$$

where $0 < \alpha \le 1$, $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ and ${}^c\mathrm{D}_a^{\alpha}$ denotes the fractional Caputo operator.
This algorithm is distinguished by the fact that it can offer more accuracy and efficiency than other algorithms. At this point, in order to achieve this approximation, we first subdivide the interval $[a,b]$ into $a = t_0 < t_1 = t_0 + h < \cdots < t_n = t_0 + nh = b$, where $h = \frac{b-a}{n}$ is the step size corresponds to the mesh points $t_j = t_0 + jh$, $j = 1, \ldots, n$. Thus, by applying the first three terms of Theorem 2.1, we can acquire

$$y(t_{j+1}) = y(t_j) + \frac{h^{\alpha}}{\Gamma(\alpha+1)} f\left(t_j + \frac{h^{\alpha}}{2\Gamma(\alpha+1)}, y(t_j) + \frac{h^{\alpha}}{2\Gamma(\alpha+1)} f(t_j, y(t_j))\right) + \frac{h^{2\alpha}}{\Gamma(2\alpha+1)} {}^c\mathrm{D}^{2\alpha} y(\xi), \tag{4.10}$$

thus,

$$\begin{aligned} &v_0 = y_0, \\ &v_{j+1} = v_j + \frac{h^{\alpha}}{\Gamma(\alpha+1)} f\left(t_j + \frac{h^{\alpha}}{2\Gamma(\alpha+1)}, v_j + \frac{h^{\alpha}}{2\Gamma(\alpha+1)} f(t_j, v_j)\right), \end{aligned} \tag{4.11}$$

where v_j denotes the numerical solution of problem (4.9), for $j = 1, 2, \cdots, n-1$.

3 Main results

The goal of this section is to solve the following fractional linear boundary value problem FLBVP:

$$\begin{cases} {}^c\mathrm{D}_a^{\alpha}\left({}^c\mathrm{D}_a^{\alpha} y\right)(t) + p(t)\, {}^c\mathrm{D}_a^{\alpha} y(t) + q(t) y(t) = r(t), \quad t \in \mathrm{J} := [a,b], \\ y(a) = \mathscr{A}, \quad y(b) = \mathscr{B}, \end{cases} \tag{4.12}$$

where $0 < \alpha \le 1$, $\mathscr{A}, \mathscr{B} \in \mathbb{R}$, p, q and r are continous functions, such that $q > 0$. The operator ${}^c\mathrm{D}_a^{\alpha}$ is the Caputo fractional derivative.

To do this, we will first use the fractional linear shooting method (FLSM) to transform our problem into two FIVPs of two-sequential Caputo derivatives. Then, we apply the modified fractional Euler method

(MFEM) to solve these problems. Finally, we deduce the general solution of the FLBVP.

3.1 Fractional Linear Shooting Technique For FLBVP

The objective of this method is to convert problem (4.12) into two equivalent initial value problems:

$$\begin{cases} {}^c\mathrm{D}_a^\alpha\left({}^c\mathrm{D}_a^\alpha y_1\right)(t) + p(t)\,{}^c\mathrm{D}_a^\alpha y_1(t) + q(t)y_1(t) = r(t), \quad t \in \mathrm{J}, \\ \\ y_1(a) = \mathscr{A}, \quad {}^c\mathrm{D}_a^\alpha y_1(a) = 0, \end{cases} \tag{4.13}$$

and

$$\begin{cases} {}^c\mathrm{D}_a^\alpha\left({}^c\mathrm{D}_a^\alpha y_2\right)(t) + p(t)\,{}^c\mathrm{D}_a^\alpha y_2(t) + q(t)y_2(t) = 0, \quad t \in \mathrm{J}, \\ \\ y_2(a) = 0, \quad {}^c\mathrm{D}_a^\alpha y_2(a) = 1, \end{cases} \tag{4.14}$$

where y_1 stands for the solution of the nonhomogeneous FIVP (4.13) and y_2 stands for the solution of the homogeneous FIVP (4.14).

Now, we suppose that the general solution y of the FLBVP (4.12) is a suitable linear combination of the solutions of the two-sequential FIVPs (4.13) and (4.14).
In other words, we assume

$$y(t) = \mathscr{C}_1 y_1(t) + \mathscr{C}_2 y_2(t), \quad t \in \mathrm{J}, \tag{4.15}$$

where $\mathscr{C}_1$ and $\mathscr{C}_2$ two constants.
Consequently, to determinate the constants $\mathscr{C}_1$ and $\mathscr{C}_2$, we suppose that

$${}^c\mathrm{D}_a^\alpha\left({}^c\mathrm{D}_a^\alpha y\right)(t) = \mathscr{C}_1\,{}^c\mathrm{D}_a^\alpha\left({}^c\mathrm{D}_a^\alpha y_1\right)(t) + \mathscr{C}_2\,{}^c\mathrm{D}_a^\alpha\left({}^c\mathrm{D}_a^\alpha y_2\right)(t), \tag{4.16}$$

and

$${}^c\mathrm{D}_a^\alpha y(t) = \mathscr{C}_1\,{}^c\mathrm{D}_a^\alpha y_1(t) + \mathscr{C}_2\,{}^c\mathrm{D}_a^\alpha y_2(t). \tag{4.17}$$

By substituting the equations (4.15), (4.16) and (4.17) into the equation associated with the problem (4.12), we get

$$\left(\mathscr{C}_1\,{}^c\mathrm{D}_a^\alpha\left({}^c\mathrm{D}_a^\alpha y_1\right)(t) + \mathscr{C}_2\,{}^c\mathrm{D}_a^\alpha\left({}^c\mathrm{D}_a^\alpha y_2\right)(t)\right) + p(t)\left(\mathscr{C}_1\,{}^c\mathrm{D}_a^\alpha y_1(t) + \mathscr{C}_2\,{}^c\mathrm{D}_a^\alpha y_2(t)\right) + q(t)\left(\mathscr{C}_1 y_1(t) + \mathscr{C}_2 y_2(t)\right) = r(t),$$

or

$${}^c\mathrm{D}_a^\alpha\left({}^c\mathrm{D}_a^\alpha y_1\right)(t) + \mathscr{C}_2\,{}^c\mathrm{D}_a^\alpha\left({}^c\mathrm{D}_a^\alpha y_2\right)(t) + \mathscr{C}_1 p(t)\mathrm{D}_a^\alpha y_1(t) + \mathscr{C}_2 p(t)\mathrm{D}_a^\alpha y_2(t) + \mathscr{C}_1 q(t) y_1(t) + \mathscr{C}_2 q(t) y_2(t) = r(t).$$

So, we have

$$\mathscr{C}_1\left({}^c\mathrm{D}_a^\alpha\left({}^c\mathrm{D}_a^\alpha y_1\right)(t) + p(t)\,{}^c\mathrm{D}_a^\alpha y_1(t) + q(t)y_1(t)\right) + \mathscr{C}_2\left({}^c\mathrm{D}_a^\alpha\left({}^c\mathrm{D}_a^\alpha y_2\right)(t) + p(t)\,{}^c\mathrm{D}_a^\alpha y_2(t) + q(t)y_2(t)\right) = r(t).$$

According to the FIVPs (4.13) and (4.14), we then obtain

$$\mathscr{C}_1\left(r(t)\right) + \mathscr{C}_2\,(0) = r(t).$$

This implies,

$$\mathscr{C}_1 = 1. \tag{4.18}$$

This would make the general solution y reported in (4.15) to be as

$$y(t) = y_1(t) + \mathscr{C}_2 y_2(t), \quad t \in \mathrm{J}. \tag{4.19}$$

In the next step, to determine the second constant $\mathscr{C}_2$, we use the boundary conditions given in (4.12), then we get

$$y(a) = y_1(a) + \mathscr{C}_2 y_2(a) = \mathscr{A}.$$

By the initial conditions of the two problems (4.13) and (4.14), we obtain

$$\mathscr{A} + \mathscr{C}_2(0) = \mathscr{A} \Longrightarrow \mathscr{A} = \mathscr{A},$$

which is true, and

$$y(b) = y_1(b) + \mathscr{C}_2 y_2(b) = \mathscr{B},$$

which implies directly that

$$\mathscr{C}_2 = \frac{\mathscr{B} - y_1(b)}{y_2(b)}. \tag{4.20}$$

Now, replacing (4.20) in (4.19) gives

$$y(t) = y_1(t) + \left(\frac{\mathscr{B} - y_1(b)}{y_2(b)}\right) y_2(t), \quad t \in \mathrm{J}, \tag{4.21}$$

which represents the solution of the main FLBVP (4.12).
It should be noted here that the solution (4.21) of (4.12) satisfies its boundary conditions. In other words, we can notice that

$$y(a) = y_1(a) + \left(\frac{\mathscr{B} - y_1(b)}{y_2(b)}\right) y_2(a) \quad = \quad \mathscr{A} + \left(\frac{\mathscr{B} - y_1(b)}{y_2(b)}\right)(0),$$

so,

$$y(a) = \mathscr{A}.$$

Also,

$$y(b) = y_1(b) + \left(\frac{\mathscr{B} - y_1(b)}{y_2(b)}\right) y_2(b) \quad = \quad \mathscr{B}.$$

3.2 Solving the Two-Sequential Fractional Initial Value Problems

In this part, the MFEM will be applied to solve the FIVPs (4.13) and (4.14), and for solve these problems, we shall first reduce each one into a one dimensional fractional differential system of order α which they are equivalent. Then, we will rewrite the systems (4.13) and (4.14) as follows:

$$\begin{cases} {}^{c}\mathrm{D}_a^{\alpha}\left({}^{c}\mathrm{D}_a^{\alpha} y_1\right)(t) = -p(t)\,{}^{c}\mathrm{D}_a^{\alpha} y_1(t) - q(t) y_1(t) + r(t), & t \in \mathrm{J} := [a,b], \\ y_1(a) = \mathscr{A}, \quad {}^{c}\mathrm{D}_a^{\alpha} y_1(a) = 0, \end{cases} \tag{4.22}$$

and

$$\begin{cases} {}^{c}\mathrm{D}_a^{\alpha}\left({}^{c}\mathrm{D}_a^{\alpha} y_2\right)(t) = -p(t)\,{}^{c}\mathrm{D}_a^{\alpha} y_2(t) - q(t) y_2(t), & t \in \mathrm{J} := [a,b], \\ y_2(a) = 0, \quad {}^{c}\mathrm{D}_a^{\alpha} y_2(a) = 1. \end{cases} \tag{4.23}$$

Now, we put

$$f_1\left(t, y_1(t), \mathrm{D}_a^\alpha y_1(t)\right) = -p(t)\, {}^c\mathrm{D}_a^\alpha y_1(t) - q(t)y_1(t) + r(t).$$
$$f_2\left(t, y_2(t), \mathrm{D}_a^\alpha y_2(t)\right) = -p(t)\, {}^c\mathrm{D}_a^\alpha y_2(t) - q(t)y_2(t). \tag{4.24}$$

According to (4.24), the problems (4.22) and (4.23) becomes

$$\begin{cases} {}^c\mathrm{D}_a^\alpha\left({}^c\mathrm{D}_a^\alpha y_1\right)(t) = f_1\left(t, y_1(t), \mathrm{D}_a^\alpha y_1(t)\right), \quad t \in \mathrm{J}, \\ \\ y_1(a) = \mathscr{A}, \quad {}^c\mathrm{D}_a^\alpha y_1(a) = 0, \end{cases} \tag{4.25}$$

and

$$\begin{cases} {}^c\mathrm{D}_a^\alpha\left({}^c\mathrm{D}_a^\alpha y_2\right)(t) = f_2\left(t, y_2(t), {}^c\mathrm{D}_a^\alpha y_2(t)\right), \quad t \in \mathrm{J}, \\ \\ y_2(a) = 0, \quad {}^c\mathrm{D}_a^\alpha y_2(a) = 1. \end{cases} \tag{4.26}$$

At this stage, we transform the previous problems into a α-order fractional system. In order to serve this objective, we assume

$$\begin{aligned} {}^c\mathrm{D}_a^\alpha y_1(t) &= v_1(t). \\ {}^c\mathrm{D}_a^\alpha y_2(t) &= v_2(t). \end{aligned}$$

By adopting these assumptions, the problems (4.25) and (4.26) can be reformulated as follows:

$$\begin{cases} {}^c\mathrm{D}_a^\alpha y_1(t) = v_1(t) = g_1\left(t, y_1(t), v_1(t)\right), \\ {}^c\mathrm{D}_a^\alpha v_1(t) = f_1\left(t, y_1(t), v_1(t)\right), \\ y_1(a) = \mathscr{A}, \quad v_1(a) = 0, \end{cases} \tag{4.27}$$

and

$$\begin{cases} {}^c\mathrm{D}_a^\alpha y_2(t) = v_2(t) = g_2\left(t, y_2(t), v_2(t)\right), \\ {}^c\mathrm{D}_a^\alpha v_2(t) = f_2\left(t, y_2(t), v_2(t)\right), \\ y_2(a) = 0, \quad v_2(a) = 1. \end{cases} \tag{4.28}$$

Thus, in order to solve these systems using MFEM, we split the interval $\mathrm{J} = [a, b]$ as follows: $a = t_0 < t_1 = t_0 + h < t_2 = t_0 + 2h < \cdots < t_n = t_0 + nh = b$ such that $t_j = t_0 + jh$ and $h = \frac{b-a}{n}$, for $j = 0, \ldots, n$.

For the convenience, we denote respectively $g_i(t, y_i(t), v_i(t))$ and $f_i(t, y_i(t), v_i(t))$ by g_i and f_i, for all $i = 1, 2$. Now, by apply the basic formula (4.11) of MFEM, we can obtain

$$\begin{cases} y_1(t_{j+1}) = y_1(t_j) + \frac{h^\alpha}{\Gamma(\alpha+1)} g_1\Big(t_j + \frac{h^\alpha}{2\Gamma(\alpha+1)}, y_1(t_j) + \frac{h^\alpha}{2\Gamma(\alpha+1)} g_1, v_1(t_i) + \frac{h^\alpha}{2\Gamma(\alpha+1)} f_1\Big), \\ v_1(t_{j+1}) = v_1(t_j) + \frac{h^\alpha}{\Gamma(\alpha+1)} f_1\Big(t_j + \frac{h^\alpha}{2\Gamma(\alpha+1)}, y_1(t_j) + \frac{h^\alpha}{2\Gamma(\alpha+1)} g_1, v_1(t_j) + \frac{h^\alpha}{2\Gamma(\alpha+1)} f_1\Big), \end{cases} \tag{4.29}$$

and

$$\begin{cases} y_2(t_{j+1}) = y_2(t_j) + \frac{h^\alpha}{\Gamma(\alpha+1)} g_2\Big(t_j + \frac{h^\alpha}{2\Gamma(\alpha+1)}, y_2(t_j) + \frac{h^\alpha}{2\Gamma(\alpha+1)} g_2, v_2(t_i) + \frac{h^\alpha}{2\Gamma(\alpha+1)} f_2\Big), \\ v_2(t_{j+1}) = v_2(t_j) + \frac{h^\alpha}{\Gamma(\alpha+1)} f_2\Big(t_j + \frac{h^\alpha}{2\Gamma(\alpha+1)}, y_2(t_j) + \frac{h^\alpha}{2\Gamma(\alpha+1)} g_2, v_2(t_j) + \frac{h^\alpha}{2\Gamma(\alpha+1)} f_2\Big), \end{cases} \tag{4.30}$$

Consequently, the systems (4.29) and (4.30) repesents an approximate solutions of systems (4.27) and (4.28). Hence, y_1 and y_2 are then the desired solutions of FIVPs (4.13) and (4.14).

As a result, the general solution y of the FLBVP (4.12) is given as follows:

$$y(t_j) = y_1(t_j) + \left(\frac{\mathscr{B} - y_1(b)}{y_2(b)} \right) y_2(t_j), \tag{4.31}$$

for $j = 0, \dots, n$

4 Physical Applications

In this part, two practical applications will be suggested in order to demonstrate the viability of the approach established in this chapter.

4.1 Application 1: Beam deflection

Consider a simply supported uniformly loaded beam of length L. Then, we consider the following differential problem

$$\begin{cases} y''(x) = \frac{rL}{2El}x - \frac{r}{2El}x^2, & x \in [0, L], \\ \\ y(0) = 0, \quad y(L) = 0, \end{cases} \tag{4.32}$$

which is modeled to find the deflection of the beam y, where

- r represents the intensity of the uniform load.
- E represents the modulus of elasticity .
- l is the moment of inertia.

We intend to apply the proposed method to obtain a numerical solution for the fractional version of the problem , which is given by

$$\begin{cases} {}^cD^{\alpha}\left({}^cD^{\alpha}y\right)(x) = \frac{rL}{2El}x - \frac{r}{2El}x^2, & x \in [0, L], \\ \\ y(0) = 0, \quad y(L) = 0, \end{cases} \tag{4.33}$$

where $0 < \alpha \leq 1$ and ${}^cD^{\alpha}$ is the Caputo fractional derivative.
At this point, using the given data

$$E = 200, \quad l = 30000, \quad r = 15, \quad L = 3,$$

then, the exact solution to the given problem (4.33) when $\alpha = 1$ is expressed by

$$y(x) = \frac{rLx^3}{12El} - \frac{rx^4}{24El} - \frac{rL^3x}{24El}.$$

Now, we note that the problem (4.33) is written in the form of our FLBVP problem (4.12), such that

$$p = q = 0 \quad \text{and} \quad t = x.$$

For this reason, we put

$$f\left(x, y(x), {}^cD^{\alpha}y(x)\right) = \frac{rL}{2El}x - \frac{r}{2El}x^2.$$

Then, the problem (4.33) becomes

$$\begin{cases} {}^{c}\mathrm{D}^{\alpha}\left({}^{c}\mathrm{D}^{\alpha} y\right)(x) = f\left(x, y(x), {}^{c}\mathrm{D}^{\alpha} y(x)\right), \quad x \in [0, \mathrm{L}], \\ \\ y(0) = 0, \quad y(\mathrm{L}) = 0, \end{cases} \tag{4.34}$$

However, to deal with this problem, we will follow the procedures described in Section 3. This will produce Figure 4.1 which include numerical comparison between the exact solution and the numerical solution obtained by the proposed method under different values of α.

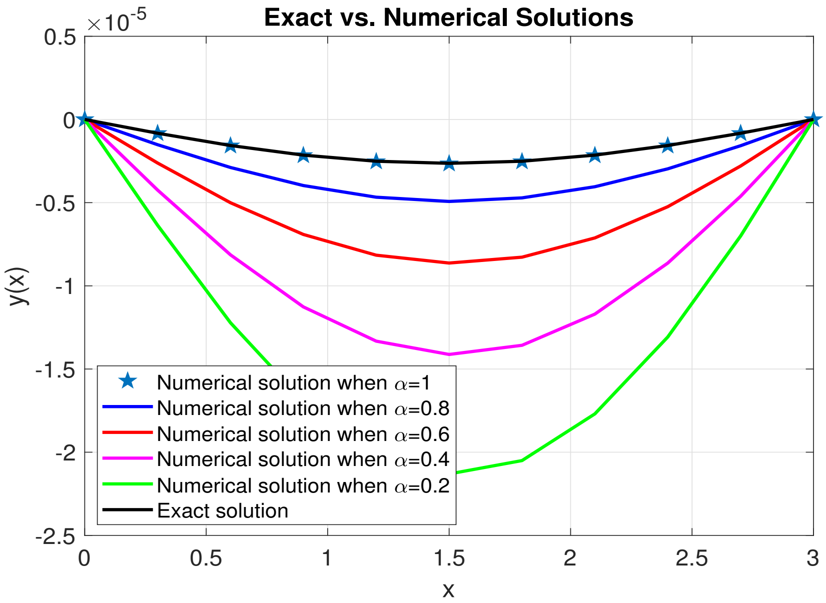

Figure 4.1 – Exact solution Vs Numerical solutions according to different values of α.

Then, by considering α = 1 in the problem (4.33), we plot in Figure 4.2 the numerical solution obtained by combined methods (FLSM-MFEM) and compared with the exact solution.

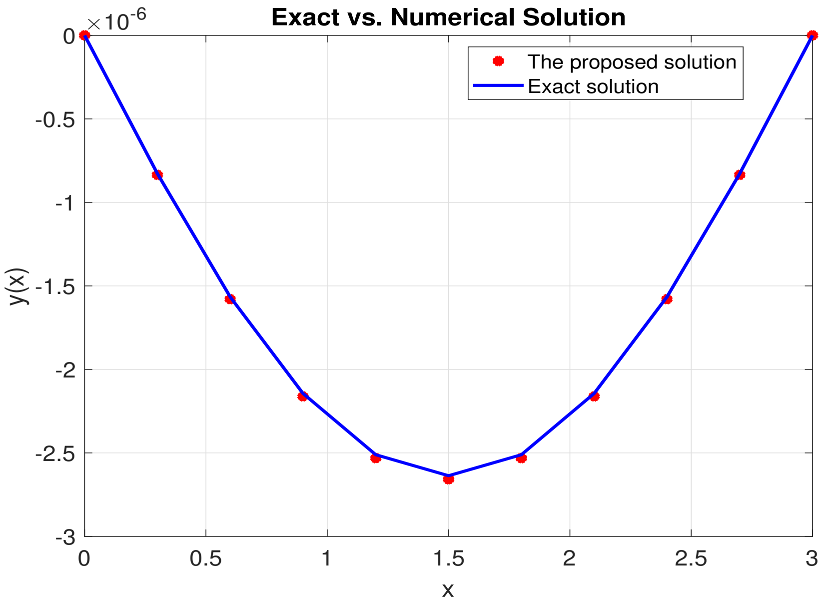

Figure 4.2 – Exact solution Vs Numerical solution according to $\alpha = 1$.

After that, we plot in Figure 4.3 the absolute error between the approximated solution and the exact solution for the problem (4.33) in the case of $\alpha = 1$.

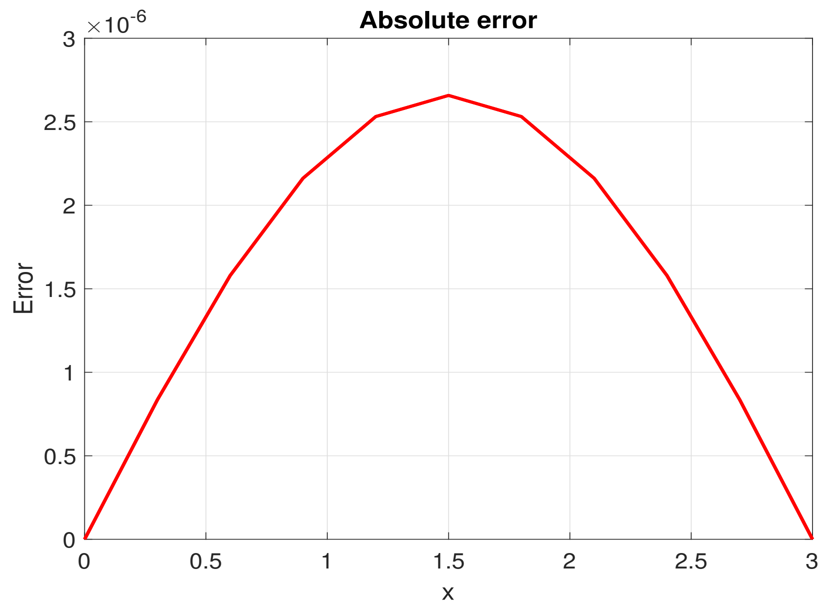

Figure 4.3 – Representation of absolute error between the exact and numerical solution of problem (4.33).

Finally, we conclude that the accuracy of the suggested approach is readily apparent when compared to the previous numerical results.

4.2 Application 2: Heat balance for a long, thin rod

We consider a long, thin rod. Then, we focus about the conversation of heat that can be used to develop a heat balance for this rod. The equation that arises if the system is in a steady state and the rod is not insulated along its length, is represented as follows

$$T''(x) + h(T_a - T(x)) = 0, \quad x \in [0, L], \tag{4.35}$$

where

- h is a heat transfer coefficient.
- T_a represents the temperature of the ambient air.

We solve the equation (4.35) in the simple case where the temperatures at the ends of the bar are at constant values. These conditions have been mathematically reformulated by

$$T(0) = \mathscr{A}, \quad T(L) = \mathscr{B}, \tag{4.36}$$

such that $\mathscr{A}, \mathscr{B} > 0$.
Similary as the first application, we aim to propose a numerical solution for the fractional formulation of the previous problem, by applying the suggested numerical method. So, we consider the following probelm

$$\begin{cases} {}^cD^\alpha \left({}^cD^\alpha T \right)(x) + h(T_a - T(x)) = 0, \quad x \in [0, L], \\ \\ T(0) = \mathscr{A}, \quad T(L) = \mathscr{B}, \end{cases} \tag{4.37}$$

where $0 < \alpha \leq 1$ and ${}^cD^\alpha$ is the Caputo fractional derivative.
We note that the exact solution of the problem (4.37) where $\alpha = 1$ and

$$T_a = 20, \quad \mathscr{A} = 40, \quad \mathscr{B} = 200, \quad h = 0.01, \quad L = 10,$$

is as follows:

$$T(x) = 73,45e^{\frac{1}{10}x} - 53,45e^{-\frac{1}{10}x} + 20.$$

At this point, it is important to highlight that the problem (4.37) is expressed in the form of our main FLBVP problem (4.12), such that

$$p = 0 \quad \text{and} \quad t = x.$$

In order to achieve this result, we take

$$f\left(x, T(x), {}^cD^\alpha T(x)\right) = -h(T_a - T(x)),$$

then, the problem (4.37) takes the form

$$\begin{cases} {}^cD^\alpha \left({}^cD^\alpha T \right)(x) = f\left(x, T(x), {}^cD^\alpha T(x)\right), \quad x \in [0, L], \\ \\ T(0) = \mathscr{A}, \quad T(L) = \mathscr{B}, \end{cases} \tag{4.38}$$

We have seen through Section 3 that in order to approximate the solution of (4.38), we must convert it into two FIVPs by FLSM, then apply MFEM to solve the problems obtained, and finally derive the numerical solution of the problem (4.38) as a linear combination.

As a result, Figure 4.4 will be produced to illustrat a numerical comparison according to different values of the fractional order α between the exact solution and the numerical solution generated by the suggested method.

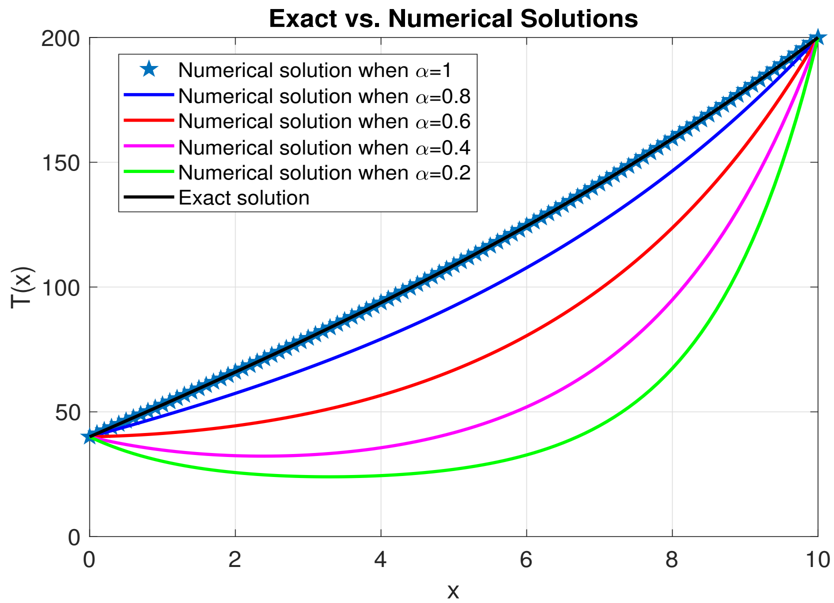

Figure 4.4 – A graphical comparison between the exact solution and the numerical solution for different values of α.

Then, for $\alpha = 1$ in (4.37), the numerical solution obtained and the exact solution are ploted in Figure 4.5.

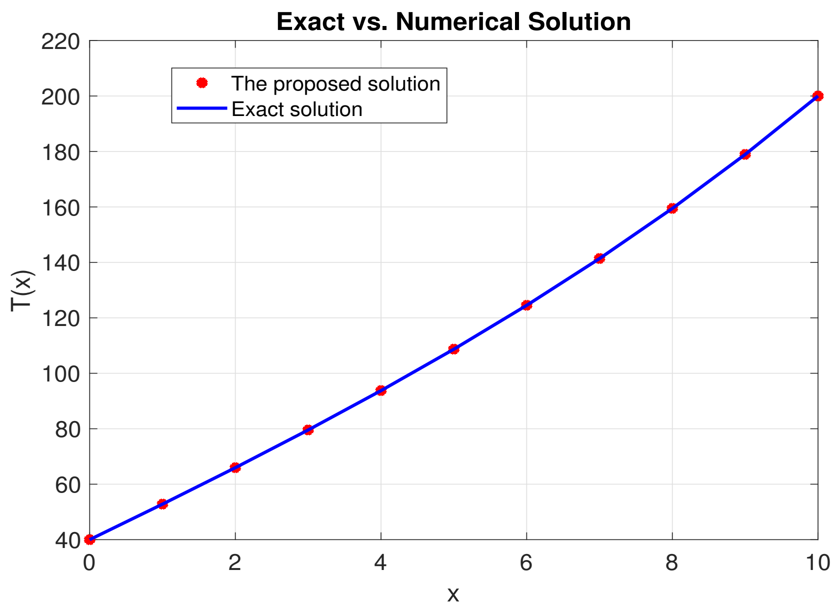

Figure 4.5 – A graphical comparison: Exact solution Vs Numerical solution for $\alpha = 1$.

Finally, in order to demonstrate the effectiveness of the current method, we plot in Figure 4.6 the absolute error between the approximated solution and the exact solution for $\alpha = 1$.

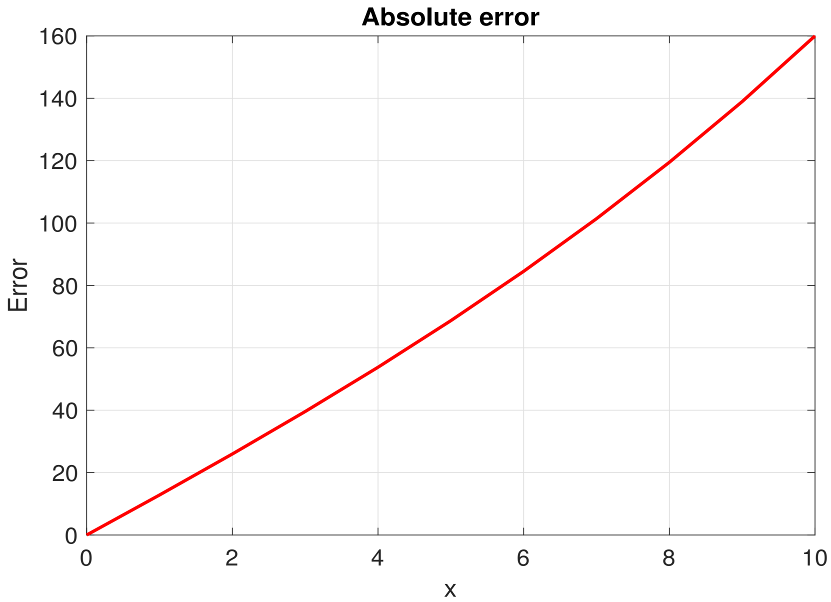

Figure 4.6 – A graphical representation for the absolute error between the exact and numerical solution of problem (4.37).

5 Conclusion

In this chapter, we have solved a class of two-sequential fractional linear differential equations with boundary conditions. The fractional linear shooting method (FLSM) and the Modified fractional euler method (MFEM) are coupled to provide the approximate solution of the problem under consideration. The accuracy of the proposed method is confirmed by the numerical results of the physical applications. To the best of the author's knowledge, this combination is a novel contribution. Some of our results from this study have been submitted for publication [3].

General Conclusion & Perspectives

In this thesis, we have presented some original results concerning both analytical and numerical solutions for some fractional boundary value problems.

To achieve the desired objectives, we have included a preliminary chapter that covers essential tools and fundamental concepts of fractional calculus, which are pertinent to our research. After that, the detailed contributions are expounded in the remaining three chapters.

In chapter 2, we have suggested a nonlinear sequential fractional problem related to the Van Der Pol-Duffing Jerk equation, where we practically studied the existence, uniqueness and stability of the solutions through the Caputo–Hadamard approach within a Banach space. This investigation employs Banach contraction principle, Krasnoselskii theorem and the Ulam-stability theorems.

In chapter 3, we have proposed a class of sequential fractional problem linked to the pantograph equation type. The existence and uniqueness througth Banach contraction principle has been analyzied. Then, the existence of at least one solution using Leray-Schauder alternative has been addressed. Finally, a different types of Ulam-stability has been established.

In chapter 4, we have proposed a combined approach of two numerical methods to deal with a class of fractional linear boundary value problems. These methods are the fractional linear shooting method (FLSM) and the modified fractional euler method (MFEM). Then, the accuracy and efficiency of the proposed scheme have been provided by performing several numerical comparisons via several illustrative applications.

As perspectives of this thesis, we propose to:

- Study a fractional systems of sequential Van Der Pol-Duffing (VDPD) Jerk equations.
- Examine the stability of the solutions by another type.
- Find more accurate and comprehensive numerical methods to solve fractional boundary value problems.

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Analytical and Numerical Study of Certain Fractional Boundary Problems

Abstract : The main objective of this thesis is to present an analytical and numerical contribution of certain fractional boundary problems according to different approaches. Original results ensuring the existence and uniqueness/existence as well as stability of solutions are discussed for some new problems involving fractional order operators. In addition, an approach for solving a type of fractional linear problems with boundary conditions is developed and some applications are presented, where the validity and accuracy of this scheme are shown.

The analytical results of this thesis focus on the application of some fixed point theorems and certain types of Ulam stability to address two proposed fractional problems. The first problem concerns the Van de Pol Duffing (VDPD)-Jerk oscillator, while the second one involves the pantograph type equation, utilizing the Caputo-Hadamard approach. Illustrative examples will be provided to demonstrate the validity of the results.

We devote a final part of our project to numerical results, where an approach is developed to approximate the solutions of a class of fractional linear boundary value problems and some applications are presented in this context.

Key Words : Fractional boundary problem, Van de Pol Duffing (VDPD)-Jerk equation, pantograph equation, Hadamard integral, Caputo-Hadamard derivative, fixed point theorems, existence, uniqueness, Ulam type stability, Modified fractional Euler method.

MSC (2020) : 26A33, 34A08, 65R99.

Etude Analytique et Numérique de Certains Problèmes aux Limites Fractionnaires

Résumé : L'objectif principal de cette thèse est de présenter une contribution analytique et numérique de certaines problèmes aux limites fractionnaires selon différentes approches. Des résultats originaux assurant l'existence et l'unicité/existence ainsi que la stabilité des solutions sont discutés pour de nouveaux problèmes impliquant des opérateurs d'ordre fractionnaire. De plus, une approche pour résoudre une classe de problèmes linéaires fractionnaires avec des conditions aux limites est développée et certaines applications sont présentées, où la validité et la précision de ce schéma sont démontrées.

Les résultats analytiques de cette thèse se concentrent sur l'application de quelques théorèmes de point fixe et de certains types de stabilité au sens d'Ulam pour aborder deux problèmes fractionnaires proposés. Le premier problème concerne l'oscillateur Van de Pol Duffing (VDPD)-Jerk, tandis que le deuxième implique l'équation de type pantographe, utilisant l'approche de Caputo-Hadamard. Des exemples illustratifs seront fournis pour démontrer la validité des résultats.

Nous consacrons une dernière partie de notre projet aux résultats numériques, où une approche est développée pour approximer les solutions d'une classe de problèmes aux limites linéaires fractionnaires et certaines applications sont présentées dans ce contexte.

Mots-Clés : Problème aux limites fractionnaires, équation de Van de Pol Duffing (VDPD)-Jerk, équation du pantographe, intégrale de Hadamard, dérivée de Caputo-Hadamard, théorèmes de point fixe, existence, unicité, stabilité de type Ulam, méthode d'Euler fractionnaire modifiée.

MSC (2020) : 26A33, 34A08, 65R99.

دراسة تحليلية وعددية لبعض المسائل الكسرية ذات شروط الحدية

الملخص: الهدف الرئيسي من هذه الأطروحة هو تقديم مساهمة تحليلية وعددية لبعض المسائل الكسرية ذات شروط الحدية وفقا لمناهج مختلفة. يتم مناقشة النتائج الأصلية التي تضمن (وجود الحلول وتفردها / وجودها و استقرارها) لبعض المشاكل الجديدة التي تتضمن عوامل تشغيل كسرية. بالإضافة إلى ذلك، تم تطوير أسلوب لحل نوع من المسائل الخطية الكسرية ذات الشروط الحدية وتم عرض بعض التطبيقات، حيث تم توضيح صحة ودقة هذا المخطط

تركز النتائج التحليلية لهذه الأطروحة على تطبيق بعض نظريات النقطة الثابتة وأنواع معينة من استقرار Ulam لمعالجة نموذجين كسريين مقترحين. يتعلق المشكل الأول بمذبذب Van Der Pol-Duffing Jerk ، في حين يتضمن المشكل الثاني معادلة Pantograph، وذلك باستخدام نهج Caputo-Hadamard، وسيتم تقديم أمثلة توضيحية لإثبات صحة النتائج.

لقد خصصنا الجزء الأخير من مشروعنا للنتائج العددية، حيث تم تطوير نهج لتقريب الحلول لفئة من المسائل الخطية الكسرية ذات الشروط الحدية ويتم تقديم بعض التطبيقات في هذا السياق

الكلمات المفتاحية: مسألة كسرية ذات شروط حدية، معادلة Van Der Pol-Duffing Jerk (VDPD) ، معادلة Pantograph ، تكامل Hadamard ، مشتق Caputo-Hadamard، نظريات النقطة الثابتة، الوجود، التفرد، استقرار Ulam، طريقة Euler الكسرية المعدلة.

تصنيف المواضيع الرياضية (2020) : 26A33، 34A08 ، 65R99.